LINEARITY OF GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

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ABSTRACT. We consider an arbitrary o-minimal structure $\mathcal{M}$ and a definably connected definable group G. The main theorem provides definable real closed fields $R_1, \ldots, R_k$ such that $G/Z(G)$ is definably isomorphic to a direct product of definable subgroups of $\mathrm{GL}_{n_1}(R_1), \ldots, \mathrm{GL}_{n_k}(R_k)$, where $Z(G)$ denotes the center of G. It follows from this result a Levi decomposition for G, and that $[G,G]Z(G)/Z(G)$ is definable and definably isomorphic to a direct product of *semialgebraic* linear groups over $R_1, \ldots, R_k$.

1. INTRODUCTION

By a result of Otero, Peterzil and Pillay [20], it is well-known that, if a group G is definable in an o-minimal expansion $\mathscr{R}$ of a real closed field $(R, <, +, \cdot, \cdots)$, and if G is definably connected, then $G/Z(G)$ is definably isomorphic to a definable subgroup of $\mathrm{GL}_n(R)$ for an integer n (Fact 5.1).

For groups definable in arbitrary o-minimal structures, a similar result was proven by Peterzil, Pillay and Starchenko [21] for *centerless* definably connected definable groups: such a group is definably isomorphic to a direct product of definable subgroups of $\mathrm{GL}_{n_1}(R_1), \ldots \mathrm{GL}_{n_k}(R_k)$ for definable real closed fields $R_1, \ldots, R_k$ and integers $n_1, \ldots, n_k$ (Fact 2.12).

In this paper, we work inside a fixed arbitrary o-minimal structure $\mathcal{M} = (M, <, \cdots)$*, and* definable *means* $\mathcal{M}$*-definable (with parameters).* Our aim is to unify the previous two theorems, which will be used extensively, and as a result, we prove the following.

Main Theorem 5.17. – *Let G be a definably connected definable group. Then $G/Z(G)$ is the direct product of definable groups $\overline{H_1}, \ldots, \overline{H_k}$ such that for every $i \in \{1, \ldots, k\}$ there are a definable real closed field R_i, an integer n_i and a definable isomorphism between $\overline{H_i}$ and a definable subgroup of* $\mathrm{GL}_{n_i}(R_i)$.

Moreover, we show that G is the central product of $H_1, \ldots, H_k$ (Corollary 5.18).

The proof of Theorem 5.17 uses numerous results on groups definable in o-minimal structures. The main new ingredients come from the theory of groups of finite Morley rank, namely *pseudo-tori*, *U_R-groups* for a definable real closed field R, and *U-groups* (Definitions 3.1, 3.9 and 3.23). Thanks to these notions, we can provide two results on the structure of definable groups, which are important for the main result.

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Theorem 3.29. – *Any nilpotent definably connected definable group G is the central product of a pseudo-tori by a U-group.*

We recall that a group is said to be *definably simple* if it has no proper non-trivial normal definable subgroup.

Theorem 4.10. – *Let G be a definably connected definable group. Then G has a normal U-group U such that G/U is a central extension of a direct product of definably simple definable groups.*

We note that, in the Main Theorem, the subgroups $\overline{H_i}$ of $\mathrm{GL}_{n_i}(R_i)$ are not necessarily semialgebraic. However, by using the analysis of linear groups in [22], we may obtain a structure result, closer to the semialgebraicity.

Corollary 5.19. – *Let G be a definably connected definable group. Then the group $G'Z(G)/Z(G)$ is a direct product of definably connected definable groups $\overline{H_1}, \ldots, \overline{H_k}$ such that for every $i \in \{1, \ldots, k\}$ there is a definable real closed field R_i and a definable isomorphism between $\overline{H_i}$ and a semialgebraic linear group over R_i.*

We show that, thanks to our main result, we may generalize the Levi decomposition, obtained by Conversano and Pillay [6] for groups definable in an o-minimal expansion of a real closed field, to groups definable in an arbitrary o-minimal structure.

For this subject, there is a problem with *semisimple groups.* Indeed, a semisimple group is defined to be a definably connected definable group with no infinite abelian normal subgroup (Definition 6.1). However, Conversano exhibited a definably connected definable group G with no semisimple subgroup S such that $G = RS$ for a normal solvable subgroup R. In order to remedy to this problem, Conversano and Pillay introduced in [6] *ind-definable semisimple subgroups*, and they provide the Levi decomposition with these subgroups (Fact 6.3).

In this paper, we introduce *quasi-semisimple groups* as definably connected definable groups with no decomposition of the form RH for a normal definable solvable subgroup R and a proper definable subgroup H (Definition 6.1). For such a group S, the derived subgroup is perfect and $S/Z(S)$ is semisimple. Then we provide a Levi decomposition for any definably connected definable group (Theorem 6.6 below). Furthermore, we show that if G is any definably connected group definable in an o-minimal expansion of a real closed field, its maximal ind-definable semisimple subgroups are precisely the derived subgroups of its maximal quasi-semisimple subgroups (Corollary 6.7).

Theorem 6.6. – *Let G be a definably connected definable group. Then G has a maximal quasi-semisimple subgroup S, unique up to conjugacy in G. Moreover, there is a normal solvable definable subgroup R such that $G = RS$ and $G \cap S \leq S$.*

The organization of this paper is as follows. In §2, we recall known results and give some useful corollaries. The purpose of §3 is the analysis of nilpotent groups (Theorem 3.29). In particular, we introduce *pseudo-tori*, *U_R-groups* and *U-groups*, which are fundamental notions for this paper. In §4, we study the group actions on a solvable group, and then we obtain a structure theorem for any definably connected definable group (Theorem 4.10). In §5, we prove the main result of this paper (Theorem 5.17). In §6, we apply the main result to Levi decomposition (Theorem 6.6).

2. Preliminaries

The basic reference for o-minimal structures is [27] (see [19] for a survey on groups definable in an o-minimal structure).

By [12], in an arbitrary o-minimal structure, every interpretable group is definably isomorphic to a definable one. Actually, any group definable in an o-minimal structure eliminates imaginaries. More precisely, the following result is due to M. Edmundo.

Fact 2.1.. – [11, Theorem 7.2] *Let G be a definable group, and let $\{T(x) : x \in X\}$ be a definable family of non-empty definable subsets of G. Then there is a definable function $t : X \to G$ such that for all $x, y \in X$ we have $t(x) \in T(x)$ and if $T(x) = T(y)$ then $t(x) = t(y)$.*

2.1. **Nilpotent definable groups.** We recall two general facts on nilpotent groups definable in an o-minimal structure, and more generally. Any group definable in an o-minimal structure $\mathscr{N}$ satisfies the descending chain condition on $\mathscr{N}$-definable subgroups [25, Remark 2.13 (ii)]. In particular, it is an *$\mathcal{M}_c$-group*, that is a group with descending chain condition on centralizers. Thus, the following two facts are satisfied by nilpotent groups definable in an o-minimal structure.

For every group G, we denote by $Z_0(G) = 1$ the trivial group, and we define $Z_i(G)$ for each integer i by $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$.

Fact 2.2.. – [13, Lemma 3.7.10] *Any infinite nilpotent $\mathcal{M}_c$-group has infinite center.*

More generally, if a group G has a finite subset X such that $Z(G) = C_G(X)$, and if H is a normal subgroup such that $H \cap Z_k(G)$ is infinite for an integer k, then $Z(G) \cap H$ is infinite.

Proof – There is a smallest integer j such that $B = Z_{j+1}(G) \cap H$ is infinite. Then $[g, B]$ is contained the finite subgroup $Z_j(G) \cap H$ for each $g \in G$, and the index of $C_B(g)$ in G is finite. Thus $B/C_B(X)$ is finite, and $Z(G) \cap B$ has finite index in B, so $Z(G) \cap H = Z(G) \cap B$ is infinite. □

Fact 2.3.. – (see [2, Lemma 6.3] for a special case) *Let G be a nilpotent group. We consider an expansion $\mathscr{G} = (G, \cdot, \cdots)$ of the group G such that G has the descending chain condition on its $\mathscr{G}$-definable subgroups. If H is a $\mathscr{G}$-definable subgroup of G of infinite index, the index of H in $N_G(H)$ is infinite too.*

Proof – We consider the normal subgroup $K = \cap_{g\in G} H^g$ of G and, for each $g \in G$, the subgroup $C_g = \{x \in G \mid [g, x] \in K\}$. By descending chain condition on $\mathscr{G}$-definable subgroups, K and $Z = \cap_{g\in G} C_g$ are $\mathscr{G}$-definable, and there is a finite subset X of G such that $Z = \cap_{g\in X} C_g$. We note that $Z/K = Z(G/K)$ is the center of G/K, so by Fact 2.2, it is infinite. Since it normalizes H/K, the subgroup Z normalizes H. Moreover, since $Z \cap H \leq Z$ contains K, the subgroup $Z \cap H$ is normal in G, and since it is contained in H, we have $Z \cap H = K$. Thus the index of H in $N_G(H) \geq Z$ is infinite. □

2.2. **Connected component.** For every definable group G, we denote by G° the *definably connected component* (of the identity) in G. It is the smallest definable subgroup of G of finite index in G [25, Proposition 2.12]. A group G is said to be *definably connected* if $G = G^\circ$.

In this section, we show that for any definable group G, this subgroup G° is definable in the pure group $(G, \cdot)$, and has no proper subgroup of finite index:

every subgroup of finite index is definable and G° is the smallest subgroup of finite index (Proposition 2.9 below). In particular, the definably connected component of a definable group is independant from the language.

The proof of Proposition 2.9 requires several facts.

It follows from [25] that any definable group G has largest definably connected definable solvable normal subgroup $R(G)$, called the *solvable radical* of G. However, another definition of solvable radical is used in [1].

Fact 2.4.. – [1, Lemma 4.5] *Let G be a definable group. The subgroup generated by all normal solvable subgroups of G is definable and solvable.*

Moreover, by [7], any definable group G has a largest nilpotent normal subgroup $F(G)$, and this subgroup is definable by [11, Lemma 6.7].

For each group G, we denote by $G' = [G, G]$ the derived subgroup. We recall that, for a definable group G, this subgroup is not necessarily definable (Conversano exhibits a counter-example [5, Example 3.1.7]). However, Baro, Jaligot and Otero [1] show its definability for a large class of definable groups.

The derived subgroup of a solvable definably connected definable group has been studied in [11, Theorem 6.9], and a precision is given in [1, Proposition 5.5].

Fact 2.5.. – *Let G be a solvable definably connected definable group. Then the following two conditions are satisfied:*

- [11, Theorem 6.9] *its derived subgroup G' is contained in $F(G)$;*
- [1, Proposition 5.5] *the group $G/F(G)^\circ$ is abelian and divisible.*

Fact 2.6 describes the structure of nilpotent groups, where a group G is the *central product* of two subgroups H and K if $G = HK$ and $[h, k] = 1$ for each $(h, k) \in H \times K$. We denote this by $G = H * K$.

Fact 2.6.. – *Let G be a nilpotent definable group.*

(1) [11, Theorem 6.10] and [1, Lemma 3.10 (c)] *G° is divisible and G has a finite characteristic subgroup F such that $G = G^\circ * F$.*
(2) [28, Theorem 4.12 (Chernikov)] and [8, Corollary 1.5.12] *The torsion subgroup of G° is central in G.*

Corollary 2.7.. – *Let G be a nilpotent definably connected definable group. Then any definable subgroup of G containing $Z(G)$ is definably connected.*

Proof – Since $Z(G)$ contains the torsion part of G by Fact 2.6 (2), the group $G/Z(G)$ is torsion-free. In particular, each definable subgroup of $G/Z(G)$ is definably connected. But G is divisible by Fact 2.6 (1), so the torsion part of $Z(G)$ is divisible, and Fact 2.6 (1) applied with $Z(G)$ shows that $Z(G)$ is definably connected. Hence, for any definable subgroup $H/Z(G)$ of $G/Z(G)$, the subgroup $Z(G)$ is contained in H°, and we have $H = H^\circ Z(G) = H^\circ$. □

The following result is a very important theorem for groups definable in an o-minimal structure. It is used in the proof of Proposition 2.9 below.

Fact 2.8.. – [21, Theorem 4.1] *Let G be an infinite $(G, \cdot)$-definably connected definable group. Assume G has no nontrivial abelian normal subgroup. Then G is the direct product of $(G, \cdot)$-definable subgroups $H_1, \ldots, H_k$ such that for every $i \in \{1, \ldots, k\}$ there is a definable real closed field R_i and a definable isomorphism between H_i and a semialgebraic subgroup of $\mathrm{GL}_{n_i}(R_i)$. Moreover, H_i is $(H_i, \cdot)$-definably simple and H_i° is definably simple.*

Proposition 2.9.. – *Let G be definable group. Then G° is definable in the pure group $(G,\cdot)$. Moreover, G° has no proper subgroup of finite index.*

Proof – Every $(G,\cdot)$-definable subgroup of G is definable, so G has a smallest $(G,\cdot)$-definable subgroup of finite index. In particular, this subgroup is definable, contains G° as a subgroup of finite index, and it has no proper $(G,\cdot)$-definable subgroup of finite index. So we may assume that G is $(G,\cdot)$-definably connected, and we have just to prove that G has no proper subgroup of finite index.

Let N be a subgroup of G of finite index n. We show that $N = G$. We may assume that N is contained in G°. Moreover, since the index of N in G is finite, N contains a G-normal subgroup of finite index, and we may assume that N is normal in G. Let $X = \{x^n \mid x \in G\}$. In particular, X is a definable subset of N.

We show that XX contains $R(G)$. By Fact 2.6 (1), the definable subgroup $F(G)^\circ$ is divisible, so it is contained in X. By Fact 2.5, the quotient group $R(G)/F(G)^\circ$ is divisible, so $XF(G)^\circ \subseteq XX$ contains $R(G)$.

Let Z be the subgroup of G° generated by all its normal solvable subgroups. It is definable and solvable (Fact 2.4), so $Z/R(G)$ is finite. In particular, G°/Z has no non-trivial abelian normal subgroup, and Fact 2.8 implies that G°/Z is the direct product of definable subgroups $H_1/Z,\dots,H_k/Z$ such that for every $i \in \{1,\dots,k\}$ there is a definable real closed field R_i and a definable isomorphism f_i between H_i/Z and a semialgebraic subgroup L_i of $\mathrm{GL}_{n_i}(R_i)$, and such that $H_i^\circ Z/Z$ is definably simple. Moreover, since G°/Z is definably connected, H_i/Z is definably connected for each i, so $L_i \simeq H_i/Z$ is definably simple.

We show that N contains a $(G,\cdot)$-definable set containing $H_i \cap N$ for each i. Let $i \in \{1,\dots,k\}$. Since L_i is a definably simple semialgebraic subgroup of $\mathrm{GL}_{n_i}(R_i)$, each element $g \in L_i$ is a product of a semisimple element $s \in L_i$ and a unipotent element $u \in L_i$. Then s is contained in a maximal torus T_i of L_i and u is contained in a unipotent subgroup U_i of L_i. In particular, T_i and U_i are divisible, so $f_i^{-1}(s)$ and $f_i^{-1}(u)$ are contained in XZ/Z, and $f_i^{-1}(g)$ belongs to XXZ/Z. Consequently H_i is contained in XXZ. Since $XX \subseteq N$ contains $R(G)$, there is a finite subset W of $Z \cap N$ such that $Z \cap N = WR(G)$. Therefore $H_i \cap N$ is contained in

$$XXZ \cap N = XX(Z \cap N) = XXWR(G) \subseteq XXWXX$$

Thus $M = (H_1 \cap N)\cdots(H_k \cap N)$ is a subgroup of finite index in N contained in $XXWXX$. We consider a finite subset E of N such that $N = EM$, and we obtain $N = EXXWXX$. So N is $(G,\cdot)$-definable, and $N = G$. □

2.3. **Definable fields.** The following fundamental results are due to Pillay, and Peterzil and Steinhorn respectively. They are crucial for us.

Fact 2.10.. – [25, Theorem 3.9 and Proposition 3.11] *Let K be an infinite definable field. Then K is real closed or algebraically closed. It is real closed if and only if its dimension is 1.*

Fact 2.11.. – [24, Theorem 4.1] *Let $\mathscr{K} = (K,+,0,\cdot)$ be an infinite definable ring without zero divisors. Then K is a division ring and there is a one-dimensional definable subring R of K which is a real closed field such that K is either R, $R(\sqrt{-1})$, or the ring of quaternions over R.*

Lemma 2.14 and Proposition 2.15 are very useful for this paper. The proof of Proposition 2.15 is based on the following very important fact (Fact 2.12), and on the study of abelian definable subgroups of the general linear group over a definable real closed field (Fact 2.13). Moreover, we note that the proof of Fact 2.12 is based on the *theory of nonorthogonality* from [21].

Fact 2.12.. – [21, Theorems 3.1 and 3.2] *Let G be a definably connected definable centerless group. Then G is definably isomorphic to a direct product $H_1 \times \cdots \times H_k$, where, for each $i = 1, \ldots, k$, there is a definable real closed field R_i such that H_i is a definable subgroup of $\mathrm{GL}_{n_i}(R_i)$ for each $i = 1, \ldots, k$.*

Fact 2.13.. – *Let G be a commutative definably connected definable subgroup of $\mathrm{GL}_n(R)$ where R is a definable real closed field. Then the following three conditions hold:*

(1) [22, Fact 3.1] *if G is semialgebraic, it is semialgebraically isomorphic to a group of the form $\mathrm{SO}_2(R)^m \times (R^*_{>0})^l \times (R_+)^k$;* [22, Lemma 3.9] *every definably connected definable subgroup H of G has a definable complement in G;*
(2) [22, Special case of Proposition 3.10] *G is definably isomorphic to a linear semialgebraic group over R.*

Lemma 2.14.. – *Let R and S be two definable real closed fields. If the groups R_+ and S_+ are definably isomorphic, then the fields R and S are definably isomorphic.*

Proof – Let $f : R_+ \to S_+$ be a definable isomorphism. In particular $f(1)$ is non-zero and we may consider the map $g : R \to S$ defined by $g(x) = f(x)f(1)^{-1}$. Then g is a definable isomorphism from R_+ and S_+ such that $g(1) = 1$.

Now, for each $\alpha \in R$, the subset $A_\alpha = \{x \in R \mid g(x\alpha) = g(x)g(\alpha)\}$ of R is a definable subgroup of R_+ containing 1. So we obtain $A_\alpha = R$ for each $\alpha \in R$ and g is a field isomorphism. □

Proposition 2.15.. – *Let $\mathscr{R} = (R, +, \cdot)$ and $\mathscr{S} = (S, \oplus, *)$ be two definable real closed fields. If there is an infinite definable $\mathscr{R}$-linear group H definably isomorphic to an $\mathscr{S}$-linear group, then the fields $\mathscr{R}$ and $\mathscr{S}$ are definably isomorphic.*

Proof – We may assume that H has no proper infinite definable subgroup. Then H is definably isomorphic either to $\mathrm{SO}_2(R)$ or to $R^*_{>0}$ or to R_+ (Fact 2.13 (1) and (3)). By the same way and by Lemma 2.14, we may assume that H is definably isomorphic either to $\mathrm{SO}_2(S)$ or to $S^*_{>0}$.

If H is definably isomorphic to $\mathrm{SO}_2(R)$, then it has torsion and it is definably isomorphic to $\mathrm{SO}_2(S)$. We consider the semi-direct product $G = (R^2_+ \times S^2_+) \rtimes H$ where $H \simeq \mathrm{SO}_2(R)$ acts $\mathscr{R}$-linearly on R^2_+ and such that $H \simeq \mathrm{SO}_2(S)$ acts $\mathscr{S}$-linearly on S^2_+. In particular, G is centerless and it has no decomposition $H = A \times B$ as a direct product of two proper subgroups. By Fact 2.12, there is a definable real closed field $\mathscr{T} = (T, \cdots)$ and a definably linear group $K \leq \mathrm{GL}_n(T)$ definably isomorphic to G. But $K \simeq G$ is definably connected and 2-solvable, so its derived subgroup $K' \simeq G'$ is definably isomorphic to T^m_+ for an integer m and to $G' \simeq R^2_+ \times S^2_+$. This implies that the groups R_+, S_+ and T_+ are definably isomorphic, so the fields $\mathscr{R}$ and $\mathscr{S}$ are definably isomorphic by Lemma 2.14.

Hence we may assume that H is torsion-free, so it is definably isomorphic either to $R^*_{>0}$ or to R_+, and to $S^*_{>0}$. If it is definably isomorphic to $R^*_{>0}$, we consider the semi-direct product $G = (R_+ \times S_+) \rtimes H$ where $H \simeq R^*_{>0}$ acts $\mathscr{R}$-linearly on R_+ and such that $H \simeq S^*_{>0}$ acts $\mathscr{S}$-linearly on S_+. As in the previous paragraph, Fact 2.12

provides a real closed field $\mathscr{T} = (T, \cdots)$ and a definably linear group $K \leq \mathrm{GL}_n(T)$ definably isomorphic to G, and we conclude that the groups R_+, S_+ and T_+ are definably isomorphic, so the fields $\mathscr{R}$ and $\mathscr{S}$ are definably isomorphic by Lemma 2.14.

Thus we may assume that H, R_+, and $S^*_{>0}$ are definably isomorphic. The group $H \times R^*_{>0} \simeq R_+ \times R^*_{>0}$ acts $\mathscr{R}$-linearly on $R_+ \times R_+$ where the action is defined by $(a,t) \cdot (x,y) = (tx, atx + ty)$. We consider the semi-direct product $G = ((R_+ \times R_+) \times S_+) \rtimes (H \times R^*_{>0})$ where $H \times R^*_{>0}$ acts as above on $R_+ \times R_+$, the group $R^*_{>0}$ acts trivially on S_+, and $H \simeq S^*_{>0}$ acts $\mathscr{S}$-linearly on S_+. Again G is centerless and has no decomposition as a direct product of two proper subgroups, so Fact 2.12 provides a real closed field $\mathscr{T} = (T, \cdots)$ and a definably linear group $K \leq \mathrm{GL}_n(T)$ definably isomorphic to G. As above we conclude that the groups R_+, S_+ and T_+ are definably isomorphic, so the fields $\mathscr{R}$ and $\mathscr{S}$ are definably isomorphic by Lemma 2.14. □

2.4. **The structure of solvable groups by Edmundo.** Edmundo gives in [11] a precise description of the structure of solvable groups. His main result, namely Fact 2.24, is very useful for a key result of the analysis of nilpotent groups (Proposition 3.22). Before stating it, we specify the terminology.

In [24], Peterzil and Steinhorn introduced the notion of *definable compactness* in o-minimal structures.

Definition 2.16.. – *Let G be a definable group. We say that G is* definably compact *if for every definable continuous embedding $\sigma : (a,b) \subseteq M \to G$, where $-\infty \leq a < b \leq +\infty$, there are $c, d \in G$ such that $\lim_{x \to a^+} \sigma(x) = c$ and $\lim_{x \to b^-} \sigma(x) = d$, where the limits are taken with respect to the topology on G.*

We recall that a *semisimple group* is defined to be a definably connected definable group with no infinite abelian normal subgroup (Definition 6.1).

Fact 2.17.. – [11, Corollary 4.8] (see also [23, Corollary 5.4]) *Let G be a definably connected definably compact definable group. Then G is either abelian or $G/Z(G)$ is definably semisimple definable group. In particular, if G is abelian then it is abelian.*

Fact 2.18.. – [11, Lemma 3.14] *Let A be a normal definable subgroup of a definable group U. Then U is definably compact if and only if A and U/A are definably compact.*

Fact 2.19.. – [24, Proof of Theorem 4.1] (see also Fact 2.11) *Let $\mathscr{K} = (K, +, 0, \cdot)$ be an infinite definable ring without zero divisors. Then K is not definably compact.*

Miller and Starchenko introduced *linearly bounded groups* in [18], and Edmundo has investigated *semi-bounded groups* in [10] (see also the introduction of [11]), a special case of linearly bounded groups. We do not give the definitions, just their main properties.

First at all, we recall that any o-minimal ordered groups is abelian by [26, Theorem 2.1].

Fact 2.20.. – [26, Theorem 2.1 and Lemma 2.2] *Let $(G, <, \cdot)$ be an o-minimal ordered group. Then the group G is abelian and divisible, and it has no non-trivial proper definable subgroup.*

Fact 2.21.. – [18, Theorem A (Growth Dichotomy)] *Let $\mathscr{R}$ be an o-minimal expansion of an ordered group $(R, <, +)$. Then exactly one of the following holds:*

- *$\mathscr{R}$ is linearly bounded;*
- *$\mathscr{R}$ defines a binary operation $\cdot$ such that $(R, <, +, \cdot)$ is an ordered real closed field.*

Corollary 2.22.. – *If $\mathscr{R} = (R, <, +, \cdot)$ is a definable real closed field, then the ordered group $(R_{>0}, <, \cdot)$ is either definably isomorphic to $(R, <, +)$ or it is linearly bounded.*

Proof – If $(R_{>0}, <, \cdot)$ is not linearly bounded, then by Fact 2.21, there is a binary operation $*$ such that $\mathscr{S} = (R_{>0}, <, \cdot, *)$ is an ordered real closed field. But its additive group $(R_{>0}, <, \cdot)$ is an $\mathscr{R}$-linear group, so the fields $\mathscr{R}$ and $\mathscr{S}$ are definably isomorphic by Proposition 2.15, hence the groups $(R_{>0}, <, \cdot)$ and $(R, <, +)$ are definably isomorphic. $\square$

Fact 2.23.. – [10, Special case of Fact 1.6] *For an o-minimal expansion $\mathscr{R} = (R, 0, 1, +, <, \cdots)$ of an ordered group the following are equivalent:*

(1) *$\mathscr{R}$ is semi-bounded.*
(2) *There is no $\mathscr{R}$-definable real closed field with domain R.*

Moreover, in the following result, two definable o-minimal structures $\mathscr{I} = (I, <, \cdots)$ and $\mathscr{J} = (J, <, \cdots)$ are said to be *globally orthogonal* if there is no definable bijection between I and J.

Fact 2.24.. – [11, Theorems 5.8 and 5.10] *Suppose that U is a definably-connected definable solvable group. Then U has a definable normal subgroup V such that U/V is a definably compact definable solvable group and $V = K \times W_1 \times \cdots \times W_s \times V_1 \times \cdots \times V_k$. Here K is the definably-connected, definably compact normal subgroup of U of maximal dimension. For each $j \in \{1, \dots, s\}$ (resp. $i \in \{1, \dots, k\}$), there is a semi-bounded definable o-minimal expansion $\mathscr{J}_j$ of a group (resp., a definable o-minimal expansion $\mathscr{I}_i$ of a real closed field) all of which are pairwise globally orthogonal such that W_j is a direct product of copies of the additive group of $\mathscr{J}_j$ and V_i is definably isomorphic to an $\mathscr{I}_i$-definable solvable group.*

Moreover, for each $i \in \{1, \dots, k\}$, we have $V_i = W \times V$ where W is the maximal $\mathscr{I}_i$-definable subgroup of V_i which is a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathscr{I}_i$-definable group. The group V is an $\mathscr{I}_i$-definable group such that $Z(V)$ has an $\mathscr{I}_i$-definable subgroup Z such that $Z(V)/Z$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathscr{I}_i$-definable group. There are $\mathscr{I}$-definable subgroups $1 < Z_1 < \cdots < Z_m = Z$ such that, for each $l \in \{1, \dots m\}$, the group Z_l/Z_{l-1} is the additive group of $\mathscr{I}_i$, and there is an $\mathscr{I}_i$-definable embedding of $V/Z(V)$ into $\mathrm{GL}_n(I)$.

3. Nilpotent groups

The structure of solvable groups by Edmundo [11] (see §2.4) provides valuable information on nilpotent groups definable in an o-minimal structure. By using these results together with methods from groups of finite Morley rank, we obtain a new decomposition of nilpotent groups based on *pseudo-tori* and *U_R-groups* (Definitions 3.1 and 3.9, Theorem 3.29).

The structure of nilpotent groups in this new language is very effective for the study of group actions in §4.

3.1. **pseudo-tori.** Cherlin defined in [4] a *good torus* and a *decent torus* as analogues of an algebraic torus for groups of finite Morley rank. These groups are defined from the torsion, and a more general notion of a torus was introduced in [17]: the *pseudo-tori* whose definition for the finite Morley rank context is very close to the following definition.

Definition 3.1.. – *A* pseudo-torus *is a definably connected nilpotent definable group T such that no definable quotient group T/N is definably isomorphic to the additive group R_+ of a definable real closed field R.*

Remark 3.2.. – Any definable quotient T/N of a pseudo-torus T is a pseudo-torus.

The following result gives examples of pseudo-tori, which encompasse *definably compact* groups, *linearly bounded* groups and *semi-bounded* groups (see §2.4). It will be useful for the proof of Proposition 3.22.

Lemma 3.3.. – *Let G be a solvable definably connected definable group. Suppose that G satisfies one of the following three conditions:*

- *G is definably compact;*
- *there is a definable expansion $\mathscr{R}$ of the group G such that $\mathscr{R}$ is a linearly bounded structure;*
- *there is a definable expansion $\mathscr{R}$ of the group G such that $\mathscr{R}$ is a semi-bounded structure.*

Then G is a pseudo-torus.

Proof – If G is definably compact, then it is abelian by Fact 2.17. Now, since any definable quotient of G is definably compact (Fact 2.18), it follows from Fact 2.19 that G is a pseudo-torus.

If there is a definable expansion $\mathscr{R} = (G, <, +, \cdots)$ of the group G such that $\mathscr{R}$ is a linearly bounded structure or a semi-bounded structure, then G is abelian and has no non-trivial proper definable subgroup (Fact 2.20). Since it follows from Facts 2.21 and 2.23 that there is no definable binary operation $\cdot$ such that $(G, <, +, \cdot)$ is a real closed field, G is a pseudo-torus. □

We start our study of pseudo-tori. The following result is used in the proof of Lemma 3.5.

Fact 3.4.. – [11, Corollary 7.3 (1)] (see also [23, Theorem 1.1] and Fact 2.11) *Let A and B be two definable abelian groups. If there is an infinite definable family of definable homomorphisms from A into B, then there is a definable real closed field whose additive group is definably isomorphic to a definable subgroup of B and a quotient of definable subgroups of A.*

Lemma 3.5.. – *Let T be a pseudo-torus and B be a nilpotent definable group. Then any definable family $\mathscr{A}$ of homomorphisms from T to B is finite.*

Proof – We proceed by induction on the dimension of B. Since for each $\alpha \in \mathscr{A}$, the image $\mathrm{Im}\,\alpha \simeq T/\mathrm{Ker}\,\alpha$ of α is definably connected, we have $\mathrm{Im}\,\alpha \leq B^\circ$ and we may assume that B is definably connected. We assume toward a contradiction that $\mathscr{A}$ is infinite. In particular, B is infinite.

We assume toward a contradiction that B has a proper infinite normal definable subgroup A. For each $\alpha \in \mathscr{A}$, we consider $\overline{\alpha} : T \to B/A$ defined by $\overline{\alpha}(t) = \alpha(t)A$. Then the definable family $\{\overline{\alpha} \mid \alpha \in \mathscr{A}\}$ is finite by induction hypothesis, and there exists $\alpha \in \mathscr{A}$ such that the definable family $\mathscr{B} = \{\beta \in \mathscr{A} \mid \overline{\beta} = \overline{\alpha}\}$ is infinite. For each $\beta \in \mathscr{B}$, the map $u_\beta : T \to A$ defined by $u_\beta(t) = \beta(t)\alpha(t)^{-1}$ is a definable group homomorphism, and since $\mathscr{B}$ is infinite, the definable family $\{u_\beta \mid \beta \in \mathscr{B}\}$ is infinite too, contradicting our induction hypothesis. Hence B has no proper infinite normal definable subgroup. In particular B is abelian (Fact 2.2), so it has no proper infinite definable subgroup.

Let K be the intersection of the subgroups $\operatorname{Ker}\alpha$ for $\alpha \in \mathscr{A}$. Since $T/\operatorname{Ker}\alpha \simeq \operatorname{Im}\alpha \leq B$ is abelian for each non-zero element $\alpha \in \mathscr{A}$, the quotient group T/K is abelian. For each $\alpha \in \mathscr{A}$, we consider $\tilde{\alpha} : T/K \to B$ defined by $\tilde{\alpha}(tK) = \alpha(t)$. Since $\mathscr{A}$ is infinite, $\overline{\mathscr{A}} = \{\tilde{\alpha} \mid \alpha \in \mathscr{A}\}$ is infinite too. Then by Fact 3.4 there is a definable subgroup B_0 of B such that B_0 is definably isomorphic to the additive group R_+ of a definable real closed field R. In particular, B_0 is infinite and we obtain $B = B_0$ by the previous paragraph. But $\overline{\mathscr{A}}$ is infinite, so there is a non-zero element $\tilde{\alpha} \in \overline{\mathscr{A}}$, and its image $\operatorname{Im}\tilde{\alpha} = \operatorname{Im}\alpha \simeq T/\operatorname{Ker}\alpha$ is definably connected. Hence $\operatorname{Im}\alpha$ is an infinite definable subgroup of B and α is a surjective homomorphism by the previous paragraph. Thus we have $T/\operatorname{Ker}\alpha \simeq \operatorname{Im}\alpha = B \simeq R_+$, contradicting that T is a pseudo-torus and that R is real closed, so $\mathscr{A}$ is finite. □

Corollary 3.6.. – *Let T be a pseudo-torus and G be a definably connected definable group acting definably on T. Then G centralizes T. In particular, T is abelian.*

Proof – By Lemma 3.5, the quotient group $G/C_G(T)$ is finite, and since G is definably connected, G centralizes T. In particular, the case where $G = T$ acts by conjugation on T shows that T is abelian. □

Proposition 3.7.. – *Any nilpotent definable group G has a unique maximal pseudo-torus $T(G)$. In particular, any pseudo-torus of G is central in G.*

Proof – We proceed by induction on the dimension of G. We may assume that G is definably connected. Let S and T be two maximal pseudo-tori of G.

We show that S and T are central in G. If $N_G(T) < G$, we have $T = T(N_G(T))$ by induction hypothesis, therefore T is a definably characteristic subgroup of $N_G(T)$ and we obtain $N_G(N_G(T)) = N_G(T)$. But G is nilpotent, hence we have $N_G(T) = G$ contradicting $N_G(T) < G$. This proves that T is normal in G, and T is central in G by Corollary 3.6. In the same way, S is central in G.

We assume toward a contradiction that ST is not a pseudo-torus. Then ST has a definable subgroup N such that ST/N is definably isomorphic to the additive group R_+ of a definable real closed field R. In particular, the quotient group TS/N is torsion-free and it has dimension one by Fact 2.10. If T is not contained in N, we have $TS = TN$ and $T/(T \cap N)$ is definably isomorphic to $TS/N \simeq R_+$, contradicting that T is a pseudo-torus. Thus T is contained in N, and in the same way, S is contained in N, contradicting $N < ST$. This proves that ST is a pseudo-torus, and that $T(G) = S = T$ is well defined. □

Proposition 3.8.. – *Let G be a nilpotent definable group, and N be a normal definable subgroup of G. Then $T(G/N) = T(G)N/N$.*

Proof – We proceed by induction on the dimension of G. Since a definable quotient of a pseudo-torus is a pseudo-torus, $T(G/N)$ contains $T(G)N/N$ and we may assume that $G/N = T(G/N)$ is a pseudo-torus.

Let T be a minimal definable subgroup of G among the ones satisfying $G = TN$. Since G/N is a pseudo-torus, G/N is definably connected and we have $G = T^\circ N$. Thus T is definably connected by the minimality of T.

We assume toward a contradiction that T is not a pseudo-torus. Then T has a definable quotient T/M definably isomorphic to R_+ for a real closed field R. In particular, T/M is torsion-free and has dimension 1 by Fact 2.10. Since $(T \cap MN)/M$ is a definable subgroup of T/M, it is either equal to T/M or trivial, so we have either $T \cap MN = M$ or $T \leq MN$. In the first case we have

$$(G/N)/(MN/N) \simeq G/MN = TN/MN \simeq T/M \simeq R_+$$

contradicting that G/N is a pseudo-torus. In the second case we have $G = TN = MN$, contradicting the minimality of T. Hence T is a pseudo-torus, and we obtain $T \leq T(G)$ and $G = T(G)N$. □

3.2. **U_R-groups.** Burdges introduced $U_{0,r}$-groups in [3] as a concept of unipotence for groups of finite Morley rank. This notion is very effective for the study of groups of finite Morley rank. Another analogue of unipotent algebraic groups, namely the *homogeneous $U_{0,r}$-groups*, was proposed in [14] in order to remedy to a weakness of $U_{0,r}$-groups, since they are not necessarily preserved by passage to definable subgroups. Later, a more precise unipotence notion was introduced in [15, §3.2], very close to Definitions 3.9 and 3.15. This last notion, together with pseudo-tori and the homogeneity of [14], is a crucial tool for some analysis as [15].

We note that N is a normal subgroup of G in the following definition (Fact 2.3).

Definition 3.9.. – *Let R be a definable real closed field. A* U_R-group *is a nilpotent definable group G such that, for every maximal proper definably connected definable subgroup N, the quotient group G/N is definably isomorphic to R_+.*

Remark 3.10.. – Any U_R-group is definably connected.

Proposition 3.11.. – *Let R be a definable real closed field, and G be a nilpotent definable group. Then any family of U_R-subgroups of G generates a U_R-subgroup.*

In particular, G has a unique maximal U_R-subgroup.

Proof – We note that we do not know if any subgroup of G generated by U_R-subgroups is definable. We have just to show that any non-necessarily definable subgroup H of G contains a unique maximal U_R-subgroup. We proceed by induction on the dimension of G. Since any U_R-group is definably connected, we may assume that G is definably connected. Let U and V be two maximal U_R-subgroups of H.

We show that U is normal in H. We may assume that U is not normal in G. By induction hypothesis, U is the unique maximal U_R-subgroup of $N_H(U) \leq N_G(U) < G$, therefore U is normal in $N_H(N_H(U))$ and we obtain $N_H(N_H(U)) = N_H(U)$. But H is nilpotent, hence $N_H(U) = H$ and U is normal in H. In particular, UV is a definably connected definable subgroup of H.

We may assume that UV is infinite, therefore UV has a maximal proper definably connected definable subgroup N. If $UN = UV$, we have $U/(U \cap N) \simeq UV/N$ and by the maximality of N in UV, the group $(U \cap N)^\circ$ is a maximal proper definably connected definable subgroup of U. Since U is a U_R-group, the group $U/(U \cap N)^\circ \simeq R_+$ is torsion-free and $U \cap N$ is definably connected, so we obtain $UV/N \simeq U/(U \cap N) \simeq R_+$. In the same way, if $VN = UV$ the groups UV/N and R_+ are definably isomorphic. But N is proper in UV, so we have either $U \not\leq N$

or $V \nleq N$, and by the maximality of N we have either $UN = UV$ or $VN = UV$. Hence UV/N is definably isomorphic to R_+, and UV is a U_R-group. Now by the maximality of U and V, we obtain $UV = U = V = U_R(H)$, as desired. □

Thus we may define a radical $U_R(\cdot)$ for each definable real closed field R.

Definition 3.12.. – *Let R be a definable real closed field. For each definable group G, we denote by $U_R(G)$ the unique maximal U_R-subgroup of $F(G)$.*

Lemma 3.13.. – *Let G be a definable group with a normal definable subgroup N such that G/N is definably isomorphic to R_+ for a real closed field R. Then $N \cap G^\circ$ is definably connected and $G = UN$ for an abelian U_R-subgroup U.*

Proof – Since $G/N \simeq R_+$ is torsion-free, it is definably connected and G/N is definably isomorphic to $G^\circ/(N \cap G^\circ)$. Therefore the torsion part of G°/N° is $(N \cap G^\circ)/N^\circ$, so it is finite, and Fact 2.6 (1) gives $N \cap G^\circ = N^\circ$.

Let U be a minimal definable subgroup of G among the ones satisfying $U \nleq N$. For any $u \in U \setminus N$ the subgroup $Z(C_U(u))$ is definable, abelian and contains u, so $Z(C_U(u)) = U$ by minimality of U, and U is abelian. Since $G/N \simeq R_+$ is torsion-free, and since its dimension is 1 (Fact 2.10), we have $G = UN$.

We show that U is a U_R-group. Since $G/N \simeq R_+$ is definably connected, we have $G = U^\circ N$, and U is definably connected by minimality of U. Now the first paragraph applied with U and $U/(U \cap N) \simeq R_+$ shows that $U \cap N$ is definably connected. But, again by the minimality of U, each proper definable subgroup of U is contained N. Hence $U \cap N$ is the unique maximal proper definably connected definable subgroup of U. Thus U is a U_R-group. □

Proposition 3.14.. – *Let R be a definable real closed field, G be a nilpotent definable group, and N be a normal definable subgroup of G. Then*

$$U_R(G/N) = U_R(G)N/N$$

Proof – We show that $U_R(G/N)$ contains $U_R(G)N/N$. Let M/N be a maximal proper definably connected definable subgroup of $U_R(G)N/N$. Then $U_R(G)N/M \simeq U_R(G)/(U_R(G) \cap M)$ has no non-trivial proper definably connected definable subgroup, and $(U_R(G) \cap M)^\circ$ is a maximal proper definably connected definable subgroup of $U_R(G)$. Thus $U_R(G)/(U_R(G) \cap M)^\circ$ is definably isomorphic to R_+ and, by Lemma 3.13, the subgroup $U_R(G) \cap M$ is definably connected. Therefore $U_R(G)N/M \simeq U_R(G)/(U_R(G) \cap M)$ is definably isomorphic to R_+, so $U_R(G)N/N$ is a U_R-group and it is contained in $U_R(G/N)$.

We show that $U_R(G/N) = U_R(G)N/N$. We denote by U the preimage in G of $U_R(G/N)$. For each maximal proper definably connected definable subgroup M/N of $U_R(G/N)$, the group U/M is definably isomorphic to R_+, so Lemma 3.13 gives $U = U_R(G)M$. Consequently $U_R(G)N/N$ is contained in no proper definably connected definable subgroup of $U_R(G/N)$. Since $U_R(G)N/N$ is definably connected, we obtain $U_R(G/N) = U_R(G)N/N$, as desired. □

3.3. **Homogeneous U_R-groups.** Similarly to the groups of finite Morley rank, we define an *homogeneous U_R-group* [14]. The purpose of this section is to show that any U_R-group is homogeneous (Proposition 3.22).

Definition 3.15.. – *Let R be a definable real closed field. A U_R-group is said to be* homogeneous *if its definable subgroups are U_R-groups.*

Remark 3.16.. – Let R be a definable real closed field.

- Any homogeneous U_R-group is definably connected and torsion-free.
- Every definable subgroup of a homogeneous U_R-group is a homogeneous U_R-group.
- By Proposition 3.14, every definable quotient group of a homogeneous U_R-group is a homogeneous U_R-group.

Lemma 3.17.. – *Let R be a definable real closed field. If a nilpotent definable group G has a normal homogeneous U_R-subgroup U such that G/U is a homogeneous U_R-group, then G is a homogeneous U_R-group.*

PROOF – Let H be a definable subgroup of G. Since G/U is a homogeneous U_R-group, HU/U is a U_R-group, and by Proposition 3.14, we have $H = U_R(H)(H \cap U)$. But U is a homogeneous U_R-group, hence $H \cap U$ and H are U_R-groups. □

For the proof of Lemma 3.20, we need *G-minimal subgroups.*

Definition 3.18.. – *Let G be a definable group. A subgroup of G is said to be* G-minimal *if it is definable, infinite, normal, and minimal for these conditions.*

Remark 3.19.. –

- By the descending chain condition on definable subgroups of G [25, Remark 2.13 (ii)], any infinite normal definable subgroup H of a definable group G contains a G-minimal subgroup.
- In a definable group G, every G-minimal subgroup is definably connected.

Lemma 3.20.. – *Let R and S be two definable real closed fields, and let G be a nilpotent definable group. If R and S are not definably isomorphic, then $[U_R(G), U_S(G)] = 1$.*

PROOF – We proceed by induction on the dimension of G. We may assume that G is infinite. In particular, $Z(G)$ is infinite (Fact 2.2) and contains a G-minimal subgroup A. Since A is G-minimal and central in G, it has no proper infinite definable subgroup. By induction hypothesis and by Proposition 3.14, the commutator $[U_R(G), U_S(G)]$ is contained in A. We assume toward a contradiction that there exist $u \in U_R(G)$ and $v \in U_S(G)$ such that $[u, v]$ is not trivial. We consider the maps $f : U_S(G) \to A$ and $g : U_R(G) \to A$ defined by $f(x) = [u, x]$ and $g(x) = [x, v]$. Since $[u, v] \neq 1$, they are two non-zero definable homomorphisms. Consequently, by the minimality of A and since $U_R(G)$ and $U_S(G)$ are definably connected, the maps f and g are surjective. Now A is both a U_R-group and a U_S-group by Proposition 3.14, and since A has no proper infinite definable subgroup, it is definably isomorphic to R_+ and S_+, contradicting Lemma 2.14. Thus we obtain $[U_R(G), U_S(G)] = 1$. □

Lemma 3.21.. – *Let R be a definable real closed field. If G is a U_R-group, then G' is a homogeneous U_R-group.*

PROOF – First we show that $G/Z(G)$ is a homogeneous U_R-group. Let $H/Z(G)$ be a definable subgroup of $G/Z(G)$. We show that $H/Z(G)$ is a U_R-group. We may assume that $H/Z(G)$ is non-trivial. Let $M/Z(G)$ be a maximal proper definably connected definable subgroup of $H/Z(G)$. Then H and M are definably connected (Corollary 2.7). By Proposition 3.7, the group $T(H) \leq T(G)$ is contained in $Z(G)$, and H/M is not a pseudo-torus (Proposition 3.8). Then there is a normal definable subgroup N/M of H/M such that H/N is definably isomorphic to S_+ for a definable real closed field S. By Lemma 3.13 and the maximality of M, we obtain $M = N$ and $H = U_S(H)M$. In particular, $U_S(G) \geq U_S(H)$ is not central in G, and Lemma 3.20 says that the fields R and S are definably isomorphic. Thus H/M is definably isomorphic to R_+ and $G/Z(G)$ is a homogeneous U_R-group.

We show by induction on the dimension of G that G' is a homogeneous U_R-group. We may assume that G is not abelian, and we consider $g \in Z_2(G) \setminus Z(G)$. Then the map $f : G \to Z(G)$ defined by $f(x) = [g, x]$ is a definable group homomorphism, and $\operatorname{Ker} f$ contains $Z(G)$. Hence, by the previous paragraph, its image $\operatorname{Im} f \simeq (G/Z(G))/(\operatorname{Ker} f/Z(G))$ is a non-trivial homogeneous U_R-subgroup of G'. Now $G'/\operatorname{Im} f$ is a homogeneous U_R-group by induction hypothesis, and the result follows from Lemma 3.17. □

Proposition 3.22.. – *For any definable real closed field R, every U_R-group is homogeneous. In particular, such a group is torsion-free.*

Proof – Let G be a U_R-group. We may assume that G is infinite. By Lemmas 3.17 and 3.21, we may assume that G is abelian. By Lemma 3.3, Proposition 3.8 and Fact 2.24, there is a definable o-minimal expansion of a real closed field $\mathscr{S} = (S, <, \cdots)$ such that G is definably isomorphic to an $\mathscr{S}$-definable group, and there are $\mathscr{S}$-definable subgroups $1 < Z_1 < \ldots < Z_m = G$ where for each $l \in \{1, \ldots, m\}$, the group Z_l/Z_{l-1} is the additive group of $\mathscr{S}$.

Thus G is a U_S-group (Proposition 3.14). Since G is a U_R-group too, the fields R and S are definably isomorphic by Lemma 2.14. Let H be a definable subgroup of G. For each $i \in \{1, \ldots, m\}$, the group $(H \cap Z_i)/(H \cap Z_{i-1})$ is either trivial or definably isomorphic to $Z_i/Z_{i-1} \simeq S_+ \simeq R_+$, so H is a U_R-group by Proposition 3.14. This proves that G is a homogeneous U_R-group. □

3.4. **Decomposition of nilpotent groups.** In this section, se state our main result on nilpotent groups (Theorem 3.29). From U_R-groups, we introduced *U-groups* as an analogue of unipotent subgroups of algebraic groups.

Definition 3.23.. – *A* U-group *is a nilpotent definable group G generated by $U_{R_1}(G), \ldots, U_{R_k}(G)$ for definable real closed fields $R_1, \ldots, R_k$.*

Remark 3.24.. –

- A U-group is generated by definably connected definable subgroup, so any U-group is definably connected.
- Since, for any definable real closed field R, every definable quotient group of a U_R-group is a U_R-group, every definable quotient of a U-group is a U-group.

Lemma 3.25.. – *Every definable group G has a unique maximal normal U-subgroup $U(G)$.*

Proof – Let U be a maximal normal U-subgroup of G. If V is another normal U-subgroup of G, then UV is a normal nilpotent definably connected definable subgroup of G. Since U and V are U-groups, UV is a U-group too. □

Lemma 3.26.. – *In a nilpotent definable group G, the subgroup $U(G)$ contains all the U-subgroups of G.*

PROOF – For each definable real closed field R, the subgroup $U_R(G)$ is definable, definably connected and normal in G, so $U(G)$ contains $U_R(G)$ for each definable real closed field R, and the result follows. $\square$

Proposition 3.27.. – *For every U-group G, there are finitely many definable real closed fields $R_1, \dots, R_k$ such that*

$$G = U_{R_1}(G) \times \cdots \times U_{R_k}(G)$$

In particular, G is torsion-free. Moreover, for each definable real closed R, if R is not definably isomorphic to R_i for $i \in \{1, \dots, k\}$, then $U_R(G)$ is trivial.

PROOF – We proceed by induction on the smallest integer k such that G is generated by $U_{R_1}(G), \dots, U_{R_k}(G)$ for definable real closed fields $R_1, \dots, R_k$. In particular, the fields $R_1, \dots, R_k$ are not definably isomorphic. We consider $H = U_{R_1}(G) \cdots U_{R_{k-1}}(G)$. By induction hypothesis, $U_R(G)$ is trivial for each real closed field R not definably isomorphic to R_i for $i \in \{1, \dots, k-1\}$, and

$$H = U_{R_1}(G) \times \cdots \times U_{R_{k-1}}(G)$$

In particular, $U_{R_k}(H)$ is trivial, so we have $H \cap U_{R_k}(G) = 1$ and G is the direct product of $U_{R_1}(G), \dots, U_{R_k}(G)$.

Let R be a definable real closed field. We show that if R is not definably isomorphic to R_i for $i \in \{1, \dots, k\}$, then $U_R(G)$ is trivial. By the previous paragraph and Proposition 3.22, the group $U_R(G)H/H \leq G/H \simeq U_{R_k}(G)$ is a U_{R_k}-group. But by Proposition 3.14, it is a U_R-group, hence it is trivial by Lemma 2.14, and $U_R(G) = U_R(H)$ is trivial, as desired. $\square$

Corollary 3.28.. – *For any U-group G, we have $T(G) = 1$.*

PROOF – By Proposition 3.27, there are finitely many definable real closed fields $R_1, \dots, R_k$ such that G is the direct product of $U_{R_1}(G), \dots, U_{R_k}(G)$. We proceed by induction of k. By induction hypothesis, the group $T(G/U_{R_1}(G)) \simeq T(U_{R_2}(G) \times \dots \times U_{R_k}(G))$ is trivial, and Proposition 3.8 gives $T(G) \leq U_{R_1}(G)$. Then $T(G)$ is a U_{R_1}-group (Proposition 3.22), so $T(G)$ is trivial. $\square$

Theorem 3.29.. – *Any nilpotent definably connected definable group G is the central product of $T(G)$ by $U(G)$. More precisely, the following decomposition holds*

$$G = T(G) * (U_{R_1}(G) \times \cdots \times U_{R_k}(G))$$

for definable real closed fields $R_1, \dots, R_k$ such that $U_R(G) = 1$ for each definable real closed field not definably isomorphic to R_i for $i = 1, \dots, k$. Moreover, $U_{R_i}(G)$ is a homogeneous U_{R_i}-group for each $i = 1, \dots, k$.

PROOF – It follows from Proposition 3.7 that $T(G)$ is central in G, so the group $T(G)U(G)$ is the central product of $T(G)$ by $U(G)$. We assume toward a contradiction that $G \neq T(G)U(G)$. Let M be a maximal definably connected definable subgroup of G containing $T(G)U(G)$. Since M contains $U(G)$, it contains $U_R(G)$ for each definable real closed field R, and Proposition 3.14 shows that no definable quotient of G/M is definably isomorphic to R_+ for a definable real closed field R. Thus G/M is a pseudo-torus and Proposition 3.8 gives $G = T(G)M$, contradicting that M contains $T(G)$. Hence we have $G = T(G)U(G)$, and the decomposition of G follows from Propositions 3.22 and 3.27. $\square$

Corollary 3.30.. – *A nilpotent definably connected definable group G is a U-group is and only if $T(G)$ is trivial.*

Proof – This follows from Corollary 3.28 and Theorem 3.29. □

Corollary 3.31.. – *Every definable subgroup H of a U-group G is a U-group.*

Proof – The group G is torsion-free by Proposition 3.27, so H is definably connected and this follows from Corollary 3.30. □

Corollary 3.32.. – *The derived subgroup G' of a definably connected definable nilpotent group G is a U-group.*

Proof – This follows from Theorem 3.29, Corollary 3.6 and Lemma 3.21. □

4. Structure of definable groups

The purpose of this section is to describe the structure of any definably connected definable group G from $U(G)$. We show that $G/U(G)$ is a central extension of a direct product of definably simple definable groups (Theorem 4.10). The proof is based on the structure of nilpotent groups (Theorem 3.29), and on the study of group actions on a solvable group.

Lemma 4.1.. – *Let G be a solvable definably connected definable group. Then G' is contained in $U(G)$.*

Proof – By Fact 2.5, the group G' is contained in $F(G)$. Since G is definably connected and $F(G)/F(G)^\circ$ is finite, G centralizes $F(G)/F(G)^\circ$. Since $\overline{T} = F(G)^\circ/U(G)$ is a pseudo-torus by Proposition 3.8 and Theorem 3.29, the group $\overline{T}$ is centralized by G too (Corollary 3.6). Consequently, $G/U(G)$ is a nilpotent definably connected definable group, and by Corollary 3.32, its derived subgroup is a normal U-subgroup of $F(G)/U(G)$.

Let R be a definable real closed field. By Proposition 3.14, we have

$$U_R(F(G)/U(G)) = U_R(F(G))U(G)/U(G),$$

and since $U(G) = U(F(G))$ contains $U_R(F(G))$, the groups $U_R(F(G)/U(G))$ and $U(F(G)/U(G))$ are trivial. But $G'U(G)/U(G)$ is contained in $U(F(G)/U(G))$ by the previous paragraph, hence G' is contained in $U(G)$, as desired. □

Lemma 4.4 generalizes Fact 2.5. Thanks to Lemma 4.3, its proof is slightly simpler than the one of [14, Theorem 6.10]. Moreover, it uses the following result.

Fact 4.2.. – [25, Corollary 2.15 (i)] *Any infinite definable group has an infinite definable abelian subgroup.*

Lemma 4.3.. – *Let G be a definably connected definable group. If H is a normal definable subgroup such that G/H is solvable, then $G = R(G)H$.*

Proof – By Fact 2.4, the subgroup R generated by all normal solvable subgroups of G is definable and solvable. Then we have $R(G) = R^\circ$, and G/R satisfies the hypotheses of Fact 2.8. Thus G/R is the direct product of definable subgroups $H_1/R, \ldots, H_k/R$ such that for every $i \in \{1, \ldots, k\}$ there is a definable real closed field R_i and a definable isomorphism between H_i/R and a semialgebraic subgroup of $\mathrm{GL}_{n_i}(R_i)$. Moreover, $H_i^\circ R/R$ is definably simple. Since G/R is definably connected, H_i/R is definably connected for each i. In particular, $H_i/R = H_i^\circ R/R$ is definably simple for each i, and G/R has no proper definable subgroup $\overline{N}$ such that $(G/R)/\overline{N}$ is solvable. Thus we obtain $G = RH$, and since G is definably connected and $R(G) = R^\circ$, this implies $G = R(G)H$. $\square$

Lemma 4.4.. – *Let G and H be two definably connected definable groups. We assume that H is solvable. If G acts definably by conjugation on H, then $[G, H]$ is contained in $U(H)$.*

Proof – We consider a minimal counter-example G acting on H. By minimality of G and Fact 4.2, the group $\overline{G} = G/C_G(H/U(H))$ is abelian. By Lemma 4.3, we have $G = R(G)C_G(H/U(H))$, so $G = R(G)$ is solvable by minimality of G. We consider the semi-direct product $H \rtimes G$ where G acts by conjugation on H. It is a solvable definably connected definable group. Then $[G, H] \leq (H \rtimes G)'$ is contained in $U(H \rtimes G) \cap H$ by Lemma 4.1. Since $U(H \rtimes G) \cap H$ is a normal U-subgroup of H by Corollary 3.31, we obtain $[G, H] \leq U(H)$, contradicting $G \neq C_G(H/U(H))$. Thus $[G, H]$ is contained in $U(H)$. $\square$

Lemma 4.5.. – *Let H be a (non-necessarily definable) subgroup of a nilpotent definable group G. Then H has a unique maximal definably connected definable subgroup.*

Proof – We proceed by induction on the dimension of G. We may assume that G is definably connected. Let M be a maximal definably connected definable subgroup of H. We show that M is normal in H. We may assume that M is not normal in G. By induction hypothesis, M is the unique maximal definably connected definable subgroup of $N_H(M) \leq N_G(M) < G$, therefore M is normal in $N_H(N_H(M))$ and we obtain $N_H(N_H(M)) = N_H(M)$. But H is nilpotent, hence $N_H(M) = H$ and M is normal in H.

Now, if N is any definably connected definable subgroup of H, then NM is a definably connected definable subgroup of H too, and it is contained in M by the maximality of M. This proves the uniqueness of M. $\square$

Corollary 4.6.. – *In any nilpotent definable group G, every family of definably connected definable subgroups of G generate a definably connected definable subgroup.*

The following result and Corollary 4.9 are in the spirit of [1].

Proposition 4.7.. – *Let G and H be two definably connected definable groups. We assume that H is solvable. If G acts definably by conjugation on H, then $[G, H]$ is a U-subgroup of H. In particular, $[G, H]$ is definable and definably connected.*

Proof – By Lemma 4.4, the group $[G, H]$ is contained in $U(H)$, so we have just to prove that $[G, H]$ is definable and definably connected. We proceed by induction on the dimension of H. Since $[G, H]$ is contained in the nilpotent definable group $U(H)$, it has a unique maximal definably connected definable subgroup M (Lemma 4.5). If M is nontrivial, then $[G, H]/M$ is definable and definably connected by induction hypothesis, so $[G, H]$ is definable and definably connected. Thus we may assume that $[G, H]$ contains no non-trivial definably connected definable subgroup.

We show that $[G, H]$ is central in $U(H)$. We may assume that $U(H)$ is non-trivial. By induction hypothesis, $[G, H]Z(U(H))/Z(U(H))$ is definable and definably connected. Since $U(H)$ is a U-group, it is torsion-free (Proposition 3.27), so $Z(U(H))$ is definably connected and $[G,H]Z(U(H))$ is a definably connected definable subgroup.

- If $U(H) = [G,H]Z(U(H))$, then $[G,H]$ contains $U(H)'$. By Corollary 3.32, the subgroup $U(H)' \leq [G,H]$ is definable and definably connected, so it is trivial and $U(H)$ is abelian.
- If $[G,H]Z(U(H)) < U(H)$, then $[U(H),[G,H]Z(U(H))]$ is a definably connected definable subgroup by induction hypothesis, and since it is contained in $[G,H]$, it is trivial. Thus $U(H)$ centralizes $[G,H]$.

Now, for each $g \in G$, the map $ad_g : U(H) \to Z(U(H))$ defined by $ad_g(x) = [g,x]$ is a definable group homomorphism. Since its image is a definably connected definable subgroup of $[G,H]$, it is trivial, so G centralizes $U(H)$.

Thus, for each $h \in H$, the map $ad_h : G \to Z(U(H))$ defined by $ad_h(x) = [x,h]$ is a definable group homomorphism. Since its image is a definably connected definable subgroup of $[G,H]$, it is trivial, we obtain $[G,H] = 1$ and $[G,H]$ is a definably connected definable subgroup of H. □

Corollary 4.8.. – *Let G be a definably connected definable group acting definably by conjugation on a nilpotent definable group H. Then $[G,H] = [G,H^\circ]$ is a U-subgroup of H.*

Proof – By Fact 2.6 (1), the group H has a finite characteristic subgroup F such that $H = H^\circ * F$. Since G is definably connected, it centralizes F and we have $[G,H] = [G,H^\circ]$. Now the result follows from Proposition 4.7. □

The following result is not useful for Theorem 4.10, it will be used in the proof of Theorem 5.17.

Corollary 4.9.. – *Let G be a definably connected definable group acting definably by conjugation on a solvable definable group H. Then $[G,H]$ is a definably connected definable subgroup of H.*

Proof – We proceed by induction on the dimension of the group $H \rtimes G$ where G acts by conjugation on H. We may assume that G acts faithfully on H. If $[G,H]$ contains a non-trivial $(H \rtimes G)$-normal definably connected definable subgroup A, then we may applied our induction hypothesis to $H/A \rtimes G$ where G acts by conjugation on H/A, and we obtain that $[G,H]$ is a definably connected definable subgroup of H. Thus we may assume that $[G,H]$ contains no non-trivial $(H \rtimes G)$-normal definably connected definable subgroup.

The group $[G,H^\circ]$ is definable and definably connected by Proposition 4.7, so its H-conjugates too. Since $[G,H^\circ]$ is normal in H°, its H-conjugates too, so the subgroup L generated by the H-conjugates of $[G,H^\circ]$ is definable and definably connected. But L is a subgroup of $[G,H]$ normal in $H \rtimes G$. Hence it is trivial by the previous paragraph, and G centralizes H°.

Since H/H° is finite and G is definably connected, $[G,H]$ is contained in H°. Then for each $h \in H$, we may consider the map $u_h : G \to H^\circ$ defined by $u_h(x) = [x,h]$. Since G centralizes H°, the map u_h is a group homomorphism, and its image is a definably connected definable subgroup of H°. Moreover, for each $a \in H^\circ$ and each $x \in G$, since G centralizes H° we have

$$u_h(x)^a = [x,h^a] = [x,a^{-1}a^{h^{-1}}h] = [x,h]$$

So the image of u_h is central in H°, and the subgroup generated by $\operatorname{Im} u_h$ for $h \in H$ is a definably connected definable subgroup of $Z(H^\circ)$. But this subgroup is equal to

$[G, H]$, and it is normalized by G and H. Hence it is trivial by the first paragraph, and G centralizes H. Thus $[G, H] = 1$ is definable and definably connected. □

Theorem 4.10.. – *Let G be a definably connected definable group. Then $G/U(G)$ is a central extension of a direct product of definably simple definable groups.*

More precisely, G has a normal solvable definable subgroup R such that the following three conditions hold:

- *R contains all the normal solvable subgroups of G;*
- *$[G, R]$ is a U-group and $[G, R] = [G, R^\circ]$;*
- *(Fact 2.8) G/R is the direct product of definably simple definable subgroups $H_1, \ldots, H_k$ such that for every $i \in \{1, \ldots, k\}$ there is a definable real closed field R_i and a definable isomorphism between H_i and a semialgebraic subgroup of $\mathrm{GL}(n_i, R_i)$.*

Proof – By Fact 2.4, the subgroup R generated by all normal solvable subgroups of G is definable and solvable. Then we have $R(G) = R^\circ$, and G/R satisfies the hypotheses of Fact 2.8. Thus, the first and the third assertions are satisfied.

Now, since $[G, R^\circ]$ is a U-group (Proposition 4.7), we have just to prove that $[G, R] = [G, R^\circ]$. Since G is definably connected and since R/R° is finite, G centralizes R/R°. In particular, R/R° is abelian. Moreover, since $[G, R^\circ]$ contains $[R, R^\circ]$, the group R centralizes $R^\circ/[G, R^\circ]$, so the group $R/[G, R^\circ]$ is nilpotent. Then Corollary 4.8 shows that the commutator $[G/[G, R^\circ], R/[G, R^\circ]]$ is trivial and we obtain $[G, R] = [G, R^\circ]$. □

5. Linearity of definable groups

We prove the main theorem in this section (Theorem 5.17). Its proof is based on the previous sections, on the study of *definably linear groups* (Definition 5.12 and Fact 5.4) and on the analysis of groups definable in an o-minimal expansion of a real closed field. In particular, the following two results are crucial for the proof of Theorem 5.17.

Fact 5.1.. – [20, Proof of Corollary 3.1] *Let $\mathscr{R} = (R, <, \cdots)$ be an o-minimal expansion of a real closed field. If G is definable in $\mathscr{R}$, then $G/Z(G)$ can be definably embedded into $\mathrm{GL}_n(R)$.*

Proposition 5.2.. – *Let $\mathscr{R}_0 = (R, <, +, \cdot)$ be a definable real closed field, and let $\mathscr{R}$ be a definable expansion of $\mathscr{R}_0$ such that, for each integer n, all the definable relations of R^n are $\mathscr{R}$-definable. Let H be a normal definable subgroup of a definable group G. If H and G/H are definably isomorphic to an $\mathscr{R}$-definable group, then G is definably isomorphic to an $\mathscr{R}$-definable group.*

Proof – First we assume that G° is definably isomorphic to an $\mathscr{R}$-definable group. We applied the method of Borovik and Cherlin [16, Proposition 4.3]. Let W be the wreath product of G° by G/G°. It is definably isomorphic to an $\mathscr{R}$-definable group, and we have just to find a definable group monomorphism from G to W. We consider a left transversal $T = \{g_1, \ldots, g_r\}$ to G° in G. For each $x \in G$ and each $i \in \{1, \ldots, r\}$, we denote by $n_i(x)$ the unique element of G° such that $n_i(x)g_i x \in T$, and we define a map $\mu : G \to W$ by $\mu(x) = ((n_1(x), \ldots, n_r(x)), xG^\circ)$. The map μ is definable, and it is a group homomorphism (see the proof of [9, Theorem 18.9 p.68]). Moreover, if x belongs to $\operatorname{Ker}\mu$, the last coordinate gives $x \in G^\circ$, and since

$g_1 x \in \{g_1, \ldots, g_r\}$, we obtain $g_1 x = g_1$ and $x = 1$. Thus μ is a definable group monomorphism from G to W, as desired. Hence we may assume that G is definably connected.

Now we proceed by induction on the dimension of H. By the structure of H° described in Theorem 4.10, we may assume that either H is finite, or H has no non-trivial normal abelian subgroup, or H is abelian, or H is a U-group. Suppose that H has no non-trivial normal abelian subgroup. By Theorem 4.10, the group G has a normal solvable definable subgroup R such that the following two conditions hold:

- R contains all the normal solvable subgroups of G;
- G/R is the direct product of definably simple definable subgroups.

Consequently, since HR/R is a normal definable subgroup of G/R, the group HR/R is a direct product of some subgroups $H_1, \ldots, H_k$, and G/R has a normal definable subgroup S/R such that $G/R = HR/R \times S/R$. Thus we have $G = HS$ and $H \cap S \leq R$. But R is solvable and H has no non-trivial normal abelian subgroup, so $R \cap H$ is trivial, and since H and S are normal in G, we obtain $G = H \times S$. Hence $G \simeq H \times G/H$ is definably isomorphic to an $\mathscr{R}$-definable group. Thus we may assume that either H is finite, or H is abelian, or H is a U-group.

However, if H is finite, then since G is definably connected, G centralizes H, and H is abelian. Moreover, if H is a non-abelian U-group, then H' is infinite and definable (Corollary 3.32), and the induction hypothesis applied with H/H' and H' shows that G is definably isomorphic to an $\mathscr{R}$-definable group. Hence we may assume that H is abelian.

For each $g \in G$, we denote by $\overline{g} = gH$ the left coset of g modulo H. By Fact 2.1, there is a definable function $t : G \to G$ such that for all $x, y \in G$, we have $x \in xH$ and if $xH = yH$ then $t(x) = t(y)$. We define a map $\Phi : \overline{G} \times \overline{G} \to H$ by $\Phi(xH, yH) = t(xy)^{-1} t(x) t(y)$. In particular, the map Φ is definable, so its graph is a definable subset of $\overline{G} \times \overline{G} \times H \subseteq R^n$ for an integer n, and Φ is $\mathscr{R}$-definable by our hypothesis over $\mathscr{R}$. We consider the set $L = G/H \times H$ and the group $\mathscr{L} = (L, \otimes)$ where for every $(\overline{g}, h) \in L$ and $(\overline{g'}, h') \in L$, the product $(\overline{g}, h) \otimes (\overline{g'}, h')$ is defined by

$$(\overline{g}, h) \otimes (\overline{g'}, h') = (\overline{g}\overline{g'}, h^{g'} h' \Phi(\overline{g}, \overline{g'}))$$

We note that, since the groups G/H and H are definably isomorphic to $\mathscr{R}$-definable groups, and since Φ is $\mathscr{R}$-definable too, the group $\mathscr{L} = (L, \otimes)$ is definably isomorphic to an $\mathscr{R}$-definable group. Moreover the map $f : G \to L$ defined by $f(g) = (\overline{g}, t(g)^{-1} g)$ is a definable group isomorphism, so G is definably isomorphic to an $\mathscr{R}$-definable group. $\square$

Corollary 5.3.. – *Let $\mathscr{R}_0 = (R, <, +, \cdot)$ be a definable real closed field, and let $\mathscr{R}$ be a definable expansion of $\mathscr{R}_0$ such that, for each integer n, all the definable relations of R^n are $\mathscr{R}$-definable. Then every U_R-group is definably isomorphic to an $\mathscr{R}$-definable group.*

Proof – This follows from Propositions 3.22 and 5.2. $\square$

The definable subgroups of $\mathrm{GL}_n(R)$ are studied in [22] whose main result is Fact 5.4. We provide below some useful complements. In particular, we show that a definable quotient of a definably connected subgroup of $\mathrm{GL}_n(R)$ is definably isomorphic to a subgroup of $\mathrm{GL}_n(R)$ (Proposition 5.11).

Fact 5.4.. – [22, Theorem 4.1] *Let $\mathscr{R}$ be an o-minimal expansion of a real closed field $(R, <, \cdots)$, and let G be a $\mathscr{R}$-definably connected $\mathscr{R}$-definable subgroup of $\mathrm{GL}_n(R)$ for an integer n. Then there are semialgebraic subgroups G_1 and G_2 of $\mathrm{GL}_n(R)$ such that $G_2 \leq G \leq G_1$, G_2 is a normal subgroup of G_1 and G_1/G_2 is abelian. Moreover, there are abelian, $\mathscr{R}$-definable, $\mathscr{R}$-definably connected subgroups $A_1, \ldots, A_k$ of G such that $G = G_2 \cdot A_1 \cdots A_k$.*

Fact 5.5.. – [22, Lemma 3.4 (ii)] *Let R be a definable real closed field, and let G be a definable subgroup of $\mathrm{GL}_n(R)$ for an integer n. If G is a definable subgroup of a semialgebraic group of the form $(R_+)^k$, then G is semialgebraic.*

Corollary 5.6.. – *Let R be a definable real closed field, and let G be a definable subgroup of $\mathrm{GL}_n(R)$ for an integer n. If G is a definable subgroup of a semialgebraic unipotent group U, then G is semialgebraic.*

Proof – We proceed by induction on the dimension of G. Since U is unipotent, it is torsion-free and its definable subgroups are definably connected. Let M be a maximal proper definable subgroup of G. By induction hypothesis, M is semialgebraic. Then G/M is a definable subgroup of the semialgebraic unipotent group $N_U(M)/M$. Thus, if M is non-trivial, G/M is semialgebraic by induction hypothesis, so G is semialgebraic. Hence we may assume that M is trivial. Now G is abelian (Fact 4.2), and it is a definable subgroup of $Z(C_U(G))$. Since $Z(C_U(G))$ is an abelian semialgebraic unipotent group, it is of the form $(R_+)^k$, and G is semialgebraic by Fact 5.5. □

We recall that a *semisimple group* is defined to be a definably connected definable group with no infinite abelian normal subgroup (Definition 6.1).

Fact 5.7.. – [22, Theorem 4.5] *Let R be a definable real closed field, and let G be a definably connected definable subgroup of $\mathrm{GL}_n(R)$ for an integer n. Then $G = NH$ for a normal solvable definable subgroup N and a semialgebraic semisimple subgroup H such that $N \cap H$ is finite.*

Lemma 5.8.. – *Any semisimple group S is perfect and satisfies $R(S) = 1$.*

Proof – Since S is a semisimple group, $Z(U(S))$ is finite, and $U(S)$ is finite too by Fact 2.2. But $U(S)$ is a U-group, so it is definably connected, and consequently it is trivial. Hence $R(S)$ is abelian (Proposition 4.7), and since S is semisimple, $R(S)$ is finite. Thus, since $R(S)$ is definably connected, $R(S)$ is trivial.

Let R be the subgroup of S generated by all normal solvable subgroups of G. It is definable and solvable (Fact 2.4), so $R^\circ = R(S)$ is trivial and R is finite. Now S/R has no non-trivial normal abelian subgroup, and Fact 2.8 with Proposition 2.9 shows that S/R is perfect. Thus we have $S = S'R$ and S' has finite index in S, so Proposition 2.9 gives $S = S'$. □

Lemma 5.9.. – *Let R be a definable real closed field, and let G be a definably connected definable subgroup of $\mathrm{GL}_n(R)$ for an integer n. Then $G' = U \rtimes S$ for a semialgebraic unipotent group U and a semialgebraic semisimple group S. In particular, G' is semialgebraic.*

Proof – By Fact 5.7, we have $G = NH$ for a normal solvable definable subgroup N and a semialgebraic semisimple subgroup H such that $N \cap H$ is finite. Since G is definably connected, we may assume that N is definably connected. Moreover, we have $R(H)=1$ and H is perfect (Lemma 5.8). Thus we have $G'=N'[H,N]H'=N'[H,N]H$, and $U=N'[H,N]$ is a U-group by Proposition 4.7. In particular, U is a definable subgroup of N, and it is torsion-free (Proposition 3.27), so $U\cap H\cap N\cap H$ is trivial.

Let $\overline{N}$ be the smallest semialgebraic subgroup of $\mathrm{GL}_n(R)$ containing N. Then $\overline{N}$ is semialgebraically connected by Proposition 2.9. Moreover, $\overline{N}'$ is contained in N by Fact 5.4, so $\overline{N}$ is solvable. This implies that $\overline{N}'$ is a semialgebraic unipotent group. Since H is definably connected, it is semialgebraically connected, and then $[H,\overline{N}]$ is a semialgebraic unipotent group too. Now $\overline{N}'[H,\overline{N}]H$ is a semialgebraic unipotent group. Since $U=N'[H,N]$ is a definable subgroup of $\overline{N}'[H,\overline{N}]$, Corollary 5.6 shows that U is semialgebraic, and we have the decomposition $G'=U\rtimes S$ with $S=H$. □

Corollary 5.10.. – *Let R be a definable real closed field, and let G be a definably connected definable subgroup of $\mathrm{GL}_n(R)$ for an integer n. Then any normal definable subgroup H of G' is semialgebraic.*

Proof – We may assume that H is definably connected. By Fact 5.7, we have $H=NT$ for a normal solvable definable subgroup N and a semialgebraic semisimple subgroup T such that $N\cap T$ is finite. Since H is definably connected, we have $H=N^\circ T$. Moreover, by Lemma 5.9, there are a semialgebraic unipotent group U and a semialgebraic semisimple group S such that $G'=U\rtimes S$. In particular, since $R(S)=1$ (Lemma 5.8), we have $R(G')=U$. Thus N° is a definable subgroup of $R(H)\leq R(G')=U$, and Corollary 5.6 implies that N° is semialgebraic. So $H=N^\circ T$ is semialgebraic. □

Proposition 5.11.. – *Let R be a definable real closed field, and let G be a definably connected definable subgroup of $\mathrm{GL}_n(R)$ for an integer n. If H is a normal definable subgroup of G, then G/H is definably isomorphic to a definable subgroup of $\mathrm{GL}_m(R)$ for an integer m.*

Proof – By Corollary 5.10, the group $H\cap G'$ is semialgebraic, so $N_{\mathrm{GL}_n(R)}(H\cap G')/(H\cap G')$ definably embeds into the $\mathrm{GL}_l(R)$ for an integer l, and $G/(H\cap G')\leq N_{\mathrm{GL}_n(R)}(H\cap G')/(H\cap G')$ is definably isomorphic to a definable subgroup of $\mathrm{GL}_l(R)$. Thus we may assume that $H\cap G'$ is trivial.

By Lemma 5.9, the group G' is semialgebraic, so $N_{\mathrm{GL}_n(R)}(G')/G'$ definably embeds into $\mathrm{GL}_k(R)$ for an integer k, and $G/G'\leq N_{\mathrm{GL}_n(R)}(G')/G'$ is definably isomorphic to a definable subgroup of $\mathrm{GL}_k(R)$. But $H^\circ G'/G'$ has a definable complement C/G' in G/G' by Fact 2.13 (2), and since $C\cap H^\circ\leq G'\cap H$ is trivial, G/H° is definably isomorphic to $H^\circ C/H^\circ\simeq C/(C\circ H^\circ)\simeq C$. Hence we may assume that H is finite. Then $N_{\mathrm{GL}_n(R)}(H)/H$ definably embeds into $\mathrm{GL}_m(R)$ for an integer m, and $G/H\leq N_{\mathrm{GL}_n(R)}(H)/H$ is definably isomorphic to a definable subgroup of $\mathrm{GL}_m(R)$, as desired. □

Definition 5.12.. – *A definable group G is said to be* definably linear *(over finitely many definable real closed fields $R_1,\dots,R_k$), if G has a definable faithful linear representation over the ring $R_1\oplus\cdots\oplus R_k$.*

In other words, G definably embeds in $H_1\times\cdots\times H_k$, where H_i is a linear semialgebraic group over R_i for each $i=1,\dots,k$.

Lemma 5.13.. – *Let $R_1, \dots, R_k$ be finitely many definable real closed fields. Any definable group G has a smallest normal definable subgroup N such that G/N is definably linear over $R_1, \dots, R_k$.*

Proof – It is sufficying to show that, if A and B are two normal definable subgroups of G such that G/A and G/B are definably linear, then $G/(A \cap B)$ is definably linear. Moreover, we may assume that $A \cap B$ is trivial. We consider definable real closed fields $R_1, \dots, R_k, S_1, \dots, S_l$ such that G/A definably embeds in $H_1 \times \dots \times H_k$, where H_i is a linear semialgebraic group over R_i for each $i = 1, \dots, k$, and such that G/B definably embeds in $K_1 \times \dots \times K_l$, where K_j is a linear semialgebraic group over S_j for each $j = 1, \dots, l$. Let $f : G \to G/A \times G/B$ be the map defined by $f(x) = (xA, xB)$. Since f is a definable group monomorphism, G is definably linear. $\square$

Lemma 5.14.. – *Let $R_1, \dots, R_k$ be finitely many definable real closed fields, and let H_i be a definable subgroup of a linear semialgebraic group over R_i for each $i = 1, \dots, k$. If $R_1, \dots, R_k$ are not definably isomorphic, then for any definably connected definable subgroup L of $H_1 \times \dots \times H_k$, we have*

$$L = (L \cap H_1) \times \dots \times (L \cap H_k)$$

In particular, if G is a definably connected definable group, and if G is definably linear over finitely many definable real closed fields $R_1, \dots, R_k$, then G is definably isomorphic to a direct product of definable subgroups of $\mathrm{GL}_{n_1}(R_1), \dots, \mathrm{GL}_{n_k}(R_k)$.

Proof – Since G is definably linear over $R_1, \dots, R_k$, the group G definably embeds into a direct product $H_1 \times \dots \times H_k$, where H_i is a linear semialgebraic group over R_i for each $i = 1, \dots, k$. It is sufficying to show that, for any definably connected definable subgroup L of $H_1 \times \dots \times H_k$, we have

$$L = (L \cap H_1) \times \dots \times (L \cap H_k)$$

We assume toward a contradiction that L is a counter-example of minimal dimension. Therefore, for each proper definably connected definable subgroup L_0 of L we have $L_0 = (L_0 \cap H_1) \times \dots \times (L_0 \cap H_k)$. This implies that, if we consider $K = (L \cap H_1) \times \dots \times (L \cap H_k)$, then K° contains all the proper definably connected definable subgroups of L.

For each $i \in \{1, \dots, k\}$, we denote by $p_i : H_1 \times \dots \times H_k \to H_i$ the ith projection map. Since K is proper in L, there is $i \in \{1, \dots, k\}$ such that $p_i(L)$ is non-trivial. If $p_j(L)$ is trivial for each $j \neq i$, then L is contained in H_i, contradicting $K < L$. Therefore there exists $j \neq i$ such that $p_j(L)$ is non-trivial. We consider $K_i = \mathrm{Ker}\, p_i$ and $K_j = \mathrm{Ker}\, p_j$. They are proper subgroups of L, so K contains K_i° and K_j°, and K_iK_j is a proper normal definable subgroup of L. But L/K_i and L/K_j are definably isomorphic to $p_i(L) \leq H_i$ and $p_j(L) \leq H_j$ respectively, so there exists two integer m and n such that L/K_iK_j is definably isomorphic to a definable subgroup P_i of $\mathrm{GL}_m(R_i)$ and to a definable subgroup P_j of $\mathrm{GL}_n(R_j)$. Hence the fields R_i and R_j are definably isomorphic by Proposition 2.15, contradicting that the fields $R_1, \dots, R_k$ are not definably isomorphic. $\square$

Lemma 5.15.. – *Let G be a definably connected definable group. If $U(Z(G))$ is trivial, then $G/Z(G)$ is centerless.*

Proof – We consider $Z/Z(G) = Z(G/Z(G))$. The subgroup Z is definable, nilpotent and normal in G. By Corollary 4.8, the group $[G, Z]$ is a U-group, and since it is contained in $Z(G)$, the subgroup Z is central in G. Thus $G/Z(G)$ is centerless. □

Lemma 5.16.. – *Let G be a definable group and let R be a definable real closed field. Then G has a smallest normal definable subgroup K such that G/K is a U_R-group.*

Proof – We have just to prove that if A and B are two normal definable subgroups such that G/A and G/B are U_R-groups, then $G/(A \cap B)$ is a U_R-group. Since G/A and G/B are nilpotent, $G/(A \cap B)$ is nilpotent too. But AB/B is a definable subgroup of G/B, so it is a homogeneous U_R-group by Proposition 3.22, and by Proposition 3.22 again, G/A is a homogeneous U_R-group. Hence, since $A/(A \cap B) \simeq AB/B$, it follows from Lemma 3.17 that $G/(A \cap B)$ is a homogeneous U_R-group. □

Theorem 5.17.. – *Let G be a definably connected definable group. Then $G/Z(G)$ is the direct product of definable groups $\overline{H_1}, \ldots, \overline{H_k}$ such that for every $i \in \{1, \ldots, k\}$ there is a definable real closed field R_i, an integer n_i and a definable isomorphism from $\overline{H_i}$ to a definable subgroup of $\mathrm{GL}_{n_i}(R_i)$.*

Proof – By Lemma 5.14, we have just to prove that $G/Z(G)$ is definably linear. We proceed by induction on the dimension of G. By Fact 2.12, we may assume $G/Z(G)$ is not centerless. In particular, $U(Z(G))$ is non-trivial by Lemma 5.15. Let R be a definable real closed field such that $U_R(Z(G))$ is non-trivial, and A be a G-minimal subgroup of $U_R(Z(G))$ (Definition 3.18). In particular, A is torsion-free (Proposition 3.22). Since A is G-minimal and central in G, it has no proper non-trivial subgroup, and since it is a U_R-group, it is definably isomorphic to R_+.

We show that we may assume that A is the unique G-minimal subgroup of G. Indeed, if G has another G-minimal subgroup $B \neq A$, we consider $Z_A/A = Z(G/A)$ and $Z_B/B = Z(G/B)$. By induction hypothesis, the groups G/Z_A and G/Z_B are definably linear, so $G/(Z_A \cap Z_B)$ is definably linear by Lemma 5.13. Since A and B are G-minimal, the group $A \cap B$ is finite, and since G is definably connected, $A \cap B$ is central in G. Thus, for each $z \in Z_A \cap Z_B$, the map $u_z : G \to A \cap B$ is a definable group homomorphism, and since G is definably connected, its image is a definably connected subgroup of the finite subgroup $A \cap B$. Therefore z is centralizes G and $Z_A \cap Z_B = Z(G)$. Now $G/Z(G)$ is definably linear, and we may assume that A is the unique G-minimal subgroup of G. In particular, $U(G)$ is a U_R-group (Proposition 3.27).

Let $Z/A = Z(G/A)$. We show that $Z = U_R(Z)Z(G)$ and that $Z/Z(G)$ is a U_R-group. By induction hypothesis, the group G/Z is definably linear. For each $g \in G$, the map $u_g : Z \to A$ defined by $u_g(x) = [g, x]$ is a group homomorphism, and since $A \simeq R_+$, we have either $\mathrm{Ker}\, u_g = Z$ or $Z/\mathrm{Ker}\, u_g \simeq R_+$. It follows from Lemma 5.16 that $Z/Z(G)$ is a U_R-group. Then, by Proposition 3.14, we have $Z = U_R(Z)Z(G)$.

Since G/Z is definably linear, Lemma 5.14 says that G is definably isomorphic to a direct product $K_1/Z \times \cdots \times K_k/Z$, where K_i/Z is a definably linear group over a definable real closed field R_i for each $i = 1, \ldots, k$. Since G is definably connected, K_i/Z is definably connected for each $i = 1, \ldots, k$. Moreover, we may assume that $R = R_1$, and that the fields $R_1, \ldots, R_k$ are not definably isomorphic.

We note that we does not say that K_1/Z is non-trivial. We show that $[K_1, K_j]$ is contained in $Z(G)$ for each $j \neq 1$. For each $g \in K_1$ and each $j \neq 1$, the map $\overline{ad_j}_g : K_j \to Z/Z(G)$ defined by $\overline{ad_j}_g(x) = [g, x]$ is a group homomorphism, and $K_j/\mathrm{Ker}\,\overline{ad_j}_g$ is definably isomorphic to a subgroup of $Z/Z(G)$, so it is a U_R-group (Proposition 3.22). Since $\mathrm{Ker}\,\overline{ad_j}_g$ contains Z, either $K_j = \mathrm{Ker}\,\overline{ad_j}_g$, or the group K_j/Z has a normal definable subgroup N/Z such that $(K_j/Z)/(N/Z)$ is definably isomorphic to R_+. In the second case, since $(K_j/Z)/(N/Z)$ is definably isomorphic to a definable linear group over R_j by Proposition 5.11, the fields R_j and $R = R_1$ are definably isomorphic by Proposition 2.15, contradicting $j \neq 1$. Thus we have $K_j = \mathrm{Ker}\,\overline{ad_j}_g$, and $[K_1, K_j]$ is contained in $Z(G)$ for each $j \neq 1$.

Let $j \neq 1$ and let $H_j/Z(G)$ be a definable subgroup of $K_j/Z(G)$ such that $K_j = ZH_j$, and minimal for this condition. We prove that K_1 centralizes H_j. Since $Z/Z(G)$ is a U_R-group, it is definably connected, and since K_j/Z is definably connected too, the group $K_j/Z(G)$ is definably connected, so such a subgroup $H_j/Z(G)$ is definably connected. By the previous paragraph, for each $g \in K_1$ and each $j \neq 1$, we may define a group homomorphism $ad_{j\,g} : H_j \to Z(G)$ by $ad_{j\,g}(x) = [g, x]$. Therefore $H_j/\mathrm{Ker}\,ad_{j\,g}$ is definably isomorphic to the subgroup $\mathrm{Im}\,ad_{j\,g}$ of $Z(G)$. In particular, $H_j/\mathrm{Ker}\,ad_{j\,g}$ is abelian. Since $\mathrm{Ker}\,ad_{j\,g}$ contains $Z(G)$ and since $H_j/Z(G)$ is definably connected, Lemma 4.3 gives $H_j = R_j\mathrm{Ker}\,ad_{j\,g}$ where $R_j/Z(G) = R(H_j/Z(G))$. Now R_j is a normal solvable subgroup of G, and $[G, R_j]$ is a U-group (Corollary 3.31 and Theorem 4.10). Thus, since $U(G)$ is a U_R-group by the second paragraph, $[G, R_j]$ and $\mathrm{Im}\,ad_{j\,g}$ are U_R-subgroups (Proposition 3.22). So, if $\mathrm{Im}\,ad_{j\,g} \simeq H_j/\mathrm{Ker}\,ad_{j\,g}$ is non-trivial, then $H_j/\mathrm{Ker}\,ad_{j\,g}$ has a proper normal definable subgroup $N/\mathrm{Ker}\,ad_{j\,g}$ such that H_j/N is definably isomorphic to R_+. By minimality of H_j, we have $K_j \neq ZN$, so we obtain $(Z \cap H_j)N < H_j$, and since $H_j/N \simeq R_+$ has no non-trivial proper definable subgroup (Fact 2.10), we have $Z \cap H_j \leq N$. Thus we obtain

$$(K_j/Z)/(NZ/Z) \simeq K_j/NZ = H_jZ/NZ \simeq H_j/(H_j \cap NZ) = H_j/N \simeq R_+$$

Now, by Propositions 2.15 and 5.11, the fields $R_1 = R$ and R_j are definably isomorphic, contradicting $j \neq 1$. Consequently $\mathrm{Im}\,ad_{j\,g}$ is trivial and every $g \in K_1$ centralizes H_j.

In particular, the previous paragraph shows that $H_2, \ldots H_k$ centralize $Z \leq K_1$, and since $G = K_1H_2 \cdots H_k$, we obtain $C_Z(K_1) = C_Z(G) = Z(G)$. Then, for each $j = 2, \ldots k$, we have $H_j \cap Z \leq C_Z(K_1) = Z(G)$, and by the previous paragraph again, we have $C_{K_j}(K_1) = C_{ZH_j}(K_1) = H_jC_Z(K_1) = H_jZ(G) = H_j$. Consequently H_j is a normal definable subgroup of G. Moreover, we note that $Z \cap (H_2 \cdots H_k)$ is contained in $Z \cap C_G(K_1) = Z(G)$. Thus, since G/Z is the direct product of $K_1/Z, \ldots K_k/Z$, and since $K_i = ZH_i$ for each $i = 2, \ldots, k$, we obtain

$$G/Z(G) = K_1/Z(G) \times H_2/Z(G) \times \cdots \times H_k/Z(G)$$

But for each $i = 2, \ldots, k$, the group $H_i/Z(G) = H_i/(H_i \cap Z) \simeq H_iZ/Z = K_i/Z$ is definably linear over R_i. Hence we have just to prove that $K_1/Z(G)$ is definably linear over R.

Let $U/U_R(G) = U_R(G/U_R(G))$. We show that $C_Z(U) = Z(G)$. For each $z \in Z$, we consider the definable group homomorphism $v_z : G \to A$ defined by $v_z(x) = [x, z]$. Since A is definably isomorphic to R_+, the group $G/\mathrm{Ker}\,v_z \simeq \mathrm{Im}\,v_z$

is a U_R-group for each $z \in Z$, and $G/C_G(Z)$ is a U_R-group by Lemma 5.16. Moreover, Lemma 4.3 shows that $G = R(G)C_G(Z)$. In particular, $R(G)/(R(G) \cap C_G(Z))$ is a nilpotent group. Let $D = U_R(R(G) \cap C_G(Z))$. Since $R(G)/U(G)$ is abelian (Proposition 4.7) and since $U(G)$ is a U_R-group, the group $[R(G), R(G) \cap C_G(Z)]$ is a definable subgroup of $U_R(G)$ (Corollary 4.9), and it is contained in D (Proposition 3.22). This implies that $R(G)/D$ is a nilpotent group. Since $R(G)/(R(G) \cap C_G(Z)) \simeq G/C_G(Z)$ is a U_R-group and D is contained in $R(G) \cap C_G(Z)$, Proposition 3.14 shows that the subgroup V defined by $V/D = U_R(R(G)/D)$ covers $R(G)/(R(G) \cap C_G(Z))$ and $G/C_G(Z)$. Moreover, $U_R(G)$ contains D and, since $U_R(G)/D$ is a U_R-group (Proposition 3.14), the group V/D contains $U_R(G)/D$, and $V/U_R(G) \simeq (V/D)/(U_R(G)/D)$ is a normal U_R-subgroup of $G/U_R(G)$. Thus V is contained in U, and we obtain $G = VC_G(Z) = UC_G(Z)$, so $C_Z(U) = Z(G)$.

Let $\mathscr{R}$ be a definable expansion of $(R, <, +, \cdot\,)$ such that, for each integer n, all the definable relations of R^n are $\mathscr{R}$-definable. By Proposition 5.2 and Corollary 5.3, the groups $U_R(G)$, U and $Z/Z(G) = U_R(G)Z(G)/Z(G)$ are definably isomorphic to $\mathscr{R}$-definable groups. Moreover, since K_1/Z is definably linear over R, Proposition 5.2 says that $K_1/Z(G)$ is definably isomorphic to an $\mathscr{R}$-definable group. We consider the semi-direct product $L = U \rtimes K_1/Z(G)$ where $K_1/Z(G)$ acts by conjugation on U. Then, by Proposition 5.2 again, the group L is definably isomorphic to an $\mathscr{R}$-definable group. Now $L/Z(L)$ is definably linear over L (Fact 5.1). Let $N/Z(G) = Z(L) \cap K_1/Z(G)$. Then K_1/N is definably linear over R. Moreover, since $C_Z(U) = Z(G)$ by the previous paragraph, $Z(L) \cap Z/Z(G)$ is trivial, so $N \cap Z = Z(G)$. Since K_1/Z and K_1/N are definably linear over R, Lemma 5.13 shows that $K_1/Z(G)$ is definably linear over R, and we conclude that $G/Z(G)$ is definably linear over R. □

We may state Theorem 5.17 under the following formulation.

Corollary 5.18.. – *Let G be a definably connected definable group. Then G is the central product of definable subgroups $H_1, \dots, H_k$ such that for every $i \in \{1, \dots, k\}$ there is a definable real closed field R_i, an integer n_i and a definable isomorphism from $H_iZ(G)/Z(G)$ to a definable subgroup of $\mathrm{GL}_{n_i}(R_i)$.*

Proof – We may assume that G is ω-saturated. We consider the groups $H_i/Z(G) = \overline{H_i}$ in Theorem 5.17, and we assume that the fields $R_1, \dots, R_k$ are not definably isomorphic. We have just to prove that the groups $H_i/Z(G) = \overline{H_i}$ in Theorem 5.17 satisfy $[H_i, H_j] = 1$ for each $j \neq i$. By Facts 2.6 (1) and 4.2, there are $a \in H_i$ and $b \in H_j$ such that $[a, b]$ is of infinite order. We consider the maps $u : H_j \to Z(G)$ defined by $u(x) = [a, x]$ and $v : H_i \to Z(G)$ defined by $v(x) = [x, b]$. They are definable group homomorphisms, and since $I = \mathrm{Im}\, u \cap \mathrm{Im}\, v$ contains $[a, b]$, the group I is infinite. But $\mathrm{Ker}\, u$ (resp. $\mathrm{Ker}\, v$) contains $Z(G)$, so $\mathrm{Im}\, u$ (resp. $\mathrm{Im}\, v$) is definably isomorphic to a definable quotient of $H_i/Z(G)$ (resp. $H_j/Z(G)$) which is definably isomorphic to a definable subgroup of $\mathrm{GL}_{n_i}(R_i)$ (resp. $\mathrm{GL}_{n_j}(R_j)$). This implies that I is an infinite definable group which is, by Proposition 5.11, definably linear over R_i and definably linear over R_j. Hence, by Proposition 2.15, the fields R_i and R_j are definably isomorphic, contradicting our hypothesis over the fields $R_1, \dots, R_k$. □

Corollary 5.19.. – *Let G be a definably connected definable group. Then the group $G'Z(G)/Z(G)$ is a definably connected definable subgroup. More precisely, $G'Z(G)/Z(G)$ is a direct product of definably connected definable groups $\overline{H_1}, \dots, \overline{H_k}$*

such that for every $i \in \{1, \ldots, k\}$ there is a definable real closed field R_i and a definable isomorphism between $\overline{H_i}$ and a semialgebraic linear group over R_i.

Proof – This follows from Theorem 5.17 and Lemma 5.9. □

6. A Levi-like decomposition

Conversano exhibited a definably connected definable group G such that $R(G) = Z(G)$ and whose derived subgroup is not definable [5, Example 3.1.7]. Moreover, this group G has no *semisimple subgroup* S such that $G = R(G)S$. This motivates the introduction of *quasi-semisimple groups.*

Definition 6.1.. – *Let S be a definably connected definable subgroup of a definable group G.*

- *S is said to be* semisimple *if it has no infinite abelian normal subgroup;*
- *S is said to be* quasi-semisimple *if $R(S)H < S$ for every proper definable subgroup H of S.*

Remark 6.2.. – It follows from Lemma 5.8 that any semisimple group S is quasi-semisimple.

Conversano and Pillay introduce in [6] *ind-definable semisimple subgroups*, and they show their existence and conjugacy in every definably connected group G definable in an o-minimal expansion $\mathscr{R}$ of a real closed field.

We refer to [6] for the definition of an *ind-definable semisimple subgroup*, and we provide just their main properties.

Fact 6.3.. – [6, Theorem 1.1] *Let $\mathscr{R}$ be an o-minimal expansion of a real closed field K, and let G be an $\mathscr{R}$-definably connected $\mathscr{R}$-definable group. Then G has a maximal ind-definable semisimple subgroup S, unique up to conjugacy in G. Moreover $G = R(G)S$, and the centre $Z(S)$ of S is finitely generated and contains $R(G) \cap S$.*

Furthermore, the following properties are satisfied:

(1) [6, Lemma 2.7] *any ind-definable semisimple subgroup of G is perfect;*
(2) [6, Proof of Theorem 1.1] *there is a maximal semisimple subgroup $T/Z(G)^\circ$ of $G/Z(G)^\circ$ such that $S = T'$.*
(3) [6, Proofs of Lemmas 4.1 and 4.2] *if G is a definable subgroup of* $\mathrm{GL}_n(K)$ *for an integer n, the maximal ind-definable semisimple subgroups of G are precisely its maximal semisimple subgroups.*

We will show that, if S is a subgroup of an $\mathscr{R}$-definably connected $\mathscr{R}$-definable group, then S is a maximal ind-definable semisimple subgroup if and only if it is the derived subgroup of a maximal quasi-semisimple subgroup (Corollary 6.7).

Lemma 6.4.. – *Let G be a definably linear definable group. If G is definably connected, then G has a maximal semisimple subgroup S, unique up to conjugacy in G. Moreover, $G = R(G)S$ and $R(G) \cap S$ is finite and contained in the centre of S.*

Proof – By Lemma 5.14, there are finitely many definable real closed fields $R_1, \ldots, R_k$ such that G is definably isomorphic to a direct product $H_1 \times \cdots \times H_k$, where H_i is a definable subgroup of a linear algebraic group over R_i for each $i = 1, \ldots, k$. By Fact 5.7, for each i, we find in H_i a maximal semisimple subgroup S_i such that $H_i = R(H_i)S_i$ and $R(H_i) \cap S_i$ is finite. Then we have $R(G) =$

$R(H_1)\times\cdots\times R(H_k)$, and $S=S_1\times\cdots\times S_k$ is a semisimple subgroup. In particular, we obtain $G=R(G)S$, and $R(G)\cap S$ is finite. Moreover, since S is definably connected, it centralizes the finite normal subgroup $R(G)\cap S$.

Let T be a maximal semisimple subgroup of G. By Lemma 5.14, we have

$$T=(T\cap H_1)\times\cdots\times(T\cap H_k)$$

In particular, for each i, the subgroup $T\cap H_i$ is a maximal semisimple subgroup of H_i, and by Fact 6.3, the subgroups S_i and $T\cap H_i$ are conjugate in H_i. Thus S and T are conjugate in G. □

Corollary 6.5.. – *Let G be a definably linear definable group. Then G is semisimple if and only if it is quasi-semisimple.*

Proof – By Remark 6.2, we may assume that G is a quasi-semisimple group, and we have just to prove that G is semisimple. By Lemma 6.4, the group G has a semisimple subgroup S such that $G=R(G)S$. Since G is quasi-semisimple, this implies that $G=S$ is semisimple. □

For each subset X of a definable group G, the intersection of all definable subgroups of G containing X is a definable subgroup by descending chain condition on definable subgroups [25, Remark 2.13 (ii)]. This subgroup is denoted by $d(X)$.

Theorem 6.6.. – *Let G be a definably connected definable group. Then G has a maximal quasi-semisimple subgroup S, unique up to conjugacy in G. Moreover*

- $G=R(G)S$;
- $R(G)\cap S$ *is central in* S.

Moreover, $SZ(G)/Z(G)$ is a maximal semisimple subgroup of $G/Z(G)$, S' is a perfect group, $S=d(S')$, and $S/Z(S)$ has no non-trivial normal abelian subgroup.

Proof – By Theorem 5.17, the group $G/Z(G)$ is definably linear. By Corollary 6.5, its semisimple subgroups are precisely its quasi-semisimple subgroups. By Lemma 6.4, it has a maximal quasi-semisimple subgroup $S_0/Z(G)$, unique up to conjugacy in $G/Z(G)$. Moreover, we have

$$G/Z(G)=R(G/Z(G))S_0/Z(G)$$

and $R(G/Z(G))\cap S_0/Z(G)$ is contained in the (finite) centre of $S_0/Z(G)$, and by Lemma 5.8, the subgroup $S_0/Z(G)$ is perfect.

We consider $S=d(S_0')$. Since $S_0/Z(G)$ is perfect, we have $S_0=S_0'Z(G)=S_0''Z(G)$, so $S_0=SZ(G)$ and $S_0'=S'$. In particular, we have $S=d(S')$. Moreover, since $S_0/S_0''=Z(G)S_0''/S_0''$ is abelian, we obtain $S'=S_0'=S_0''=S''$ and S' is perfect.

We show that S is a quasi-semisimple subgroup and that $R(S)$ is contained in $Z(G)\cap S$. Since $SZ(G)/Z(G)=S_0/Z(G)$ is quasi-semisimple, it is definably connected, and we have $S=S^\circ(S\cap Z(G))$. Therefore $S_0'=S'=(S^\circ)'$ is contained in S°, and $S=d(S_0')$ is contained in S° too, so S is definably connected. Since $S/(Z(G)\cap S)\simeq SZ(G)/Z(G)=S_0/Z(G)$ is semisimple, the radical $R(S/(Z(G)\cap S))$ is trivial (Lemma 5.8) and $R(S)$ is contained in $Z(G)\cap S$. Thus, if H is a definable subgroup of S such that $R(S)H=S$, then we have $(Z(G)\cap S)H=S$ and $H'=S'=S_0'$. This implies that H contains $S=d(S_0')=d(H')$, so $H=S$ and S is quasi-semisimple.

We show that any quasi-semisimple subgroup of G is contained in a conjugate of S. Let T be such a subgroup. We may assume that no quasi-semisimple subgroup of G contains properly T. If $H/Z(G)$ is a definable subgroup of $TZ(G)/Z(G)$ such that $R(TZ(G)/Z(G))H/Z(G) = TZ(G)/Z(G)$, then we have

$$TZ(G) = R(TZ(G))H = R(T)H$$

and $T = R(T)(T \cap H)$, so $T \cap H = T$ because T is quasi-semisimple. Therefore H contains T, we have $H/Z(G) = TZ(G)/Z(G)$, and $TZ(G)/Z(G)$ is quasi-semisimple. Now $TZ(G)/Z(G)$ is a semisimple subgroup of $G/Z(G)$, and it is contained in a conjugate of $S_0/Z(G) = SZ(G)/Z(G)$ by Lemma 6.4, so we may assume that $TZ(G)/Z(G)$ is contained in $SZ(G)/Z(G)$. In particular, we have $T' = (TZ(G))' \leq (SZ(G))' = S'$. But $TZ(G)/Z(G)$ is a semisimple group, so it is perfect (Lemma 5.8), and we obtain $TZ(G) = T'Z(G)$ and

$$T = T'(T \cap Z(G)) = d(T')(T \cap Z(G))^\circ = d(T')R(T)$$

Hence, since T is quasi-semisimple, we have $T = d(T')$ and T is contained in $d(S') \leq S$, as desired.

We show that $S/Z(S)$ has no non-trivial normal abelian subgroup. If $A/Z(S)$ is a normal abelian subgroup of $S/Z(S)$, then $Z/Z(S) = Z(C_{G/Z(S)}(A/Z(S)))$ is a definable normal abelian subgroup of $S/Z(S)$, and Z is a definable normal nilpotent subgroup of S. But $R(S)$ is contained in $Z(G) \cap S$, so we have $Z^\circ \leq Z(G)$. Hence Corollary 4.8 implies that

$$[S, A] \leq [S, Z] = [S, Z^\circ] \leq [S, Z(G)] = 1$$

and A is central in S. Thus $S/Z(S)$ has no non-trivial normal abelian subgroup.

We prove that $G = R(G)S$ and that $R(G) \cap S$ is central in S. Since $G/Z(G) = R(G/Z(G))S_0/Z(G)$, we have $G = R(G)S_0 = R(G)SZ(G)$, and since G is definably connected and $R(G)$ contains $Z(G)^\circ$, we obtain $G = R(G)S$. Moreover, $(R(G) \cap S)Z(S)/Z(S)$ is a normal solvable subgroup of $S/Z(S)$. Thus, since the previous paragraph says that $S/Z(S)$ has no non-trivial normal abelian subgroup, $R(G) \cap S$ is contained in $Z(S)$. $\square$

Corollary 6.7.. – *Let $\mathscr{R}$ be an o-minimal expansion of a real closed field, and let G be an $\mathscr{R}$-definably connected $\mathscr{R}$-definable group. Then, for any subgroup S of G, the following conditions are equivalent:*

- *S is a maximal ind-definable semisimple subgroup (in the sense of* [6]*);*
- *S is the derived subgroup of a maximal quasi-semisimple subgroup.*

Proof – Let S be a maximal ind-definable semisimple subgroup of G. By Fact 6.3 (2), there is a maximal semisimple subgroup $T/Z(G)^\circ$ of $G/Z(G)^\circ$ such that $S = T'$. Since $Z(G)/Z(G)^\circ$ is finite, $TZ(G)/Z(G)$ is a maximal semisimple subgroup of $G/Z(G)$. But the maximal semisimple subgroups of $G/Z(G)$ are conjugate by Facts 5.1 and 6.3, so Theorem 6.6 provides a maximal quasi-semisimple subgroup L of G such that $TZ(G)/Z(G) = LZ(G)/Z(G)$. Hence we have

$$S = T' = (TZ(G))' = (LZ(G))' = L'$$

Now the result follows from the conjugacy of the maximal ind-definable semisimple subgroups in G (Fact 6.3) and from the conjugacy of the maximal quasi-semisimple subgroups in G (Theorem 6.6). $\square$

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