O-minimal Čech cohomology

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April 22, 2006

Abstract

We prove the existence of a Čech cohomology theory in arbitrary o-minimal structures with definable Skolem functions satisfying the Eilenberg-Steenrod axioms.

*With partial support from the FCT (Fundação para a Ciência e Tecnologia), program POCTI (Portugal/FEDER-EU).

†Supported by a postdoctoral fellowship from CMAF, Universidade de Lisboa. *MSC:* 03C64; 55N05. *Keywords and phrases:* O-minimal structures, Čech cohomology. Revised: June 6, 2007.

1 Introduction

We fix an arbitrary o-minimal structure $\mathcal{N} = (N, <, \ldots)$ and work in the category of definable sets, X, in $\mathcal{N}$ with the o-minimal site on X, with morphisms being definable continuous maps. The o-minimal site on X is the site whose underlying category is the set of all relatively open definable subsets of X (open in the strong, o-minimal topology) with morphisms the inclusions and *admissible covers* being finite covers by open definable sets. We also use the category of pairs (X, A) of definable sets such that $A \subseteq X$ whose morphisms $f : (X, A) \longrightarrow (Y, B)$ are continuous definable maps $f : X \longrightarrow Y$ such that $f(A) \subseteq B$.

Our main result is the following:

Theorem 1.1 *In the category of pairs $A \subseteq X$ of definable sets in an arbitrary o-minimal structure $\mathcal{N}$ we have a Čech cohomology $(\check{H}^*, d^*)$ functor with coefficients in a fixed abelian group Z such that the following hold:*
Exactness Axiom. *For each $n \in \mathbb{N}$, if $i : (A, \emptyset) \longrightarrow (X, \emptyset)$ and $j : (X, \emptyset) \longrightarrow (X, A)$ are the inclusions of pairs of definable sets then we have a natural exact sequence*

$$\cdots \longrightarrow \check{H}^n(X, A; Z) \stackrel{\check{j}^*}{\to} \check{H}^n(X; Z) \stackrel{\check{i}^*}{\to} \check{H}^n(A; Z) \stackrel{d^n}{\to} \check{H}^{n+1}(X, A; Z) \longrightarrow \cdots .$$

Excision Axiom. *For every pair of definable sets $A \subseteq X$ and definable open subset $U \subseteq X$ such that $\overline{U} \subseteq \mathring{A}$, the inclusion $(X - U, A - U) \longrightarrow (X, A)$ induces isomorphisms*

$$\check{H}^n(X, A; Z) \longrightarrow \check{H}^n(X - U, A - U; Z)$$

for all $n \in \mathbb{N}$.
Homotopy Axiom. *Assume that $\mathcal{N}$ has definable Skolem functions and let $[a, b] \subseteq N$ be a closed interval. If for $c \in [a, b]$,*

$$i_c : (X, A) \longrightarrow (X \times [a, b], A \times [a, b])$$

is the continuous definable map given by $i_c(x) = (x, c)$ for all $x \in X$, then the induced homomorphisms are equal

$$\check{i}_a^* = \check{i}_b^* : \check{H}^n(X \times [a, b], A \times [a, b]; Z) \longrightarrow \check{H}^n(X, A; Z)$$

for all $n \in \mathbb{N}$.
Dimension Axiom. *If X is a one point set, then $\check{H}^n(X; Z) = 0$ for all $n \neq 0$ and $\check{H}^0(X; Z) = Z$.*

The o-minimal setting generalises the semi-algebraic and sub-analytic contexts, and so this theorem generalises the existence of Čech cohomology in semi-algebraic geometry, as described in the appendix of the book on real algebraic geometry by Bochnak, Coste and Roy [1]. Other cohomology theories have been constructed for o-minimal structures of special types in the past. Simplicial and singular cohomologies were constructed in o-minimal expansions of fields by A.Woerheide in his doctoral thesis, a report of which can be found in [10]. A sheaf cohomology has been constructed in [9] for o-minimal structures (with certain extra technical assumptions for the homotopy axiom), which generalised the sheaf cohomology for real algebraic geometry of Delfs, for which he proved the homotopy axiom in [2]. The theory presented here generalises all of these, at least in the case of constant coefficients.

We now explain what is new here compared to the classical theory. The definition of o-minimal Čech cohomology is standard. Given a definable set X, we associate to each admissible cover $\mathcal{U}$ of X its nerve $X_{\mathcal{U}}$ which is a finite abstract simplicial complex; given a continuous definable map $f : X \longrightarrow Y$ between definable sets, we associate to f and each admissible cover $\mathcal{V}$ of Y an abstract simplicial map $f_{\mathcal{V}} : X_{f^{-1}\mathcal{V}} \longrightarrow Y_{\mathcal{V}}$. The next step is to compute the simplicial cohomology of this abstract simplicial data and take the direct limit of the simplicial cohomology data over the directed system of admissible covers. The only difference here to the classical setting is that our admissible covers are finite (and belong to the category of definable sets). This construction will be explained in more detail in Section 2.

With this definition of o-minimal Čech cohomology the verification of the exactness and dimension axioms are immediate and similar to the classical topological case since they are pure homological algebra (see [11] Chapter IX, Section 7 and Section 3, Theorem 3.4 respectively). Similarly we can prove the excision axiom as in [11] Chapter IX, Section 6, Theorem 6.1, since conditions (1) to (4) in this proof clearly hold for admissible covers of a definable set (with the same proof) and the rest of the argument there is pure homological algebra.

The homotopy axiom for topological Čech cohomology relies on the fact that stacked (open) covers over (open) covers of a topological space X are cofinal in the category of (open) covers of $X \times I$ where I is a closed interval in $\mathbb{R}$. See [11] Chapter IX, Section 5, Lemma 5.6. There are two important features in the proof of this fact: (i) it uses the compactness of I and (ii) the stacked refinement of a given cover of $X \times I$ is indexed by X (and so it is usually infinite!). Hence, the classical proof of the cofinality of stacked covers does not work in the o-minimal context: (i) closed intervals $[a, b] \subseteq N$ are usually not compact (e.g. N is non standard) and (ii) admissible covers are finite. Our solution to this problem appears in Section 3 and follows the solution of the corresponding problem in real algebraic geometry (see the appendix of the book by Bochnak, Coste and Roy [1]). More precisely, we use the o-minimal spectrum of definable sets which is a model theoretic generalization of the real spectrum of commutative rings from real algebra and real algebraic geometry.

For basic o-minimal geometry we refer the reader to the book [3] by van den Dries. Since semi-algebraic geometry is a special case of o-minimal geometry, the book [1] is also a good reference. For other o-minimal structures see for example [4], [5], [6], [7], [8] and [13]. In [4] and [5] the reader can find explicit constructions of non standard o-minimal structures.

2 O-minimal Čech cohomology

We define the o-minimal Čech cohomology of an arbitrary definable set X in $\mathcal{N}$ with respect to the o-minimal site with coefficients in a fixed arbitrary abelian group Z.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be any admissible open cover of X with index set I and for $\overline{\alpha} = (\alpha_0, \dots, \alpha_{n-1}) \in I^n$ let $U_{\overline{\alpha}} := U_{\alpha_0} \cap \dots \cap U_{\alpha_{n-1}}$, and $E_n(\mathcal{U}) = \{\overline{\alpha} \in I^n | U_{\overline{\alpha}} \neq \emptyset\}$. Then let $\check{C}^{\,n}(\mathcal{U}; Z)$ be the group of all functions, c from $E_{n+1}(\mathcal{U})$ to Z. The group $\check{C}^{\,n}(\mathcal{U}; Z)$ is called the n'th Čech group of $\mathcal{U}$ with coefficients in Z, and its elements are called the n-cochains of $\mathcal{U}$. The Čech groups form a cochain complex:

$$\cdots \xrightarrow{\delta_{n-1}} \check{C}^{\,n-1}(\mathcal{U}; Z) \xrightarrow{\delta_n} \check{C}^{\,n}(\mathcal{U}; Z) \xrightarrow{\delta_{n+1}} \cdots$$

where, for $c \in \check{C}^{\,n-1}(\mathcal{U}; Z)$ (acting on n-tuples), we let

$$(\delta_n c)(\alpha_0, \dots, \alpha_n) := \sum_{i=1}^{n} (-1)^i c(\alpha_0, \dots, \widehat{\alpha_i}, \dots \alpha_n)$$

(where the hat means that this element of the $(n+1)$-tuple is omitted). We have, due to the reversing effect of the power of -1, that $\delta_{n+1} \circ \delta_n = 0$, so that Im $(\delta_n) \subseteq$ Ker (δ_{n+1}). Thus we can form the o-minimal Čech cohomology groups with respect to this cover:

$$\check{H}^{\,n}(\mathcal{U}; Z) := \frac{\mathrm{Ker}(\delta_{n+1})}{\mathrm{Im}(\delta_n)}.$$

The equivalence classes of elements of Ker δ_n which determine these groups are called cocycles.

As any two admissible covers $\mathcal{U}$ and $\mathcal{V}$ have a common admissible refinement, the collection of all admissible covers of X form a direct system with respect to refinement. The maps induced by the inclusions on the n'th Čech groups with respect to these different admissible covers then turns them, and thus the n'th o-minimal Čech cohomology groups into a directed system (with respect to the different admissible covers). We can thus take the direct limit of these groups, to form

$$\check{H}^{\,n}(X, Z) := \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}; Z),$$

the n'th o-minimal Čech cohomology group of X with coefficients in Z.

To get the full o-minimal Čech cohomology theory, as a contravariant functor from the category of definable sets with the o-minimal site to the category of abelian groups, we need, given a definable continuous map $f : X \longrightarrow Y$ to construct, for each n, a group homomorphism, $\check{f}^{\,n} : \check{H}^{\,n}(Y; Z) \longrightarrow \check{H}^{\,n}(X; Z)$.

First, for a given admissible cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of Y, we define the homomorphism $f^n_{\mathcal{U}} : \check{C}^{\,n}(\mathcal{U}; Z) \longrightarrow \check{C}^{\,n}(f^{-1}\mathcal{U}; Z) : c \mapsto f^n_{\mathcal{U}}c$, where $f^{-1}\mathcal{U} = \{f^{-1}(U_\alpha)\}_{\alpha\in I}$. Let $c \in \check{C}^{\,n}(\mathcal{U}; Z)$ and $\overline{\alpha} = (\alpha_0, \dots, \alpha_n) \in E_{n+1}(f^{-1}\mathcal{U})$. Then $f^{-1}(U_{\alpha_0})\cap\cdots\cap f^{-1}(U_{\alpha_n}) \neq \emptyset$ and so $\emptyset \neq f(f^{-1}(U_{\alpha_0})\cap\cdots\cap f^{-1}(U_{\alpha_n})) \subseteq U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$ and $\overline{\alpha} = (\alpha_0, \dots, \alpha_n) \in E_{n+1}(\mathcal{U})$. Thus we can define $(f^n_{\mathcal{U}}c)(\alpha_0, \dots \alpha_n) := c(\alpha_0, \dots \alpha_n)$. This $f^n_{\mathcal{U}}$ commutes with the boundary map on the Čech groups, δ, and so sends kernels and images to the correct places and so can be combined with the quotient map to give a homomorphism $\check{f}^n_{\mathcal{U}} : \check{H}^{\,n}(\mathcal{U}; Z) \longrightarrow \check{H}^{\,n}(f^{-1}\mathcal{U}; Z)$. We can then take the direct limit over the admissible covers, as before, to give the required homomorphism

$$\check{f}^{\,n} : \check{H}^{\,n}(Y; Z) \longrightarrow \check{H}^{\,n}(X; Z).$$

We now give a combinatorial characterization of o-minimal Čech cohomology with coefficients in the abelian group Z as in the classical case in [11] Chapter IX. In fact we show that, given an admissible cover $\mathcal{U}$, the cochain complex $\{\check{C}^{\,*}(\mathcal{U}; Z), \delta\}$ is naturally chain equivalent to the cochain complex $\{C^*(X_{\mathcal{U}}, Z), \partial\}$, where $X_{\mathcal{U}}$ is the *nerve* of $\mathcal{U}$. This clearly gives the required result. The nerve $X_{\mathcal{U}}$ is the abstract simplicial complex consisting of all simplices whose vertices are a subset J of the index set I of $\mathcal{U} = \{U_i\}_{i\in I}$ and are such that $\bigcap_{j\in J} U_j \neq \emptyset$. The cochain complex $\{C^*(X_{\mathcal{U}}, Z), \partial\}$ is defined to be the dual of the chain complex $C_*(X_{\mathcal{U}})$ which is defined as usual for an abstract simplicial complex, so that, $C^n(X_{\mathcal{U}}, Z) = \mathrm{Hom}(C_n(X_{\mathcal{U}}), Z)$.

First note that, letting $E_n(\mathcal{U}) = \{\overline{\alpha} \in I^n | U_{\overline{\alpha}} \neq \emptyset\}$ and $G(E_n(\mathcal{U}))$ be the free abelian group generated by $E_n(\mathcal{U})$, each element $c \in \check{C}^{\,n}(\mathcal{U}; Z)$ is a map from $E_{n+1}(\mathcal{U})$ to Z and so determines a homomorphism, which we also call c, from $G(E_{n+1}(\mathcal{U}))$ to Z. In fact it is clear that we can identify $\check{C}^{\,n}(\mathcal{U}; Z)$ and $\mathrm{Hom}(G(E_{n+1}(\mathcal{U})), Z)$.

Now define a homomorphism from $G(E_{n+1}(\mathcal{U}))$ into $C_n(X_{\mathcal{U}})$ by specifying its values on the base: the image of $\overline{\alpha} = (\alpha_0, \dots, \alpha_n) \in E_{n+1}(\mathcal{U})$ is the elementary n-chain $\langle\alpha_0, \dots, \alpha_n\rangle$ (which is in $C_n(X_{\mathcal{U}})$ since all its vertice span the n-simplex s with vertices $\alpha_0, \dots, \alpha_n$, which is in $X_{\mathcal{U}}$ since $\bigcap_{j=0}^n U_{\alpha_j} \neq \emptyset$ by the definition of $E_n(\mathcal{U})$). The homomorphism we just defined is clearly actually an isomorphism, and so induces a natural chain equivalence between $\mathrm{Hom}(G(E_{n+1}(\mathcal{U})), Z)$ and $\mathrm{Hom}(C_n(X_{\mathcal{U}}), Z)$, which, by the last paragraph, gives the natural chain equivalence between $\check{C}^{\,*}(\mathcal{U}; Z)$ and $\check{C}^{\,*}(X_{\mathcal{U}}; Z)$.

We now observe that we can view morphisms in this combinatorial way also. Given a definable continuous map $f : X \longrightarrow Y$ between definable sets and an admissible cover $\mathcal{U}$ of Y, we get an abstract simplicial map $f_{\mathcal{U}} : X_{f^{-1}\mathcal{U}} \longrightarrow Y_{\mathcal{U}}$. Taking the induced homomorphism, $f^*_{\mathcal{U}}$, in simplicial cohomology we get that it coincides with the map $\check{f}^*_{\mathcal{U}}$ defined above. Thus the two cohomologies define identical functors.

Let $A \subseteq X$ be a pair of definable sets. If $\mathcal{U}$ is an admissible cover of X and $\mathcal{U}_A$ is the subcollection from $\mathcal{U}$ of those elements that intersect A, then we can form the pair of abstract simplicial complexes $(X_{\mathcal{U}}, A_{\mathcal{U}_A})$ with $A_{\mathcal{U}_A}$ a subcomplex of $X_{\mathcal{U}}$. A refinement $\mathcal{V}$ of $\mathcal{U}$ by open definable subsets gives an abstract simplicial map $(X_{\mathcal{V}}, A_{\mathcal{V}_A}) \longrightarrow (X_{\mathcal{U}}, A_{\mathcal{U}_A})$. We can thus take the direct limit of these groups, to form

$$\check{H}^n(X, A, Z) := \varinjlim_{\mathcal{U}} H^n(X_{\mathcal{U}}, A_{\mathcal{U}_A}; Z),$$

the relative n'th o-minimal Čech cohomology group of the pair (X, A) with coefficients in Z, where the $H^n(X_{\mathcal{U}}, A_{\mathcal{U}_A}; Z)$ are the relative simplicial cohomology groups with coefficients in Z.

Given pairs of definable sets $A \subseteq X$ and $B \subseteq Y$, a definable continuous map $f : X \longrightarrow Y$ such that $f(A) \subseteq B$, and an admissible cover $\mathcal{U}$ of Y with the subcover $\mathcal{U}_B$ associated to B, we get an abstract simplicial map $f_{\mathcal{U}} : X_{f^{-1}\mathcal{U}} \longrightarrow Y_{\mathcal{U}}$ such that $f_{\mathcal{U}}(A_{f^{-1}\mathcal{U}_B}) \subseteq B_{\mathcal{U}_B}$. Thus we can take the induced homomorphism, $f^*_{\mathcal{U}} : H^*(Y_{\mathcal{U}}, B_{\mathcal{U}_B}; Z) \longrightarrow H^*(X_{f^{-1}\mathcal{U}}, A_{f^{-1}\mathcal{U}_B}; Z)$, of the relative simplicial cohomology groups and take limits as before to get the homomorphism

$$\check{f}^* : \check{H}^*(Y, B; Z) \longrightarrow \check{H}^*(X, A; Z).$$

We end this section by observing that if X is defined in an o-minimal expansion of a field and (ϕ, K) is a definable triangulation of X ([3]) then letting $\mathcal{U}_K$ be the (admissible) cover of X given by the open stars of all the simplices of K then we clearly have the identity $K \cong X_{\mathcal{U}_K}$ of abstract simplicial complexes.

Also note that, by the comments in the introduction to Chapter VI of [11] (and Section 6 of that chapter) we can use the ordered chain complex as defined above and get that the resulting cohomology theory is the same as the one we would obtain using the alternating chain complex defined there.

3 The Eilenberg-Steenrod Axioms

As we explained in the introduction, we can easily prove the exactness, the excision and the dimension axioms for the o-minimal Čech cohomology defined in Section 2 by following the proofs in the topological case. For the proof of the homotopy axiom we recall the definition of the o-minimal spectrum of definable sets and use it to prove the o-minimal analogue of the lemma on stacked covers ([11] Chapter IX, Section 5, Lemma 5.6). With this lemma available we finish the proof using standard arguments.

For definable $X \subseteq N^m$ let $\widetilde{X}$ be the *o-minimal spectrum* of X, that is the set of complete m-types of the first-order theory $\mathrm{Th}_N(\mathcal{N})$ which imply a formula defining X, equipped with the topology generated by the basic open sets of the form $\widetilde{U} = \{\alpha \in \widetilde{X} : U \in \alpha\}$ where U is a definable, relatively open subset of X and $U \in \alpha$ means that a formula defining U is in α. See [9] or [12] for basic facts about $\widetilde{X}$ but recall especially that $\widetilde{X}$ is spectral topological space and so in particular it is quasi-compact.

For α a complete m-type over N we let $\mathcal{N}(\alpha)$ be the prime model of the first-order theory of $\mathcal{N}$ over $N \cup \{e\}$, where e is an element realising α.

For definable $X \subseteq N^m$ let $X(N(\alpha))$ be the realization of the definition of X in $N(\alpha)^m$. If $S \subseteq X \times Y$ are definable sets, and $\alpha \in \widetilde{X}$ let

$$S^\alpha = \{y \in N(\alpha)^m | (e, y) \in S(N(\alpha))\}.$$

If $f, g : U \longrightarrow N$ are definable continuous functions such that $f < g$, where U is definable open in N^m, then let $[f, g]_U = \{(x, t) | x \in U \text{ and } f(x) \leq t \leq g(x)\}$ and let $(f, g)_U = \{(x, t) | x \in U \text{ and } f(x) < t < g(x)\}$.

Definition 3.1 Given an admissible cover $\mathcal{U} = \{U_i\}_{i=1}^p$ of a definable set X and continuous definable functions $a = \varphi_{i,0} < \ldots < \varphi_{i,r_i} = b$ mapping U_i to N, the cover $\widehat{\mathcal{U}} = \{U_{i,k} | (i, k) \in W\}$ of $X \times [a, b]$ given by $U_{i,k} = (\varphi_{i,k-1}, \varphi_{i,k+1})_{U_i}$ and $W = \{(i, k) | 1 \leq i \leq p, 1 \leq k \leq r_i - 1\}$ is called a *towered cover* over $\mathcal{U}$. We also call $\mathcal{U}$ the base of $\widehat{\mathcal{U}}$.

Given a towered cover $\widehat{\mathcal{U}}$ of a definable set X as above we say that it is a *nice towered cover* if for any $F \subseteq W$ such that $\bigcap_{(i,k)\in F} U_{i,k} \neq \emptyset$ we have $\bigcap_{(i,k)\in F} U_{i,k} \in \widehat{\mathcal{U}}$ (i.e. any (nonempty) intersection of sets in the cover is in the cover.)

Note that a (nice) towered cover is, by definition, admissible. These covers will play the role of stacked covers in our context.

From now on we assume that $\mathcal{N}$ has definable Skolem functions. This assumption certainly holds when $\mathcal{N}$ expands an ordered group (see [3]). We now prove our analogue of the stacked covers lemma ([11] Chapter IX, Section 5, Lemma 5.6). We will require the following claim which is the o-minimal generalization of 11.5.7 in [1] (for convenience our statement of this is slightly modified).

Claim 3.2 *Given any finite cover of the definable set $X \times [a, b]$ by definable open sets, $\{V_j\}_{j=1}^p$, there is a finite cover of X by definable open sets, $\{U_i\}_{i=1}^q$ and, for each $i = 1, \ldots q$, continuous definable functions $a = \varphi_{i,0} < \ldots < \varphi_{i,r_i} = b$ mapping U_i to N such that for each pair (i, k) with $1 \leq i \leq q$ and $1 \leq k < r_i$ there is some j such that $[\varphi_{i,k-1}, \varphi_{i,k+1}]_{U_i} \subseteq V_j$.*

Proof If $\{V_j\}_{j=1}^p$ is an admissible cover of $X \times [a,b]$ and $\alpha \in \widetilde{X}$ then $\{(V_j)^\alpha\}_{j=1}^p$ is a cover of $[a,b](N(\alpha))$. Since the $(V_j)^\alpha \subseteq [a,b](N(\alpha))$ are open we can find a sequence $a = b_0^\alpha < b_1^\alpha < b_2^\alpha < \ldots < b_{r_\alpha}^\alpha = b$ of elements of $N(\alpha)$ such that for every $1 \leq k \leq r_\alpha$ there is some j such that $[b_{k-1}^\alpha, b_{k+1}^\alpha] \subseteq (V_j)^\alpha$.

Now also note that $[a,b](N(\alpha)) = (X \times [a,b])^\alpha$. By the assumption on definable Skolem functions, the definition of $N(\alpha)$ and Proposition 2.1 in [12] we have that there is a definable open set $U_\alpha \subseteq X$ with $\alpha \in \widetilde{U_\alpha}$ and continuous definable functions $\varphi_1^\alpha, \ldots, \varphi_{r_\alpha-1}^\alpha : U_\alpha \longrightarrow N(\alpha)$ such that for each i, $\varphi_i^\alpha(e) = b_i^\alpha$, where e is a fixed realization of α. Taking $\varphi_0^\alpha = a$ and $\varphi_{r_\alpha}^\alpha = b$, and shrinking U_α if necessary we get that $a = \varphi_0^\alpha < \varphi_1^\alpha \ldots < \varphi_{r_\alpha}^\alpha = b$, by continuity. Again by continuity, and shrinking the U_α more if necessary, we get that for each $1 \leq k \leq r_\alpha$ there is some j such that $[\varphi_{k-1}^\alpha, \varphi_{k+1}^\alpha]_{U_\alpha} \subseteq V_j$, since this holds at e.

Now, since $\alpha \in \widetilde{X}$ was arbitrary the $\widetilde{U_\alpha}$'s cover $\widetilde{X}$. But $\widetilde{X}$ is quasi-compact, so finitely many of them, $\widetilde{U_{\alpha_1}}, \ldots, \widetilde{U_{\alpha_q}}$ cover $\widetilde{X}$. Then taking $U_i = U_{\alpha_i}$ and $\varphi_j^{\alpha_i} = \varphi_{i,j}$ we get an admissible cover, $\{U_i\}_{i=1}^q$, of X and, for each i, continuous definable functions $a = \varphi_{i,0} < \ldots < \varphi_{i,r_i} = b$ from U_i to N as required. □

Lemma 3.3 *The nice towered covers of a definable set X are cofinal in the collection of all admissible covers, ordered by refinement.*

Proof By Claim 3.2 the towered covers of X are cofinal in the collection of all admissible covers, ordered by refinement. Thus we only need show that any towered cover of X has a nice refinement. So take a cover as in Definition 3.1.

For any $F \subseteq W$ such that $\bigcap_{(i,k)\in F} U_{i,k} \neq \emptyset$ we note that the set $U^F = \{x \in X | \exists t((x,t) \in \bigcap_{(i,k)\in F} U_{i,k})\}$ is just the projection of an intersection of definable open sets, and so is open and definable. Also each definably connected component of U^F is definable and open, so we assume U^F is definably connected.

Consider the definable continuous functions $\varphi_{\text{high}}^F = \sup\{\varphi_{i,k|U^F} | (i,k) \in F\}$ and $\varphi_{\text{low}}^F = \inf\{\varphi_{i,k|U^F} | (i,k) \in F\}$. Observe that, in fact both φ_{high}^F and φ_{low}^F are of the $\varphi_{j,l|U^F}$. Now choose any i_0 in $\{i | \exists k(i,k) \in F\}$ and note that for some k_0, k_1 we have $(\varphi_{\text{low}}^F, \varphi_{i_0,k_0})_{U^F} \subseteq U_{i_0,k_0}$ and $(\varphi_{i_0,k_1}, \varphi_{\text{high}}^F)_{U^F} \subseteq U_{i_0,k_1}$. Thus we can define a tower above U^F to be given by the sets $(\varphi_{i_0,k-1}, \varphi_{i_0,k+1})_{U^F}$ for $k_0 \leq k \leq k_1$ and the set $(\varphi_{\text{low}}^F, \varphi_{\text{high}}^F)_{U^F}$.

Now let $\mathcal{U}' = \{U_i\}_{i=1}^q$ be the cover of X obtained by taking all the sets U^F as above. Note that this refines the cover $\mathcal{U}$ since we include the case where F is a singleton, which gives the sets in the original cover. Then define the nice towered cover $\widehat{\mathcal{U}'}$ which refines $\widehat{\mathcal{U}}$ by adding to it the towers above all the U^F's with $|F| > 1$, as described above. □

The next lemma is the o-minimal analogue of [11] Chapter IX, Section 5, Lemma 5.7. We include its proof since nice towered covers are slightly different from stacked covers.

Lemma 3.4 *Suppose $\mathcal{U}$ is an admissible cover of a definable set X. If $X_{\mathcal{U}}$ is a simplex and $\widehat{\mathcal{U}}$ is a nice towered cover over $\mathcal{U}$, then $(X \times [a,b])_{\widehat{\mathcal{U}}}$ is acyclic.*

Proof Note that since $X_{\mathcal{U}}$ is a simplex the elements of $\mathcal{U}$ all intersect. Take any $x \in \bigcap_{U \in \mathcal{U}} U$. Suppose $\widehat{\mathcal{U}} = \{U_i\}_{i \in W}$ and note that $\mathcal{U}_x = \{I_x \cap U_i | i \in W\}$ is an admissible cover of $I_x = \{x\} \times [a,b]$ by definably connected definable subsets, also indexed by W. By the definition of nice towered covers we then get, for any $i, j \in W$, that $U_i \cap U_j \neq \emptyset$ if and only if $(I_x \cap U_i) \cap (I_x \cap U_j) \neq \emptyset$. But by the definition of nerves and the fact that no three distinct members of $\widehat{\mathcal{U}}$ intersect, this means that we have the following identity

$$(I_x)_{\mathcal{U}_x} = (X \times [a,b])_{\widehat{\mathcal{U}}}.$$

Then just note that $I_x = \{x\} \times [a,b]$ is definably homeomorphic to $[a,b]$, so that $(I_x)_{\mathcal{V}}$ is acyclic for any cover $\mathcal{V}$ by definably connected open definable subsets since we have the following claim whose classical analogue is Lemma 5.2 of Chapter IX of [11] and whose proof clearly also hold for our definition of Čech cohomology: If $\mathcal{V}$ is an admissible cover of $[a,b]$ by definably connected open definable sets then $[a,b]_{\mathcal{V}}$ (the nerve of $\mathcal{V}$) is acyclic. □

Proof the homotopy axiom: With Lemma 3.4 available the rest of the proof of the o-minimal homotopy axiom is pure holomogical algebra as in [11]. Indeed, replacing the use of Lemma 5.7 in Chapter IX of [11] by our Lemma 3.4 we get the analogue of Lemma 5.8 in Chapter IX of [11]: if $\widehat{\mathcal{U}}$ is a nice towered cover of X over $\mathcal{U}$ and $A \subseteq X$ is a definable subset, then the abstract simplicial maps

$$(i_a)_{\widehat{\mathcal{U}}}, (i_b)_{\widehat{\mathcal{U}}} : (X_{\mathcal{U}}, A_{\mathcal{U}_A}) \longrightarrow ((X \times [a,b])_{\widehat{\mathcal{U}}}, (A \times [a,b])_{\widehat{\mathcal{U}}_A})$$

determined by the definable maps $i_c : (X, A) \longrightarrow (X \times [a,b], A \times [a,b]) : x \mapsto (x, c)$ with $c = a, b$, induce the same homomorphism in simplicial cohomology. Since nice towered covers are cofinal (Lemma 3.3), we can go to the limit to get the required result. □

We end the paper by pointing out that, as in [11] Chapter I, we get from the axioms for o-minimal Čech cohomology interesting results such as the exactness for triples and the Mayer-Vietoris theorem.

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