

Superrosy dependent groups having finitely satisfiable generics

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Abstract

We develop a basic theory of rosy groups and we study groups of small U^b -rank satisfying NIP and having finitely satisfiable generics: U^b -rank 1 implies that the group is abelian-by-finite, U^b -rank 2 implies that the group is solvable-by-finite, U^b -rank 2, and not being nilpotent-by-finite implies the existence of an interpretable algebraically closed field.

0 Introduction

Motivation

If one hopes to apply geometric stability theory methods to study groups which are not necessarily stable, the weakest possible assumption that seems to be necessary, is rosiness, i.e. the assumption that we have an independence relation satisfying a minimal list of nice properties necessary to develop forking calculus.

A general goal is to apply techniques from stable groups to the much wider class of rosy groups. During the last ten years, significant progress in the studies of groups in simple theories (which are always rosy) has been made. In this paper, following [6], we concentrate on another generalization of stable groups, namely, we will study rosy groups satisfying NIP and having finitely satisfiable generics (definitions to follow). In particular, our results generalize the appropriate theorems about stable groups and definably compact groups definable in o-minimal expansions of real closed fields.

Another motivation is the fact that in the same way as groups of finite Morley rank generalize algebraic groups over algebraically closed fields, the groups that we consider (i.e. superrosy groups satisfying NIP and having finitely satisfiable generics) are a common generalization of algebraic groups over algebraically closed fields and compact Lie groups.

There is a long history of structural theorems about groups in model theory. For instance, one has the following [4]:

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Theorem 0.1. *Let G be a superstable group. Then*
 (a) *if it is of U -rank 1, it is abelian-by-finite, and*
 (b) *if it is of U -rank 2, it is solvable-by-finite.*

In comparison, here are the corresponding theorems about groups in o-minimal structures [11, 8]:

Theorem 0.2. *Let G be a definably connected group definable in a o-minimal structure. Then*

(a) *(Razenj) if it is of dimension 1, it is isomorphic to either $\bigoplus_{p \in P} \mathbb{Z}_{p^\infty} \oplus \bigoplus_\delta \mathbb{Q}$ or $\bigoplus_\delta \mathbb{Q}$, and, in particular, it is abelian.*

(b) *(Nesin, Pillay, Razenj) if it is of dimension 2, and non-abelian, it is $R^+ \rtimes R_{>0}^*$, for some real closed field R , and, in particular, it is solvable.*

Our initial goal was to obtain results along these lines in the more general setting of superrosy groups with hereditary fsg and the NIP. Indeed, we have proved the following theorems.

Theorem 1. *Let G be superrosy with NIP. Then if G is of U^b -rank 1 and has fsg, it is abelian-by-finite.*

Theorem 2. *Let G be superrosy with NIP. Then if G is of U^b -rank 2 and has hereditarily fsg, it is solvable-by-finite.*

While in the case of groups, there are a rich class of non-stable examples of dependent, superrosy groups with finitely satisfiable generics, the same is not true of fields. In fact, we show

Theorem 3. *Suppose that K is a superrosy field and K^+ has fsg. Then K is algebraically closed.*

Finally, we show

Theorem 4. *Assume that G has NIP, hereditarily fsg, $U^b(G)=2$ and G is not nilpotent-by-finite. Then, after possibly passing to a definable subgroup of finite index and quotienting by its finite center, G is (definably) the semidirect product of the additive and multiplicative groups of an algebraically closed field F interpretable in G , and moreover $G = G^{00}$.*

Preliminaries

Throughout, we may assume that we work in a monster model (i.e. large κ -saturated model) \mathfrak{C} of a theory T . G will always denote a definable group in this model, and all collections of parameters, A, B, C etc., are assumed to be of size less than κ . (With the one exception, of course, being when we consider a global type, \mathbf{p} , over all of \mathfrak{C} .) We write G multiplicatively, with identity e .

When we write “definable” or “type definable” without further qualification, we will always mean “definable with parameters” or “type definable with parameters” respectively.

Rosy Theories

A model, M , is said to be rosy if it admits a notion of independence which extends to M^{eq} . More precisely, we have the following definition (See, for instance, [9], or [1] for an alternate treatment):

Definition 0.3. T is rosy if and only if there exists a ternary relation, \perp^* , on both real and imaginary subsets (we treat tuples as subsets) of models of T such that:

1. \perp^* is automorphism invariant.
2. If $c \in \text{acl}(aB) \setminus \text{acl}(B)$, then $a \not\perp_B^* c$.
3. If $a \perp_B^* C$ and $BUC \subseteq D$, then there is some $\tilde{a} \models \text{tp}(a/BC)$ such that $\tilde{a} \perp_B^* D$.
4. There is some λ such that for all a , whenever one has $(B_i)_{i < \alpha}$ with $B_i \subset B_j$ for $i < j$ and $a \not\perp_{B_i}^* B_j$ for $i < j < \alpha$, then $\alpha < \lambda$.
5. If $B \subseteq C \subseteq D$, then $a \perp_B^* D$ if and only if $a \perp_B^* C$ and $a \perp_C^* D$.
6. $C \perp_A^* B$ if and only if $c \perp_A^* B$ for any finite $c \subseteq C$.
7. $a \perp_C^* b$ if and only if $b \perp_C^* a$.

We refer to a relation that satisfies (1) to (7) an independence relation. If $a \not\perp_C^* b$, we say that $\text{tp}(a/Cb)$ $*$ -forks over C . A model is rosy if its theory is rosy.

Remark 0.4. Any theory with a ternary relation on \mathfrak{C}^{eq} satisfying (1) to (4) together with the left to right direction of (5) is rosy, but these need not imply that the ternary relation is an independence relation. We restrict our attention to \perp^* .

Alternatively, in any rosy theory, one may define a particular, well-behaved, notion of independence, namely \mathfrak{b} -forking, and give an equivalent definition of rosiness based on the behaviour of this notion of independence [5].

Definition 0.5. A formula $\delta(x, a)$ strongly divides over A if the formula is not almost over A and $\{\delta(x, a')\}_{a' \models \text{tp}(a/A)}$ is k -inconsistent for some $k \in \mathbb{N}$.

We say that $\delta(x, a)$ \mathfrak{b} -divides over A if we can find some tuple c such that $\delta(x, a)$ strongly divides over Ac .

A formula \mathfrak{b} -forks over A if it implies a (finite) disjunction of formulas which \mathfrak{b} -divide over A .

We say that the type $p(x)$ \mathfrak{b} -divides over A if there is a formula in $p(x)$ which \mathfrak{b} -divides over A ; \mathfrak{b} -forking is similarly defined. We say that a is \mathfrak{b} -independent from b over A , denoted $a \perp_A^{\mathfrak{b}} b$, if $\text{tp}(a/Ab)$ does not \mathfrak{b} -fork over A .

Fact 0.6. A theory is rosy if and only if \mathfrak{b} -forking is a symmetric relation.

Not only is \mathfrak{b} -forking symmetric, in fact we have (e.g. [9]):

Fact 0.7. *In a rosy theory, \mathfrak{b} -independence is an independence relation.*

Of all independence relations, \mathfrak{b} -independence is the weakest in the following sense:

Fact 0.8. *Let \perp^* be an independence relation (or, in fact, any ternary relation satisfying (1)-(4) and the left to right direction of (5)), then $a \perp_C^* b \Rightarrow a \perp_C^{\mathfrak{b}} b$.*

Example 0.9. *1. Any simple theory is rosy.
2. Any o-minimal theory is rosy.
3. The theory of p -adically closed valued fields is not rosy.*

The structures in which we are interested are not just rosy, but *superrosy*. We will define superrosy theories in terms of the $U^{\mathfrak{b}}$ -rank [9].

Definition 0.10. *We define the U^* -rank inductively as follows. Let p be a complete type over some set A . Then,*

1. $U^*(p) \geq 0$ if p is consistent.
2. For any ordinal α , $U^*(p) \geq \alpha + 1$ if there is some tuple a and some type $q \in S(Aa)$ such that $q \supset p$, $U^*(q) \geq \alpha$ and q $*$ -forks over A .
3. For any λ limit ordinal, $U^*(p) \geq \lambda$ if and only if $U^*(p) \geq \sigma$ for all $\sigma < \lambda$.

An important property of U^* is that it satisfies the Lascar inequalities:

Proposition 0.11. $U^*(a/bA) + U^*(b/A) \leq U^*(ab/A) \leq U^*(a/bA) \oplus U^*(b/A)$

Proof. As Theorem 5.1.6 in [12]. ■

Definition 0.12. *T is superrosy if and only if $U^{\mathfrak{b}}(p) < \infty$ for every type p .*

Clearly, a theory is superrosy if and only if there is some independence relation such that that $U^*(p) < \infty$ for every p (recalling, of course, that we insist that an independence relation extends to \mathfrak{C}^{eq}).

Definition 0.13. *For a definable set $X := \varphi(\mathfrak{C}, a)$ we define $U^*(X) := \sup\{U^*(p) : p \in S(a), \varphi(x, a) \in p\}$.*

If T is of finite U^* -rank, then for every X as above, $U^*(X) = U^*(p)$ for some $p \in S(a)$ containing $\varphi(x, a)$. The same conclusion is also true when X is a definable group, as we will show later.

Example 0.14. *1. Any supersimple theory is superrosy, and $U^{\mathfrak{b}}$ -rank equals SU -rank.
2. Any o-minimal theory is superrosy, and $U^{\mathfrak{b}}$ -rank equals dimension.*

Finally, the following lemma about rosy theories is quite useful: namely that although the definition of \mathfrak{b} -dividing requires that one produce a k -inconsistent set, to get \mathfrak{b} -forking one need only find an *almost* k -inconsistent set. That is one need only to find a uniform family of formulas so that the intersection of any k members of the family is finite. There seems to be no reference for this fact, so we give a proof below.

Lemma 0.15. *Suppose that $\varphi(a, b)$, and suppose that for any k distinct realizations, b_1, \dots, b_k , of the non-algebraic $\text{tp}(b/C)$ the conjunction $\bigwedge_{i \leq k} \varphi(x, b_i)$ defines a finite set, but $a \notin \text{acl}(C)$. Then $a \not\downarrow_C^* b$.*

Proof. Suppose that for any k distinct realizations, b_1, \dots, b_k , of $\text{tp}(b/C)$ the conjunction $\bigwedge_{i \leq k} \varphi(x, b_i)$ defines a finite set. By compactness, there is m such that each such conjunction defines a set of size less than m . Thus there is a maximal n (possibly $n = 0$) such that there is $\vec{a} = (a_1, \dots, a_n)$ for which one may find infinitely many $b_i \models \text{tp}(b/C)$ such that each $\varphi(\mathfrak{C}, b_i)$ contains \vec{a} . We may assume, after possibly moving \vec{a} by a C -automorphism, that $\text{tp}(b/C\vec{a})$ is not algebraic and $\varphi(\mathfrak{C}, b)$ contains \vec{a} .

By the maximality of n , we see that for every $a' \in \varphi(\mathfrak{C}, b) \setminus \{a_1, \dots, a_n\}$, $b \in \text{acl}_C(a', \vec{a})$. If $a \in \text{acl}(bC)$, we are done. Otherwise, we can choose $a' \models \text{tp}(a/bC)$ different from a_1, \dots, a_n so that

$$(*) \quad a' \downarrow_{bC}^* \vec{a}.$$

Since $b \in \text{acl}_C(a', \vec{a})$ and $b \notin \text{acl}_C(\vec{a})$, we get $b \not\downarrow_{C\vec{a}}^* a'$. This together with $(*)$ implies $a' \not\downarrow_C^* b$. Hence $a \not\downarrow_C^* b$. ■

Dependence

In addition to rosiness, the groups we will consider satisfy a second condition: *dependence* (often called the non independence property and denoted by NIP):

Definition 0.16. *T is said to have the independence property if there is $\varphi(x, y)$, an $\mathfrak{M} \models T$, and an infinite set $A \subseteq \mathfrak{M}$ such that for any $B \subseteq A$ one can find a c_B such that $\varphi(\mathfrak{M}, c_B) \cap A = B$. Otherwise, T is said to be dependent.*

The only consequence of dependence in groups that we will use is the following, from [2].

Fact 0.17. *If G is a group defined in a dependent theory, for each φ there is some n such that the intersection of any finite family of φ -definable subgroups is an intersection of n members of the family.*

Example 0.18. *1. Any stable theory is dependent. In fact, simple and dependent is the same as stable, and supersimple and dependent is the same as superstable.*

2. p -adically closed fields are dependent.

3. Pseudo-algebraically closed but not algebraically closed fields are not dependent.

Finitely Satisfiable Generics

The final condition satisfied by the groups we consider is that they (and sometimes their definable subgroups) have *finitely satisfiable generics*.

First let us give a precise definition of translations of formulas and types which we will use in this paper.

Definition 0.19. *We assume that G is a group definable in \mathfrak{C} by a formula $G(x)$. Let $g \in G$, $\varphi(x, y)$ be any formula, $\pi(x)$ any partial type (over a small set) and $\mathbf{p}(x)$ any global type containing $G(x)$. We define:*

1. $\varphi^*(x, wy) := (\exists u)(\varphi(u, y) \wedge G(w) \wedge G(u) \wedge x = w \cdot u)$ and $g\varphi(x, y) := \varphi^*(x, gy)$,
2. $g\pi(x) := \{g\varphi(x) : \varphi(x) \in \pi(x)\}$,
3. $g\mathbf{p}(x)$ is the unique global type implied by $\{g\varphi(x) : \varphi(x) \in \mathbf{p}(x)\}$.

It is obvious that $g\varphi(x)$ and $g\pi(x)$ define the sets $g \cdot \varphi(G)$ and $g \cdot \pi(G)$, respectively. If $p \in S(A)$ extends $G(x)$ and $g \in A$, then gp implies the unique complete type over A , and in some situations gp will denote this complete type.

Definition 0.20. *A formula $\varphi(x)$ (or the set $\varphi(G)$) is left generic if there are $g_1, \dots, g_n \in G$ such that $g_1\varphi(G) \cup \dots \cup g_n\varphi(G) = G$. We say that a type is left generic if every formula in it is left generic.*

Definition 0.21. *G has finitely satisfiable generics (or fsg) if there is a global type \mathbf{p} containing $G(x)$ and a model $M \prec \mathfrak{C}$, of cardinality less than the degree of saturation of \mathfrak{C} , such that for all g , $g\mathbf{p}$ is finitely satisfiable in M (i.e. each formula in $g\mathbf{p}$ defines a set which intersects M).*

We say that G has hereditarily fsg if every definable subgroup of G also has fsg.

When we consider a definable subgroup G of some non-saturated model (as in the example below), we say that G has (hereditarily) fsg if the set defined by the same formula in a large saturated model of theory has (hereditarily) fsg.

Remark 0.22. *It is not difficult to see that if $Y \subseteq G$ is generic, and N is a normal definable subgroup, then the image of Y under the quotient map is generic in G/N . Thus, if G has fsg, and N is a normal definable subgroup, then G/N has fsg.*

Example 0.23. 1. *Algebraic groups have hereditarily fsg, as in fact do all stable groups, or even stably dominated groups.*

2. *Compact Lie groups (which are interpretable in o-minimal structures) have hereditarily fsg, by [6].*

3. *The complex numbers, in the language of fields together with a predicate defining the algebraic closure of rational numbers, has fsg, but not hereditarily fsg.*

The following fact (from [6]) is the central result about groups with finitely satisfiable generics:

Fact 0.24. *Suppose that G has fsg as witnessed by \mathbf{p} . Then*

1. *A formula is left generic if and only if it is right generic (so we will skip the words*

‘left’ and ‘right’).

2. \mathfrak{p} is generic.

3. The family of nongeneric subsets of G form an ideal, so every partial generic type can be extended to a global one.

4. G^{00} (i.e. the smallest type definable subgroup of bounded index in G) exists, is type definable over empty set, and is the stabilizer of every global generic type.

We mention here a couple of additional facts to be used later:

Remark 0.25. Note that for every $X \subseteq G$ if $g_1X \cup \dots \cup g_nX$ covers G , then $X^{-1}g_1^{-1} \cup \dots \cup X^{-1}g_n^{-1}$ covers G as well. Thus X generic implies X^{-1} is generic.

Proposition 0.26. Assume G has fsg. If G^{00} is definable, it has a unique global generic type.

Proof. We may assume $G = G^{00}$. By Fact 0.24, at least one global generic type exists. Denote it by \mathfrak{p} . Take any generic formula $\varphi(x)$. It is enough to show that $\varphi(x) \in \mathfrak{p}$.

There is $g \in G$ such that $g\varphi(x) \in \mathfrak{p}$. So $\varphi(x) \in g^{-1}\mathfrak{p}$. But by Fact 0.24, $G = G^{00}$ stabilizes \mathfrak{p} , i.e. $g^{-1}\mathfrak{p} = \mathfrak{p}$. So $\varphi(x) \in \mathfrak{p}$. \blacksquare

1 Rosy groups

The facts about rosy groups we will use are all quite straightforward, and proofs in general follow the proofs about groups in simple and stable theories. However, there is no prior exposition of these facts in the case of rosy groups, so we will provide proofs, or in the case where the proof is identical to that in simple theories, a reference.

We recall the definition of local \mathfrak{p} -ranks, which we will use briefly in 1.3 below and then repeatedly during our proof of the existence of \mathfrak{p} -generics.

Throughout this section, G will denote a group definable by a formula $G(x)$ (over \emptyset) in a monster model \mathfrak{C} of a rosy theory T . Such a group will be called a rosy group.

Definition 1.1. Given a formula $\psi(x)$, a finite set Φ of formulas with object variables x and parameter variables y , a finite set of formulas Θ in the variables y, z , and natural number $k > 0$, we define the $\mathfrak{p}_{\Phi, \Theta, k}$ -rank of ψ inductively as follows:

1. $\mathfrak{p}_{\Phi, \Theta, k}(\psi) \geq 0$ if ψ is consistent.
2. For λ limit ordinal, $\mathfrak{p}_{\Phi, \Theta, k}(\psi) \geq \lambda$ if and only if $\mathfrak{p}_{\Phi, \Theta, k}(\psi) \geq \alpha$ for all $\alpha < \lambda$.
3. $\mathfrak{p}_{\Phi, \Theta, k}(\psi) \geq \alpha + 1$ if and only if there is a $\varphi \in \Phi$, some $\theta(y, z) \in \Theta$ and parameter c such that
 - (a) $\mathfrak{p}_{\Phi, \Theta, k}(\psi \wedge \varphi(x, a)) \geq \alpha$ for infinitely many $a \models \theta(y, c)$, and
 - (b) $\{\varphi(x, a)\}_{a \models \theta(y, c)}$ is k -inconsistent.

Given a (partial) type $\pi(x)$ we define $p_{\Phi, \Theta, k}(\pi(x))$ to be the minimum of $p_{\Phi, \Theta, k}(\psi)$ for $\psi \in \pi(x)$. When Φ and Θ each contain only one formula, we will write $p_{\varphi, \theta, k}(\psi)$.

We recall that a theory is rosy if and only if for each ψ, Φ, Θ, k , the local thorn rank $p_{\Phi, \Theta, k}(\psi)$ is finite. Given a partial type $\pi(x)$, and a $p_{\Phi, \Theta, k}$ -rank, one can always extend $\pi(x)$ to a complete type of the same $p_{\Phi, \Theta, k}$ -rank as $\pi(x)$. Moreover, $a \downarrow_C^b b$ if and only if for each Φ, Θ, k , one has that $p_{\Phi, \Theta, k}(\text{tp}(a/bC)) = p_{\Phi, \Theta, k}(\text{tp}(a/C))$.

First, and easiest, we have a collection of chain conditions. Definitions, (from, e.g., [13]) are repeated here for convenience.

Definition 1.2. A family of groups $\{H_i : i \in I\}$ is called uniformly definable if there is a formula φ and parameters b_i such that H_i is defined by $\varphi(x, b_i)$.

A group satisfies the uniform chain condition, or ucc, if for any formula φ there is an $m_\varphi < \omega$ such that each chain of φ -definable groups has length at most m_φ .

Proposition 1.3. A rosy group has the ucc.

Proof. Note that if there is a chain of length ω of groups $H_i := \varphi(\mathfrak{C}, b_i)$, then by compactness, there is a descending chain of arbitrary length. Let $\{H_i : i < \omega \cdot \omega\}$ be such that for $i > j$, $H_i < H_j$. Then for $k < \omega$, $[H_{k\omega} : H_{(k+1)\omega}] \geq \aleph_0$. Let $\tilde{\varphi}(x, b, d)$ be the formula that says “ x is in the coset of $\varphi(\mathfrak{C}, b)$ that corresponds to the element d of M^{eq} .” This shows that G has infinite $p_{\tilde{\varphi}, \theta, 2}$ -rank for appropriate θ . ■

Proposition 1.4. A superrosy group has the ω dcc, i.e. any descending chain of definable groups, each with infinite index in its predecessor, is finite.

Proof. Suppose such a descending chain $G = H_0 > H_1 > \dots$ exists. Let A be such that each H_i is A -definable. We can choose a sequence $(a_i : i \in \omega)$ such that

1. $b_0 \notin \text{acl}(A)$, $b_1 \notin \text{acl}_A(b_0)$, $b_2 \notin \text{acl}_A(b_0, b_1)$, \dots , where $b_i \in \mathfrak{C}^{eq}$ is a name of the coset $a_i H_{i+1}$,
2. $a_0 H_0 \supseteq a_1 H_1 \supseteq a_2 H_2 \supseteq \dots$

Now take any $a \in \bigcap_{i \in \omega} a_i H_i$. Then $b_i \in \text{acl}_A(a) \setminus \text{acl}_A(b_{<i})$. So $a \not\downarrow_{Ab_{<i}}^b b_i$ for $i \in \omega$, contradicting superrosiness. ■

The proof also shows the following:

Proposition 1.5. A superrosy group of finite U^b -rank also satisfies the ω acc: any ascending chain of definable groups, each with infinite index in its successor, is finite.

Definition 1.6. A group has the intersection chain condition, or icc, if for each φ there is some $m_\varphi < \omega$ such that any chain of intersections of φ -definable groups has length at most m_φ .

Proposition 1.7. A rosy dependent group has the icc.

Proof. Dependent means we may apply Fact 0.17, and rosy means we may apply the ucc. Together, they clearly give the icc. \blacksquare

We never use the full icc in any proof. Rather we use the following:

Corollary 1.8. *Rosy dependent groups have the icc on centralizers.*

Of particular use is the centralizer connected component of G .

Definition 1.9. *The centralizer connected component is the intersection of all centralizers of finite index.*

Corollary 1.10. *The centralizer connected component of a rosy dependent group has finite index in this group and is \emptyset -definable.*

\mathfrak{p} -Generics

We define now a different notion of generic for a given independence relation \perp^* , modeled after the notion of a generic type in simple theories. We first introduce a definition of a $*$ -generic type, and prove several facts about them assuming that they exist, and then show that they exist in the particular case where the independence relation is \perp^b . Most proofs are simply obtained from the standard proof in simple theories by replacing \perp by \perp^* . In these cases, we give a reference rather than a proof.

It is important to note that we deal with two different notions, both regularly called “generic”. If we say simply “generic”, we will always mean generic in the sense that finitely many translates cover G . The notion of genericity that arises from an independence relation \perp^* , which we are about to introduce, will always be referred to as $*$ -generic.

Definition 1.11. *We say that a type, $p \in S(A)$, extending $G(x)$ is left $*$ -generic over A if for all $a, b \in G$ with $a \models p$ and $a \perp_A^* b$, one has that $b \cdot a \perp^* Ab$. We say that it is right $*$ -generic over A if, for a, b as above, we have $a \cdot b \perp^* Ab$. A type is $*$ -generic if it is both right and left $*$ -generic.*

Lemma 1.12. 1. *If p is left (right) $*$ -generic then p does not $*$ -fork over the empty set.*

2. *Let $a, b \in G$. If $\text{tp}(a/A)$ is left $*$ -generic and $b \in \text{acl}(A)$, then $\text{tp}(b \cdot a/A)$ is also left $*$ -generic.*

3. *Let $p(x)$ be a type containing $G(x)$, and let q be a non- $*$ -forking extension of p . Then p is left $*$ -generic if and only if q is left $*$ -generic.*

4. *If $p \in S(A)$ and $B \subseteq A$, and p is $*$ -generic, then so is $p|_B$.*

5. *If $p \in S(A)$ is left $*$ -generic, then p^{-1} is as well, where we define p^{-1} to be $\text{tp}(a^{-1}/A)$ where a is any realization of p .*

6. A type is left $*$ -generic if and only if it is right $*$ -generic.
7. If $a \perp_A^* b$ and $\text{tp}(a/A)$ is $*$ -generic, then so is $\text{tp}(b \cdot a/A)$.

Proof. 1. Let $b = e$.

2. Lemma 4.1.2.1 of [12].
3. Lemma 4.1.2.2 and Lemma 4.1.2.3 of [12].
4. Left (right) $*$ -generic types do not $*$ -fork over the empty set.
5. Lemma 4.1.2.4 of [12].
6. Note that p is left $*$ -generic if and only if p^{-1} is right $*$ -generic.
7. Note that $\text{tp}(a/Ab)$ is $*$ -generic, and thus so is $\text{tp}(b \cdot a/Ab)$. From this we see that $\text{tp}(b \cdot a/A)$ is also $*$ -generic. ■

Now we prove the existence of \mathfrak{b} -generic types. As in the corresponding proof in the case of simple theories, the existence of \mathfrak{b} -generics will follow from the definition and examination of translation invariant local ranks.

We introduce the translation invariant stratified \mathfrak{b} -ranks as a specific kind of local \mathfrak{b} -rank:

Definition 1.13. *The stratified $\mathfrak{b}_{\Phi, \Theta, k}^G$ -rank of a partial type $\pi(x)$ is defined as $\mathfrak{b}_{\Phi^*, \tilde{\Theta}, k}(\pi(x) \cup \{G(x)\})$, where $\Phi^* = \{\varphi^*(x, wy) : \varphi(x, y) \in \Phi\}$ and $\tilde{\Theta}(wy, w'z) = \{\theta(y, z) \wedge w = w' \wedge G(w) : \theta \in \Theta\}$.*

While the stratified ranks are defined for formulas, it is clear that if two different formulas define the same set, they have the same rank, so we may speak of the stratified ranks of definable sets as well.

Lemma 1.14. *The stratified rank is translation invariant.*

Proof. By induction on rank: It is clear that for every $\psi(x)$, for every Φ, Θ , and k , and for every $g \in G$, we have $\mathfrak{b}_{\Phi, \Theta, k}^G(\psi(x)) \geq 0$ if and only if $\mathfrak{b}_{\Phi, \Theta, k}^G(g\psi(x)) \geq 0$.

Now suppose that $\tilde{\theta}(wy, hc) \in \tilde{\Theta}$ and $\varphi \in \Phi$ witness that the $\mathfrak{b}_{\Phi, \Theta, k}^G$ -rank of $\psi(x)$ is greater than n . That is, suppose

1. $\mathfrak{b}_{\Phi, \Theta, k}^G(\psi(x) \wedge \varphi^*(x, hb)) \geq n$ for infinitely many $hb \models \tilde{\theta}(wy, hc)$, and
2. $\{\varphi^*(x, hb)\}_{hb \models \tilde{\theta}(wy, hc)}$ is k -inconsistent.

Then $\tilde{\theta}(wy, (g \cdot h)c)$ and $\varphi \in \Phi$ witness that the $\mathfrak{b}_{\Phi, \Theta, k}^G$ -rank of $g\psi(x)$ is greater than n . ■

Lemma 1.15. *For $a \in G$, $a \perp_A^b b$ if and only if for each Φ, Θ, k , we have that $\mathfrak{b}_{\Phi, \Theta, k}^G(\text{tp}(a/Ab)) = \mathfrak{b}_{\Phi, \Theta, k}^G(\text{tp}(a/A))$.*

Proof. For the left to right direction, if $a \perp_A^b b$, then all the local ranks of $\text{tp}(a/Ab)$ and $\text{tp}(a/A)$ are equal, and in particular all of the stratified ranks are equal.

For the right to left direction, first suppose that $\text{tp}(a/Ab)$ \mathfrak{b} -divides over A . This is witnessed by some $\varphi(x, b)$ and $\theta(y, c)$. Thus one has $\{\varphi^*(x, e\tilde{b}) : e\tilde{b} \models \tilde{\theta}(x, ec)\}$ is k -inconsistent. Hence,

$$\mathfrak{b}_{\varphi, \theta, k}^G(\text{tp}(a/A)) > \mathfrak{b}_{\varphi, \theta, k}^G(\text{tp}(a/A) \cup \{\varphi^*(x, eb)\}) \geq \mathfrak{b}_{\varphi, \theta, k}^G(\text{tp}(a/Ab)).$$

Now suppose that $a \not\downarrow_A^{\mathfrak{b}} b$. Thus for some n , $\text{tp}(a/Ab)$ implies a disjunction of formulas $\varphi_i(x, e_i)$, $i \leq n$, such that each \mathfrak{b} -divides over A (witnessed, say, by θ_i and k_i respectively). As only the type of the e_i over Ab matters, we may assume that $a \downarrow_{Ab}^{\mathfrak{b}} e_1$. Thus by the left to right direction of this lemma,

$$\mathfrak{b}_{\varphi_1, \theta_1, k_1}^G(\text{tp}(a/Ab e_1)) = \mathfrak{b}_{\varphi_1, \theta_1, k_1}^G(\text{tp}(a/Ab)).$$

But φ_1, θ_1 , and k_1 witness that $\text{tp}(a/Ab e_1)$ \mathfrak{b} -divides over A , so by the previous paragraph,

$$\mathfrak{b}_{\varphi_1, \theta_1, k}^G(\text{tp}(a/A)) > \mathfrak{b}_{\varphi_1, \theta_1, k_1}^G(\text{tp}(a/Ab e_1)).$$

Thus \mathfrak{b} -forking implies that not all of the stratified \mathfrak{b} -ranks can be equal. \blacksquare

Now it is a simple matter to show the existence of \mathfrak{b} -generic types.

Theorem 1.16. *There is a \mathfrak{b} -generic type for G over A .*

Proof. This is the same proof as the existence of generic types in simple theories in Proposition 4.1.7 in [12], after one replaces $D^*(\pi_i, \varphi_i, k_i)$ with $\mathfrak{b}_{\Phi_i, \Theta_i, k_i}^G(\pi_i)$. \blacksquare

We say that a partial type (or a set defined by this type) is $*$ -generic if it can be extended to a complete $*$ -generic type.

Proposition 1.17. *Let $\pi(x, A)$ be a partial type extending $G(x)$. The the following are equivalent:*

1. π is \mathfrak{b} -generic for G ,
2. $\mathfrak{b}_{\Phi, \Theta, k}^G(\pi)$ is the maximal possible among types in $S_G(A)$, for all Φ, Θ and all k ,
3. For any $g \in G$ the partial type $g\pi$ does not \mathfrak{b} -fork over \emptyset ,
4. For any $g \in G$ the partial type $g\pi$ does not \mathfrak{b} -fork over A .

Proof. The proofs (1) \Rightarrow (2), (3) \Rightarrow (4), and (4) \Rightarrow (1) are the same as those given in Lemma 4.1.9 of [12]. We consider (2) \Rightarrow (3).

Suppose that $g\pi$ \mathfrak{b} -forks over \emptyset . Then for some n , there are $\varphi_i(x, b_i)$, $i \leq n$, whose disjunction is implied by $g\pi$, and such that each \mathfrak{b} -divides over \emptyset . Thus there are θ_i, k_i such that $\{\varphi_i(x, \tilde{b}_i) : \tilde{b}_i \models \theta_i(y, c_i)\}$ is k_i -inconsistent for each i . Let $\Phi = \{\varphi_1, \dots, \varphi_n\}$ and let $\Theta = \{\theta_1, \dots, \theta_n\}$ and let k be the max of k_1, \dots, k_n . Thus for each extension of $g\pi$ to b_1, \dots, b_n one has (as in the proof of Lemma 1.15) that the $\mathfrak{b}_{\Phi, \Theta, k}^G$ -rank of the extension is less than that of $G(x)$. But since for a given Φ, Θ, k there is always some extension of the same $\mathfrak{b}_{\Phi, \Theta, k}^G$ -rank to any set, it must be that the $\mathfrak{b}_{\Phi, \Theta, k}^G$ -rank of $g\pi$ is less than that of $G(x)$. As the stratified ranks are translation invariant, this contradicts (2). \blacksquare

Remark 1.18. *The above was rather easier than the corresponding proofs in the case of simple theories because we restrict our attention to definable groups. In a similar fashion, one could define stratified ranks also for type-definable groups, and prove all the above results in this wider context.*

Question 1.19. *We have obtained \mathfrak{p} -generics by analyzing the local ranks. If one instead works with an abstract independence relation \perp^* without associated local ranks, do $*$ -generic types still exist?*

As mentioned earlier, one may always find a type whose $U^{\mathfrak{p}}$ -rank is equal to that of G . In fact, this is true for any independence relation \perp^* for which $*$ -generics exist.

Remark 1.20. *If p is a $*$ -generic type of G , then $U^*(p) = U^*(G)$. If $U^*(G) < \infty$ and p is such that $U^*(p) = U^*(G)$, then p is $*$ -generic. In particular, since a \mathfrak{p} -generic type exists, $U^{\mathfrak{p}}(G) = U^{\mathfrak{p}}(p)$ where p is a complete \mathfrak{p} -generic type.*

Proof. The proof that every $*$ -generic type has maximal U^* -rank is the same as the proof of a similar fact for SU -rank (e.g. in the remarks at the beginning of Section 5.4 of [12]). For the other direction, consider any type $p \in S(A)$ extending $G(x)$ of maximal U^* -rank. Take $a \models p$ and $b \in G$ such that $a \perp_A^* b$. Then $U^*(p) = U^*(a/A) = U^*(a/Ab) = U^*(ba/Ab) \leq U^*(ba) \leq U^*(p)$. So $ba \perp_A^* Ab$, i.e. p is $*$ -generic. ■

It is easy to see that our two notions of generics are not the same:

Example 1.21. *Let R be an ω -saturated real closed field. We may think of R^2 as C , the algebraic closure of R . Consider the unit circle in R^2 as a multiplicative subgroup of C . Then consider the circle intersect an infinitesimal neighborhood of the point $(1,0)$. This is of $U^{\mathfrak{p}}$ -rank one as is S^1 . Thus it is \mathfrak{p} -generic, but clearly it is not generic.*

On the other hand, we do have the converse:

Proposition 1.22. *A generic type is \mathfrak{p} -generic.*

Proof. Let $X \subseteq G$ be a generic definable set. We must show that each stratified rank of X is maximal. Say $G \subseteq g_1X \cup \dots \cup g_nX$. Thus $G(x)$ implies $\bigvee_{i \leq n} (x \in g_iX)$. Thus the $\mathfrak{p}_{\Phi, \Theta, k}^G$ -rank of G is equal to the maximum of the $\mathfrak{p}_{\Phi, \Theta, k}^G$ -ranks of g_iX . But these are all the same, as the stratified ranks are translation invariant. ■

As in Question 1.19, it is not clear whether every generic type is $*$ -generic. One can say slightly more about the relation between generic and \mathfrak{p} -generic in the case of subgroups. The following is clear, but useful.

Proposition 1.23. *1. Let H be a type definable subgroup of G . H has bounded index in $G \Leftrightarrow H$ is generic in $G \Rightarrow H$ is \mathfrak{p} -generic in G .*

2. Let H be a definable subgroup of G . H has finite index in $G \Leftrightarrow H$ is generic in $G \Leftrightarrow H$ is \mathfrak{p} -generic in G .

In the following considerations, we will often use cosets modulo a definable subgroup H of G . Then, for every $g \in G$, \bar{g} will always denote the coset gH treated as an element of \mathfrak{C}^{eq} .

The following will be useful later.

Proposition 1.24. *Let H be an A -definable normal subgroup of G . If $\text{tp}(g/A)$ is a $*$ -generic of G , then $\text{tp}(\bar{g}/A)$ is a $*$ -generic of G/H . Furthermore, if at least one $*$ -generic type of G exists (e.g. it is the case when $*$ = \mathfrak{p}), then all $*$ -generics of G/H over A arise in this fashion.*

Proof. Assume for simplicity that $A = \emptyset$. Take any $\bar{h} \in G/H$ such that $\bar{g} \downarrow^* \bar{h}$. We need to show that $\bar{h}\bar{g} \downarrow^* \bar{h}$. First we can find $g' \models \text{tp}(g/\bar{g})$ so that $g' \downarrow_{\bar{g}}^* \bar{h}$. Then $g' \downarrow^* \bar{h}$. Now we choose $h' \models \text{tp}(h/\bar{h})$ so that $h' \downarrow_{\bar{h}}^* g'$. So $g' \downarrow^* h'$. Since $\text{tp}(g') = \text{tp}(g)$ is $*$ -generic, we get $h'g' \downarrow^* h'$. As $\bar{h} = \bar{h}' \in \text{acl}(h')$ and $\bar{g} = \bar{g}'$ imply $\bar{h}\bar{g} = \bar{h}'\bar{g}' = \overline{h'g'} \in \text{acl}(h'g')$, we get $\bar{h}\bar{g} \downarrow^* \bar{h}$.

For the converse, take any \bar{g} which is $*$ -generic in G/H . Now choose $h \in G$ which is $*$ -generic in G and $h \downarrow^* g$. Then $\bar{g} \downarrow^* h$, so \bar{g} is $*$ -generic of G/H over h . Since $\bar{h} \in \text{acl}(h)$, we get $\bar{h}\bar{g} \downarrow^* h$. Choose $g' \models \text{tp}(hg/\bar{h}\bar{g})$ so that $g' \downarrow_{\bar{h}\bar{g}}^* h$. Then $h \downarrow^* g'$. Since $\text{tp}(h)$ is $*$ -generic, we get that $g_1 := h^{-1}g'$ is also $*$ -generic. On the other hand, $\bar{g}' = \bar{h}g$ so $\bar{g}_1 = \overline{h^{-1}g'} = \overline{h^{-1}hg} = \bar{g}$. \blacksquare

Definition 1.25. *Suppose H is an A -definable subgroup of G and a coset aH is definable over B . A type $p \in S(B)$ is a $*$ -generic of aH over B if there is a non- $*$ -forking extension $q \in S(ABa)$ of p such that $a^{-1}q$ is $*$ -generic of H .*

It is obvious that p is \mathfrak{p} -generic for aH if and only if it is a type extending $x \in aH$ of maximal possible stratified ranks (equal to the stratified ranks of H). If $U^*(G) < \infty$, then p is $*$ -generic of aH if and only if it is a type extending $x \in aH$ of maximal possible U^* -rank (and $U^*(p) = U^*(H)$). One can also apply the proof of [12, Lemma 4.3.12] to conclude that if g is \mathfrak{p} -generic for G over A , then gH is definable over A, \bar{g} and g is \mathfrak{p} -generic for gH over A, \bar{g} (the proof uses the existence of a \mathfrak{p} -generic in H).

In the final part of this section, we check some basic properties of U^* -rank in groups. In particular, we show a version of Lascar inequalities for groups.

Proposition 1.26. *Assume H is an A -definable subgroup of G . If $\text{tp}(g/A)$ is a $*$ -generic type of G , then $U^*(G/H) = U^*(\bar{g}/A)$. Furthermore, if $U^*(G) < \infty$ and at least one $*$ -generic type of G exists, then all elements of maximal U^* -rank in G/H over A arise in this way.*

Proof. Assume for simplicity that $A = \emptyset$. Take any $h \in G$. We need to show that $U^*(\bar{g}) \geq U^*(\bar{h})$. Wlog $h \downarrow^* g$. Then $\bar{g} \downarrow^* h$. Since g is $*$ -generic, we get that $hg \downarrow^* h$. Let $g_1 = hgh^{-1}$. Since $g_1h = hg$, we see that $g_1h \downarrow^* h$ and hence $\overline{g_1h} \downarrow^* h$. Using this, we get $U^*(\bar{g}) = U^*(\bar{g}/h) = U^*(\overline{h^{-1}g_1h}/h) = U^*(\overline{g_1h}/h) = U^*(\overline{g_1h}) \geq U^*(\overline{g_1h}/g_1) = U^*(\bar{h}/g_1) \geq U^*(\bar{h})$.

The proof of the second part is similar as in Proposition 1.24. Suppose that $\bar{g} \in G/H$ is such that $U^*(\bar{g}) = U^*(G/H)$. Now choose $h \in G$ which is $*$ -generic in G and $h \downarrow^* g$. Then $\bar{g} \downarrow^* h$, so $U^*(\bar{g}/h) = U^*(G/H)$. Since \bar{g} and $\bar{h}g$ are interalgebraic over h , we get that $U^*(\bar{h}g/h) = U^*(G/H) \geq U^*(\bar{h}g)$. So $\bar{h}g \downarrow^* h$. Choose $g' \models \text{tp}(hg/\bar{h}g)$ so that $g' \downarrow_{\bar{h}g}^* h$. Then $h \downarrow^* g'$. Since $\text{tp}(h)$ is $*$ -generic, we get that $g_1 := h^{-1}g'$ is also $*$ -generic. On the other hand, $\bar{g}' = \bar{h}g$ so $\bar{g}_1 = \bar{h}^{-1}g' = \bar{h}^{-1}hg = \bar{g}$. \blacksquare

Proposition 1.27 (Lascar inequalities for groups). *Let H be a definable subgroup of G . Then*

1. $\alpha + U^*(G/H) \leq U^*(G) \leq U^*(H) \oplus U^*(G/H)$ for every $\alpha < U^*(H)$ or $\alpha = U^*(H)$ if there is a $*$ -generic type of H ,
2. $U^b(H) + U^b(G/H) \leq U^b(G) \leq U^b(H) \oplus U^b(G/H)$.

Proof. (2) follows from (1) and the existence of \mathfrak{b} -generics.

(1). Wlog H is \emptyset -definable. First let us prove the right inequality. Take any $\lambda < U^*(G)$ or $\lambda = U^*(G)$ if there is a $*$ -generic type of G . Then, by Remark 1.20, there is $g \in G$ such that $U^*(g) \geq \lambda$. In the following computation, gH denotes a subset of G and $\bar{g} = gH$ an element of \mathfrak{C}^{eq} . Using Lascar inequalities, we get $\lambda \leq U^*(g) = U^*(\bar{g}/g) + U^*(g) \leq U^*(g, \bar{g}) \leq U^*(g/\bar{g}) \oplus U^*(\bar{g}) \leq U^*(gH) \oplus U^*(G/H) = U^*(H) \oplus U^*(G/H)$.

Now we turn to the left inequality. Take any α as in the proposition and $\beta < U^*(G/H)$ or $\beta = U^*(G/H)$ there is an element in G/H of maximal U^* -rank. It is enough to show that $\alpha + \beta \leq U^*(G)$. By Remark 1.20, there are $h \in H$ and $\bar{g} \in G/H$ such that $U^*(h) \geq \alpha$ and $U^*(\bar{g}) \geq \beta$. Wlog we can assume that $h \downarrow^* g$. Then $U^*(gh/\bar{g}) \geq U^*(gh/g) = U^*(h/g) = U^*(h) \geq \alpha$. Let $g_1 = gh$. Then $\bar{g} = \bar{g}_1$ and so, by Lascar inequalities, we get $U^*(G) \geq U^*(g_1) = U^*(\bar{g}_1/g_1) \oplus U^*(g_1) \geq U^*(g_1, \bar{g}_1) \geq U^*(g_1/\bar{g}_1) + U^*(\bar{g}_1) = U^*(gh/\bar{g}) + U^*(\bar{g}) \geq \alpha + \beta$. \blacksquare

\mathfrak{b} -orthogonality and \mathfrak{b} -regular types

We work in our rosy theory T . We define all notions with respect to an arbitrary independence relation \downarrow^* . In particular, everything applies to \mathfrak{b} -independence. We define $*$ -orthogonality and $*$ -regular types in the same way as the corresponding notions are defined in stable theories.

Definition 1.28. *Let p and q be complete types, and A be a set containing each of their domains. If $a \downarrow_A^* b$ for any a, b realizing non- $*$ -forking extensions to A of p, q respectively, then we say that p and q are $*$ -orthogonal. We say that $p \in S(A)$ is $*$ -regular if it is $*$ -orthogonal to all its $*$ -forking extensions.*

Remark 1.29. *$*$ -regularity is preserved under non- $*$ -forking extensions.*

As in the stable case, using Lascar inequalities one can show the following:

Remark 1.30. *Each type of U^* -rank ω^α is $*$ -regular.*

Now we study $*$ -regularity in our group G definable in \mathfrak{C} .

Lemma 1.31. *Let $p, q \in S(A)$ extend $G(x)$ and let p be $*$ -generic. Then there is some g such that $gp \cup q$ is a non- $*$ -forking extension of q (i.e. there is a complete type extending $gp \cup q$ which does not $*$ -fork over A).*

Proof. Take $a \models p$ and $b \perp_A^* a$ with $b \models q$. Since p is $*$ -generic, so is $\text{tp}(a^{-1}/A)$. Since a and a^{-1} are interdefinable, we also have $a^{-1} \perp_A^* b$. Thus we see that $ba^{-1} \perp^* Ab$. We note that $b \models ba^{-1}p \cup q$, a partial type over $A(ba^{-1})$. Since $b \perp_A^* ba^{-1}$, we may extend this partial type to a complete type, q' , over $A(ba^{-1})$ extending q which does not $*$ -fork over A . We let $g := ba^{-1}$. ■

Proposition 1.32. *If $p \in S(A)$ is a $*$ -regular $*$ -generic type of G , then it is $*$ -orthogonal to every non- $*$ -generic type $q \in S(A)$ of G .*

Proof. Wlog $A = \emptyset$. By Lemma 1.31, we can choose g and a non- $*$ -forking extension $r \in S(B)$ of q so that $gp \subseteq r$, where $B = \text{dcl}(g)$. Then, by Lemma 1.12.2 and 1.12.3, we get $g^{-1}r \in S(B)$ is a $*$ -forking extension of p , so by $*$ -regularity of p , $g^{-1}r$ is $*$ -orthogonal to p . Hence

(*) r is $*$ -orthogonal to p .

Now suppose for a contradiction that p and q are not $*$ -orthogonal. Then there is $C \subseteq G$, $a \models p$, and $b \models q$ such that $a \perp^* C$, $b \perp^* C$, and $a \not\perp_C^* b$.

Take any $b' \models r$. Since $\text{tp}(b) = \text{tp}(b')$, we can choose $g' \models \text{tp}(g)$ so that $\text{tp}(gb') = \text{tp}(g'b)$ and $g' \perp_b^* Ca$. As $g \perp^* b'$, we get $g' \perp^* b$ and hence $g' \perp^* Cab$. Thus $a \perp^* Cg'$, $b \perp^* Cg'$, and $a \not\perp_{Cg'}^* b$. It follows that $\text{tp}(b/g')$ is not $*$ -orthogonal to p . This yields a contradiction with (*). ■

2 Groups of U^b -rank 1

In this section G is a definable group in a monster model \mathfrak{C} of a rosy theory T . In all results of this section in which we assume that T satisfies NIP, one can replace this assumption by the weaker condition that G has icc on centralizers (see Corollary 1.8).

Proposition 2.1. *If G contains a \mathfrak{b} -generic involution, then it contains a \mathfrak{b} -generic element g such that $[G : C(g)] < \omega$.*

Proof. Let i be a \mathfrak{b} -generic involution. Choose $j \models \text{tp}(i)$ so that $j \perp^b i$. Then j is \mathfrak{b} -generic over i . So $i \perp^b ij$. Thus $\text{tp}(i/ij)$ is \mathfrak{b} -generic.

On the other hand, $(ij)^i = (ij)^{-1}$. So for every $i' \models \text{tp}(i/ij)$ we also have $(ij)^{i'} = (ij)^{-1}$. Hence $(ij)^i = (ij)^{i'}$, so $i' \in C(ij)i$. We conclude that $C(ij)i$ is definable over ij and a formula defining this set belongs to $\text{tp}(i/ij)$. Since $\text{tp}(i/ij)$ is \mathfrak{b} -generic, we get that $C(ij)i$ and hence $C(ij)$ is \mathfrak{b} -generic. Hence $[G : C(ij)] < \omega$. Moreover, as i is \mathfrak{b} -generic and $i \perp^b j$, we get that ij is also \mathfrak{b} -generic. ■

Corollary 2.2. *If T satisfies NIP and G has a \mathfrak{b} -generic involution, then G is abelian-by-finite.*

Proof.

Let H be the centralizer connected component of G (a definable set by the icc on centralizers). It is of finite index, and we will show that it is abelian. Let g be a \mathfrak{b} -generic involution; by Proposition 2.1, $C(g)$ is a finite index subgroup of G . Thus H is a subgroup of $C(g)$, and hence $g \in C_G(H)$. Since $C_G(H)$ may be extended to a \mathfrak{b} -generic type, it is of finite index in G , and thus $C_G(H) = H$, and the latter is abelian. \blacksquare

The next two corollaries follow immediately from Proposition 2.1 and Corollary 2.2.

Corollary 2.3. *Assume $U^{\mathfrak{b}}(G) = \alpha < \infty$. If G contains an involution of $U^{\mathfrak{b}}$ -rank α , then it contains an element g such that $[G : C(g)] < \omega$ and $U^{\mathfrak{b}}(g) = \alpha$.*

Corollary 2.4. *If T satisfies NIP, and G has an involution i such that $U^{\mathfrak{b}}(i) = U^{\mathfrak{b}}(G) < \infty$, then G is abelian-by-finite.*

Theorem 2.5. *If T satisfies NIP, G has hereditarily fsg, and $0 < U^{\mathfrak{b}}(G) < \infty$, then G contains an infinite definable abelian subgroup.*

Proof. We can replace G by an infinite definable subgroup of least possible $U^{\mathfrak{b}}$ -rank and we can assume that G is centralizer connected. We will show that G is abelian.

If $Z(G)$ is infinite, then $[G : Z(G)] < \omega$, so $G = Z(G)$, and we are done. So we can assume that $Z(G)$ is finite. Let $H = G/Z(G)$. Then H is infinite and each non-trivial element of H has a finite centralizer. Now we will show that this leads to a contradiction.

Claim *There are finitely many conjugacy classes in H .*

Proof. Take any $g \in H \setminus \{e\}$. Since $C(g)$ is finite, we get that $U^{\mathfrak{b}}(g^H) = U^{\mathfrak{b}}(H)$. Now the relation of being in the same conjugacy class is a \emptyset -definable equivalence relation on $H \setminus \{e\}$ whose classes are \mathfrak{b} -generic. Hence there must be only finitely many of them. \square

By the above claim, we get that H^{00} is definable. So wlog $H = H^{00}$. By Proposition 0.26, we get that there is a unique generic type in H . So in virtue of the claim, we get that there is a unique generic conjugacy class a^H .

Case 1. *There is no involution in H .*

Since a^H is generic, $(a^{-1})^H$ is also generic. By uniqueness, $a^H = (a^{-1})^H$. Hence there is $g \in H$ such that $a^{-1} = g^{-1}ag$. Thus $a = g^{-2}ag^2$. Since a is not an involution, we get $a \in C(g^2) \setminus C(g)$. So $C(g) \subsetneq C(g^2)$. Since $C(g)$ is finite and there are no involutions, we get that g has an odd exponent. This implies $C(g^2) = C(g)$, a contradiction.

Case 2. *There is an involution $i \in H$.*

There are two ways to get a contradiction. By [7, Theorem 2.1], we get that there is a non-trivial element with infinite centralizer, a contradiction. Alternatively, we can argue as follows: there is $g \in H$ such that i^g is \mathfrak{p} -generic, but i^g is also an involution, so by Proposition 2.1 (or Corollary 2.2), we get a non-trivial element with an infinite centralizer, a contradiction. \blacksquare

If G has fsg, then every definable subgroup of finite index in G also has fsg. So if every definable subgroup of G is either finite or of finite index in G and G has fsg, then G has hereditarily fsg. Thus we have the following immediate corollary:

Corollary 2.6. *If T satisfies NIP, G has fsg, $U^{\mathfrak{p}}(G) < \infty$, and each definable subgroup of G is either finite or of finite index in G , then G is abelian-by-finite.*

In particular, we have proven Theorem 1:

Theorem 1. *If T has NIP, G has fsg, and $U^{\mathfrak{p}}(G) = 1$, then G is abelian-by-finite.*

Now we are going to modify the proof of Theorem 2.5 (using the idea of the proof of [10, Proposition 7.2]) to generalize Theorem 1 to the case of groups with a \mathfrak{p} -regular \mathfrak{p} -generic type.

Theorem 2.7. *If T satisfies NIP, G has fsg, and at least one \mathfrak{p} -regular \mathfrak{p} -generic type, then G is abelian-by-finite.*

Proof. First of all we can assume that G is centralizer connected.

Claim 1 *The conjugacy class of every non-central element is \mathfrak{p} -generic (i.e. there is an element in this conjugacy class which is \mathfrak{p} -generic over a name of this class).*

Proof. Take any $a \in G$ such that a^G is not \mathfrak{p} -generic. Let $\text{tp}(b/a)$ be a \mathfrak{p} -regular \mathfrak{p} -generic type. Then $b^{-1}ab$ is not \mathfrak{p} -generic over a . By Proposition 1.32, we get $b \not\downarrow_a^{\mathfrak{p}} b^{-1}ab$. Thus b is \mathfrak{p} -generic over $\{a, b^{-1}ab\}$. So the set defined by the formula $x^{-1}ax = b^{-1}ab$ is \mathfrak{p} -generic and definable over $\{a, b^{-1}ab\}$. But this set is equal to $C(a)b$. Hence $C(a)$ is \mathfrak{p} -generic. Thus $[G : C(a)] < \omega$. Since G is centralizer connected, we get $a \in Z(G)$. \square

Claim 2 *G is the union of $Z(G)$ and finitely many \mathfrak{p} -generic conjugacy classes.*

Proof. By Claim 1, G is the union of $Z(G)$ and some number of \mathfrak{p} -generic conjugacy classes. Suppose for a contradiction that there are infinitely many of them. Then the relation of being in the same conjugacy class is \emptyset -definable and it divides $G \setminus Z(G)$ into infinitely many classes. So at least one of these classes, say C , is non-algebraic over \emptyset (as an element of \mathfrak{C}^{eq}). Now we can choose $c \in C$ which is \mathfrak{p} -generic over C . Since $C \in \text{dcl}(c)$ and $C \notin \text{acl}(\emptyset)$, we get $c \not\downarrow^{\mathfrak{p}} C$, a contradiction with Lemma 1.12.1. \square

We will show that $Z(G) = G$. Suppose for a contradiction that it is false. Then $[G : Z(G)] \geq \omega$. By Claim 2 and Proposition 1.24, $G/Z(G)$ is the union of $\{e\}$ and finitely many \mathfrak{p} -generic conjugacy classes. Hence $H := (G/Z(G))^{\text{cl}}$ is definable

and it is also the union of $\{e\}$ and finitely many \mathfrak{b} -generic conjugacy classes. In the same way as in the proof of Theorem 2.5, we conclude that there is a unique generic conjugacy class a^H .

As in the proof of Theorem 2.5, we get that there is an involution $i \in H$. Since i^H is \mathfrak{b} -generic, we get that there is an involution which is \mathfrak{b} -generic. We finish using Proposition 2.1. \blacksquare

By Theorem 2.7 and Remark 1.30, we get the following strengthening of Theorem 1, which generalizes the appropriate result about superstable groups.

Corollary 2.8. *If T satisfies NIP, G has fsg, and $U^{\mathfrak{b}}(G) = \omega^\alpha$, then G is abelian-by-finite.*

At the end let us make a few remarks. Proposition 2.1 and Corollaries 2.2, 2.3, and 2.4 are true for an arbitrary independence relation, \downarrow^* (the same proofs work). Since $U^{\mathfrak{b}}$ -rank is less or equal than the U^* -rank, Theorem 2.5, Corollary 2.6 and Theorem 1 are also true for \downarrow^* . Since Proposition 1.32 is true for \downarrow^* , one can easily check that the whole proof of Theorem 2.7 also works for \downarrow^* .

As to Corollary 2.8, it is true for an arbitrary independence relation \downarrow^* under the additional assumption that there is a type of U^* -rank ω^α (in other words a $*$ -generic exists), in which case it follows immediately from the fact that Theorem 2.7 is true for \downarrow^* ; an alternative way to prove this is to modify the proof of Theorem 2.5 and use Lascar inequalities for groups.

By Proposition 1.22 and Theorem 2.7, we get that if T satisfies NIP, G has fsg and at least one \mathfrak{b} -regular generic type, then G is abelian-by-finite.

Notice that the proofs of Theorems 1 and 2.7 produce a definable abelian subgroup of finite index in G , namely the centralizer connected component of G .

Notice also that when A is an arbitrary abelian subgroup of finite index in G , we may follow the proof of Theorem 3.17 of [10] in considering the intersection of $C(a)$ for $a \in A$. This is definable, as is its center, which is abelian of finite index in G and contains A .

Finally, we should mention a conjecture generalizing the results above. The existence of finitely satisfiable generics implies the weaker condition that G is *definably amenable*. (This means, roughly, that there is a left invariant probability measure on the definable sets of G . See [6] for a precise definition.)

Conjecture 2.9. *In each result in this section, the hypothesis “ G has fsg” may be replaced with “ G is definably amenable”.*

3 Groups of $U^{\mathfrak{b}}$ -rank 2

In this section G is a group definable in a monster model \mathfrak{C} of an arbitrary theory T .

Theorem 3.1. *If G has hereditarily fsg, the definable quotients of definable subgroups of G satisfy icc on centralizers, and $U^{\mathfrak{b}}(G)=2$, then G is solvable-by-finite.*

By Corollary 1.8, Theorem 3.1 implies Theorem 2:

Theorem 2. *If T satisfies NIP, G has hereditarily fsg, and $U^{\mathfrak{b}}(G) = 2$, then G is solvable-by-finite.*

The rest of this section is devoted to the proof of Theorem 3.1. Some ideas are taken from the proof that each connected group of Morley rank 2 is solvable [10, Theorem 3.16]. The main obstacle in comparison with the Morley rank 2 case is that a \mathfrak{b} -generic may not be generic and there may be many \mathfrak{b} -generics even in the connected component.

The structure of the proof is as follows. We suppose for a contradiction that G is not solvable-by-finite. First we define Borels (albeit in a slightly different manner from that which is used in the Morley rank 2 case because we do not have definable connected components), and we study their properties. Then we use them to find involutions. In the last part of the proof we use Borels, involutions and some particular function that comes from the theory of black box groups to get a final contradiction.

Proof of Theorem 3.1. By icc, we can assume that G is centralizer connected. If $Z(G)$ is infinite, then either $[G : Z(G)] < \omega$ or $U^{\mathfrak{b}}(Z(G)) = U^{\mathfrak{b}}(G/Z(G)) = 1$ and then we are done by Theorem 1. So we can assume that $Z(G)$ is finite. Then $G/Z(G)$ is centerless and centralizer connected. So wlog G is centerless. Suppose for a contradiction that G is not solvable-by-finite.

If centralizers of all non-trivial elements in G are finite, we can argue in the same way as in the proof of Theorem 2.5. So we can assume that there is a non-trivial element in G with an infinite centralizer.

Definition 3.2. *We say that a subgroup B of G is a Borel if it is a minimal infinite intersection of centralizers.*

By icc and the last paragraph, we have that at least one Borel exists and all Borels are intersections of finitely many infinite centralizers, so they are definable.

Claim 1. *(i) Every Borel has infinite index in G .*

(ii) All Borels are abelian.

(iii) Any two Borels are either equal or they have trivial intersection.

(iv) Every Borel has finite index in its normalizer.

(v) Every non-trivial element $a \in G$ with an infinite centralizer centralizes exactly one Borel, which will be denoted by $B(a)$. In fact, $B(a)$ is the centralizer connected component of $C(a)$ computed in G .

Proof. (i) This follows from the fact that the centralizer of any non-trivial element has infinite index in G .

(ii) By (i), $U^{\mathfrak{b}}(B) = 1$. So by Theorem 1, there is an abelian subgroup B_0 of finite index in B . By the definition of Borels, B is centralizer connected. Hence $B_0 \leq Z(B)$. So for every $a \in B$, $[B : C_B(a)] < \omega$. Since B is centralizer connected, it must be abelian.

(iii) Suppose $B_1 \neq B_2$ are Borels. Then $B_1 \cap B_2$ is finite. Let $a \in B_1 \cap B_2$. Then $B_1, B_2 \subseteq C(a)$. So $[C(a) : B_1] \geq \omega$, which implies $U^b(C(a)) = 2$, so $[G : C(a)] < \omega$. Hence $a = e$.

(iv) If B is a Borel and $[N(B) : B] \geq \omega$, then $U^b(N(B)/B) = 1$ and $[G : N(B)] < \omega$, so we finish using Theorem 1.

(v) Since $U^b(C(a)) = 1$, if a centralizes a Borel B , then B is a subgroup of finite index in $C(a)$. So by (iii), there is a unique Borel centralized by a , and, of course, it must be the centralizer connected component of $C(a)$ computed in G . \square

The following is an easy corollary of Claim 1.

Claim 2. *If $B_1 \neq B_2$ are Borels, then:*

(i) $C(B_1) \cap C(B_2) = \{e\}$,

(ii) $N(B_1) \cap N(B_2)$ is finite.

Proof. The first part follows from the fact that every non-trivial element centralizes at most one Borel. The second part is an easy consequence of the fact that $B_1 \cap B_2 = \{e\}$ and B_1 and B_2 are subgroups of finite index in $N(B_1)$ and $N(B_2)$, respectively. \square

Claim 3. *If B is a Borel and a is generic over a name of B , then BaB is generic.*

Proof. We have that there are $m, n \in \omega$ such that:

$$(1) \quad BN(B)B = Bb_1B \cup \dots \cup Bb_mB$$
 for some $b_1, \dots, b_m \in N(B)$,

$$(2) \quad B(G \setminus N(B))B = Ba_1B \cup \dots \cup Ba_nB$$
 for some $a_1, \dots, a_n \in G \setminus N(B)$.

Item (1) follows from Claim 1(iv). To see (2), notice that if $a \notin N(B)$, then $B^a \neq B$, hence $B^a \cap B = \{e\}$. So $f : B \times B \rightarrow BaB$ defined by $f(x, y) = xay$ is a definable bijection. Thus $U^b(BaB) = 2$. Moreover, for all $a_1, a_2 \in G$ we have that Ba_1B and Ba_2B are either equal or disjoint. Since $U^b(G) = 2$, we get that there are only finitely many sets of the form BaB for $a \notin N(B)$.

Let d be a name for B . Take any generic a over d . Let $p = \text{tp}(a/d)$. By (1) and (2), we get that $p(G)$ is covered by Ba'_1B, \dots, Ba'_kB for some $a'_1, \dots, a'_k \in p(G)$

Let \mathbf{p}' be a global generic extension of p . Then there is i such that a formula defining the set Ba'_iB is in \mathbf{p}' . So we have proved that there is $a' \models p$ such that $Ba'B$ is generic. Hence BaB is generic. \square

In the remainder of the proof, we analyze $B \cap G^{00}$. The next claim is not necessary to finish the proof, but we include it, since it gives us a better understanding of Borels (and lets us simplify our notation).

Claim 4. *Every Borel is relatively connected in G^{00} , i.e. $B^{00} = B \cap G^{00}$.*

Proof. The inclusion $B^{00} \subseteq B \cap G^{00}$ is obvious. We will prove $B^{00} \supseteq B \cap G^{00}$.

Suppose for a contradiction that there is $c \in (B \cap G^{00}) \setminus B^{00}$. Let $a \in G^{00}$ be generic over a name of B .

By Claim 3, we easily get that boundedly many two-sided translates of $B^{00}aB^{00}$ cover G , so all these translates are defined by partial generic types. Let \mathbf{p} be a global generic type extending a partial generic type defining the set $cB^{00}aB^{00}$. Then a partial type defining the set $B^{00}aB^{00}$ is contained in $c^{-1}\mathbf{p} = \mathbf{p}$ (as $c \in G^{00}$ and $\text{Stab}(\mathbf{p}) = G^{00}$). Hence we get

$$B^{00}aB^{00} \cap cB^{00}aB^{00} \neq \emptyset.$$

Combining this with the fact that the function $f : B \times B \rightarrow G$ defined by $f(b_1, b_2) = b_1ab_2$ is injective and that $cB^{00} \cap B^{00} = \emptyset$, we get a contradiction. \square

Claim 5. *If B is a Borel and $a \in N(B) \setminus C(B)$, then $C(a)$ is finite.*

Proof. First notice that we have

(*) $a^{N(B)}$ is infinite.

Otherwise, a^B would also be finite. So $C(a) \cap B$ would be infinite. Hence $C(a) \cap B = B$, which would mean that $a \in C(B)$, a contradiction.

For any g we have $a^{N(B)g} = (a^{N(B)})^g \subseteq N(B)^g = N(B^g)$. Using this and Claim 2(ii) we see that

(**) if $N(B)g_1 \neq N(B)g_2$, then $a^{N(B)g_1} \cap a^{N(B)g_2}$ is finite.

Now add a and a name of B to the language. Take g so that $N(B)g \notin \text{acl}^{eq}(\emptyset)$. Let $\phi(x, y)$ be a formula over \emptyset such that $\phi(\mathfrak{C}, N(B)g) = a^{N(B)g}$ (on the left hand side $N(B)g$ is treated as an element of the sort $N(B) \setminus G$ and on the right hand side it is treated as a set). By (*), (**) and Lemma 0.15, we get that there is $b \in \phi(\mathfrak{C}, N(B)g) = a^{N(B)g}$ such that $U^b(b) = 2$. Hence, working in the original language, $U^b(b/a) = 2$. Since $a^G = b^G$, we get that $U^b(a^G) = 2$, so $C(a)$ is finite. \square

Now we will combine the above result and the proof of Claim 4 to get that Borels are relatively self-normalizing in G^{00} , i.e for every Borel B we have $N_{G^{00}}(B \cap G^{00}) = B \cap G^{00}$. By Claim 1(iii), this is equivalent to the condition $N(B) \cap G^{00} = B \cap G^{00}$, and by Claim 4, to the statement $N_{G^{00}}(B^{00}) = B^{00}$.

Claim 6. *All Borels are relatively self-normalizing in G^{00} .*

Proof. Take any Borel B and $a \in G^{00}$ generic over a name of B . Suppose for a contradiction that there is $c \in (N(B) \cap G^{00}) \setminus (B \cap G^{00})$. The same argument as in the proof of Claim 4 yields $B^{00}aB^{00} \cap cB^{00}aB^{00} \neq \emptyset$. Hence

$$(!) \quad BaB \cap cBaB \neq \emptyset.$$

Subclaim *The function $f : N(B) \times B \rightarrow G$ defined by $f(b_1, b_2) = b_1ab_2$ is injective.*

Proof. Suppose $b_1ab_2 = c_1ac_2$ for some $b_1, c_1 \in N(B)$ and $b_2, c_2 \in B$. Let $b = c_1^{-1}b_1$. We see that $b^a \in B$, so $b \in B^{a^{-1}}$. Hence $C(b)$ is infinite. On the other hand, $b \in N(B)$, so by Claim 5, we get that $b \in C(B)$.

Since $a \notin N(B)$, we have that $B^{a^{-1}} \neq B$, so by Claim 2(i), $C(B) \cap B^{a^{-1}} = \{e\}$. Since $b \in C(B) \cap B^{a^{-1}}$, we get $b = e$. So $b_1 = c_1$ and then also $b_2 = c_2$. \square

Subclaim and (!) yield a contradiction. \square

So far we have been studying various properties of Borels. Now we will use Borels to find an involution in G^{00} .

Claim 7. *There is an involution in G^{00} .*

Proof. By fsg, there is a generic element $a \in G^{00}$. Moreover, we have at least one Borel B . So we can choose $a \in G^{00}$ generic over a name of B .

By Claim 3, we get

(*) BaB and $Ba^{-1}B$ are generic.

Subclaim. $B^{00}aB^{00} = B^{00}a^{-1}B^{00}$.

Proof. It is clear that $B^{00}aB^{00}$ and $B^{00}a^{-1}B^{00}$ are contained in G^{00} . Let π be a partial type defining the set $B^{00}a^{-1}B^{00}$.

Since by (*), boundedly many two-sided translates of $B^{00}a^{-1}B^{00}$ cover G , π is a partial generic type. Let \mathbf{p} be any extension of π to a global generic.

Consider any definable subset X of B containing B^{00} . Then finitely many right translates of BaX cover BaB , so by (*), BaX is generic. Hence boundedly many left translates of $B^{00}aX$ cover G , which implies that one of them, say $gB^{00}aX$, is defined by a partial type which is contained in \mathbf{p} .

Since boundedly many right translates of $gB^{00}aB^{00}$ by elements from X cover $gB^{00}aX$, one of them, say $gB^{00}aB^{00}x$ where $x \in X$, is defined by a partial type which is contained in \mathbf{p} .

By the definition of π , we have that \mathbf{p} extends the partial type $x \in G^{00}$, so $gB^{00}aB^{00}x \cap G^{00} \neq \emptyset$. Hence $gx(x^{-1}B^{00}aB^{00}x) \cap G^{00} \neq \emptyset$. Thus $gx \in G^{00}$, which implies that $g = hx^{-1}$ for some $h \in G^{00}$.

So a partial type defining $B^{00}aX$ is contained in $g^{-1}\mathbf{p} = xh^{-1}\mathbf{p} = x\mathbf{p}$ (as $\text{Stab}(\mathbf{p}) = G^{00}$). On the other hand, a partial type defining $xB^{00}a^{-1}B^{00}$ is also contained in $x\mathbf{p}$. Hence

$$(!) \quad B^{00}aX \cap xB^{00}a^{-1}B^{00} \neq \emptyset.$$

So we have proved that for every definable subset X of B containing B^{00} there is $x \in X$ such that (!) holds. Thus by the compactness theorem, $B^{00}aB^{00} \cap B^{00}a^{-1}B^{00} \neq \emptyset$ and hence $B^{00}aB^{00} = B^{00}a^{-1}B^{00}$. \square

By the subclaim, there is $b \in B^{00}$ such that $(ab)^2 \in B$. Put $i := ab$. Then, of course, $i \in G^{00}$. We will show that i is an involution.

Since a is generic over a name of B and $\text{U}^b(B) = 1$, we get that $i \neq e$. Suppose for a contradiction that $i^2 \neq e$. As $i^2 \in B$, we have $B(i^2) = B$. But i centralizes i^2 , so $B^i = B(i^2)^i = B(i^2) = B$. Thus $i \in N(B)$, which implies $a \in N(B)$, a contradiction

with the fact that a is generic over a name of B . \square

Notice that by Corollary 2.3, we get that all involutions have infinite centralizers. Indeed, if there was an involution i with a finite centralizer, then $U^b(i^G) = 2$, so there would be an involution of U^b -rank 2. Then by Corollary 2.3, we would get a non-trivial element whose centralizer has a finite index in G , a contradiction.

Claim 8. *If i and j are involutions such that $B(i) \neq B(j)$, then $C(ij)$ is finite.*

Proof. The argument is similar as in the Morley rank 2 case. Indeed, we have $(ij)^i = (ij)^j = ji = (ij)^{-1}$. Hence $B(ij)^i = B((ij)^{-1}) = B(ij)$, which implies that $i \in N(B(ij))$. Similarly, $j \in N(B(ij))$. But i and j have infinite centralizers. Hence by Claim 5, $i, j \in C(B(ij))$. Similarly, by Claim 5, we easily conclude that $i \in C(B(i))$ and $j \in C(B(j))$. In virtue of Claim 2(i), we get $B(i) = B(ij) = B(j)$, a contradiction. \square

Claim 9. *Every Borel B contains at most finitely many involutions.*

Proof. Let I be the set of all involutions. Suppose for a contradiction that $I \cap B$ is infinite. Then $I \cap B^g$ is also infinite for every $g \in G$. Now add a name of B to the language.

Take g so that $N(B)g \notin \text{acl}^{eq}(\emptyset)$. Let $\phi(x, y)$ be a formula over \emptyset such that $\phi(\mathfrak{C}, N(B)g) = I \cap B^{N(B)g} = I \cap B^g$. We see that for any $N(B)g_1 \neq N(B)g_2$ realizing $\text{tp}(N(B)g)$ we have $\phi(\mathfrak{C}, N(B)g_1) \cap \phi(\mathfrak{C}, N(B)g_2) = I \cap B^{g_1} \cap B^{g_2} = \{e\}$.

So by Lemma 0.15, there is an involution $b \in \phi(\mathfrak{C}, N(B)g) = I \cap B^g$ such that $U^b(b) = 2$, a contradiction with Corollary 2.3. \square

Let X be the set of all elements of G^{00} with finite centralizers. By Claim 4 and Claim 6, we see that $G^{00} \setminus X$ is a union of connected components of Borels.

Claim 10. (i) *For every $a \in X$, if $a^n \in G^{00} \setminus X$, then $a^n = e$.*

(ii) *There is no $a \in X$ such that $a^2 \in G^{00} \setminus X$.*

(iii) *Every element of X has an odd exponent.*

Proof. (i) Suppose $a \in X$, $a^n \in G^{00} \setminus X$ and $a^n \neq e$. Since $a^{-1}a^na = a$, we get $B(a^n)^a = B(a^n)$, so by Claim 6, we get $a \in N_{G^{00}}(B(a^n)^{00}) = B(a^n)^{00}$, a contradiction.

(ii) If $a \in X$ and $a^2 \in G^{00} \setminus X$, then by (i), we see that $a^2 = e$. So we get an involution with a finite centralizer, a contradiction.

(iii) Suppose that some $a \in X$ has an even exponent, say $2n$. Then $a^n \neq e$, so by (i), $a^n \in X$. But we also have $(a^n)^2 = e$, a contradiction with (ii). \square

Claim 11. *For every $a \in X$ there is a unique $b \in G^{00}$ such that $b^2 = a$. Moreover, $b \in \langle a \rangle$.*

Proof. By Claim 10(iii), we have that $\langle a \rangle$ is an abelian group of odd order, say n . So there is a unique $b \in \langle a \rangle$ such that $b^2 = a$.

Now suppose $b_1^2 = a$ for some $b_1 \in G^{00}$. Then $b_1 \in X$. Since $b_1^{2n} = a^n = e$, we get $(b_1^n)^2 = e$, so by Claim 10(i, ii), $b_1^n = e$. But $\langle a \rangle$ is a subgroup of $\langle b_1 \rangle$ of cardinality n . Thus $\langle a \rangle = \langle b_1 \rangle$, which implies $b = b_1$. \square

Now we will introduce and use a function similar to that appearing in [3], which comes from the theory of so-called black box groups.

By Claim 7, we can choose an involution $i \in G^{00}$. Let $B := B(i)$. By Claim 6, we have $N_{G^{00}}(B^{00}) = B^{00}$.

Notice that if $g \in G^{00} \setminus B$, then $B(i^g) \neq B(i)$, so by Claim 8, we get $ii^g \in X$. Hence $\sqrt{ii^g}$ is well-defined in G^{00} by Claim 11. So we can define a function $f : G^{00} \setminus B \rightarrow G^{00}$ putting

$$f(g) = \sqrt{ii^g}g^{-1}.$$

Claim 12. $\text{rng}(f) = B^{00}$.

Proof. First we check that $\text{rng}(f) \subseteq B^{00}$. By Claim 6, we get $B^{00} = C(i) \cap G^{00}$, so it is enough to show that for $g \in G^{00} \setminus B$, $i\sqrt{ii^g}g^{-1} = \sqrt{ii^g}g^{-1}i$. We have the following sequence of equivalent conditions: $i\sqrt{ii^g}g^{-1} = \sqrt{ii^g}g^{-1}i$ iff $i\sqrt{ii^g}i^g = \sqrt{ii^g}$ iff $i\sqrt{ii^g}i^g i\sqrt{ii^g}i^g = ii^g$ iff $\sqrt{ii^g}(ii^g)^{-1}\sqrt{ii^g} = e$. Since ii^g and $\sqrt{ii^g}$ commute, we see that the last condition is true. So $\text{rng}(f) \subseteq B^{00}$. Now we easily check that for $b \in B^{00}$ we have $f(bg) = f(g)b^{-1}$. So $\text{rng}(f) = B^{00}$. \square

Take any $g \in G^{00} \setminus B$. By Claim 12, we get $\sqrt{ii^g} = b_1g$ for some $b_1 \in B^{00}$. Hence $ii^g = b_1gb_1g$, so $g^{-1} = ib_1gb_1i$. Let $b = ib_1$. Since $i, b_1 \in B^{00}$, we get $b \in B^{00}$ and $(bg)^2 = e$. But $g \notin B^{00}$, so bg is an involution.

Hence we have proved the following statement:

(*) for every $g \in G^{00}$, $B^{00}g \cap I \neq \emptyset$ where I is the set of involutions in G^{00} .

In other words, $B^{00}I = G^{00}$.

Notice that everything that we have proved about B (including (*)) holds for any Borel whose connected component contains an involution.

Let d be a name of B . Replacing B by B^h for some $h \in G^{00}$, if necessary, we can assume that $d \notin \text{acl}^{eq}(\emptyset)$. Now we can easily choose an element $g \in G^{00} \setminus (X \cup I \cup \{e\})$ so that $g \perp^b d$. Then $g \in G^{00} \setminus B^{00}$.

Claim 13. (i) $U^b(g^G) = 1$

(ii) g^B is infinite.

Proof. Point (i) follows from Lascar inequalities and the fact that $C(g)$ is infinite. To see (ii), notice that if g^B were finite, then $C(g) \cap B$ would be infinite, so $C(g) \cap B = B$. Then $g \in G^{00} \cap C(B) = B^{00}$, a contradiction. \square

Now add g to the language. Then $d \notin \text{acl}^{eq}(\emptyset)$ and there is a formula $\phi(x, y)$ over \emptyset such that $\phi(\mathfrak{C}, d) = g^B$.

Claim 14. *If $d_1 \neq d_2$ are any realizations of $\text{tp}(d)$, then $\phi(\mathfrak{C}, d_1) \cap \phi(\mathfrak{C}, d_2)$ is finite.*

Proof. There are automorphisms f_1 and f_2 such that $f_1(d) = d_1$ and $f_2(d) = d_2$. Let $B_1 = f_1[B]$ and $B_2 = f_2[B]$. Then $\phi(\mathfrak{C}, d_1) = g^{B_1}$, $\phi(\mathfrak{C}, d_2) = g^{B_2}$, and $B_1 \neq B_2$. So by (*) (applied to B_1 and B_2), we get that there are $b_1 \in B_1^{00}$, $b_2 \in B_2^{00}$, and $j_1, j_2 \in I$ such that $g = b_1 j_1 = b_2 j_2$. Since $g \notin I$, we have $b_1, b_2 \neq e$.

Suppose for a contradiction that $g^{B_1} \cap g^{B_2}$ is infinite. Then $b_1 j_1^{B_1} \cap b_2 j_2^{B_2}$ is infinite. So there are infinite subsets $\{k_n : n \in \omega\}$ and $\{l_n : n \in \omega\}$ of I such that $c := b_2^{-1} b_1 = k_n l_n$ for all $n \in \omega$. Since $B_1 \neq B_2$ and $b_1, b_2 \neq e$, we get $c \neq e$.

Notice that there is $n \in \omega$ such that $B(k_n) \neq B(l_n)$. Otherwise, $k_n, l_n \in B(k_n)^{00}$ and so $c \in B(k_n)^{00}$ for all $n \in \omega$. Hence $k_n, l_n \in B(k_0)^{00}$ for all $n \in \omega$, a contradiction with Claim 9.

So by Claim 8, we get that $C(c)$ is finite. On the other hand, $c^{k_n} = (k_n l_n)^{k_n} = (k_n l_n)^{-1} = c^{-1}$, so $k_n \in N(C(c))$ for all $n \in \omega$, which implies that $N(C(c))$ is infinite. This is a contradiction. \square

By Lemma 0.15, Claim 13(ii), and Claim 14, we get that there is an element $h \in \phi(\mathfrak{C}, d)$ such that $U^{\text{p}}(h) = 2$. But $h \in g^G$ which is \emptyset -definable (as we have added g to the language). Thus $U^{\text{p}}(g^G) = 2$, a contradiction to Claim 13(i). This completes the proof of Theorem 3.1. \blacksquare

Remark 3.3. *We may see, by examining the proof above, that not only is there a solvable group of finite index, but that we may take this group to be definable. However, in any case, given a solvable group of finite index, we may find a definable solvable group also of finite index by the argument of, for instance, Theorem 3.17 of [10].*

Notice that Theorem 3.1 is true for an arbitrary independence relation \downarrow^* , because U^{p} -rank is less or equal to U^* -rank.

From the proof of Theorem 3.1, we get that, after possibly passing to a definable subgroup of finite index and quotienting by its finite center, the group is solvable of solvability degree at most 2.

Let us finish this section with the following conjecture which generalizes Theorem 2.

Conjecture 3.4. *Suppose G is superrosy and has NIP. Then if G , and every definable subgroup of G , is definably amenable (in the sense of [6]) and of U^{p} -rank 2, it is solvable-by-finite.*

4 Superrosy fields

First we make several remarks about (absolute) connected components and generic types in fields. Then we adapt Macintyre's proof of [10, Theorem 3.1] to prove Theorem 3. The last part of this section is devoted to the proof of Theorem 4.

Suppose K is an infinite field definable in a monster model $\mathfrak{C} \models T$. The additive and multiplicative groups of K will be denoted by K^+ and K^* , respectively.

Proposition 4.1. (i) If $(K^+)^{00}$ exists (e.g. if K^+ has fsg or T satisfies NIP), then $(K^+)^{00} = K^+$.

(ii) If K^+ has fsg, then it has a unique global generic type \mathbf{p} and $\text{Stab}_{K^*}(\mathbf{p}) = K^* = (K^*)^{00}$.

(iii) If there is a global type \mathbf{p} such that $\text{Stab}_{K^*}(\mathbf{p}) = K^*$ and K^* has a global generic type (e.g. if K^* has fsg), then \mathbf{p} is the unique global generic type of K^* .

Proof. (i) Since $(K^+)^{00}$ is the smallest subgroup of bounded index in K^+ , we see that for any $k \in K^*$, $k(K^+)^{00}$ also has this property. Hence $k(K^+)^{00} = (K^+)^{00}$, which means that $(K^+)^{00}$ is a nontrivial ideal of K . Thus $(K^+)^{00} = K^+$.

(ii) Assume that K^+ has fsg. By (i) and Proposition 0.26, we get the existence and uniqueness of a global generic type \mathbf{p} of K^+ . Then we see that for every $k \in K^*$, $k\mathbf{p}$ is also generic, so $k\mathbf{p} = \mathbf{p}$. Hence $\text{Stab}_{K^*}(\mathbf{p}) = K^*$. On the other hand, it is clear that if H is a type-definable subgroup of bounded index in K^* , then $\text{Stab}_{K^*}(\mathbf{p}) \leq H$. So $(K^*)^{00}$ exists and is equal to K^* .

(iii) As in the proof of Proposition 0.26, we show that every generic formula in K^* belongs to \mathbf{p} . So, if there is a global generic type for K^* , it must be \mathbf{p} . ■

Proposition 4.2. Assume that T is superrosy, $(K^+)^{00} = K^+$ and $(K^*)^{00} = K^*$. Then for every $n > 0$, the function $f(x) = x^n$ is onto and, if $\text{char}(K) = p$ is finite, the function $g(x) = x^p - x$ is also onto.

Proof. The proof is completely standard. Let us show the first part (the second one is similar). Let a be \mathfrak{b} -generic over \emptyset . Since $a \in \text{acl}(a^n)$, Lascar inequalities give us that a^n is also \mathfrak{b} -generic. So we get that K^n has finite index in K^* , and by our assumption, we conclude that $K^n = K$. ■

Proposition 4.3. If K^+ has fsg and L is a finite extension of K , then L is definable in \mathfrak{C} and L^+ also has fsg. In particular, $(L^+)^{00} = L^+$ and $(L^*)^{00} = L^*$. If T is additionally superrosy, then L satisfies the conclusion of Proposition 4.2.

Proof. Of course L is definable in $K^{\times n}$ for some n . That L^+ has fsg follows by induction from [6, Proposition 4.5] and the fact that L^+ can be identified with $(K^+)^{\times n}$. The rest is a consequence of Propositions 4.1 and 4.2. ■

Theorem 3. Suppose that T is superrosy and K^+ has fsg. Then K is algebraically closed.

Proof. If not, then (as in Macintyre's proof, see [10, Theorem 3.1]) by Galois theory, there are finite extensions $K \subseteq L \subseteq F$ such that F is a cyclic extension of L of prime degree, q , and if q is different from the characteristic of K , then a primitive q th root of 1 belongs to L ; then we get a contradiction with the last part of Proposition 4.3. ■

The following conjecture seems more difficult to prove. But it may be easier than the corresponding conjecture concerning supersimple fields, namely that each supersimple field is pseudo-algebraically closed.

Conjecture 4.4. *Suppose T is superrosy and has NIP. Then K is either an algebraically closed or a real closed field.*

Suppose G is a group definable in $\mathfrak{C} \models T$. We will prove the following theorem which, by Corollary 1.8, implies Theorem 4.

Theorem 4.5. *Assume that G has hereditarily fsg, the definable quotients of definable subgroups of G satisfy icc on centralizers, $U^b(G)=2$ and G is not nilpotent-by-finite. Then, after possibly passing to a definable subgroup of finite index and quotienting by its finite center, G is (definably) the semidirect product of the additive and multiplicative groups of an algebraically closed field F interpretable in \mathfrak{C} , and moreover $G = G^{00}$.*

Proof. By Theorem 3.1, we know that G is solvable-by-finite. In fact, it has a definable solvable subgroup of finite index. We may assume that G is centralizer connected, centerless and solvable. Let U be a definable normal commutative subgroup of G of U^b -rank 1. We may assume that U is centralizer connected in the sense of G , namely that for no $g \in G$ is $C(g) \cap U$ a proper subgroup of U of finite index. Note that $C(U)$ (centralizer of U in G) has infinite index in G (otherwise, G is nilpotent-by-finite). Also $G/C(U)$ being of U^b -rank 1 is abelian-by-finite. It follows that there is $b \in G \setminus C(U)$ such that $C(b)$ is infinite (and so of U^b -rank 1). (Otherwise, every conjugacy class in $G \setminus C(U)$ has U^b -rank 2 so there are only finitely many of them, but then $G/C(U)$ has only finitely many conjugacy classes, contradicting it being infinite and abelian-by-finite.) By choice of U and b , $C(b) \cap U$ is finite, hence $C(b)U$ (the group generated by $C(b)$ and U which is definable) has U^b -rank 2 so finite index in G . We claim that $C(b) \cap U = \{e\}$. For otherwise, the centralizer of every element in $C(b) \cap U \neq \{e\}$ is of U^b -rank 2, which contradicts G being centralizer connected and centerless. Let T be a commutative definable subgroup of $C(b)$ of finite index. It follows likewise that $T \cap C(U)$ is also trivial and for every $u \in U \setminus \{e\}$, $T \cap C(u)$ is finite. Now the group UT has U^b -rank 2, so we may assume it equals G . So to summarize the situation we have:

- (*) $G = UT$ (semidirect product of U and T), U, T are commutative of U^b -rank 1, $C(U) \cap T = \{e\}$ and for every $u \in U \setminus \{e\}$, $T \cap C(u)$ is finite.

We will write U additively, and T multiplicatively. T acts by conjugation on U , and we sometimes let $t \cdot u$ denote $u^t = tut^{-1}$. By the last part of (*), each orbit except $\{0\}$ is infinite. So by (*), T acts regularly on each orbit except $\{0\}$. (Otherwise, there are $u \in U \setminus \{0\}$ and $t \in T \setminus \{e\}$ such that $u^t = u$. Then every element of the infinite orbit u^T is stabilized by t , so $C(t) \cap U$ is infinite, and hence $t \in T \cap C(U)$, a contradiction.) Hence as U has U^b -rank 1, there are only finitely many orbits. But U^{00} is clearly a union of such orbits. Hence U^{00} is definable. So we may assume:

- (**) $U = U^{00}$ (hence we know there is a unique generic type in U).

Note that all our previous assumptions remain valid. In particular T acts freely on $U \setminus \{0\}$. Let C_1, \dots, C_r be the orbits of $U \setminus \{0\}$ under T . By (**), exactly one of them, without loss C_1 is generic in U . If it so happened that $r = 1$ then we easily get our desired conclusion (as in the finite Morley rank case). In fact we will undertake an analysis which will eventually show that $r = 1$ anyway.

The important thing we will use is that each element of U is a sum of generics and thus a sum of elements of C_1 . Each element of T defines an endomorphism (in fact automorphism) of U . Let S be the ring of endomorphisms of U generated by T . Note that two different elements of T define different endomorphisms (in fact automorphisms) of U , so we can and will identify an element of T with the endomorphism it defines. Moreover, multiplication in T is just the restriction to T of multiplication (composition) in S . As T is commutative, so is S . For $s \in S$ we still write the action on U as \cdot . We write $T + T$ for the subset of S consisting of elements $s + t$ for $s, t \in T$. Fix an element $u \in C_1$.

Claim (i) Any $s \in S$ is determined by $s \cdot u$.

(ii) $S = T + T$.

(iii) The ring S is an (interpretable) field.

(iv) Let $i : S \rightarrow U$ be given by $i(s) = s \cdot u$. Then i induces an isomorphism between the field S and $(U, +, \otimes)$ (some definable \otimes) and moreover this field is algebraically closed.

Proof. This is actually quite routine and implicit or explicit in the literature, but we will give some details anyway.

(i) As every element of U is a sum of elements of C_1 and as $C_1 = T \cdot u$, we see that every element of U is of the form $s \cdot u$ for some $s \in T + T$. Let $s_1, s_2 \in S$, and suppose that $s_1 \cdot u = s_2 \cdot u$. Let $x \in U$, and suppose $s \in T + T$ is such that $x = s \cdot u$. Then $s_1 \cdot x = s_1 \cdot (s \cdot u) = s \cdot (s_1 \cdot u) = s \cdot (s_2 \cdot u) = s_2 \cdot (s \cdot u) = s_2 \cdot x$.

(ii) follows from (i) as $(T + T) \cdot u = U$.

(iii) Suppose first that $s \in S$, $x \in U$ and $s \cdot x = 0$. Now $x \in C_i$ for some $i = 1, \dots, r$ and $C_i = T \cdot x$. As S is commutative, s is 0 on C_i , so $\ker(s)$ (a definable subgroup of U) is infinite, thus as U is connected of U^b -rank 1, $\ker(s) = U$ and $s = 0$. So we have shown that any nonzero $s \in S$ is injective, hence also surjective (by connectedness of U again). The existence of inverses follows easily: let $s \in S$ be nonzero. From what we have seen let $x \in U$ be such that $s \cdot x = u$. Let $s' \in S$ be such that $s' \cdot u = x$. So $s \cdot (s' \cdot u) = u$, and by part (i), $s \cdot s'$ is the identity. So s' is the inverse of s . We have shown that S is a field. Identifying S with $T \times T/E$ for a suitable definable equivalence relation E (using (i) and (ii)), we see that addition and multiplication are definable.

So we have an interpretable definable field. But we only have a theorem telling us the structure of such a field when the underlying additive group has fsg. This is the point of:

(iv) We know by (i) that i is a bijection. Moreover i clearly takes addition on S to addition on U . So we have a definable field structure $(U, +, \otimes)$ on U whose additive part has the fsg (by our hypothesis on G). By Theorem 3 the field is algebraically

closed. □

Let us now complete the proof of the theorem. Consider the definable field structure $F = (U, +, \otimes)$ expanding $(U, +)$. From Proposition 4.1 (U^*, \otimes) is (absolutely) connected, in particular has no proper definable subgroup of finite index. But T embeds definably in (U^*, \otimes) via $t \rightarrow t \cdot u$, and T as well as U^* has U^b -rank 1. This forces $T \cdot u$ to equal $U \setminus \{0\}$. So in fact the action of T on U is the action of F^* on F^+ .

For absolute connectedness, we have that $U = U^{00}$ and that $T = T^{00}$ (being isomorphic to the multiplicative group of F). It follows that G , the semidirect product of U and T is absolutely connected: $G^{00} \cap U = U$ and $G^{00} \cap T = T$, hence $G = G^{00}$. ■

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