

DJ4

Adrien Deloro and Éric Jaligot

*Ma Virgilio n'avea lasciati scemi
di sé, Virgilio dolcissimo patre,
Virgilio a cui per mia salute die' mi.*

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1 Introduction

This is the final article of the series [DJ12, DJ10, DJ13]. It has a sad story but at least it has the merit to exist. For the last time we deal with $*$ -locally $^\circ$ soluble groups of finite Morley rank. And although we have already used some phrases that our prospective reader may not be familiar with we hope the present work to be of interest to some experts in finite group theory.

1.1 The Context

Let us first say a few words of groups of finite Morley rank. The present subsection is deliberately vague as we only hope to catch the reader's attention (possibly through provocation). Should we succeed we can suggest three books. The first monograph dealing with groups of finite Morley rank, among other groups, was [Poi87]. An excellent and thorough reference textbook is [BN94b] which has no pictures but many exercises instead. The more recent [ABC08] quickly focuses on the specific topic of the classification of infinite simple groups of finite Morley rank of so-called even type, a technical assumption. For the moment let us be quite unspecific.

Morley rank is a notion invented by model theorists for pure mathematical logic. It turned out to be an abstract form of the Zariski dimension in algebraic geometry. But for the sake of the introduction we wish to suggest a completely different, anachronistic motivation.

The classification of simple Lie groups, the classification of simple algebraic groups, and the classification of finite simple groups are facets of a same truth: in certain categories, simple groups are matrix groups in the naive sense. The case of the finite simple groups reminds us that we are at the level of an erroneous truth, but still there must be something common to Lie groups, algebraic groups, and finite groups beyond the mere group structure that forces them to fall into the same class.

In a sense, groups of finite Morley rank answer this; Morley rank is a form of common structural layer, or methodological least common denominator to the Lie-theoretic, algebraic geometric, and finite group-theoretic worlds. They are groups equipped with a dimension on subsets enabling the most basic computations; the expert in finite group theory will be delighted to read that matching involutions against cosets, for instance, is possible. On the other hand, no analysis, no geometry, and no number theory are available. But the existence of a rudimentary dimension is a common though thin structural layer.

Although we gave no definition we hope to have motivated the Cherlin-Zilber conjecture, which postulates in particular that infinite simple groups of finite Morley rank are matrix groups. The conjecture goes back to the seventies and is still open. [ABC08] gives a complete answer in a special case where there are “many” involutions, but apart from that there is little hope to prove the conjecture in full generality. Yet after all, not all finite simple groups are groups of Lie type, so refuting the Cherlin-Zilber conjecture would certainly not show that it is not interesting.

The present work deals with *small* groups of finite Morley rank; the notion of smallness under consideration, viz. $*$ -local $^\circ$ solubility defined in §2, is borrowed from finite group theory. The decorations are here to remind the reader that it has nothing to do with local solubility in the classical sense, but with the solubility of local subgroups in the sense of Thompson's N -groups [Tho68].

Another, even more restrictive, notion of smallness in [Tho68] was minimal simplicity; its finite Morley rank analogue is named minimal connected simplicity and also defined in §2. Minimal connected simple groups of finite Morley rank have already been studied at length (see §§1.2 and 1.3) and the present article completes the move from the minimal connected simple to the $*$ -locally $^\circ$ soluble setting.

We do not provide a full classification of $*$ -locally $^\circ$ soluble groups of finite Morley rank but we delineate major cases and delimit the insertion of possible pathological groups in context – just in case the conjecture would be false, but not that false.

1.2 The Result

Our notations are explained in §2; a hasty expert already looking for the structure of the proof will find a few words at the beginning of §4.

Theorem. Let \hat{G} be a connected, U_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup.

Then the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$, isomorphic to that of $\mathrm{SL}_2(\mathbb{C})$, or is a 2-torus of Prüfer 2-rank at most 2.

Suppose in addition that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.

Then $m_2(\hat{G}) \leq 2$, G or \hat{G}/G is 2^\perp , and involutions are conjugate in \hat{G} . Moreover one of the following cases occurs:

- **PSL₂**: $G \simeq \mathrm{PSL}_2(\mathbb{K})$ in characteristic not 2; \hat{G}/G is 2^\perp ;
- **CiBo₀**: G is 2^\perp ; $m_2(\hat{G}) \leq 1$; for $\iota \in I(\hat{G})$, $C_G(\iota) = C_G^\circ(\iota)$ is a self-normalising Borel subgroup of G ;
- **CiBo₁**: $m_2(G) = m_2(\hat{G}) = 1$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G ;
- **CiBo₂**: $\mathrm{Pr}_2(G) = 1$ and $m_2(G) = m_2(\hat{G}) = 2$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G^\circ(i)$ is an abelian Borel subgroup of G inverted by any involution in $C_G(i) \setminus \{i\}$ and satisfies $\mathrm{rk} G = 3 \mathrm{rk} C_G^\circ(i)$;
- **CiBo₃**: $\mathrm{Pr}_2(G) = m_2(G) = m_2(\hat{G}) = 2$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G .

There is at present no hope to remove any of the non-algebraic configurations of type **CiBo** (“Centralisers of Involutions are Borel subgroups”; unlike the cardinal, these configurations are far from innocent). Three of them were first described in [CJ04] under much stronger assumptions of both group-theoretic and model-theoretic natures, and the goal of [Del07b, Del07a, Del08] merely was to carry the same analysis with no model-theoretic restrictions. Despite progress in technology, nothing new could be added on the **CiBo** configurations, nor was since. So it is likely they will linger for a while; one may even imagine that they ultimately might be proved consistent.

But beyond porting the description of non-algebraic configurations from the minimal connected simple setting [Del07b] to the broader $*$ -locally $^\circ$ soluble context, our theorem gives strong limitations on how these potential counterexamples embed into bigger groups. This line of thought goes back to Delahan and Nesin proving that so-called simple bad groups have no involutory automorphisms ([DN93]; [BN94b, Proposition 13.4]; a simple bad group would be the most dramatic kind of counterexample to the Cherlin-Zilber conjecture).

The present result therefore replaces a number of earlier (pre)publications: [BCJ07, Del07b, Del07a, Del08, DJ08, BCD09], the contents of which are described in §1.3 hereafter. We can however *not* do better than Frécon’s analysis.

Fact ([Fré10, Theorem 3.1]). *Let G be a minimal connected simple group with a nontrivial Weyl group. Then each connected definable automorphism group of G is inner.*

Parenthetically said, the reason why we cannot extend the latter analysis is the following. [Fré10] uses the conjugacy of Carter subgroups in minimal connected simple groups obtained by Frécon himself [Fré08], and there is a technical gap between the minimal connected simple and $*$ -locally $^\circ$ soluble classes preventing us from replicating methods.

What next then? The author would be delighted to share some thoughts on the possibility to classify simple groups of finite Morley rank no simple subquotients of which have a connected Sylow 2-subgroup. But at the moment we are looking towards the past.

1.3 Version History

The present subsection will be of little interest to the non-expert; we include it mostly because the present article is meant to be our last on the topic.

The project of classifying $*$ -locally $^\circ$ soluble groups with involutions started as early as 2007 under the suggestion of Borovik and yet is only the last chapter of an older story: the identification of $\mathrm{PSL}_2(\mathbb{K})$ among small groups of odd type.

- We could go back to Cherlin’s seminal article on groups of small Morley rank [Che79] which identified PSL_2 , considered bad groups, and formulated the algebraicity conjecture. Other important results on PSL_2 in the finite Morley rank context were found by Hrushovski [Hru89], Nesin et Ali(i) [Nes90a, BDN94, DN95]. But we shall not go this far.
- Jaligot was the first to do something specifically in so-called odd type [Jal00], adapting computations from [BDN94] (we say a bit more in §4.2 and 4.3).

- Another preprint by Jaligot [Jal01], then at Rutgers University, deals with *tame* minimal connected simple groups of Prüfer rank 1. (Tameness is a model-theoretic assumption on fields arising in a group; it is already used for instance in [DN95].) In this context, either the group is isomorphic to $\mathrm{PSL}_2(\mathbb{K})$, or centralisers^o of involutions are Borel subgroups.

Quite interestingly the tameness assumption, viz. “NO BAD FIELDS”, appears there in small capitals and bold font each time it is used; it seems clear that Jaligot already thought about removing it.

- Jaligot’s time at Rutgers resulted in a monumental article with Cherlin [CJ04] where tame minimal connected simple groups were thoroughly studied and potential non-algebraic configurations carefully described. The very structure of our Theorem reflects the main result of [CJ04].
- Using major advances by Burdges, the author then writing his dissertation under the supervision of Jaligot could remove the tameness assumption from [CJ04] and reach essentially the same conclusions ([Del07b], published as [Del07a, Del08]).

- A few months before the author’s PhD completion the present project of classifying \ast -locally^o soluble groups of finite Morley rank was suggested by Borovik, a task the author and Jaligot undertook with great enthusiasm and which along the years resulted in the series [DJ12, DJ10, DJ13].

A 2008 preprint [DJ08] was close to fully porting [Del07b] to the \ast -locally^o soluble context. Involutions were however confined inside the group. (This amounts to supposing $\hat{G} = G$ in the Theorem.)

- When a post-doc at Rutgers University the author in an unpublished joint work with Burdges and Cherlin [BCD09] went back to the minimal connected simple case but with external involutions. (This amounts to supposing G minimal connected simple and 2^\perp in the Theorem.)
- Delays and shifts in interests postponed both [DJ08] and [BCD09]. In the Spring of 2013 the author tried to convince Jaligot that time had come to redo [DJ08] in full generality, that is with external involutions. The present Theorem was an ideal statement we vaguely dreamt of but we never discussed nor even mentioned to each other anything beyond as it looked distant enough. In April and May we were trying to fix earlier proofs with all possible repair patches, and unequal success.

The author remembers how Jaligot was taking notes in a small red “Rutgers” notebook when visiting Paris. He only recovered a file of little relevance [D.J] but none of the above mentioned handwritten sketches after Jaligot’s untimely death.

And this is how a project started with great enthusiasm was completed in grief and sorrow; nonetheless completed. The author feels he is now repaying his debt for the care he received as a student, for an auspicious dissertation topic, and for all the friendly confidence his advisor trusted him with.

In short I hope that the present work is the kind of monument Éric’s shadow begs for. I dare write that the article is much better than last envisioned in the Spring of 2013. Offended reader, understand – that *there* precisely lies my tribute to him.

Such a reconstruction would never have been even imaginable without the hospitality of the Mathematics Institute of NYU Shanghai during the Fall of 2013. The good climate and supportive staff made it happen. Last but not least and despite the author’s lack of taste for mixing genres, Lola’s immense patience is most thankfully acknowledged.

2 Prerequisites and Facts

One word on general terminology: the author supports linguistic minorities.

Definition ([DJ12, Definition 3.1(4)]). *A group G of finite Morley rank is $*$ -locally $^\circ$ soluble if $N_G^\circ(A)$ is soluble for every nontrivial, definable, abelian, connected subgroup $A \leq G$.*

Remark (and Definition). An extreme case of $*$ -local $^\circ$ solubility is when all definable, connected, proper subgroups of G are soluble; G is then said to be *minimal connected simple*. As opposed to past work (see §1.3) the present article does not rely on minimal connected simplicity.

We have tried to make the article as self-contained as possible, an uneasy task as the study of groups of finite Morley rank has grown to a substantial body. Reading the other articles in the series [DJ12, DJ10, DJ13] is however not necessary to understand this one. In the introduction we already mentioned three general references [Poi87, BN94b, ABC08]. Yet we highly recommend the preliminaries of a research article, [ABF13, §2]; the reader may wish to first look there before picking a book from the shelves.

We denote by $d(X)$ the definable hull of X , i.e. the smallest definable group containing X . If H is a definable group, we denote by H° its connected component. If H fails to be definable we then set $H^\circ = H \cap d^\circ(H)$.

As one imagines, involutions will play a major role. We denote by $I(G)$ the set of involutions in G ; i, j, k, ℓ will stand for some of them. We also use ι, κ, λ for involutions of the bigger, ambient group \hat{G} . We are not very happy with this notation. When a group has no involutions, we call it 2^\perp . We shall refer to the following as “commutation principles”.

Fact 1. *Suppose that there exists some involutive automorphism ι of a semi-direct product $H \rtimes K$, where K is 2-divisible, and that ι centralises or inverts H , and inverts K . Then $[H, K] = 1$.*

2.1 Semi-simplicity

The principle below may be the only general statement one can make about torsion in groups of finite Morley rank; p stands for a prime number.

Fact 2 (torsion lifting, [BN94b, Exercise 11 p.98]). *Let G be a group of finite Morley rank, $H \trianglelefteq G$ be a normal, definable subgroup and $x \in G$ be such that xH is a p -element in G/H . Then $d(x) \cap xH$ contains a p -element of G .*

Apart from the above principle, most of our knowledge of torsion relies either on the assumption that $p = 2$, on some solubility assumption, or on a U_p^\perp -ness assumption explained below.

- To emphasize the first aspect ($p = 2$), recall that in groups of finite Morley rank, maximal 2-subgroups (also known as Sylow 2-subgroups) are conjugate [BN94b, Theorem 10.11]. As a matter of fact, their structure is known [BN94b, Corollary 6.22]. If S is a Sylow 2-subgroup then $S^\circ = T * U_2$ where T is a 2-torus and U_2 a 2-unipotent group. Let us explain the terminology:

- T is a sum of finitely many copies of the Prüfer 2-group, $T \simeq \mathbb{Z}_{2^\infty}^d$, and d is called the Prüfer 2-rank of T , $\text{Pr}_2(T) = d$. By conjugacy, $\text{Pr}_2(G) = \text{Pr}_2(T)$ is well-defined. Interestingly enough, $N_G^\circ(T) = C_G^\circ(T)$ [BN94b, Theorem 6.16, “rigidity of tori”]; this actually holds for any prime.
- U_2 in turn has bounded exponent. We shall mostly deal with groups having no infinite such subgroups, and we call them U_2^\perp groups.

The 2-rank $m_2(G)$ is the maximal rank of an elementary abelian 2-subgroup of G ; again this is well-defined by conjugacy. A U_2^\perp assumption implies finiteness of $m_2(G)$; one always has $\text{Pr}_2(G) \leq m_2(G)$; see [Del12] for some reverse inequality.

- Actually the same holds for any prime p provided the ambient group of finite Morley rank is soluble [BN94b, Theorem 6.19 and Corollary 6.20]. In case the ambient group is also connected, then Sylow p -subgroups are connected [BN94b, Theorem 9.29]. We call this fact the structure of torsion in definable, connected, soluble groups.

- A group of finite Morley rank is said to be U_p^\perp (also: of p^\perp -type) if it contains no infinite, elementary abelian p -group. A word on Sylow p -subgroups of U_p^\perp groups is said in §2.2.

For the moment we give another example of how we often rely either on some specific assumption on involutions, or on solubility.

Fact 3 (special case of [BC08, Theorem 2.1]). *Let \hat{G} be a U_p^\perp group of finite Morley rank. Suppose that \hat{G} contains a non-trivial, definable, connected, normal subgroup $G \trianglelefteq \hat{G}$ and some elementary abelian p -group of rank 2 $\hat{V} \leq \hat{G}$. If G is soluble, or if $p = 2$ and G has no involutions, then $G = \langle C_G^\circ(v) : v \in V \setminus \{1\} \rangle$.*

We finish with a property of repeated use.

Fact 4 (Steinberg's torsion theorem; [Del09]). *Let G be a connected, U_p^\perp group of finite Morley rank and $\zeta \in G$ be a p -element such that $\zeta^{p^n} \in Z(G)$. Then $C_G(\zeta)/C_G^\circ(\zeta)$ has exponent dividing p^n .*

As the argument essentially relies on the connectedness of centralisers of *inner* tori obtained by Altinel and Burdges [AB08, Theorem 1], one should not expect anything similar for outer automorphisms of order p , not even for outer toral automorphisms.

2.2 Sylow Theory

Recall that by definition, a *Sylow p -subgroup* of some group of finite Morley rank is a maximal, *soluble* p -subgroup. It turns out that for p -subgroups of groups of finite Morley rank, solubility is equivalent to local solubility (in the classical sense of finitely generated subgroups being soluble) [BN94b, Theorem 6.19], so every soluble p -subgroup is contained in some Sylow p -subgroup alright. But the solubility requirement is not for free: even if the ambient group of finite Morley rank G is assumed to be U_p^\perp , it is not known whether every p -subgroup of G is soluble; as a matter of fact it is apparently not known whether G can embed a Tarski monster. In short, a Sylow p -subgroup is not necessarily a maximal p -subgroup, even in the U_p^\perp case.

Fact 5 ([BC09, Theorem 4]). *Let G be a U_p^\perp group of finite Morley rank. Then Sylow p -subgroups of G are conjugate.*

Remarks. Let \hat{G} be a U_p^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, normal subgroup.

- The Sylow p -subgroups of G are exactly the traces of the Sylow p -subgroups of \hat{G} .
A Sylow p -subgroup of G is obviously the trace of some Sylow p -subgroup of \hat{G} . The opposite direction is immediate by conjugacy of the Sylow p -subgroups in the U_p^\perp group \hat{G} .
- The Sylow p -subgroups of \hat{G}/G are exactly the images of the Sylow p -subgroups of \hat{G} . The following argument was suggested by Gregory Cherlin.

Let φ be the projection modulo G . Suppose that \hat{S} is a Sylow p -subgroup of \hat{G} but $\varphi(\hat{S})$ is not a Sylow p -subgroup of \hat{G}/G . Then there is a p -element $\alpha \in N_{\hat{G}/G}(\varphi(\hat{S})) \setminus \varphi(\hat{S})$, which we lift to a p -element a of \hat{G} . Observe that $\alpha \in N_{\hat{G}/G}(\varphi(\hat{S}^\circ))$, so $\varphi([a, \hat{S}^\circ G]) = [\alpha, \varphi(\hat{S}^\circ)] \leq \varphi(\hat{S}^\circ G)$ and a normalises $\hat{S}^\circ G$.

Now $N = N_{\hat{G}}(\hat{S}^\circ G)$ is definable since it is the inverse image of $N_{\hat{G}/G}(\varphi(\hat{S}^\circ))$ which is definable as the normaliser of a p -torus by the rigidity of tori. In particular, N conjugates its Sylow p -subgroups, and a Frattini argument yields $N \leq \hat{S}^\circ G \cdot N_{\hat{G}}(\hat{S}) \leq GN_{\hat{G}}(\hat{S})$. Write $a = gn$ with $g \in G$ and $n \in N_{\hat{G}}(\hat{S})$; n is a p -element modulo G , so lifting torsion there is a p -element $m \in nG \cap d(n)$. Now $m \in N_{\hat{G}}(\hat{S})$ and therefore $m \in \hat{S}$. Hence $a = gn \in nG = mG \subseteq \hat{S}G$ and $\alpha = \varphi(a) \in \varphi(\hat{S})$, a contradiction.

As a consequence any Sylow p -subgroup of \hat{G}/G is the image of a Sylow p -subgroup of \hat{G} . For the converse fix a Sylow p -subgroup Σ of \hat{G}/G . Then $N_{\hat{G}/G}(\Sigma^\circ)$ is definable; let $\hat{S}_2 \leq \varphi^{-1}(N_{\hat{G}/G}(\Sigma^\circ))$ be a Sylow p -subgroup of $\varphi^{-1}(N_{\hat{G}/G}(\Sigma^\circ))$ and $\hat{S} \leq \hat{G}$ be a Sylow

p -subgroup of \hat{G} containing \hat{S}_2 . Then by the first part of the argument $\varphi(\hat{S})$ is a Sylow p -subgroup of \hat{G}/G . So by conjugacy in the latter there is $\hat{g} \in \hat{G}$ with $\varphi(\hat{S}^{\hat{g}}) = \Sigma$. Hence $\hat{S}_2^{\hat{g}} \leq \hat{S}^{\hat{g}} \leq \varphi^{-1}(N_{\hat{G}/G}(\Sigma^\circ))$ and by definition of \hat{S}_2 , $\hat{S}_2^{\hat{g}} = \hat{S}^{\hat{g}}$ is a Sylow p -subgroup of \hat{G} , mapping onto Σ .

- Without the U_p^\perp assumption this remains quite obscure. The reader will find in [PW93, PW00] a model-theoretic discussion.

We shall refer to the many consequences of the following fact as “torality principles”.

Fact 6 ([BC09, Corollary 3.1]). *Let \underline{p} be a set of primes. Let G be a connected group of finite Morley rank with a \underline{p} -element a such that $C(a)$ is $U_{\underline{p}}^\perp$. Then a belongs to any maximal \underline{p} -torus of $C(a)$.*

We add some unrelated remarks, just for the sake of it.

Remarks.

- Let \hat{G} be a connected, U_p^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected subgroup. Let $\hat{T} \leq \hat{G}$ be a maximal p -torus of \hat{G} . Then $T = \hat{T} \cap G$ is a maximal p -torus of G .

Let $\hat{S} \geq \hat{T}$ be a Sylow p -subgroup of \hat{G} . Then $S = \hat{S} \cap G$ is a Sylow p -subgroup of G . So $T = G \cap \hat{T} \leq G \cap \hat{S}^\circ \leq C_S(S^\circ) = S^\circ$ by torality principles. Hence $T \leq S^\circ \leq \hat{S}^\circ \cap G = \hat{T} \cap G = T$.

- This is not true for an arbitrary p -torus $\hat{\tau} \leq \hat{G}$: take two copies T_1, T_2 of \mathbb{Z}_{2^∞} with respective involutions i and j ; now let $\hat{G} = (T_1 \times T_2)/\langle ij \rangle$ and G be the image of T_1 . Then the intersection of (the image of) T_2 with G is $\langle \bar{i} \rangle$.
- The question deserves to be asked for so-called maximal decent tori.

2.3 Unipotence

For a complete exposition of Burdges’ unipotence theory, see Burdges’ Ph.D. [Bur04b], its first formally published expositions [Bur04a, Bur06], or the first article in the present series [DJ12].

Notation. *We order unipotence parameters as follows:*

$$(2, \infty) \succ (3, \infty) \succ \cdots \succ (p, \infty) \succ \cdots (0, \text{rk}(G)) \succ \cdots \succ (0, 0)$$

We denote unipotence parameters by ρ, σ, τ .

Notation.

- For any group of finite Morley rank H , ρ_H will denote the greatest unipotence parameter it admits, i.e. with $U_{\rho_H}(H) \neq 1$; we simply call it the parameter of H .
- For ι a definable involutory automorphism of some group of finite Morley rank, we let $\rho_\iota = \rho_{C^\circ(\iota)}$.

With these notations at hand let us review a few classical properties. Bear in mind that by definition, a ρ -group is always definable, connected, and nilpotent. The reader should be familiar with the following before venturing further.

Fact 7.

- If N is a connected, nilpotent group of finite Morley rank then $N = *_\rho U_\rho(N)$ (central product) where ρ ranges over all unipotent parameters (Burdges’ decomposition: [Bur04b, Theorem 2.31], [Bur06, Corollary 3.6], [DJ12, Fact 2.3]);
- if H is a connected, soluble group of finite Morley rank, one has $U_{\rho_H}(H) \leq F^\circ(H)$ ([Bur04b, Theorem 2.21], [Bur04a, Theorem 2.16], [DJ12, Fact 2.8]);

- (iii) if a σ -group V_σ normalises a ρ -group V_ρ with $\rho \preceq \sigma$ then $V_\rho V_\sigma$ is nilpotent ([Bur04b, Lemma 4.10], [Bur06, Proposition 4.1], [DJ12, Fact 2.7]);
- (iv) the image and preimage of a ρ -group under a definable homomorphism are ρ -groups (push-forward and pull-back: [Bur04b, Lemma 2.12], [Bur04a, Lemma 2.11]);
- (v) if G is a soluble group of finite Morley rank, $S \subseteq G$ is any subset, and $H \trianglelefteq G$ is a ρ -subgroup, then $[H, S]$ is a ρ -group ([Bur04b, Lemma 2.32], [Bur06, Corollary 3.7]);
- (vi) generalising the latter Frécon obtained a remarkable homogeneity result we shall not use (to our surprise):
 if G is a connected group of finite Morley rank acting definably on a ρ -group then $[G, H]$ is a homogeneous ρ -group, i.e. all its definable, connected subgroups are ρ -groups ([Fré06, Theorem 4.11], [DJ12, Fact 2.1]).

Recall that a Sylow ρ -subgroup is a maximal ρ -subgroup. Also recall from Burdges' decidedly inspiring thesis ([Bur04b, §4.3], oddly published only in [FJ08, §3.2]) that if π denotes a set of unipotence parameters, then a Carter π -subgroup of some ambient group G is a definable, connected, nilpotent subgroup L which is U_π -self-normalising, i.e. with $U_\pi(N_G^\circ(L)) = L$. Carter subgroups, i.e. definable, connected, nilpotent, quasi-self-normalising subgroups are examples of the latter where π is the set of all unipotence parameters. All this is very well-understood in a soluble context [Wag94, Fré00a].

So we now move to another topic: intersections of Borel subgroups of a $*$ -locally $^\circ$ soluble group (a Borel subgroup is a definable, connected, soluble subgroup which is maximal as such). We shall refer to the following as “uniqueness principles”.

Fact 8 ([DJ12, Corollary 4.3]). *Let G be a $*$ -locally $^\circ$ soluble group of finite Morley rank, ρ be a unipotence parameter and B be a Borel subgroup of G with $\rho_B = \rho$. Let $U_1 \leq U_\rho(B)$ be a ρ -subgroup containing some subset X with $\rho_{C_G^\circ(X)} \preceq \rho$. Then $U_\rho(B)$ is the only Sylow ρ -subgroup of G containing U_1 . Furthermore B is the only Borel subgroup of G containing U_1 with parameter ρ .*

In particular, if $G \trianglelefteq \hat{G}$ where \hat{G} is another (not necessarily $*$ -locally $^\circ$ soluble) group of finite Morley rank, $N_{\hat{G}}(U_1) \leq N_{\hat{G}}(B)$.

2.4 The Bender Method

For reference we list below the facts from Burdges' monumental rewriting of Bender's Method [Bur04b, §9], [Bur07] that we shall use.

The method was devised to study intersections of Borel subgroups; unfortunately it is very technical. It will play an important role throughout the proof of our main “Maximality” Proposition 6. As a matter of fact it does not appear elsewhere in the present article with the only exception of Step 2 of Proposition 3.

It must be noted that the Bender method does *not* finish any job; it merely helps treat non-abelian cases on the same footing as the abelian case. This will be clear during Step 7 of Proposition 6. So the reader who feels lost here must keep in mind the following:

- non-abelian intersections complicate the details but do not alter in the least the skeleton of the proof of Proposition 6;
- the utter technicality is, in Burdges' own words [Bur04b], “motivated by desperation”;
- non-abelian intersections are not supposed to exist in the first place.

Since Burdges' original work was in the context of so-called “minimal connected simple groups” we need to quote [DJ12] which merely reproduced Burdges' work in the $*$ -locally $^\circ$ soluble case.

Fact 9 ([DJ12, 4.46(2)]). *Let G be a $*$ -locally $^\circ$ soluble group of finite Morley rank. Then any nilpotent, definable, connected subgroup of G contained in two distinct Borel subgroups is abelian.*

Yet past the nilpotent case it is not always possible to prove abelianity of intersections of Borel subgroups. The purpose of the Bender method is at least to extract as much information as possible from non-abelian intersections. Unfortunately, “as much as possible” means much more than reasonable. This is the analysis of so-called *maximal pairs* [DJ12, Definition 4.12], a terminology we shall avoid.

Fact 10 (from [DJ12, 4.50]). *Let G be a $*$ -locally $^\circ$ soluble group of finite Morley rank. Let $B \neq C$ be two distinct Borel subgroups of G . Suppose that $H = (B \cap C)^\circ$ is non-abelian.*

Then the following are equivalent:

[DJ12, 4.50(1)] *B and C are the only Borel subgroups of G containing H ;*

[DJ12, 4.50(2)] *H is maximal among connected components of intersections of distinct Borel subgroups;*

[DJ12, 4.50(3)] *H is maximal among intersections of the form $(B \cap D)^\circ$ where $D \neq B$ is another Borel subgroup;*

[DJ12, 4.50(6)] $\rho_B \neq \rho_C$.

In the following, subscripts ℓ and h stand for *light* and *heavy*, respectively.

Fact 11 (from [DJ12, 4.52]). *Let G, B_ℓ, B_h, H be as in the assumptions and conclusions of Fact 10. For brevity let $\rho' = \rho_{H'}$, $\rho_\ell = \rho_{B_\ell}$, $\rho_h = \rho_{B_h}$; suppose $\rho_\ell < \rho_h$.*

Then the following hold:

[DJ12, 4.52(2)] *any Carter subgroup of H is a Carter subgroup of B_h ;*

[DJ12, 4.38, 4.51(3) and 4.52(3)] $U_{\rho'}(F(B_h)) = (F(B_h) \cap F(B_\ell))^\circ$ is ρ' -homogeneous; ρ' is the least unipotence parameter in $F(B_h)$;

[DJ12, 4.52(6)] $U_{\rho'}(H) \leq F^\circ(B_\ell)$ and $N_G^\circ(U_{\rho'}(H)) \leq B_\ell$;

[DJ12, 4.52(7)] $U_\sigma(F(B_\ell)) \leq Z(H)$ for $\sigma \neq \rho'$;

[DJ12, 4.52(8)] *any Sylow ρ' -subgroup of G containing $U_{\rho'}(H)$ is contained in B_ℓ .*

And we finish with an addendum.

Lemma A. *Let \hat{G} be a connected group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup. Let $B_1 \neq B_2$ be two distinct Borel subgroups of G such that $H = (B_1 \cap B_2)^\circ$ is maximal among connected components of intersections of distinct Borel subgroups and non-abelian. Let $Q \leq H$ be a Carter subgroup of H . Then:*

- $N_{\hat{G}}(H) = N_{\hat{G}}(B_1) \cap N_{\hat{G}}(B_2)$;
- $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B_1) \cup N_{\hat{G}}(B_2)$.

Proof. By [DJ12, 4.50 (1) and (6)], B_1 and B_2 are the only Borel subgroups of G containing H , and they have distinct unipotence parameters. This proves the first item. Let ρ' be the parameter of H' and $Q_{\rho'} = U_{\rho'}(Q)$. Then $N_{\hat{G}}(Q) \leq N_{\hat{G}}(Q_{\rho'}) \leq N_{\hat{G}}(N_G^\circ(Q_{\rho'}))$ and three cases can occur, following [DJ12, 4.51].

- In case (4a), $N_{\hat{G}}(Q) \leq N_{\hat{G}}(H) = N_{\hat{G}}(B_1) \cap N_{\hat{G}}(B_2)$; we are done.
- In case (4b), B_1 is the only Borel subgroup of G containing $N_G^\circ(Q_{\rho'})$, so $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B_1)$.
- Case (4c) is similar to case (4b) and yields $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B_2)$. □

3 Requisites (General Lemmas)

Our theorem requires extending some well-known facts, so let us revisit a few classics. All lemmas below go beyond the $*$ -locally $^\circ$ soluble setting.

3.1 Normalisation Principles

The results in the present subsection are folklore; it turns out that none was formally published. They originate either in [Del07b, Chapitre 2] or in [Bur09]. We shall use them with no reference, merely invoking “normalisation principles”.

Lemma B (cf. [Del07b, Lemmes 2.1.1 and 2.1.2] and [Del07a, §3.4]). *Let \hat{G} be a group of finite Morley rank, $G \leq \hat{G}$ be a definable subgroup, $P \leq G$ be a Sylow p -subgroup of G , and $\hat{S} \leq N_{\hat{G}}(G)$ be a soluble p -subgroup normalising G . If $p \neq 2$ suppose that \hat{G} is U_p^\perp . Then some G -conjugate of \hat{S} normalises P .*

Proof. Since G is definable, $d(\hat{S}) \leq N_{\hat{G}}(G)$, so we may assume $\hat{G} = G \cdot d(\hat{S})$ and $G \trianglelefteq \hat{G}$. We may assume that \hat{S} is a Sylow p -subgroup of \hat{G} . Recall that $S = \hat{S} \cap G$ is then a Sylow p -subgroup of G (see for instance §2.2). Since G is definable and U_p^\perp if $p \neq 2$, it conjugates its Sylow p -subgroups; there is $g \in G$ with $P = S^g$. Hence \hat{S}^g normalises $\hat{S}^g \cap G = S^g = P$. \square

Remarks. The argument is slightly subtler than it looks.

- The original version [Del07b, Lemmes 2.1.1 and 2.1.2] made the unnecessary assumption that \hat{S} , there denoted K , be definable. Its proof used only conjugacy in \hat{G} ; but when $K^{\hat{g}} \leq N_{\hat{G}}(P)$ for some $\hat{g} \in \hat{G}$, why should $K^{\hat{g}}$ be a G -conjugate of K ? [Del07b] then used definability of K to continue: we may assume $\hat{G} = G \cdot K \leq G \cdot N_{\hat{G}}(K)$, so $K^{\hat{g}}$ is actually a G -conjugate of K . Alas it is false in general that $d(\hat{S}) \leq N_{\hat{G}}(\hat{S})$ (consider the Sylow 2-subgroup of $\mathrm{PSL}_2(\mathbb{C})$). So without definability of \hat{S} one is forced to conjugate in G like we do here.
- In particular, if G is not supposed to be definable (and one then needs to assume $G \trianglelefteq \hat{G}$ to save the beginning of the proof), the statement is not clear at all. An arbitrary subgroup of a U_p^\perp group of finite Morley rank need not conjugate its Sylow p -subgroups, take $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{3}]) \leq \mathrm{PSL}_2(\mathbb{C})$ for instance. But for a *normal* subgroup, I do not know. This could even depend on the Cherlin-Zilber conjecture.

Lemma C ([Del07b, Corollaires 2.1.5 and 2.1.6]). *Let \hat{G} be a group of finite Morley rank, $H \leq \hat{G}$ be a soluble, definable subgroup, π be a set of unipotence parameters, $L \leq H$ be a Carter π -subgroup of H , and $\hat{S} \leq N_{\hat{G}}(H)$ be a soluble p -subgroup normalising H . Suppose that H is U_p^\perp . Then some H -conjugate of \hat{S} normalises L .*

Proof. We first deal with the case where $L = Q$ is a Carter subgroup of H ; the last paragraph will handle the general case. We may suppose that H is connected; we may suppose that $\hat{G} = H \cdot d(\hat{S})$ is soluble and that $H \trianglelefteq \hat{G}$; we may suppose that \hat{S} is a Sylow p -subgroup of \hat{G} . Since H is soluble it conjugates its Carter subgroups, so $\hat{G} = H \cdot N_{\hat{G}}(Q)$.

First assume that H is p^\perp . Let $\hat{R} \leq N_{\hat{G}}(Q)$ be a Sylow p -subgroup of $N_{\hat{G}}(Q)$. Since H is p^\perp , $\hat{R} \simeq \hat{R}H/H$ is a Sylow p -subgroup of $N_{\hat{G}}(Q)H/H = \hat{G}/H$. Hence \hat{R} is in fact a Sylow p -subgroup of \hat{G} : by conjugacy of Sylow p -subgroups in the definable, soluble group \hat{G} , \hat{S} normalises some Carter subgroup of H .

If we no longer assume that H is p^\perp , then since H is U_p^\perp the structure of torsion in definable, connected, soluble groups implies that Sylow p -subgroups of H are tori. By Lemma B, \hat{S} normalises a Sylow p -subgroup P of H ; by the rigidity of tori it centralises it, so it centralises $d(P)$ as well. Up to conjugacy in H , Q contains P and then again centralises $d(P)$. So we may work in $C_{\hat{G}}(d(P))$ and factor out $d(P)$, which reduces to the first case. Then \hat{S} normalises some Carter subgroup \bar{C} of $H/d(P)$, and normalises its preimage $\varphi^{-1}(\bar{C}) \leq H$ which is a Carter subgroup of H [Fré00a, Corollaire 5.20].

The reader has observed that for the moment, \hat{S} normalises some Carter subgroup of H . But by conjugacy of such groups in H , there is an H -conjugate of \hat{S} normalising Q .

We now go back to the general case of a Carter π -subgroup L of H . By [FJ08, Corollary 5.9] there is a Carter subgroup Q of H with $U_\pi(Q) \leq L \leq U_\pi(Q) \cdot U_\pi(H')$; by what we just proved and up to conjugating over H we may suppose that Q is \hat{S} -invariant. So we consider the generalised centraliser $E = E_H(U_\pi(Q))$ [Fré00a, Définition 5.15], a definable, connected, and

\hat{S} -invariant subgroup of H satisfying $U_\pi(Q) \leq F^\circ(E)$ [Fr 00a, Corollaire 5.17]; by construction of E and nilpotence, $\langle L, Q \rangle \leq E$. If $E < H$ then noting that L is a Carter π -subgroup of E we apply induction. So we may suppose $E = H$. But in this case $U_\pi(Q) \leq F^\circ(H)$ so actually $L \leq U_\pi(F^\circ(H))$ and equality holds as the former is a Carter π -subgroup of H . It is therefore \hat{S} -invariant. \square

The following Lemma is entirely due to Burdges who cleverly adapted the Fr con-Jaligot construction of Carter subgroups [FJ05]. We reproduce it here with Burdges' kind permission. The lemma is not used anywhere in the present article but included for possible future reference.

Lemma D ([Bur09]). *Let \hat{G} be a U_2^\perp group of finite Morley rank, $G \leq \hat{G}$ be a definable subgroup, and $\hat{S} \leq N_{\hat{G}}(G)$ be a 2-subgroup. Then G has a \hat{S} -invariant Carter subgroup.*

Proof. We may assume that every definable, \hat{S} -invariant subquotient of G of smaller rank has a \hat{S} -invariant Carter subgroup; we may assume that $C_{\hat{S}}(G) = 1$; we may assume that G is connected.

We first find an infinite, definable, abelian, \hat{S} -invariant subgroup. Let $\iota \in Z(\hat{S})$ be a central involution; then $C_G^\circ(\iota) < G$. If $C_G^\circ(\iota) = 1$ then G is abelian and there is nothing to prove. So we may suppose that $C_G^\circ(\iota)$ is infinite and find some \hat{S} -invariant Carter subgroup of $C_G^\circ(\iota)$ by induction; it contains an infinite, definable, abelian, \hat{S} -invariant subgroup.

Let ρ be the minimal unipotence parameter such that there exists a non-trivial \hat{S} -invariant ρ -subgroup of G (possibly $\rho = (0, 0)$); this makes sense since there exists an infinite, definable, abelian, \hat{S} -invariant subgroup. Let $P \leq G$ be a maximal \hat{S} -invariant ρ -subgroup; $P \neq 1$. Let $N = N_G^\circ(P)$.

If $N < G$ then induction applies: N has a \hat{S} -invariant Carter subgroup Q . So far PQ is soluble; moreover for any parameter σ , $U_\sigma(Q)$ is \hat{S} -invariant as well. So by definition of ρ and [DJ12, Fact 2.7], PQ is actually nilpotent, hence $PQ = Q$, $P \leq Q$, and $P \leq U_\rho(Q)$. By maximality of P , $P = U_\rho(Q)$ is characteristic in Q so $N_G^\circ(Q) \leq N_N^\circ(Q) = Q$ and Q is a Carter subgroup of G .

Now suppose that $N = G$, that is, P is normal in G . By induction, $\overline{G} = G/P$ has a \hat{S} -invariant Carter subgroup \overline{C} . Let H be the preimage of \overline{C} in G ; H is soluble. By Lemma C, H has a \hat{S} -invariant Carter subgroup Q . Here again PQ is soluble and even nilpotent, so $P \leq Q$. Since H is soluble, $Q/P = PQ/P$ is a Carter subgroup of $H/P = \overline{C}$ [Fr 00a, Corollaire 5.20], so $Q/P = \overline{C}$ and $Q = H$. Now $N_G^\circ(Q)/P \leq N_{\overline{C}}^\circ(\overline{C}) = \overline{C} = Q/P$, so $N_G^\circ(Q) = Q$ and Q is a Carter subgroup of G . \square

Remarks.

- Burdges left the assumption that \hat{G} is U_2^\perp implicit from the title of his prepublication and the original statement must therefore be taken with care: the Sylow 2-subgroup of $(\mathbb{F}_2)_+ \rtimes (\mathbb{F}_2)^\times$ certainly does not normalise any Carter subgroup.
- The assumption that $p = 2$ is used only to find an infinite, definable, abelian \hat{S} -invariant subgroup. It is not known whether all connected groups of finite Morley rank having a definable automorphism of order $p \neq 2$ with finitely many fixed points are soluble, a classical property of algebraic groups though.

3.2 Involutive Automorphisms

The need for the present subsection is the following. [DJ10, Section 5] collected various well-known facts in order to provide a decomposition for a connected, soluble group of odd type under an *inner* involutive automorphism. But in the present article we shall consider the case of “outer” automorphisms, more precisely the action of abstract 2-tori on our groups. So the basic discussion of [DJ10] must take place in a broader setting; this is what we do here.

Notation. *If α is an involutory automorphism of some group G , we let $G^+ = C_G(\alpha) = \{g \in G : g^\alpha = g\}$ and $G^- = \{g \in G : g^\alpha = g^{-1}\}$. We also let $\{G, \alpha\} = \{[g, \alpha] : g \in G\}$ (in context there is no risk of confusion with the usual notation for unordered pairs).*

If G and α are definable, so are G^+ , G^- , and $\{G, \alpha\}$; in general only the first need be a group. $\{G, \alpha\}$ is nevertheless stable under inversion, since $[g^\alpha, \alpha] = [g, \alpha]^{-1}$. Observe that $\{G, \alpha\} \subseteq G^-$ but equality may fail to hold: for instance if α centralises G and G contains an involution i , then $i \in G^+ \cap G^-$ but $i \notin \{G, \alpha\} = \{1\}$. Notice further that $G = G^+ \cdot G^-$ iff $\{G, \alpha\} \subseteq (G^-)^{\wedge 2}$ and $G = G^+ \cdot \{G, \alpha\}$ iff $\{G, \alpha\} \subseteq \{G, \alpha\}^{\wedge 2}$, where $X^{\wedge 2}$ denotes the set of squares of X . Finally remark that $\deg\{G, \alpha\} = \deg \alpha^G \alpha = \deg \alpha^G \leq \deg G$.

Lemma E (cf. [DJ10, Theorem 19]). *Let G be a group of finite Morley rank with Sylow 2-subgroup a (possibly trivial) central 2-torus S , and α be a definable involutive automorphism of G . Then $G = G^+ \cdot \{G, \alpha\}$ where the fibers of the associated product map are in bijection with $I(\{G, \alpha\}) \cup \{1\} = \Omega_2([S, \alpha])$. Furthermore one has $G = (G^+)^{\circ} \cdot \{G, \alpha\}$ whenever G is connected.*

Proof. The proof follows that of [DJ10, Theorem 19] closely and for some parts a minor adjustment would suffice. We prefer to give a complete proof and discard [DJ10]. Bear in mind that if $a^b = a^{-1}$ for two elements of G , then a has order at most 2 (this is [DJ10, Lemma 20], an easy consequence of torsion lifting). Also remember from [DJ10, Lemma 18] that G is 2-divisible: merely because 2-torsion is divisible and central.

Step 1. $S \cap \{G, \alpha\} = [S, \alpha]$.

Proof of Step. This is the argument from [DJ10, Theorem 19, Step 1] with one more remark. One inclusion is trivial. Now let $\zeta \in S \cap \{G, \alpha\}$, and write $\zeta = [g, \alpha]$. Since G is 2-divisible we let $h \in H$ satisfy $h^2 = g$. Let $n = 2^k$ be the order of ζ . Then $[h^2, \alpha] = [h, \alpha]^h [h, \alpha] = \zeta \in Z(G)$ so $[h, \alpha]$ and $[h, \alpha]^h$ commute. Hence $1 = \zeta^n = [h, \alpha]^n [h, \alpha]^{nh}$. It follows that h inverts $[h, \alpha]^n$ which must have order at most 2: so $\xi = [h, \alpha]^{-1}$ is a 2-element inverted by α , and since it is central it commutes with h . Finally $[\xi, \alpha] = \xi^{-2} = [h, \alpha]^2 = [h^2, \alpha] = \zeta$. \diamond

It immediately follows that $I(\{G, \alpha\}) \cup \{1\} = \Omega_2([S, \alpha])$.

Step 2. $\{G, \alpha\}$ is 2-divisible and $G = G^+ \cdot \{G, \alpha\}$.

Proof of Step. Here again this is the argument from [DJ10, Theorem 19, Step 2]; 2-divisibility of $\{G, \alpha\}$ was announced but not explicitly proved.

Let $x = [g, \alpha] \in \{G, \alpha\}$. Like in [DJ10, Theorem 19, Step 2], write the definable hull of x as $d(x) = \delta \oplus \langle \gamma \rangle$ where δ is connected and γ has finite order; rewrite $\gamma = \varepsilon \zeta$ where ε has odd order and ζ is a 2-element; let $\Delta = \delta \oplus \langle \varepsilon \rangle$, so that $d(x) = \Delta \oplus \langle \zeta \rangle$ where Δ is 2-divisible and inverted by α . Now let $y \in \Delta$ satisfy $y^4 = x \zeta^{-1}$. Then $[gy^2, \alpha] = [g, \alpha]^{y^2} [y^2, \alpha] = xy^{-4} = \zeta \in S \cap \{G, \alpha\} = [S, \alpha]$ by Step 1, so there is $\xi \in S$ with $[\xi^2, \alpha] = \zeta$. Now $[y^{-1}\xi, \alpha] = [y^{-1}, \alpha]^\xi [\xi, \alpha] = y^2 [\xi, \alpha]$ squares to $y^4 [\xi, \alpha]^2 = x \zeta^{-1} [\xi^2, \alpha] = x$. The set $\{G, \alpha\}$ is therefore 2-divisible; as observed this implies $G = G^+ \cdot \{G, \alpha\}$. \diamond

Step 3. *Fibers in Step 2 are in bijection with $\Omega_2([S, \alpha])$.*

Proof of Step. Let $k = [s, \alpha]$ have order at most 2, where $s \in S$. Fix any decomposition $\gamma = a \cdot [g, \alpha]$ with $a \in G^+$ and $g \in G$. Since α inverts (hence centralises) k , one has $ka_+ \in G^+$. Moreover $[sg, \alpha] = [s, \alpha]^g [g, \alpha] = k^g [g, \alpha] = k [g, \alpha] \in \{G, \alpha\}$. So $a[g, \alpha] = (ka) \cdot (k[g, \alpha])$ is yet another decomposition for γ .

Conversely work like in [DJ10, Theorem 19, Step 3]: suppose that $ax = by$ are two decompositions, with $a, b \in G^+$ and $x = [g, \alpha], y = [h, \alpha] \in \{G, \alpha\}$. Then $(a^{-1}b)^y = (xy^{-1})^y = y^{-1}x = (yx^{-1})^\alpha = (b^{-1}a)^\alpha = b^{-1}a = (a^{-1}b)^{-1}$ so $a^{-1}b$ has order at most 2, say $k = a^{-1}b$. More precisely, $k = xy^{-1} = [g, \alpha][h, \alpha]^{-1} = [g, \alpha]h^{-\alpha}h$ is central, so $k = h[g, \alpha]h^{-\alpha} = [gh^{-1}, \alpha] \in \{G, \alpha\}$; it follows from Step 1 that $k \in \Omega_2([S, \alpha])$. \diamond

Step 4. *Left G^+ -translates of the set $(G^+)^{\circ} \cdot \{G, \alpha\}$ are disjoint or equal.*

Proof of Step. Like in [DJ10, Theorem 19, Step 4]: suppose that $a(G^+)^{\circ} \cdot \{G, \alpha\}$ meets $b(G^+)^{\circ} \cdot \{G, \alpha\}$, in say $ag_+[g, \alpha] = bh_+[h, \alpha]$ with natural notations. By Step 3, $k = (ag_+)^{-1}(bh_+)$ is in

$\Omega_2([S, \alpha])$, therefore central in G and inverted (hence centralised) by α . So $k = (bh_+)(ag_+)^{-1} = (ag_+)(bh_+)^{-1}$. Hence for any $b\gamma_+[\gamma, \alpha] \in b(G^+)^{\circ} \cdot \{G, \alpha\}$ one finds:

$$b\gamma_+[\gamma, \alpha] = k^2 b\gamma_+[\gamma, \alpha] = a(g_+ h_+^{-1} \gamma_+)([\gamma, \alpha]k)$$

Since $k \in \Omega_2([S, \alpha])$, there is $s \in S$ with $k = [s, \alpha]$. So $[\gamma s, \alpha] = [\gamma, \alpha]^s [s, \alpha] = [\gamma, \alpha]k \in \{G, \alpha\}$: hence $b\gamma_+[\gamma, \alpha] \in a(G^+)^{\circ} \cdot \{G, \alpha\}$. This shows $b(G^+)^{\circ} \{G, \alpha\} \subseteq a(G^+)^{\circ} \{G, \alpha\}$ and the converse inclusion holds too. \diamond

Step 5. *Exactly deg G left G^+ -translates of $(G^+)^{\circ} \cdot \{G, \alpha\}$ cover G . In particular, if G is connected, then $G = (G^+)^{\circ} \cdot \{G, \alpha\}$.*

Proof of Step. We consider such left translates. They all have rank $\text{rk } G$ by Step 3 and their degree is 1. As they are disjoint or equal by Step 4, exactly deg G of them suffice to cover G . \diamond

This ends the proof of Lemma E. \square

Remarks.

- If G is a connected group of finite Morley rank of odd type whose Sylow 2-subgroup S is central, then S is a 2-torus as $S = C_S(S^{\circ}) = S^{\circ}$ by torality principles.
- The Lemma fails if S is not 2-divisible, even at the abelian level: let α invert $\mathbb{Z}/4\mathbb{Z}$.

As a consequence we deduce another useful decomposition. It will be used repeatedly.

Lemma F (cf. [DJ10, Lemma 24]). *Let H be a U_2^{\perp} , connected, soluble group of finite Morley rank, and α be a definable involutive automorphism of H . Suppose that $\{H, \alpha\} \subseteq F^{\circ}(H)$. Then $H = (H^+)^{\circ} \cdot \{H, \alpha\}$ with finite fibers.*

Proof. By normalisation principles, H admits an α -invariant Carter subgroup Q ; by the theory of Carter subgroups of soluble groups, $H = Q \cdot F^{\circ}(H)$ [Fré00a, Corollaire 5.20]. Now both Q and $F^{\circ}(H)$ are definable, connected, nilpotent, and U_2^{\perp} : so Lemma E applies to them. Hence $Q = (Q^+)^{\circ} \cdot \{Q, \alpha\} \subseteq (H^+)^{\circ} \cdot F^{\circ}(H)$, and $H = Q \cdot F^{\circ}(H) \subseteq (H^+)^{\circ} \cdot F^{\circ}(H) \subseteq (H^+)^{\circ} \cdot (F^{\circ}(H)^+)^{\circ} \cdot \{F^{\circ}(H), \alpha\} \subseteq (H^+)^{\circ} \cdot \{H, \alpha\}$.

The fibers are finite: this works like in [DJ10, Lemma 24] since if $c_1 b_1 = c_2 b_2$ with $c_i \in H^+$, $b_i \in \{H, \alpha\}$, then $c_2^{-1} c_1 = b_2 b_1^{-1} \in H^+$ so $b_2 b_1^{-1} = b_2^{-1} b_1$ and $b_1^2 = b_2^2$, but by assumption $b_i \in \{H, \alpha\} \subseteq F^{\circ}(H)$ so b_1 and b_2 differ by an element of $\Omega_2(F^{\circ}(H))$. Unlike in Lemma E we cannot be too precise about the cardinality of the fiber. \square

Remarks.

- We can show $\{H, \alpha\} \subseteq \Omega_2(F^{\circ}(H)) \cdot \{F^{\circ}(H), \alpha\}$. For let $h \in H$; then $[h, \alpha] \in \{H, \alpha\} \subseteq F^{\circ}(H)$. Applying Lemma E in $F^{\circ}(H)$, we write $[h, \alpha] = f_+[f, \alpha]$ with $f_+ \in F^{\circ}(H)^+$ and $f \in F^{\circ}(H)$. Taking the commutator with α we find $[h, \alpha]^2 = [f, \alpha]^2$. But in $F^{\circ}(H)$, the equation $x^2 = y^2$ results in $x^{-1} \cdot x^{-1} y \cdot x = y^{-1} x = (x^{-1} y)^{-1}$ and $x^{-1} y$ has order at most 2. Hence $[h, \alpha] = k[f, \alpha]$ for some $k \in \Omega_2(F^{\circ}(H))$.
- Without the crucial assumption that $\{H, \alpha\} \subseteq F^{\circ}(H)$ one still has $H = \{H, \alpha\} \cdot (H^+)^{\circ} \cdot \{H, \alpha\}$ and therefore $H = H^- \cdot H^+ \cdot H^-$, but one can hardly say more.

Consider two copies $A_1 = \{a_1 : a \in \mathbb{C}\}$, $A_2 = \{a_2 : a \in \mathbb{C}\}$ of \mathbb{C}_+ and let $Q = \{t : t \in \mathbb{C}^{\times}\} \simeq \mathbb{C}^{\times}$ act on A_1 by $a_1^t = (t^2 a)_1$ and on A_2 by $a_2^t = (t^{-2} a)_2$. Form the group $H = (A_1 \oplus A_2) \rtimes Q$. Let α be the definable, involutive automorphism of H given by:

$$(a_1 b_2 t)^{\alpha} = b_1 a_2 t^{-1}$$

that is, “ α swaps the ± 2 weight spaces while inverting the torus”. The reader may check that α is an automorphism of H , and perform the following computations:

- $[a_1 b_2 t, \alpha] = (t^2 b - t^2 a)_1 (t^{-2} a - t^{-2} b)_2 t^{-2}$ (so $\{H, \alpha\} \not\subseteq F^{\circ}(H)$);
- $H^+ = \{a_1 a_2 \cdot \pm 1 : a \in \mathbb{C}_+\}$ (incidentally $(H^+)^{\circ} \leq F^{\circ}(H)$);

- $H^- = \{a_1(-t^2a)_2t : a \in \mathbb{C}_+, t \in \mathbb{C}^\times\}$ (incidentally $H^- = \{H, \alpha\}$);
- $H^+ \cdot H^- = \{(a+b)_1(a-t^2b)_2 \cdot \pm t : a, b \in \mathbb{C}_+, t \in \mathbb{C}^\times\}$ does *not* contain $0_1a_2 \cdot i$ for $a \neq 0$ (here i is a complex root of -1).

- Rewriting [DJ10, Theorem 19] is necessary for the argument; one can't simply use the idea of Lemma F together with the original decomposition.

Let $Q = \mathbb{C}^\times$ act on $A = \mathbb{C}_+$ by $a^t = (t^2 \cdot a)$ and form $H = A \rtimes Q$. Consider α the involutive automorphism doing $(at)^\alpha = (-a)t$ (α inverts the Fitting subgroup while centralising the Carter subgroup). The reader will check that $H^+ = Q$, $H^- = A \cdot \pm 1$, $\{H, \alpha\} = A$, and of course $H = H^+ \cdot H^-$.

Running the argument in Lemma F using the (naive) $G = G^+ \cdot G^-$ decomposition of [DJ10, Theorem 19], one finds $Q = (Q^+)^\circ \cdot Q^-$, but Q^- is not in $F^\circ(H)$. One could then wish to apply the decomposition to $F(H)$ instead, but the Sylow 2-subgroup of the latter is not a 2-torus!

Extending [DJ10, Theorem 19] into Lemma E was therefore needed for Lemma F.

3.3 U_p^\perp actions and centralisers

Let p denote a set of prime numbers.

Notation. If A and B are two subgroups of some ambient abelian group, we write $A(+)B$ to denote the quasi-direct sum, i.e. in order to mean that $A \cap B$ is finite.

Lemma G (cf. [ABC08, Corollaries 9.11 and 9.14 on pp.87-88] and [CD12, Facts 1.15 and 1.16]). *In a universe of finite Morley rank let A be a definable, abelian, connected, U_p^\perp group and R be a definable, soluble p -group acting on A . Then $A = [A, R](+)C_A^\circ(R)$. Moreover if $A_0 \leq A$ is a definable, connected, R -invariant subgroup, then $([A, R] \cap A_0)^\circ = [A_0, R]$; $C_A^\circ(R)$ covers $C_{A/A_0}^\circ(R)$, and $C_R(A) = C_R(A_0, A/A_0)$.*

The author does not consider the proof to be necessary as it follows [CD12, Facts 1.15 and 1.16] closely. Here it goes anyway.

Proof. By connectedness of A and using Zilber's indecomposibility theorem, $[A, r]$ is connected for all $r \in R$. By the ascending chain condition on definable, connected subgroups, there is a finite set $X \subseteq R$ such that $[A, R] = [A, X]$; by the descending chain condition on centralisers there is another finite set $Y \subseteq R$ such that $C_A^\circ(R) = C_A^\circ(Y)$. But R is a soluble, periodic group, whence locally finite; so taking $\langle X \cup Y \rangle$ we may as well assume that R is finite.

We show that $A = [A, R] + C_A^\circ(R)$ by induction on the order of R . By solubility, there exist $S \triangleleft R$ and $r \in R$ with $R = \langle S, r \rangle$. By induction, $A = [A, S] + C_A^\circ(S)$. But r normalises $C = C_A^\circ(S)$ which is a U_p^\perp -group. Consider the definable homomorphisms $\text{ad}_r : C \rightarrow C$ and $\text{Tr}_r : C \rightarrow C$ respectively given by:

$$\text{ad}_r(a) = [a, r], \quad \text{Tr}_r(a) = \sum_{r^i \in \langle r \rangle} a^{r^i}$$

Since $\text{ad}_r \circ \text{Tr}_r = \text{Tr}_r \circ \text{ad}_r = 0$, one has $\text{im ad}_r \leq \ker \text{Tr}_r$ and $\text{im Tr}_r \leq \ker \text{ad}_r$. But since $\ker \text{Tr}_r \cap \ker \text{ad}_r$ consists of elements of order dividing $|r|$, it is finite by assumption. It follows that $\text{rk } C \geq \text{rk } \ker \text{Tr}_r + \text{rk } \ker \text{ad}_r \geq \text{rk } \text{im ad}_r + \text{rk } \ker \text{ad}_r = \text{rk } C$. Hence $\text{rk } \text{im ad}_r = \text{rk } \ker \text{Tr}_r$ and $[C, r] = \text{im ad}_r = \text{im}^\circ \text{ad}_r = \ker^\circ \text{Tr}_r$, which implies $C = \ker^\circ \text{Tr}_r + \ker^\circ \text{ad}_r = \text{im}^\circ \text{ad}_r + \ker^\circ \text{ad}_r = [C, r] + C_C^\circ(r)$. This shows that $A = [A, S] + C_A^\circ(S) = [A, R] + [C, r] + C_C^\circ(r) = [A, R] + C_A^\circ(R)$.

We now prove that $[A, R] \cap C_A^\circ(R)$ is finite. Consider the definable homomorphism $\text{Tr}_R : A \rightarrow A$ given by:

$$\text{Tr}_R(a) = \sum_{r \in R} a^r$$

Since Tr_R vanishes on any $[A, r]$, it vanishes on $[A, R]$; notice that it coincides with multiplication by $|R|$ on $C_A(R)$. It follows that $[A, R] \cap C_A(R)$ consists of elements of order dividing $|R|$, so by assumption it is finite.

We shall say a bit more: $\ker^\circ \text{Tr}_R = [A, R]$ and $\text{im Tr}_R = C_A^\circ(R)$. Indeed $A = [A, R] + C_A^\circ(R)$ and $[A, R] \leq \ker^\circ \text{Tr}_R$, so $\ker^\circ \text{Tr}_R \leq [A, R] + C_{\ker^\circ \text{Tr}_R}^\circ(R)$. But $C_{\ker^\circ \text{Tr}_R}^\circ(R)$ consists of elements of order dividing $|R|$, therefore it is finite. It follows that $\ker^\circ \text{Tr}_R = [A, R]$. Similarly, $\text{im Tr}_R \leq C_A^\circ(R)$; by connectedness, $\text{im Tr}_R \leq C_A^\circ(R)$. From $([A, R] \cap C_A^\circ(R))^\circ = 1$ one knows $\text{rk } A \geq \text{rk}[A, R] + \text{rk } C_A^\circ(R) \geq \text{rk } \ker^\circ \text{Tr}_R + \text{rk } \text{im Tr}_R$, so we have equalities and $\text{im Tr}_R = C_A^\circ(R)$.

We turn our attention to the definable, connected, R -invariant subgroup $A_0 \leq A$; for our purpose R may still be taken to be finite. One sees that:

$$([A, R] \cap A_0)^\circ = (\ker^\circ \text{Tr}_R \cap A_0)^\circ = \ker^\circ (\text{Tr}_R)|_{A_0} = [A_0, R]$$

and letting φ stand for projection modulo A_0 :

$$\varphi(C_A^\circ(R)) = \varphi \circ \text{Tr}_R(A) = \text{Tr}_R \circ \varphi(A) = \text{Tr}_R(A/A_0) = C_{A/A_0}^\circ(R)$$

Finally let $S = C_R(A_0, A/A_0)$. We apply our results to the action of S on A and find $A \leq [A, S] + C_A^\circ(S) \leq C_A^\circ(S)$ so $S = C_R(A)$. \square

Remark. The Lemma does not hold for non-connected A since it fails at the finite level: let $R = \mathbb{Z}/2\mathbb{Z}$ act by inversion on $A = \mathbb{Z}/4\mathbb{Z}$; one has $C_A(R) = 2A = [A, R]$.

After obtaining the following Lemma the author realised it was already proved by Burdges and Cherlin using a different argument.

Lemma H ([BC08, Lemma 2.5]). *Let G be a group of finite Morley rank, R be a soluble p -group acting on G , and let $H \trianglelefteq G$ be a definable, connected, soluble, U_p^\perp , R -invariant subgroup. Then $C_{G/H}^\circ(R) = C_G^\circ(R)H/H$.*

Proof. As in Lemma G, using chain conditions and local finiteness, we may assume that R is finite. Let $L = \varphi^{-1}(C_{G/H}^\circ(R))$, where φ denotes projection modulo H . Since φ is surjective, $\varphi(L) = C_{G/H}^\circ(R)$ which is connected and a finite extension of $\varphi(L^\circ)$ which is connected as well: so $\varphi(L) = \varphi(L^\circ)$ and $L = L^\circ H = L^\circ$ by connectedness of H . Hence L itself is connected. We now proceed by induction on the solubility class of H .

First suppose that H is abelian; we proceed by induction on the solubility class of R .

- First suppose that $R = \langle r \rangle$. Be careful that the definable map $\text{Tr}_r : G \rightarrow G$ given by:

$$\text{Tr}_r(g) = \prod_{i=0}^{|r|-1} g^{r^i}$$

is *not* a group homomorphism, but $(\text{Tr}_r)|_H$ is one.

Since $[L, r] \leq H \cap \text{Tr}_r^{-1}(0) = \ker(\text{Tr}_r)|_H$, one has by connectedness and Zilber's indecomposability theorem $[L, r] \leq \ker^\circ(\text{Tr}_r)|_H = [H, r]$ by the proof of Lemma G. Bear in mind that H is abelian; it follows that $L \leq HC_G(r)$, so by connectedness $L \leq HC_G^\circ(r)$ as desired.

- Now suppose $R = \langle S, r \rangle$ with $S \triangleleft R$. By induction, $L \leq HC_G^\circ(S)$ and since $H \leq L$, one has $L \leq HC_L^\circ(S)$. Let $G_S = C_G^\circ(S)$ and $H_S = C_H^\circ(S)$; also let φ_S be the projection $G_S \rightarrow G_S/H_S$, and $L_S = \varphi_S^{-1}(C_{G_S/H_S}^\circ(r))$.

By the cyclic case, $L_S \leq H_S C_{G_S}^\circ(r) \leq HC_G^\circ(R)$. But $[C_L^\circ(S), r] \leq H \cap C_G^\circ(S)$ so by connectedness $[C_L^\circ(S), r] \leq C_H^\circ(S) = H_S$. It follows that $C_L^\circ(S) \leq L_S \leq HC_G^\circ(R)$ and $L \leq HC_L^\circ(S) \leq HC_G^\circ(R)$.

We now let $K = H'$, which is a definable, connected, R -invariant subgroup normal in G . Let $\varphi_K : G \rightarrow G/K$ and $\psi : G/K \rightarrow G/H$ be the standard projections, so that $\varphi = \psi\varphi_K$. By induction, $\varphi_K(C_G^\circ(R)) = C_{\varphi_K(G)}^\circ(R)$. But $\varphi_K(H) \trianglelefteq \varphi_K(G)$ and $\varphi_K(H)$ is abelian, so by the abelian case we just covered, $\psi(C_{\varphi_K(G)}^\circ(R)) = C_{\psi\varphi_K(G)}^\circ(R)$. Therefore:

$$\varphi(C_G^\circ(R)) = \psi(\varphi_K(C_G^\circ(R))) = \psi(C_{\varphi_K(G)}^\circ(R)) = C_{\psi\varphi_K(G)}^\circ(R) = C_{\varphi(G)}^\circ(R) \quad \square$$

Let ρ denote a unipotence parameter. We wish to generalise [Bur04a, Lemma 3.6] relaxing the \underline{p}^\perp assumption to $U_{\underline{p}}^\perp$. This will considerably simplify some arguments; in particular we shall no longer care whether Burdges' unipotent radicals of Borel subgroups contain involutions or not when taking centralisers. This will spare us the contortions of [Del07b, Lemmes 5.2.33, 5.2.39, 5.3.20, 5.3.23].

Lemma I (cf. [Bur04a, Lemma 3.6]). *Let G be a group of finite Morley rank, R be a soluble \underline{p} -group acting on G , and let $U \trianglelefteq G$ be a $U_{\underline{p}}^\perp$, R -invariant ρ -subgroup. Then $C_U^\circ(R)$ is a ρ -group.*

The author does not consider the proof to be necessary as it follows [Bur04a, Lemma 3.6] closely. Here it goes anyway.

Proof. The proof is by induction on the nilpotence class of U . First suppose that U is abelian. Then by Lemma G one has $U = [U, R](+)C_U^\circ(R)$. Let K stand for the finite intersection. Then $C_U^\circ(R)/K \simeq U/[U, R]$ which by push-forward [Bur04a, Lemma 2.11] is a ρ -group. It follows that $C_U^\circ(R)$ itself is a ρ -group. (Since we could not locate a proof of this trivial fact in the literature, here it goes: let $V = C_U^\circ(R)$ and $\varphi: V \rightarrow V/K$ be the standard projection. By pull-back [Bur04a, Lemma 2.11], $\varphi(U_\rho(V)) = V/K = \varphi(V)$, and since φ has finite kernel, $\text{rk } U_\rho(V) = \text{rk } V$. By connectedness, $V = U_\rho(V)$.)

Now let $1 < A \triangleleft U$ be an abelian definable, connected, characteristic subgroup. By induction, $C_A^\circ(R)$ and $C_{U/A}^\circ(R)$ are ρ -groups. Now by Lemma H,

$$\begin{aligned} C_{U/A}^\circ(R) &\simeq C_U^\circ(R)A/A \\ &\simeq C_U^\circ(R)/(A \cap C_U^\circ(R)) \\ &\simeq (C_U^\circ(R)/C_A^\circ(R)) / ((A \cap C_U^\circ(R))/C_A^\circ(R)) \\ &= (C_U^\circ(R)/C_A^\circ(R)) / L \end{aligned}$$

where $L = (A \cap C_U^\circ(R))/C_A^\circ(R)$ is finite. Since $C_{U/A}^\circ(R)$ is a ρ -group, so is $C_U^\circ(R)/C_A^\circ(R)$. But $C_A^\circ(R)$ is a ρ -group, so by pull-back, so is $C_U^\circ(R)$. \square

One could of course do the same with a set of unipotence parameters instead of a single parameter ρ .

Remark. As opposed to the usual setting of \underline{p}^\perp groups [Bur04a, Lemma 3.6], connectedness of $C_U(R)$ is not granted in the $U_{\underline{p}}^\perp$ case.

As a consequence, if inside a group of odd type some involution i acts on a σ -group H with $\rho_{C(i)} \prec \sigma$, then i inverts H . We shall use this fact with no reference.

3.4 Carter π -subgroups

The maybe not-so-familiar notion of a Carter π -subgroup was reminded in §2.3.

Lemma J. *Let H be a connected, soluble group of finite Morley rank, π be a set of parameters such that $U_\pi(H') = 1$, and $L \leq H$ be a (nilpotent) π -subgroup. Then there is a Carter subgroup $Q \leq H$ of H containing L .*

Proof. Induction on $|\pi|$. If $|\pi| = 1$ then we are actually dealing with a single unipotence parameter ρ , and the result follows from the theory of Sylow ρ -subgroups ([Bur04b, Lemma 4.19], [Bur06, Theorem 5.7]).

For the inductive step, write Burdges' decomposition of $L = L_\rho * M$, where M is a (nilpotent) $(\pi \setminus \{\rho\})$ -group. By induction there is a Carter subgroup Q of H containing M . Now $X = (M \cap Z^\circ(Q))^\circ \neq 1$ satisfies $C_H^\circ(X) \geq \langle L, Q \rangle$ and there are two cases.

- If $C_H^\circ(X) < H$ then by induction on the Morley rank L is contained in some Carter subgroup of $C_H^\circ(X)$. Since $Q \leq C_H^\circ(X)$, the former also is a Carter subgroup of H .
- If $C_H^\circ(X) = H$ then we go modulo X : there by push-forward $\bar{L} = U_\pi(\bar{L})$ and $U_\pi(\bar{H}') = 1$. By induction on the Morley rank again, there is a Carter subgroup \bar{Q} of \bar{H} containing \bar{L} . The preimage Q of \bar{Q} is a Carter subgroup of H containing L [Fré00a, Corollaire 5.20]. \square

3.5 W_p^\perp Groups

Notation. Let G be a U_p^\perp group of finite Morley rank. Let $W_p(G) = S/S^\circ$ for any p -Sylow subgroup S of G (these are conjugate by [BC09, Theorem 4], our Fact 5, so this is well-defined).

Lemma K. Let G be a U_p^\perp group of finite Morley rank.

- If $H \leq G$ is a definable, connected subgroup, then $W_p(H) \hookrightarrow W_p(G)$.
- If $H \trianglelefteq G$ is a definable, connected, normal subgroup, then $W_p(G/H) \simeq W_p(G)/W_p(H)$.

Proof.

- Let S_H be a Sylow p -subgroup of H and extend it to a Sylow p -subgroup S_G of G . To $w \in W_p(H)$ associate $hS_G^\circ \in W_p(G)$ where $h \in S_H$ is such that $hS_H^\circ = w$. This is a well-defined group homomorphism as $S_H^\circ \leq S_G^\circ$. It is injective since if $h \in S_H \cap S_G^\circ$, then $h \in C_{S_H}(S_H^\circ) = S_H^\circ$ by torality principles and connectedness of H .
- Let $S_H \leq S_G$ be as above and denote projection modulo H by $\bar{}$; we know that $\Sigma = \overline{S_G} \simeq S_G/S_H$ is a Sylow p -subgroup of G/H . To $w \in W_p(G)$ associate $\bar{g}\Sigma^\circ \in W_p(G/H)$ where $g \in S_G$ is such that $gS_G^\circ = w$. This is a well-defined group homomorphism as $\overline{S_G^\circ} = \Sigma^\circ$. It is clearly surjective. Now if w is in the kernel then $g \in S_G^\circ H$, and we may suppose $g \in H$ (the converse is obvious). Hence the kernel coincides with the image of $W_p(H)$ in $W_p(G)$. \square

We wish to suggest a bit of terminology.

Definition. A U_p^\perp group of finite Morley rank is W_p^\perp if its Sylow p -subgroup is connected.

As a consequence of Lemma K, when $H \trianglelefteq G$ where both are definable and connected, if H and G/H are W_p^\perp then so is G . We aim at saying a bit more about extending tori. The following result is not used anywhere in the present article.

Lemma L. Let \hat{G} be a connected, U_p^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected subgroup. Suppose that \hat{G}/G is W_p^\perp . Let $\hat{S} \leq \hat{G}$ be a Sylow p -subgroup and $S = \hat{S} \cap G$. Then there exist:

- a p -torus $\hat{T} \leq \hat{G}$ with $\hat{S} = S \rtimes \hat{T}$ (semi-direct product);
- a p -torus $\hat{\Theta} \leq \hat{G}$ with $\hat{S} = S(\times)\hat{\Theta}$ (central product over a finite intersection).

Proof. We know that S is a Sylow p -subgroup of G and that $\hat{S}/S \simeq \hat{G}/G$ is a Sylow p -subgroup of \hat{G}/G ; as the latter is W_p^\perp it is a p -torus. In particular $\hat{S} = \hat{S}^\circ S$. Note that $S \cap \hat{S}^\circ \leq C_S(S^\circ) = S^\circ$ by torality principles.

Bear in mind that p -tori are injective as \mathbb{Z} -modules. Inside \hat{S}° take a direct complement \hat{T} of S° , so that $\hat{S}^\circ = S^\circ \oplus \hat{T}$. Then $\hat{S} = S\hat{S}^\circ = S\hat{T}$, but $S \cap \hat{T} \leq S \cap \hat{S}^\circ \cap T \leq S^\circ \cap T = 1$. Hence $\hat{S} = S \rtimes \hat{T}$.

We now consider the action of \hat{S} on \hat{S}° ; observe that \hat{S} as a pure group has finite Morley rank, so Lemma G applies and yields $\hat{S}^\circ = [\hat{S}^\circ, \hat{S}](+C_{\hat{S}^\circ}(\hat{S}))$. Since \hat{S}/S is a p -torus, it is abelian, so $[\hat{S}^\circ, \hat{S}] \leq \hat{S}' \leq S$, and by Zilber's indecomposibility theorem $[\hat{S}^\circ, \hat{S}] \leq S^\circ$. Inside $C_{\hat{S}^\circ}(\hat{S})$ take a direct complement $\hat{\Theta}$ of $C_{S^\circ}(\hat{S})$, so that $C_{\hat{S}^\circ}(\hat{S}) = C_{S^\circ}(\hat{S}) \oplus \hat{\Theta}$. Then $\hat{S} = S\hat{S}^\circ = SC_{\hat{S}^\circ}(\hat{S}) = S\hat{\Theta}$, and $\hat{\Theta} \leq C_{\hat{S}^\circ}(\hat{S})$ commutes with S . Moreover $(S \cap \hat{\Theta})^\circ \leq (C_S(\hat{S}) \cap \hat{\Theta})^\circ \leq C_{S^\circ}(\hat{S}) \cap \hat{\Theta} = 1$ by construction, so $\hat{S} = S(\times)\hat{\Theta}$. \square

Remark. One may not demand that $\hat{S} = S \times \hat{T}$ (direct product). Consider the two groups $\mathrm{SL}_2(\mathbb{C})$ with involution i and \mathbb{C}^\times with involution j . Let $\hat{G} = (\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times) / \langle ij \rangle$ and $\varphi : \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times \rightarrow \hat{G}$ be the standard projection. Let $G = \varphi(\mathrm{SL}_2(\mathbb{C})) \simeq \mathrm{SL}_2(\mathbb{C})$ and $\hat{\Theta} = \varphi(\mathbb{C}^\times) \simeq \mathbb{C}^\times$. Fix any Sylow 2-subgroup \hat{S} of \hat{G} . Then with $S = \hat{S} \cap G$ one has $S\hat{\Theta} = S(\times)\hat{\Theta} = \hat{S}$, and $S \cap \hat{\Theta} = \langle \varphi(i) \rangle$.

If one asks for a semi-direct complement \hat{T} , the latter must contain its own involution, which will be $\varphi(ab)$ (or possibly $\varphi(iab)$, a similar case), where $a \in \varphi^{-1}(S) \leq \mathrm{SL}_2(\mathbb{C})$ satisfies $a^2 = i$ and $b^2 = j$ in \mathbb{C}^\times . Remember that inside a fixed Sylow 2-subgroup of $\mathrm{SL}_2(\mathbb{C})$, every element of order

four (be it toral inside the fixed Sylow 2-subgroup or not) is inverted by another element of order four. So let $\zeta \in \varphi^{-1}(S)$ invert a . Then:

$$\varphi(\zeta^{ab}) = \varphi(\zeta^a) = \varphi(i\zeta) \neq \varphi(\zeta)$$

so the action of \hat{T} on S is always non-trivial.

One may not demand $\hat{S} = S \times \hat{T}$, and in any case nothing can apparently prevent $d(\hat{T})$ from intersecting G non-trivially, so why bother?

3.6 A Counting Lemma

The following quite elementary Lemma was devised in Kapadokya in 2007 as an explanation of [Del07b, Corollaire 5.1.7] (or [Del08, Corollaire 4.7]).

Lemma M (Göreme). *Let G be a connected, U_2^\perp , W_2^\perp group of finite Morley rank. Then the number of conjugacy classes of involutions is odd (or zero).*

Proof. By torality principles, every class is represented in a fixed Sylow 2-subgroup $S = S^\circ$. We group involutions of S° by classes γ_k , and assume we find an even number of these: $I(S^\circ) = \sqcup_{k=1}^{2m} \gamma_k$. Since the number of involutions in S° is however odd, some class, say γ , has an even number of involutions. Now $N = N_G(S)$ acts on γ ; by definition of a conjugacy class and by a classical fusion control argument [BN94b, Lemma 10.22], N acts transitively on γ . Hence $[N : C_N(\gamma)] = |\gamma|$ is even. Lifting torsion, there is a non-trivial 2-element ζ in $N \setminus C_N(\gamma)$. Since $S \trianglelefteq N$, $\zeta \in S = S^\circ \leq C_N(\gamma)$, a contradiction. \square

The author hoped to be able to use this Lemma without any form of bound on the Prüfer 2-rank. He failed as one shall see in Step 8 of the Theorem. The general statement remains as a relic of happier times past.

4 The Proof – Before the Maximality Proposition

Theorem. *Let \hat{G} be a connected, U_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup.*

Then the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$, isomorphic to that of $\mathrm{SL}_2(\mathbb{C})$, or is a 2-torus of Prüfer 2-rank at most 2.

Suppose in addition that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.

Then $m_2(\hat{G}) \leq 2$, G or \hat{G}/G is 2^\perp , and involutions are conjugate in \hat{G} . Moreover one of the following cases occurs:

- **PSL₂**: $G \simeq \mathrm{PSL}_2(\mathbb{K})$ in characteristic not 2; \hat{G}/G is 2^\perp ;
- **CiBo₀**: G is 2^\perp ; $m_2(\hat{G}) \leq 1$; for $\iota \in I(\hat{G})$, $C_G(\iota) = C_G^\circ(\iota)$ is a self-normalising Borel subgroup of G ;
- **CiBo₁**: $m_2(G) = m_2(\hat{G}) = 1$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G ;
- **CiBo₂**: $\mathrm{Pr}_2(G) = 1$ and $m_2(G) = m_2(\hat{G}) = 2$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G^\circ(i)$ is an abelian Borel subgroup of G inverted by any involution in $C_G(i) \setminus \{i\}$ and satisfies $\mathrm{rk} G = 3 \mathrm{rk} C_G^\circ(i)$;
- **CiBo₃**: $\mathrm{Pr}_2(G) = m_2(G) = m_2(\hat{G}) = 2$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G .

The proof consists of eight Propositions all strongly relying on the $*$ -local $^\circ$ solubility assumption, the deepest of which will be the maximality Proposition 6. Let us briefly describe the global outline. More detailed information will be found before each proposition.

In Proposition 1 (§4.1) we determine the 2-structure of $*$ -local $^\circ$ soluble groups by elementary methods. Proposition 2 (§4.2) is a classical rank computation required both by the Algebraicity

Proposition 3 (§4.3) which identifies $\mathrm{PSL}_2(\mathbb{K})$ through reconstruction of its BN-pair, and by the Maximality Proposition 6 which shows that in non-algebraic configurations centralisers^o of involutions are Borel subgroups. The latter will take all of §5 but actually requires two more technical preliminaries: Propositions 4 (§4.4) and 5 (§4.5), which deal with actions of involutions and torsion, respectively. After Proposition 6 things go faster. We study the action of an infinite dihedral group in Proposition 7 (§6.1) and a strong embedding configuration in Proposition 8 (§6.2). Both are rather classical, methodologically speaking; Proposition 7 is more involved than Proposition 8; they can be read in any order but both rely on Maximality. The final assembling takes place in §6.3 where all preliminary propositions 1, 2, 4 and 5 reappear as independent themes.

The resulting outline surprised us. In the original, minimal connected simple setting one proceeded by first bounding the Prüfer 2-rank [BCJ07] and then studying the remaining cases [Del07a, Del08]. There maximality propositions had to be proved three times in order to complete the analysis. The reason for such a clumsy treatment, with one part of the proof being repeated over and over again, was that torsion arguments were systematically based on some control on involutions. Here we do the opposite. By providing careful torsion control in Proposition 5 and relaxing our expectations on conjugacy classes of involutions we shall be able to run maximality without prior knowledge of the Prüfer 2-rank. This seems to be the right level both of elegance and generality. Bounding the Prüfer 2-rank then follows by adapting a small part of [BCJ07].

Before the curtain opens one should note that bounding the Prüfer 2-rank of \hat{G} a priori is possible if one assumes G to be 2^\perp as Burdges noted for [BCD09]. We do not follow this line.

4.1 The 2-structure Proposition

The following Proposition comes directly from [Del07b, Chapitre 4 and Addendum], published as [Del08, §2]. It is the most elementary of our propositions, and together with the Strong Embedding Proposition 8 one of the two not requiring quasi-solubility of centralisers of involutions.

Proposition 1 (2-structure). *Let G be a connected, U_2^\perp , $*$ -locally^o soluble group of finite Morley rank. Then the Sylow 2-subgroup of G has the following form:*

- *connected, i.e. a possibly trivial 2-torus;*
- *isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$;*
- *isomorphic to that of $\mathrm{SL}_2(\mathbb{C})$, in which case $C_G^\circ(i)$ is non-soluble for any involution i of G .*

Proof. If the Prüfer rank is 0 this is a consequence of the analysis of degenerate type groups [BBC07]. If it is 1, this is well-known, see for reference [DJ10, Proposition 27]. Notice that if the Sylow 2-subgroup is like in $\mathrm{SL}_2(\mathbb{C})$ and i is any involution, then by torality principles all Sylow 2-subgroups of $C_G(i)$ are in $C_G^\circ(i)$, but none is connected: this, and the structure of torsion in connected, soluble groups of finite Morley rank prevents $C_G^\circ(i)$ from being soluble.

So we suppose that the Prüfer 2-rank is at least 2 and show that a Sylow 2-subgroup S of G is connected. Let G be a minimal counterexample to this statement. Then G is non-soluble. By $*$ -local^o solubility $Z(G)$ is finite, but we actually may suppose that G is centreless. For if the result holds of $G/Z(G)$, then $SZ(G)/Z(G)$ is a Sylow 2-subgroup of $G/Z(G)$, and therefore connected, so that $S \leq S^\circ Z(G) \cap S \leq C_S(S^\circ) = S^\circ$ by torality principles. So we may assume $Z(G) = 1$.

Still assuming that the Prüfer 2-rank is at least 2 we let $\zeta \in S \setminus S^\circ$ have minimal order, so that $\zeta^2 \in S^\circ$. Let $\Theta_1 = C_{S^\circ}^\circ(\zeta)$. If $\Theta_1 \neq 1$ then $\langle S^\circ, \zeta \rangle \leq C_G(\Theta_1)$ which is connected by [AB08, Theorem 1] and soluble by $*$ -local^o solubility. The structure of torsion in such groups yields $\zeta \in S^\circ$, a contradiction. So $\Theta_1 = C_{S^\circ}^\circ(\zeta) = 1$ and ζ therefore inverts S° . In particular it centralises $\Omega = \Omega_2(S^\circ)$, and Ω normalises $C_G^\circ(\zeta)$. By normalisation principles Ω normalises therefore a maximal 2-torus T of $C_G^\circ(\zeta)$; by torality, $\zeta \in T$ and T has the same Prüfer 2-rank as S . Now $|\Omega| \geq 4$ so there is $i \in \Omega$ such that $\Theta_2 = C_T^\circ(i)$ is non-trivial. Then $\langle T, i \rangle \leq C_G(\Theta_2)$ which is soluble and connected as above, implying $i \in T$. This is not a contradiction yet, but now $\zeta \in T \leq C_G^\circ(i)$ and of course $S^\circ \leq C_G^\circ(i)$. Hence $C_G^\circ(i) < G$ is a smaller counterexample, a contradiction. Connectedness is proved. \square

Remark. One can show that if $\alpha \in G$ is a 2-element with $\alpha^2 \neq 1$, then $C_G(\alpha)$ is connected.

For let $\alpha \in G$ have order 2^k with $k > 1$. By Steinberg's torsion theorem, $C_G(\alpha)/C_G^\circ(\alpha)$ has exponent dividing 2^k . Using torality, fix a maximal 2-torus T of G containing α . If the Sylow 2-subgroup of G is connected, then T is a Sylow 2-subgroup of G included in $C_G^\circ(\alpha)$: hence $C_G(\alpha) = C_G^\circ(\alpha)$. If the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$ or of $\mathrm{SL}_2(\mathbb{C})$, then any 2-element $\zeta \in C_G(\alpha)$ normalising T centralises α of order at least 4, so it also centralises T . It follows from torality principles that $\zeta \in T \leq C_G^\circ(\alpha)$, and $C_G(\alpha)$ is connected again.

We shall not use this remark.

4.2 The Genericity Proposition

Considerations concerning the distribution of involutions in the cosets of a given subgroup are often useful in the study of groups of even order.

So wrote Bender in the beginning of [Ben74b]. The first instance of this method in the finite Morley rank context seems to be [BDN94, after Lemma 7] which with [BN94a] aimed at identifying $\mathrm{SL}_2(\mathbb{K})$ in characteristic 2. Jaligot brought it to the odd type setting [Jal00]. This subsection is the cornerstone of Propositions 3 and 6 and is used again when conjugating involutions in Step 7 of the final argument.

Notation. For κ an involutory automorphism and H a subgroup of some ambient group, we let $T_H(\kappa) = \{h \in H : h^\kappa = h^{-1}\}$. (This set is definable as soon as κ and H are.)

The following is completely classical; the proof will not surprise the experts and is included for the sake of self-containedness. It will be applied only when H is a Borel subgroup of G .

Proposition 2 (Genericity). *Let \hat{G} be a connected, U_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup.*

Suppose that $\hat{G} = G \cdot d(\hat{S}^\circ)$ for some maximal 2-torus \hat{S}° of \hat{G} .

Let $\iota \in I(\hat{G})$ and $H \leq G$ be a definable, infinite, soluble subgroup of G . Then $K_H = \{\kappa \in \iota^{\hat{G}} \setminus N_{\hat{G}}(H) : \mathrm{rk} T_H(\kappa) \geq \mathrm{rk} H - \mathrm{rk} C_G(\iota)\}$ is generic in $\iota^{\hat{G}}$.

Proof. The statement is invariant under conjugating \hat{S}° so by torality principles we may assume $\iota \in \hat{S}^\circ$; in particular $\iota^{\hat{G}} = \iota^G$. We shall first show that $\iota^{\hat{G}} \setminus N_{\hat{G}}(H)$ is generic in $\iota^{\hat{G}}$. [DJ12, Lemmas 2.16 and 3.33] were supposed to do it instantly, but they only apply when $\iota \in G$. Minor work remains to be done.

Suppose that $\iota^{\hat{G}} \setminus N_{\hat{G}}(H)$ is not generic in $\iota^{\hat{G}}$. Then by a degree argument, $\iota^{\hat{G}} \cap N_{\hat{G}}(H)$ is generic in $\iota^{\hat{G}}$. Inside \hat{G} apply [DJ12, Lemma 2.16] with $X = \iota^{\hat{G}}$ and $M = N_{\hat{G}}(H)$: $X \cap M$ contains a definable, \hat{G} -invariant subset X_1 which is generic in X . Note that X is infinite as otherwise ι inverts \hat{G} , so X_1 is infinite as well. We cannot directly apply [DJ12, Lemma 3.33] as \hat{G} itself need not be $*$ -locally $^\circ$ soluble. So let $X_2 = \{\kappa\lambda : \kappa, \lambda \in X_1\}$, which is an infinite, \hat{G} -invariant subset of $N_{\hat{G}}(H)$. Since $X_1 \subseteq \iota^{\hat{G}} = \iota^G \subseteq \iota G = G\iota$, X_2 is actually a subset of G . Hence $X_2 \subseteq N_G(H)$. The latter need not be soluble but is a finite extension of $N_G^\circ(H)$, which is. Since X_2 is infinite and has degree 1, there is a generic subset X_3 of X_2 which is contained in some translate $nN_G^\circ(H)$ of $N_G^\circ(H)$, where $n \in N_G(H)$. Then $X_3 \subseteq N_G^\circ(H) \cdot \langle n \rangle$ which is a definable, soluble group we denote by M_2 ; X_3 itself may fail to be G -invariant. But X_2 is a G -invariant subset such that $X_3 \subseteq X_2 \cap M_2$ is generic in X_2 . By [DJ12, Lemma 3.33] applied in $G = G^\circ$ to X_2 and M_2 , G is soluble: a contradiction.

The end of the proof is rather worn-out. Consider the definable function $\varphi : \iota^{\hat{G}} \setminus N_{\hat{G}}(H) \rightarrow G \cdot \langle \iota \rangle / H$ which maps κ to κH . The domain has rank $\mathrm{rk} \iota^{\hat{G}} = \mathrm{rk} \iota^G = \mathrm{rk} G - \mathrm{rk} C_G(\iota)$. The image set has rank at most $\mathrm{rk} G - \mathrm{rk} H$. So the generic fiber has rank at least $\mathrm{rk} H - \mathrm{rk} C_G(\iota)$. But if κ, λ are in the same fiber, then $\kappa H = \lambda H$ and $\kappa\lambda \in T_H(\kappa)$. So for generic κ , $\mathrm{rk} T_H(\kappa) \geq \mathrm{rk} \varphi^{-1}(\varphi(\kappa)) \geq \mathrm{rk} H - \mathrm{rk} C_G(\iota)$. \square

Notation. For κ an involutory automorphism and H a subgroup of some ambient group, we let $\mathbb{T}_H(\kappa) = \{h^2 \in H : h^\kappa = h^{-1}\} \subseteq T_H(\kappa)$. (This set is definable as soon as κ and H are.)

The \mathbb{T} sets were denoted τ in [Del07b]; interestingly enough, they were already used in [BCJ07, Notation 7.4]. There is no a priori estimate on $\mathrm{rk} \mathbb{T}_H(\kappa)$, and Proposition 5 will remedy this.

4.3 The Algebraicity Proposition

Proposition 3 below is the historical core of the subject.

Identifying $\mathrm{SL}_2(\mathbb{K})$ is a classical topic in finite group theory. So Proposition 3 is a very weak form of the Brauer-Suzuki-Wall Theorem [BSW58] in odd characteristic. However [BSW58] heavily relied on character theory, a tool not available in and perhaps not compatible in spirit with the context of groups of finite Morley rank. A character-free proof of outstanding elegance was found by Goldschmidt. Yet his article [Gol75] dealt only with the characteristic 2 case, and ended on the conclusive remark:

Finally, some analogues of Theorem 2 [Goldschmidt's version of BSW] may hold for odd primes but [...] this problem seems to be very difficult.

Bender's investigations in odd characteristic [Ben74a] and [Ben81] both require some character theory.

In the finite Morley rank context various results identifying $\mathrm{PSL}_2(\mathbb{K})$ exist, starting with Cherlin's very first article in the field [Che79] and Hrushovski's generalisation [Hru89]. For groups of even type, [BDN94, BN94a] provide identification using heavy rank computations. In a different spirit, the reworking of Zassenhaus' classic [Zas35] by Nesin [Nes90a] and its extension [DN95] identify $\mathrm{PSL}_2(\mathbb{K})$ among 3-transitive groups; the latter gives a very handy statement.

Most of the ideas in the proof hereafter are in [Del07a] and in many other articles before. Only two points need be commented on.

- First, we shift from the tradition as in [CJ04, Del07a] of invoking the results on permutation groups Nesin had ported to the finite Morley rank context ([DN95], see above).

We decided to use final identification arguments based on the theory of Moufang sets instead. At that point of the analysis the difference may seem essentially cosmetic but the Moufang setting is in our opinion more appropriate as it focuses on the BN-pair. We now rely on recent work by Wiscons [Wis11].

(Incidentally, Nesin had started thinking about BN-pairs [Nes90b] but stopped before reaching an identification theorem for $\mathrm{PSL}_2(\mathbb{K})$ in this context; not returning to jail he apparently never returned to the topic.)

- Second, we refrained from using Frécon homogeneity, just for the thrill of it. This makes the proof only marginally longer in Step 6.

Proposition 3 (Algebraicity). *Let \hat{G} be a connected, U_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup. Suppose that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.*

Suppose that there exists $\iota \in I(\hat{G})$ such that $C_G^\circ(\iota)$ is included in two distinct Borel subgroups. Then G has the same Sylow 2-subgroup as $\mathrm{PSL}_2(\mathbb{K})$. If in addition $\iota \in G$, then $G \simeq \mathrm{PSL}_2(\mathbb{K})$, where \mathbb{K} is an algebraically closed field of characteristic not 2.

Proof. Since \hat{G} is connected, every involution ι is toral: say $\iota \in \hat{S}^\circ$ a 2-torus. We may therefore assume that $\hat{G} = G \cdot d(\hat{S}^\circ)$, so that the standard rank computations of the Genericity Proposition 2 apply. Moreover, \hat{G}/G is connected and abelian, hence W_2^\perp .

Notation 1.

- Let $B \geq C_G^\circ(\iota)$ be a Borel subgroup maximising ρ_B ; let $\rho = \rho_B$.
- Let $K_B = \{\kappa \in \iota^{\hat{G}} \setminus N_{\hat{G}}(B) : \mathrm{rk} T_B(\kappa) \geq \mathrm{rk} B - \mathrm{rk} C_G^\circ(\iota)\}$; by the Genericity Proposition 2, K_B is generic in $\iota^{\hat{G}}$.
- Let $\kappa \in K_B$.

Step 2. $U_\rho(C_G^\circ(\iota)) = 1$. If $U \leq B$ is a non-trivial ρ -group, $H \leq G$ is a definable, connected subgroup of G containing U , and $\lambda \in \iota^{\hat{G}}$ normalises H , then λ normalises B .

Proof of Step. Suppose $U_\rho(C_G^\circ(\iota)) \neq 1$. Let $D \neq B$ be a Borel subgroup containing $C_G^\circ(\iota)$ and maximising $H = (B \cap D)^\circ$: such a Borel subgroup exists by the assumption on $C_G^\circ(\iota)$. If H is not abelian then by [DJ12, 4.50(3) and (6) (our Fact 10)] $\rho_B \neq \rho_D$, but by construction $\rho_D \succeq \rho_\iota = \rho_B \succeq \rho_D$, a contradiction. Hence H is abelian, and in particular $C_G^\circ(\iota) \leq H \leq C_G^\circ(U_\rho(H))$ which is a soluble group; by definition of B , the parameter of $C_G^\circ(U_\rho(H))$ is ρ . It follows from uniqueness principles that $U_\rho(H)$ is included in a unique Sylow ρ -subgroup of G . This must be $U_\rho(B) = U_\rho(D)$, so $B = D$: a contradiction.

We just proved $\rho_\iota \prec \rho$. It follows that for any $\sigma \succeq \rho$, any ι -invariant σ -group is inverted by ι . Now let U , H , and λ be as in the statement. There is a Sylow ρ -subgroup V of H containing U . By normalisation principles λ has an H -conjugate μ normalising V : so μ inverts $V \geq U$.

Let $C = C_G^\circ(U)$, a definable, connected, soluble group. Since $U \leq U_\rho(B)$, $U_\rho(Z(F^\circ(B))) \leq C$. So there is a Sylow ρ -subgroup W of C containing $U_\rho(Z(F^\circ(B)))$. As μ inverts U it normalises C ; by normalisation principles μ has a C -conjugate ν normalising W : so ν inverts $W \geq U_\rho(Z(F^\circ(B)))$. Now ν also inverts $U_{\rho_C}(C)$, and commutation principles yield $[U_{\rho_C}(C), U_\rho(Z(F^\circ(B)))] = 1$, whence $U_{\rho_C}(C) \leq C_G^\circ(U_\rho(Z(F^\circ(B)))) \leq B$. At this point it is clear that $\rho_C = \rho$ and $U_\rho(Z(F^\circ(B))) \leq U_{\rho_C}(C)$. Moreover $U_\rho(B)$ is the only Sylow ρ -subgroup of G containing U by uniqueness principles.

On the other hand μ inverts $U_{\rho_H}(H)$ and U , so by commutation principles $[U_{\rho_H}(H), U] = 1$ and $U_{\rho_H}(H) \leq C$, meaning that $\rho_H = \rho$ as well. Hence λ inverts $U_{\rho_H}(H) = U_\rho(H) \geq U$. Since $U_\rho(B)$ is the only Sylow ρ -subgroup of G containing U , λ normalises B . \diamond

Notation 3. Let $L_\kappa = B \cap B^\kappa$ and $\Theta_\kappa = \{\ell \in L_\kappa : \ell\ell^\kappa \in L'_\kappa\}$.

Step 4. L_κ and Θ_κ are infinite, definable, κ -invariant, abelian-by-finite groups. Moreover $\Theta_\kappa^\circ \subseteq T_B(\kappa) \subseteq \Theta_\kappa$.

Proof of Step. L'_κ is finite since we otherwise let $H = N_G^\circ(L'_\kappa) \geq U_\rho(Z(F^\circ(B)))$ which is definable, connected, and soluble by $*$ -local $^\circ$ solubility: Step 2 shows that κ normalises B , contradicting its choice in Notation 1. It follows that L_κ° is abelian and L_κ is abelian-by-finite. Θ_κ is clearly a definable, κ -invariant subgroup of L_κ , so it is abelian-by-finite as well. By construction $T_B(\kappa) \subseteq \Theta_\kappa$, and Θ_κ is therefore infinite.

We now consider the action of κ on Θ_κ° and find according to Lemma G the decomposition $\Theta_\kappa^\circ = C_{\Theta_\kappa^\circ}^\circ(\kappa)(+)[\Theta_\kappa^\circ, \kappa]$. Now the definable function $\varphi : C_{\Theta_\kappa^\circ}^\circ(\kappa) \rightarrow L'_\kappa$ which maps t to $t\kappa = t^2$ is a group homomorphism, so by connectedness and since L'_κ is finite, $C_{\Theta_\kappa^\circ}^\circ(\kappa)$ has exponent 2: it is trivial. So κ inverts Θ_κ° , meaning $\Theta_\kappa^\circ \subseteq T_B(\kappa)$. \diamond

Notation 5. Let $U \leq [U_\rho(Z(F^\circ(B))), \Theta_\kappa^\circ]$ be a non-trivial, Θ_κ° -invariant ρ -subgroup minimal with these properties.

Step 6. U does exist and $C_U^\circ(\iota) = 1$; $C_{\Theta_\kappa^\circ}(U)$ is finite and there exists an algebraically closed field structure \mathbb{K} with $U \simeq \mathbb{K}_+$ and $\Theta_\kappa^\circ/C_{\Theta_\kappa^\circ}(U) \simeq \mathbb{K}^\times$. Moreover G has the same Sylow 2-subgroup as $\text{PSL}_2(\mathbb{K})$.

Proof of Step. If Θ_κ° centralises $U_\rho(Z(F^\circ(B)))$ then the κ -invariant, definable, connected, soluble group $C_G^\circ(\Theta_\kappa^\circ)$ contains $U_\rho(Z(F^\circ(B)))$ and Step 2 forces κ to normalise B , against its choice in Notation 1. Hence $[U_\rho(Z(F^\circ(B))), \Theta_\kappa^\circ] \neq 1$; it is a ρ -group (Fact 7 (v); no need for Frécon homogeneity here).

We show that $C_U^\circ(\iota) = 1$; be careful that ι need not normalise U nor even B a priori. Yet if $C_U^\circ(\iota)$ is infinite then Step 2 applied to $C_G^\circ(C_U^\circ(\iota)) \geq U_\rho(Z(F^\circ(B)))$ forces ι to normalise B , whence ι inverts $U_\rho(B) \geq U \geq C_U^\circ(\iota)$: a contradiction.

Suppose that $C_{\Theta_\kappa^\circ}(U)$ is infinite; Step 2 applied to $C_G^\circ(C_{\Theta_\kappa^\circ}(U)) \geq U$ forces κ to normalise B : a contradiction. We now wish to apply Zilber's Field Theorem. It may look like we fall short of Θ_κ° -minimality but fear not. Follow for instance the proof in [BN94b, Theorem 9.1]. It suffices to check that any non-zero r in the subring of $\text{End}(U)$ generated by Θ_κ° is actually an automorphism. But by push-forward [Bur04a, Lemma 2.11] $\text{im } r \simeq U/\ker r$ is a non-trivial, Θ_κ° -invariant ρ -subgroup. By minimality of U as such, r is surjective. In particular $\ker r$ is finite. Suppose it is non-trivial and form, like in [BN94b, Theorem 9.1], the chain $(\ker r^n)$. Each term is Θ_κ° -central by connectedness, so $C_U^\circ(\Theta_\kappa^\circ)$ contains an infinite torsion subgroup A . If there is some torsion unipotence then $A = U$ by minimality as a ρ -group, and Θ_κ° centralises U : a contradiction.

So A contains a non-trivial q -torus for some prime number q . This means that there is a q -torus in $[U_\rho(Z(F^\circ(B))), \Theta_\kappa] \leq B'$ which contradicts, for instance, [Fré00b, Proposition 3.26]. Hence every $r \in (\Theta_\kappa^\circ)_{\text{End}(U)}$ is actually an automorphism of U : field interpretation applies (it also follows, a posteriori, that U is Θ_κ° -minimal).

A priori $\Theta_\kappa^\circ/C_{\Theta_\kappa^\circ}(U)$ simply embeds into \mathbb{K}^\times . But one has by Step 4 and the definition of κ :

$$\begin{aligned} \text{rk } \Theta_\kappa^\circ/C_{\Theta_\kappa^\circ}(U) &= \text{rk } \Theta_\kappa^\circ = \text{rk } T_B(\kappa) \geq \text{rk } B - \text{rk } C_G^\circ(\iota) = \text{rk } B - \text{rk } C_B^\circ(\iota) = \text{rk } \iota^B \\ &\geq \text{rk } \iota^U = \text{rk } U - \text{rk } C_U(\iota) = \text{rk } U = \text{rk } \mathbb{K}_+ \end{aligned}$$

It follows that $\Theta_\kappa^\circ/C_{\Theta_\kappa^\circ}(U) \simeq \mathbb{K}^\times$. At this point Θ_κ° contains a non-trivial 2-torus. By the 2-structure Proposition 1 and in view of the assumption on centralisers of involutions, the Sylow 2-subgroup of G is either connected or isomorphic to that of $\text{PSL}_2(\mathbb{K})$. Suppose it is connected. Then G is W_2^\perp ; since \hat{G}/G is as well, so is \hat{G} by Lemma K. This contradicts the fact that κ inverts the 2-torus of Θ_κ° . \diamond

For the remainder of the proof we now suppose that ι is already in G . So we may assume $\hat{G} = G$. Bear in mind that since the Prüfer 2-rank is 1 by Step 6, all involutions are conjugate.

Notation 7.

- Let for consistency of notations $i = \iota \in G$ and $k = \kappa \in G$. (By totality principles, $i \in C_G^\circ(i) \leq B$.)
- Let j_k be the involution in Θ_k° .

Since i, j_k are in B they actually are B -conjugate. In particular $C_G^\circ(j_k) \leq B$.

Step 8. $\Theta_k^\circ = C_G^\circ(j_k)$. Moreover $\text{rk } U = \text{rk } C_G^\circ(i) = \text{rk } \Theta_k$, $\text{rk } B \leq 2 \text{rk } U$, and $\text{rk } G \leq \text{rk } B + \text{rk } U$.

Proof of Step. One inclusion is clear by abelianity of Θ_k° obtained in Step 4. Now let $N = N_G^\circ(C_G^\circ(k, j_k))$. Since L_k° is abelian by Step 4, so are $C_G^\circ(j_k) \leq L_k^\circ$ and its conjugate $C_G^\circ(k)$. Hence $\Theta_k^\circ \leq C_G^\circ(j_k) \leq N$ and by totality $k \in C_G^\circ(k) \leq N$. So N contains a non-trivial 2-torus and an involution inverting it: by the structure of torsion in definable, connected, soluble groups, N is not soluble. *-Local $^\circ$ solubility of G forces $C_G^\circ(k, j_k) = 1$, so k inverts $C_G^\circ(j_k)$. Hence $C_G^\circ(j_k) \leq \Theta_k^\circ$.

We now compute ranks. By Steps 6 and 8, $\text{rk } C_G^\circ(i) = \text{rk } \Theta_k^\circ = \text{rk } \mathbb{K}^\times = \text{rk } \mathbb{K}_+ = \text{rk } U$. By definition of $k \in K_B$ and Step 4, $\text{rk } \Theta_k^\circ = \text{rk } T_B(k) \geq \text{rk } B - \text{rk } C_B(i)$, so $\text{rk } B \leq 2 \text{rk } U$.

Now remember that k varies in a set K_B generic in i^G . Let $f : K_B \rightarrow i^B$ be the definable function mapping k to j_k . If $j_k = j_\ell$ then $\ell \in C_G(j_k)$ and the latter has the same rank as $C_G(i)$ so we control fibers. Hence:

$$\text{rk } G - \text{rk } C_G(i) = \text{rk } i^G = \text{rk } K_B \leq \text{rk } i^B + \text{rk } C_G(i) = \text{rk } i^B + \text{rk } C_B(i) = \text{rk } B$$

that is, $\text{rk } G \leq \text{rk } B + \text{rk } C_G(i)$. \diamond

For the end of the proof k will stay fixed; conjugating again in B we may therefore suppose that $j_k = i$.

Notation 9. Let $N = C_G(i)$ and $H = B \cap N$.

Step 10. $H = B \cap N$; (B, N, U) forms a split BN -pair of rank 1.

Proof of Step. We must check the following:

- $G = \langle B, N \rangle$;
- $[N : H] = 2$;
- for any $\omega \in N \setminus H$, $H = B \cap B^\omega$, $G = B \sqcup B\omega B$, and $B^\omega \neq B$;
- $B = U \rtimes H$.

First, $H = B \cap N = C_B(i) = C_B^\circ(i)$ by Steinberg's torsion theorem and the structure of torsion in B . By the structure of the Sylow 2-subgroup obtained in Step 6, $H < N$, so using Steinberg's torsion theorem again $[N : H] = 2$. Hence for any $\omega \in N \setminus H = Hk$ one has $B^\omega = B^k \geq H^k = H$ and $H \leq B \cap B^k$. Now by the structure of torsion in B , $B \cap B^k$ centralises the 2-torus in the abelian group $(B \cap B^k)^\circ = L_k^\circ$ so $B \cap B^k \leq C_B(i) = H$.

Recall that the action of $H = C_G^\circ(i) = \Theta_k^\circ$ on U induces a full field structure; in particular $H \cap U \leq C_U(\Theta_k^\circ) = 1$. So $U \cdot H = U \rtimes H$ has rank $2 \operatorname{rk} U \geq \operatorname{rk} B$ by Step 8 and therefore $B = U \rtimes H$.

It remains to obtain the Bruhat decomposition. But first note that if $C_{N_G(B)}(i) > C_B(i)$ then $C_{N_G(B)}(i) = N$ contains k , which contradicts $k \notin N_G(B)$ from Notation 1. So $C_{N_G(B)}(i) = C_B(i)$ and since B conjugates its involutions a Frattini argument yields $N_G(B) \subseteq B \cdot C_{N_G(B)}(i) = B$.

Finally let $g \in G \setminus B$; g does not normalise B . Let $X = (U \cap B^g)^\circ$ and suppose $X \neq 1$. In characteristic p this contradicts uniqueness principles. In characteristic 0, $U \simeq \mathbb{K}_+$ is minimal [Poi87, Corollaire 3.3], so $X = U$; at this point $U = U_\rho(B^g) = U^g$, a contradiction again. In any case $X = 1$. In particular UgB has rank $\operatorname{rk} U + \operatorname{rk} B = \operatorname{rk} G$ by Step 8 and UgB is generic in G . This also holds of UkB so $g \in BkB$ and $G = B \sqcup BkB = B \sqcup B\omega B$ for any $\omega \in N \setminus H$. This certainly implies $G = \langle B, N \rangle$. \diamond

We finish the proof with [Wis11, Theorem 1.2] or [DMT08, Theorem 2.1], depending on the characteristic. If U has exponent p , then $U_p(H) = 1$ as $H \simeq \mathbb{K}^\times$, so [Wis11, Theorem 1.2] applies. If not, then U is torsion-free: we use [DMT08, Theorem 2.1] instead. In any case, $G / \bigcap_{g \in G} B^g \simeq \operatorname{PSL}_2(\mathbb{K})$ for some field structure \mathbb{K} which on the face of it need not be the same as in Step 6 but could easily be proved to. Since $\bigcap_{g \in G} B^g$ is a normal, soluble subgroup, it is finite and central by $*$ -local $^\circ$ solubility. But central extensions of finite Morley rank of quasi-simple algebraic groups are known [AC99, Corollary 1], so $G \simeq \operatorname{SL}_2(\mathbb{K})$ or $\operatorname{PSL}_2(\mathbb{K})$, and the first is impossible by assumption on the centralisers of involutions. \square

Remark. In order to prove non-connectedness of the Sylow 2-subgroup of G , one only needs solubility of $C_G^\circ(\iota)$ regardless of how centralisers of involutions in other classes may behave. But in order to continue one needs much more.

- One cannot work with j_κ as all our rank computations rely on the equality $\operatorname{rk} C_G(j_\kappa) = \operatorname{rk} C_G(\iota)$, for which there is no better reason than conjugacy with ι . This certainly implies $\iota \in G$ to start with.
- One cannot entirely drop ι and focus on j_κ , since there is no reason why $C_G^\circ(j_\kappa)$ should be soluble.

4.4 The Devil's Ladder

The following comes from [Del07b, Proposition 5.4.9] with first clear appearances in [DJ08] and [BCD09]. This technical statement will be used three times: in order to control torsion which is the very purpose of Proposition 5, at a rather convoluted moment in Step 9 of Maximality Proposition 6, and in order to conjugate involutions in the very end of the proof of our Theorem, Step 7. It may be viewed as an extreme form of Proposition 3, Step 2; the effective contents of the proof are not perfectly clear but it suffices to hold on longer than the group.

The argument was found and named in 2007 after a Ligeti study and greatly simplified since: in [DJ] the proof still took three pages.

Proposition 4 (The Devil's Ladder). *Let \hat{G} be a connected, U_2^\perp, W_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup. Suppose that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.*

Let $\kappa, \lambda \in I(\hat{G})$ be two involutions. Suppose that for all $\mu \in I(\hat{G})$ such that $\rho_\mu \succ \rho_\kappa$, $C_G^\circ(\mu)$ is a Borel subgroup of G .

Let $B \geq C_G^\circ(\kappa)$ be a Borel subgroup of G and $1 \neq X \leq F^\circ(B)$ be a definable, connected subgroup which is centralised by κ and inverted by λ .

Then $C_G^\circ(X) \leq B$ and B is the only Borel subgroup of G of parameter ρ_B containing $C_G^\circ(X)$; in particular κ and λ normalise B .

Proof. First observe that $\kappa \in C_{\hat{G}}(X)$ which is λ -invariant, so by normalisation principles λ has a $C_{\hat{G}}(X)$ -conjugate λ' which normalises some Sylow 2-subgroup of $C_{\hat{G}}(X)$ containing κ . By the W_2^\perp assumption the Sylow 2-subgroup of \hat{G} is abelian, so $[\kappa, \lambda'] = 1$; also observe that λ' inverts X . Let $C = C_G^\circ(X)$, a definable, connected, and soluble group by $*$ -local $^\circ$ solubility.

First suppose $\rho_C \succ \rho_\kappa$. Then κ inverts $U_{\rho_C(C)}$, which is therefore abelian. Since the four-group $\langle \kappa, \lambda' \rangle$ normalises $U_{\rho_C(C)}$, one of the two involutions λ' or $\kappa\lambda'$, call it μ , satisfies $Y = C_{U_{\rho_C(C)}}^\circ(\mu) \neq 1$. Note that Y is a ρ_C -group. Let $D = C_G^\circ(Y) \geq U_{\rho_C(C)}$; it is a definable, connected, soluble, κ -invariant subgroup. Since $\rho_D \succeq \rho_C \succ \rho_\kappa$, κ inverts D . On the other hand, $Y \leq C_G^\circ(\mu)$ so $\rho_\mu \succ \rho_\kappa$ and by assumption, $C_G^\circ(\mu)$ is a Borel subgroup of G , say B_μ . Since κ and μ commute, κ normalises B_μ and inverts $U_{\rho_\mu(B_\mu)} \trianglelefteq B_\mu$. It also inverts $Y \leq B_\mu$, so by commutation principles $[U_{\rho_\mu(B_\mu)}, Y] = 1$ and $U_{\rho_\mu(B_\mu)} \leq C_G^\circ(Y) = D$.

We are still assuming $\rho_C \succ \rho_\kappa$. The involution κ inverts $U_{\rho_D(D)} \trianglelefteq D$ and $U_{\rho_\mu(B_\mu)} \leq D$; so by commutation principles $[U_{\rho_\mu(B_\mu)}, U_{\rho_D(D)}] = 1$ and $U_{\rho_D(D)} \leq N_G^\circ(U_{\rho_\mu(B_\mu)}) = B_\mu$. At this stage it is clear that $\rho_D = \rho_\mu$ and $U_{\rho_\mu(B_\mu)} = U_{\rho_D(D)}$. In particular $D \leq N_G^\circ(U_{\rho_\mu(B_\mu)}) = B_\mu$. As a conclusion,

$$X \leq C_G^\circ(U_{\rho_C(C)}) \leq C_G^\circ(Y) = D \leq B_\mu = C_G^\circ(\mu)$$

against the fact that μ inverts X .

This contradiction shows that $\rho_C \preceq \rho_\kappa$. Now $X \leq F^\circ(B)$, so $U_{\rho_B(Z(F^\circ(B)))} \leq C_G^\circ(X) = C$; hence $\rho_B \preceq \rho_C \preceq \rho_\kappa \preceq \rho_B$ and equality holds. Since by uniqueness principles $U_{\rho_B(B)}$ is the only Sylow ρ_B -subgroup of G containing $U_{\rho_B(Z(F^\circ(B)))}$, it also is unique as such containing $U_{\rho_C(C)}$. Hence $N_{\hat{G}}(C) \leq N_{\hat{G}}(U_{\rho_B(B)}) = N_{\hat{G}}(B)$. Therefore κ and λ normalise B . \square

4.5 Inductive Torsion Control

It will be necessary to control torsion in the $T_B(\kappa)$ -sets. In [Del07b] this was redone for each conjugacy class of involutions by *ad hoc* arguments which could, in high Prüfer rank, get involved (the ‘‘Birthday Lemmas’’ [Del07b, Lemmes 5.3.9 and 5.3.10] published as [Del08, Lemmes 6.9 and 6.10]). We proceed more uniformly although some juggling is required. Like in [Del08] the argument will be applied twice: to start the proof of the Maximality Proposition 6, and later to conjugate involutions in Step 7 of the final argument. This accounts for the disjunction in the statement.

There was nothing equally technical in [BCD09] as controlling involutions there was trivial. An ‘‘inner’’ version of the argument was found in Yanartaş in the Spring of 2007 and added to [DJ08]. Externalising involutions is no major issue.

Proposition 5 (Inductive Torsion Control). *Let \hat{G} be a connected, U_2^\perp , W_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup. Suppose that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.*

Let $\iota \in I(\hat{G})$ and $B \geq C_G^\circ(\iota)$ be a Borel subgroup. Suppose that for all $\mu \in I(\hat{G})$ such that $\rho_\mu \succ \rho_\iota$, $C_G^\circ(\mu)$ is a Borel subgroup of G . Let $\kappa \in I(\hat{G}) \setminus N_{\hat{G}}(B)$ be such that $T_B(\kappa)$ is infinite.

Suppose either that $B = C_G^\circ(\iota)$ or that ι and κ are \hat{G} -conjugate. Then $\mathbb{T}_B(\kappa)$ has the same rank as $T_B(\kappa)$, and contains no torsion elements.

Proof. First remember that since \hat{G} is W_2^\perp , if some involution $\omega \in I(\hat{G})$ inverts a toral element $t \in \hat{G}$, then $t^2 = 1$. One may indeed take a maximal decent torus [Che05] \hat{T} of \hat{G} containing t ; then ω normalises $C_{\hat{G}}^\circ(t)$ which contains \hat{T} and its 2-torus \hat{T}_2 , so by normalisation principles ω has a $C_{\hat{G}}^\circ(t)$ -conjugate ω' normalising \hat{T}_2 . By the W_2^\perp assumption, the latter already is a Sylow 2-subgroup of \hat{G} , whence $\omega' \in \hat{T}_2 \leq C_{\hat{G}}^\circ(t)$. It follows that ω centralises t ; it also inverts it by assumption, so $t^2 = 1$. The proof starts here.

We first show that B has no torsion unipotence. The argument is a refinement of Step 4 of Proposition 3. Suppose that there is a prime number p with $U_p(B) \neq 1$. Let $L_\kappa = B \cap B^\kappa$ (be careful that we do *not* consider the connected component). Since $C_G^\circ(L'_\kappa)$ contains both $U_p(Z(F^\circ(B)))$ and $U_p(Z(F^\circ(B^\kappa)))$, uniqueness principles imply that L'_κ is finite. Unfortunately L_κ need not quite be abelian so let us introduce:

$$\Theta_\kappa = \{\ell \in L_\kappa : \ell\ell^\kappa \in L'_\kappa\}$$

which is a definable, κ -invariant subgroup of B containing $T_B(\kappa)$; in particular it is infinite. Also note that Θ_κ° is abelian. Now let $A \leq U_p(B)$ be a Θ_κ° -minimal subgroup. Θ_κ° cannot centralise A since otherwise $C_G^\circ(\Theta_\kappa^\circ) \geq \langle A, A^\kappa \rangle$, against uniqueness principles. So by Zilber's field theorem the action induces an algebraically closed field of characteristic p structure. By Wagner's theorem on fields [Wag01, consequence of Corollary 9] Θ_κ° contains a q -torus T_q for some $q \neq p$. Up to taking the maximal q -torus of Θ_κ° we may assume that κ normalises T_q . Write if necessary T_q as the sum of a κ -centralised and a κ -inverted subgroup; by the first paragraph of the proof, κ centralises T_q . So for any $t \in T_q$ one has $tt^\kappa = t^2 \in L'_\kappa$, therefore $T_q \leq L'_\kappa$ against finiteness of the latter.

We have disposed of torsion unipotency inside B , and every element of prime order in B is toral by the structure of torsion in definable, connected, soluble groups. By the first paragraph of the proof, no element of finite order $\neq 2$ of B is inverted by *any* involution (this will be used in the next paragraph with an involution not κ). In particular $d(t^2)$ is torsion-free for any $t \in T_B(\kappa)$; hence the definable hull of any element of $\mathbb{T}_B(\kappa)$ is torsion-free.

We now show that $T_B(\kappa)$ can contain but finitely many involutions (possibly none). Suppose that it contains infinitely many. Since B has only finitely many conjugacy classes of involutions, there are $i, j \in T_B(\kappa)$ which are B -conjugate. Now $i \in B$ so $\{B, i\} \subseteq F^\circ(B)$; by Lemma F (although [DJ10, Lemma 24] would do here) $B = B^{+i} \cdot \{B, i\}$ so there is $x \in \{B, i\} \subseteq (F^\circ(B))^{-i}$ with $j = i^x$. Since i inverts x , $d(x^2)$ is torsion-free. Also, $1 \neq ij = ii^x = x^2 \in F^\circ(B)$. Let $X = d(x^2)$ which is an abelian, definable, connected, infinite subgroup; like ij it is centralised by κ and inverted by i . There are two cases.

- If $B = C_G^\circ(\iota)$ then ι centralises X whereas κi inverts it (yes, κ and i do commute). Since $X \leq F^\circ(B)$ with $C_G^\circ(\iota) \leq B$, the Devil's Ladder (Proposition 4) applied to the pair $(\iota, \kappa i)$ leads to $\kappa i \in N_{\hat{G}}(B)$ and $\kappa \in N_{\hat{G}}(B)$: a contradiction.
- If κ is \hat{G} -conjugate to ι , say $\kappa = \iota^\gamma$ for some $\gamma \in \hat{G}$, we work in $B^\gamma \geq C_G^\circ(\kappa)$. Since κ centralises X , $X \leq B^\gamma$. Since $i \in C_G^\circ(\kappa) \cap B \leq C_G(\kappa)$, and by connectedness of the Sylow 2-subgroup of \hat{G} , one has $i \in C_G^\circ(\kappa) \leq B^\gamma$. Since i inverts X , $X \leq F^\circ(B^\gamma)$. Finally by conjugacy $\rho_\kappa = \rho_\iota$ so climbing the Devil's Ladder for the pair (κ, i) we find $C_G^\circ(X) \leq B^\gamma = B^{\gamma\kappa}$. Since $X \leq F^\circ(B)$ this implies $U_{\rho_B}(Z(F^\circ(B))) \leq C_G^\circ(X) \leq B^\gamma$. Uniqueness principles now yield $B = B^\gamma$. Hence $\kappa \in N_{\hat{G}}(B)$: a contradiction.

We conclude to rank equality. Let i_1, \dots, i_n be the finitely many involutions in $T_B(\kappa)$ (possibly $n = 0$) and set $i_0 = 1$. If $t \in T_B(\kappa)$ then the torsion subgroup of $d(t)$ is some $\langle i_m \rangle$, so $d(i_m t)$ is 2-divisible, and $i_m t \in \mathbb{T}_B(\kappa)$. Hence $T_B(\kappa) \subseteq \cup i_m \mathbb{T}_B(\kappa)$, which proves $\text{rk } \mathbb{T}_B(\kappa) = \text{rk } T_B(\kappa)$. \square

Remarks.

- One needs $T_B(\kappa)$ to be infinite only to show $U_p(B) = 1$; if one were to assume the latter, the rest of the argument would still work with finite $T_B(\kappa)$, and yield $\mathbb{T}_B(\kappa) = \{1\}$.
- The fact that $U_p(B) = 1$ is a strong indication of the moral inconsistency of the configuration.

5 The Proof – The Maximality Proposition

The following Proposition forms the technical core of the present article; we would be delighted to learn of a finite group-theoretic analogue. It was first devised in the context of minimal connected simple groups of odd type [Del07b], then ported to $*$ -locally $^\circ$ soluble groups of odd type [DJ08], and to actions on minimal connected simple groups of degenerate type [BCD09]. The main idea and the final contradiction have not changed but every generalisation has required new technical arguments. So neither of the above mentioned adaptations was routine; nor was combining them. We can finally state a general form.

Proposition 6 (Maximality). *Let \hat{G} be a connected, U_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, W_2^\perp , non-soluble, $*$ -locally $^\circ$ soluble subgroup. Suppose that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.*

Then for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is a Borel subgroup of G .

Proof. The proof is longer and more demanding than others in the article, but one should be careful to distinguish two levels.

- At a superficial level, all arguments resorting to local analysis in G and to the Bender method (Steps 5 and 7) would be much shorter and more intuitive if one knew that Borel subgroups of G have abelian intersections. The assumption is of course for the mere sake of exposition as there is no hope to prove such a thing, but it may be a good idea to have a quick look at the structure of the proof in this ideally-behaved case.
- At a deeper level, assuming abelianity of intersections does not make the statement of the proposition obvious and the reader is invited to think about it. Even with abelian intersections of Borel subgroups there would still be something to prove; this certainly uses the $T_B(\kappa)$ sets and rank computations of §4.2 as nothing else is available. As a matter of fact, even under abelian assumptions, we cannot think of a better strategy than the following.

The long-run goal (Step 10) is to collapse the configuration by showing that G -conjugates of some subgroup of G generically lie inside B . This form of contradiction was suggested by Jaligot to the author then his PhD student for [Del07a]. It is typical of Jaligot's early work in odd type [Jal00, Lemme 2.13]. (The author's original argument based on the distribution of involutions was both doubtful and less elegant; even recently he could feel the collapse in terms of involutions, but failed to write it down properly.)

Controlling generic G -conjugates of an arbitrary subgroup is not an easy task. The surprise (Step 9) is that the $T_B(\kappa)$ sets (or more precisely, the $\mathbb{T}_B(\kappa)$ sets) form the desired family. Seeing this requires a thorough analysis of $\mathbb{T}_B(\kappa)$, and embedding it into some abelian subgroup of B with pathological rigidity properties (Step 7). The crux of the argument involves some intersection of Borel subgroups. Interestingly enough, abelian intersections could be removed from [Del07b, Del07a, Del08, DJ08] by a somehow artificial observation on torsion; abelian intersections started playing a non-trivial role in [BCD09] but as a result the global proof then divided into two parallel lines. We could find a more uniform treatment, although the proof of Step 7 still divides into two along the line of abelianity.

The beginning of the argument (Steps 5, 4, 2) simply prepares for the analysis, showing that $\mathbb{T}_B(\kappa)$ behaves like a semi-simple group. Of course controlling torsion with Proposition 5 is essential in the first place; studying torsion separately thus allowing inductive treatment was the main success of [DJ08]. The proof starts here.

5.1 The Reactor

Since \hat{G} is connected, by torality principles every involution has a conjugate in some fixed 2-torus \hat{S}° . We may therefore assume that $\hat{G} = G \cdot d(\hat{S}^\circ)$, so that the standard rank computations of the Genericity Proposition 2 apply. Moreover, \hat{G}/G is connected and abelian, hence W_2^\perp . Since G is W_2^\perp as well, so is \hat{G} by Lemma K.

We then proceed by descending induction on ρ_ι and fix some involution $\iota_0 \in I(\hat{G})$ such that for any $\mu \in I(\hat{G})$ with $\rho_\mu \succ \rho_{\iota_0}$, $C_G^\circ(\mu)$ is a Borel subgroup. Notice that induction will not be used as such in the current proof but merely in order to apply Propositions 4 and 5.

Be warned that there will be some running ambiguity on ι_0 starting from Notation 3 onwards.

Notation 1.

- Let $B \geq C_G^\circ(\iota_0)$ be a Borel subgroup; we suppose $B > C_G^\circ(\iota_0)$. Let $\rho = \rho_B$.
- Let $K_B = \{\kappa \in \iota_0^{\hat{G}} \setminus N_{\hat{G}}(B) : \text{rk } T_B(\kappa) \geq \text{rk } B - \text{rk } C_G^\circ(\iota_0)\}$; by the Genericity Proposition 2, K_B is generic in $\iota_0^{\hat{G}}$.
- Let $\kappa \in K_B$.
- For the moment we simply write $\mathbb{T} = \mathbb{T}_B(\kappa)$.

By Inductive Torsion Control (Proposition 5), one has $\text{rk } \mathbb{T} \geq \text{rk } B - \text{rk } C_G^\circ(\iota_0)$, and \mathbb{T} contains no torsion elements.

Step 2 (uniqueness). (i) B is the only Borel subgroup of G containing $C_G^\circ(\iota_0)$; (ii) $N_{\hat{G}}(B)$ contains a Sylow 2-subgroup \hat{S}_0 of \hat{G} . (iii) In particular if $\lambda \in I(\hat{G}) \cap N_{\hat{G}}(B)$, then $[B, \lambda] \leq F^\circ(B)$ and $B = B^{+\lambda} \cdot (F^\circ(B))^{-\lambda}$ with finite fibers. (iv) Moreover $(N_G(B))^{-\lambda} \subseteq B$.

Proof of Step. Since G is W_2^\perp , by Algebraicity Proposition 3 there is a unique Borel subgroup containing $C_G^\circ(\iota_0)$; in particular $C_G^\circ(\iota_0)$ normalises B . By totality principles, $N_{\hat{G}}(B)$ contains a full Sylow 2-subgroup \hat{S}_0 of \hat{G} , which is a 2-torus as \hat{G} is W_2^\perp . Now let $\lambda \in I(\hat{G}) \cap N_{\hat{G}}(B)$. Conjugating in $N_{\hat{G}}(B)$ we may suppose $\lambda \in \hat{S}_0$. Then $\hat{B} = B \cdot d(\hat{S}_0)$ is a definable, connected, soluble group, so $\hat{B}' \leq F^\circ(\hat{B})$. Using Zilber's indecomposibility theorem, $[B, \lambda] \leq [B, \hat{S}_0] \leq (B \cap F^\circ(\hat{B}))^\circ \leq F^\circ(B)$. So Lemma F yields $B = (B^+)^\circ \cdot \{B, \lambda\}$. Of course $\{B, \lambda\} \subseteq (F^\circ(B))^{-\lambda}$.

It remains to prove (iv). The 2-torus \hat{S}_0 also acts on $N_G(B)$, so it centralises the finite set $N_G(B)/B$. It follows that if $n \in (N_G(B))^{-\lambda}$, then $nB = n^\lambda B = n^{-1}B$, that is, $n^2 \in B$. If G has no involutions then neither does $N_G(B)/B$ by torsion lifting. But if G does have involutions, then by totality principles $B \geq C_G^\circ(\iota_0)$ already contains a maximal 2-torus of G , which is a Sylow 2-subgroup of G : hence in that case again, $N_G(B)/B$ has no involutions. In any case $n \in B$, which proves $(N_G(B))^{-\lambda} \subseteq B$. \diamond

The most important points for the moment are (i) and (iii). Point (iv) will play no role before the final Step.

5.2 The Fuel

Controlling $\iota_0^G \cap N_{\hat{G}}(B)$ was claimed to be essential in [Del08, after Corollaire 5.37]. We can actually do without but this will result in some interesting ambiguity on involutions which will rise to a diverting polyphony at the very end of the proof of Step 9.

Notation 3. Let $I_B = \{\iota \in \iota_0^{\hat{G}} : C_G^\circ(\iota) \leq B\}$.

Remarks. $I_B = \iota_0^{N_G(B)}$ and any maximal 2-torus $\hat{S} \leq N_{\hat{G}}(B)$ intersects I_B , two facts we shall use with no reference. A proof and an observation follow.

- If $\iota \in I_B$ then there is $x \in \hat{G} = G \cdot d(\hat{S}_0)$ with $\iota = \iota_0^x$, where \hat{S}_0 is a 2-torus containing ι_0 ; one may clearly assume $x \in G$. Now by Uniqueness Step 2 (i) and definition of I_B , B^x is the only Borel subgroup of G containing $C_G^\circ(\iota) \leq B$, whence $x \in N_G(B)$ and $I_B \subseteq \iota_0^{N_G(B)}$. The converse inclusion is obvious.

By Step 2 (ii), $N_{\hat{G}}(B)$ contains a Sylow 2-subgroup of \hat{G} so any maximal 2-torus $\hat{S} \leq N_{\hat{G}}(B)$ is in fact a Sylow 2-subgroup of $N_{\hat{G}}(B)$, and contains an $N_{\hat{G}}(B)$ -conjugate ι of ι_0 ; then $\iota \in \hat{S} \cap I_B$.

- On the other hand it is not clear at all whether equality holds in $I_B \subseteq \iota_0^G \cap N_{\hat{G}}(B)$. As a matter of fact we cannot show that B is self-normalising in G ; this is easy when G is 2^\perp but not in general. At this point, using $C_G^\circ(\iota) < B$, there is a lovely little argument showing that $C_G(\iota)$ is connected (which is not obvious if $G < \hat{G}$ as Steinberg's torsion theorem no longer applies), but one cannot go further. Moreover, self-normalisation techniques *à la* [ABF13] do not work in the $*$ -locally $^\circ$ soluble context.

The first claim below will remedy this.

Step 4 (action).

- (i) If $\lambda \in \iota_0^G \cap N_{\hat{G}}(B)$ but $\lambda \notin I_B$, then λ inverts $U_\rho(Z(F^\circ(B)))$.
- (ii) $[U_\rho(Z(F^\circ(B))), \mathbb{T}] \neq 1$.

Proof of Step. Let λ be as in the statement and suppose that $X = C_{U_\rho(Z(F^\circ(B)))}^\circ(\lambda)$ is non-trivial. Then X is a ρ -group. By uniqueness Step 2 (iii), $B = B^{+\lambda} \cdot (F^\circ(B))^{-\lambda}$; obviously both terms normalise X so $X \trianglelefteq B$. It follows from uniqueness principles that $U_\rho(B)$ is the only Sylow ρ -subgroup of G containing X . Since $X \leq C_G^\circ(\lambda)$ is contained in some conjugate B^x of B , $U_\rho(B^x) = U_\rho(B)$ so $C_G^\circ(\lambda) \leq B$ and $\lambda \in I_B$: a contradiction.

We move to the second claim. Suppose that \mathbb{T} centralises $U_\rho(Z(F^\circ(B)))$. Let $C = C_G^\circ(\mathbb{T})$, a definable, connected, soluble, κ -invariant subgroup; let U be a Sylow ρ -subgroup of C containing $U_\rho(Z(F^\circ(B)))$. By normalisation principles κ has a C -conjugate λ normalising U . Since $U_\rho(B)$ is the only Sylow ρ -subgroup of G containing $U_\rho(Z(F^\circ(B)))$, λ normalises B . We see two cases.

First suppose $\lambda \notin I_B$. Then by claim (i), λ inverts $U_\rho(Z(F^\circ(B)))$. If $\rho_C = \rho$, then apply uniqueness principles: $U_\rho(B)$ is the only Sylow ρ -subgroup of G containing $U_\rho(Z(F^\circ(B)))$, so it also is the only Sylow ρ -subgroup of G containing $U_\rho(C)$. As the latter is κ -invariant, so is B : a contradiction. Therefore $\rho_C \succ \rho$. It follows that λ inverts $U_{\rho_C}(C)$, whence $[U_{\rho_C}(C), U_\rho(Z(F^\circ(B)))] = 1$ by commutation principles. This forces $U_{\rho_C}(C) \leq C_G^\circ(U_\rho(Z(F^\circ(B)))) \leq B$, against $\rho_C \succ \rho$.

So $\lambda \in I_B$, i.e. $C_G^\circ(\lambda) \leq B$. But by uniqueness, Step 2 (iii), $\mathbb{T} \subseteq (F^\circ(B))^{-\lambda}$, so $\mathbb{T} \subseteq F^\circ(B) \cap F^\circ(B)^\kappa$. Since all elements in \mathbb{T} are torsion-free by the Torsion Control Proposition 5, one even has $\mathbb{T} \subseteq (F^\circ(B) \cap F^\circ(B)^\kappa)^\circ$. The latter is abelian by [DJ12, 4.46(2) (our Fact 10)], and \mathbb{T} is therefore a definable, connected, abelian subgroup. Now always by the Torsion Control and Genericity Propositions 5 and 2, and by the decomposition of B obtained in Step 2 (iii), one has:

$$\mathrm{rk} \mathbb{T} = \mathrm{rk} T_B(\kappa) \geq \mathrm{rk} B - \mathrm{rk} C_G^\circ(\iota_0) = \mathrm{rk} B - \mathrm{rk} C_G^\circ(\lambda) = \mathrm{rk}(F^\circ(B))^{-\lambda}$$

A definable set contains at most one definable, connected, generic subgroup, so \mathbb{T} is the only definable, connected, generic group included in $(F^\circ(B))^{-\lambda}$: hence $N_{\hat{G}}((F^\circ(B))^{-\lambda}) \leq N_{\hat{G}}(\mathbb{T})$ and $B^{+\lambda}$ normalises \mathbb{T} . Moreover $\mathbb{T} \cap B^{+\lambda} = 1$ since λ inverts \mathbb{T} and \mathbb{T} contains no torsion elements. So $\mathbb{T} \cdot B^{+\lambda} = \mathbb{T} \rtimes B^{+\lambda}$ is a definable subgroup of rank $\geq \mathrm{rk}(F^\circ(B_\lambda))^{-\lambda} + \mathrm{rk} B^{+\lambda} = \mathrm{rk} B$ by Step 2 (iii). Hence $B = \mathbb{T} \rtimes B^{+\lambda}$ normalises \mathbb{T} ; $B = N_G^\circ(\mathbb{T})$ by $*$ -local $^\circ$ solubility. In particular κ normalises B : a contradiction. \diamond

Claim (i) will be used only once more, in the next Step.

5.3 The Fuel, refined

Step 5 (abelianity).

(i) If $\iota \in I_B$ then $\mathbb{T} \cap C_G(\iota) = 1$.

(ii) There is no definable, connected, soluble, κ -invariant group containing $U_\rho(Z(F^\circ(B)))$ and \mathbb{T} .

(iii) \mathbb{T} is a definable, abelian, torsion-free group.

Proof of Step. The first claim is easy. Let $\iota \in I_B$ and $t \in \mathbb{T} \setminus \{1\}$ be such that $t^\iota = t$. Then $\iota \in C_{\hat{G}}(t)$ which is κ -invariant; by normalisation principles and abelianity of the Sylow 2-subgroup, κ has a $C_{\hat{G}}(t)$ -conjugate λ commuting with ι (we do not use that by $*$ -local $^\circ$ solubility and lack of torsion of \mathbb{T} , Proposition 5, $C_{\hat{G}}(t)$ is soluble-by-finite). By uniqueness Step 2 (i), B is the only Borel subgroup of G containing $C_G^\circ(\iota)$, so λ normalises B . Recall from Inductive Torsion control Proposition 5 that t is torsion-free. By uniqueness Step 2 (iii), $t^\lambda = t^\kappa = t^{-1}$ forces $t^2 = [t^{-1}, \lambda] \in F^\circ(B)$ and $t \in F^\circ(B)$. We then apply the Devil's Ladder, Proposition 4, to the action of $\langle \iota, \kappa \rangle$ on $d(t)$ and find that κ normalises B : a contradiction.

As the proof of the second claim is a little involved let us first see how it entails the third one. Suppose that $X = (F^\circ(B) \cap F^\circ(B)^\kappa)^\circ$ is non-trivial and let $H = N_G^\circ(X)$; then $*$ -local $^\circ$ solubility and the second claim yield a contradiction. Hence $X = 1$ which proves abelianity of $(B \cap B^\kappa)^\circ$. Then, since elements of $\mathbb{T} \subseteq B \cap B^\kappa$ contain no torsion in their definable hulls by Proposition 5, one has $\mathbb{T} \subseteq (B \cap B^\kappa)^\circ$ and \mathbb{T} is therefore an abelian group, obviously definable and torsion-free. So we now proceed to proving the second claim.

Let L be a definable, connected, soluble, κ -invariant group containing $U_\rho(Z(F^\circ(B)))$ and \mathbb{T} . We shall show that $U_\rho(Z(F^\circ(B)))$ and \mathbb{T} commute, which will contradict Step 4 (ii). To do this we proceed piecewise in the following sense. Bear in mind that for $t \in \mathbb{T}$, $d(t)$ is torsion-free by inductive torsion control, Proposition 5, so one may take Burdges' decomposition of the definable, connected, abelian group $d(t)$. As a result, the set \mathbb{T} is a union of products of various abelian τ -groups for various parameters τ . We shall show that each of them centralises $U_\rho(Z(F^\circ(B)))$, which will be the contradiction.

So we let τ be a parameter and Θ be an abelian τ -group included in the set \mathbb{T} . If $\tau = \rho$ then we are done as $\Theta \leq U_\rho(B)$. So suppose $\tau \prec \rho$ and prepare to use the Bender method (§2.4). Since $L \geq \langle U_\rho(Z(F^\circ(B))), \mathbb{T} \rangle$, L is not abelian by Step 4 (ii).

Let $C \leq G$ be a Borel subgroup of G containing $N_G^\circ(L') \geq L$ and maximising ρ_C . Notice that

$$U_{\rho_C}(Z(F^\circ(C))) \leq C_G^\circ(F^\circ(C)) \leq C_G^\circ(C') \leq C_G^\circ(L') \leq N_G^\circ(L')$$

so by uniqueness principles and definition of C , C is actually the only Borel subgroup of G containing $N_G^\circ(L')$. As the latter is κ -invariant, so is C ; in particular $C \neq B$. Moreover $U_\rho(Z(F^\circ(B))) \leq C$ so uniqueness principles force $\rho_C \succ \rho$, and $H = (B \cap C)^\circ \geq \langle U_\rho(Z(F^\circ(B))), \mathbb{T} \rangle$ is non-abelian. So we are under the assumptions of Fact 11 with $B_\ell = B$ and $B_h = C$.

We determine the linking parameter ρ' , i.e. the only parameter of the homogeneous group H' [DJ12, 4.51(3)]. But Fact 7 (v) (no need for Frécon homogeneity here) shows that the by Step 4 (ii) non-trivial commutator $[U_\rho(Z(F^\circ(B))), \mathbb{T}]$ is a ρ -subgroup of H' , hence $\rho' = \rho$.

We now construct a most remarkable involution. Let $V_\rho \leq C$ be a Sylow ρ -subgroup of C containing $U_\rho(Z(F^\circ(B)))$. Since κ normalises C , it has by normalisation principles a C -conjugate λ normalising V_ρ . By uniqueness principles, $U_\rho(B)$ is the only Sylow ρ -subgroup of G containing $U_\rho(Z(F^\circ(B)))$, so λ normalises B . If $\lambda \notin I_B$ then by Step 4 (i) λ inverts $U_\rho(Z(F^\circ(B)))$; since $\rho_C \succ \rho$ it certainly inverts $U_{\rho_C}(C)$ as well, whence by commutation principles $[U_\rho(Z(F^\circ(B))), U_{\rho_C}(C)] = 1$ and $U_{\rho_C}(C) \leq C_G^\circ(U_\rho(Z(F^\circ(B)))) \leq B$, contradicting $\rho_C \succ \rho$. Hence $\lambda \in I_B$; it normalises B and C (hence H).

We return to our abelian τ -group Θ included in the set \mathbb{T} , with $\tau \prec \rho$. Let $V_\tau \leq H$ be a Sylow τ -subgroup of H containing Θ . By normalisation principles λ has an H -conjugate μ normalising V_τ . We shall prove that μ actually centralises V_τ ; little work will remain after that. Observe that V_τ is a definable, connected, nilpotent group contained in two different Borel subgroups of G so by [DJ12, 4.46(2) (Fact 9)] it is abelian. By Fact 7 (v) (no need for Frécon homogeneity here), $[V_\tau, \mu]$ is a τ -group inverted by μ .

Now note that μ , like λ , is in I_B , and normalises B and C . Moreover by Step 2 (iii), $[V_\tau, \mu] \leq F^\circ(B)$. We shall prove that $[V_\tau, \mu] \leq F^\circ(C)$ as well by making it commute with all of $F^\circ(C)$, checking it on each term of Burdges' decomposition of $F^\circ(C)$. Keep Fact 11 in mind.

First, by [DJ12, 4.38], $\rho' = \rho$ is the least parameter in $F^\circ(C)$; we handle it as follows. Recall that $[V_\tau, \mu] \leq F^\circ(B)$ is a τ -group, so $[V_\tau, \mu] \leq U_\tau(F^\circ(B))$. By [DJ12, 4.52(7)] and since $\rho' = \rho \neq \tau$, the latter is in $Z(H)$. But by [DJ12, 4.52(3)], $U_\rho(F^\circ(C)) = U_{\rho'}(F^\circ(C)) = (F^\circ(B) \cap F^\circ(C))^\circ \leq H$, so $[V_\tau, \mu]$ does commute with $U_\rho(F^\circ(C))$. Now let $\sigma \succ \rho$ be another parameter. Remember that μ normalises C ; since $\mu \in \iota_0^{\hat{G}}$, $\sigma \succ \rho_\mu$ and μ inverts $U_\sigma(F^\circ(C))$. It inverts $[V_\tau, \mu]$ as well so commutation principles force $[V_\tau, \mu]$ to centralise $U_\sigma(F^\circ(C))$.

As a consequence $[V_\tau, \mu] \leq C$ centralises $F^\circ(C)$. Unfortunately this is not quite enough to apply the Fitting subgroup theorem as literally stated in [BN94b, Proposition 7.4] due to connectedness issues. The first option is to note that with exactly the same proof as in [BN94b, Proposition 7.4]: in any connected, soluble group K of finite Morley rank one has $C_K^\circ(F^\circ(K)) = F^\circ(K)$. Another option is to observe that by [DJ12, 4.52(1)], $F^\circ(C)$ has no torsion unipotence: in particular, the torsion in $F(C)$ is central in C [DJ12, 2.14]. All together $[V_\tau, \mu]$ commutes with $F(C)$ and we then use the Fitting subgroup theorem stated in [BN94b, Proposition 7.4] to conclude $[V_\tau, \mu] \leq F(C)$. Either way we find $[V_\tau, \mu] \leq F^\circ(C)$, and we already knew $[V_\tau, \mu] \leq F^\circ(B)$. By connectedness $[V_\tau, \mu] \leq (F^\circ(B) \cap F^\circ(C))^\circ$. But the latter as we know [DJ12, 4.51(3)] is $\rho' = \rho$ -homogeneous: since $\rho \succ \tau$, this shows $[V_\tau, \mu] = 1$.

In particular $\mu \in I_B$ centralises $\Theta \leq V_\tau$. By claim (i), $\Theta = 1$ which certainly commutes with $U_\rho(Z(F^\circ(B)))$. This contradiction finishes the proof of claim (ii). \diamond

Remark. It is possible to avoid using the Devil's Ladder in the proof of claim (i). Postpone and finish the proof of claim (ii) as follows:

$\mu \in I_B$ centralises Θ , so $\mu \in C_{\hat{G}}(\Theta)$ which is κ -invariant. By normalisation principles and abelianity of the Sylow 2-subgroup, κ has a $C_{\hat{G}}(\Theta)$ -conjugate ν commuting with μ . Since $\mu \in I_B$, by uniqueness Step 2 (i) ν normalises B . By Step 2 (iii) $\Theta = [\Theta, \nu] \leq F^\circ(B)$ commutes with $U_\rho(Z(F^\circ(B)))$. Hence all of \mathbb{T} commutes with $U_\rho(Z(F^\circ(B)))$, against Step 4 (ii).

Then prove claim (i):

Now let $t \in \mathbb{T} \setminus \{1\}$ be centralised by $\iota \in I_B$. Like in the previous paragraph, $t \in F^\circ(B)$; t has infinite order and is inverted by κ . But we prove in the third claim that $(F^\circ(B) \cap F^\circ(B)^\kappa)^\circ = 1$, a contradiction.

Both claims (i) and (iii) are crucial. Claim (ii) is a gadget used in the proof of claim (iii) and in the next step.

5.4 The Core

Notation 6.

- Let π be the set of parameters occurring in \mathbb{T} .
- Let $J_\kappa = U_\pi(C_B^\circ(\mathbb{T}))$ (one has $\mathbb{T} \leq J_\kappa$) by Step 5 (iii).

We feel extremely uncomfortable with the next step. The question of why to maximise over $C_B^\circ(\mathbb{T})$ is a mystery and always was. Nine years before writing these lines, the author then a PhD student produced an incorrect study of some similar maximal intersection configuration, and after noticing a well-hidden flaw had to reassemble his proof by trying all possible maximisations. Exactly the same happened to him again. We feel like one piece of the puzzle is still missing, or more confusingly that we are playing with incomplete sets of pieces from distinct puzzles. There are many ways to get it wrong and the step works by miracle.

Step 7 (rigidity). J_κ is an abelian Carter π -subgroup of B . There is a maximal 2-torus \hat{S} of \hat{G} contained in $N_{\hat{G}}(B) \cap N_{\hat{G}}(J_\kappa)$, and for any $\iota \in I_B \cap \hat{S}$, one has: $C_{U_\pi(N_G^\circ(J_\kappa))}(\iota) \leq C_G^\circ(\mathbb{T})$.

Proof of Step. First of all, observe that by totality principles there is a maximal 2-torus \hat{S}_0 of \hat{G} containing ι_0 ; by uniqueness Step 2 (i) \hat{S}_0 normalises B . Bear in mind that any maximal 2-torus in $N_{\hat{G}}(B)$ contains an involution in I_B .

We need more structure now, so let $C \neq B$ be a Borel subgroup of G containing $C_B^\circ(\mathbb{T})$ and maximising $H = (B \cap C)^\circ$. There is such a Borel subgroup indeed since $C_G^\circ(\mathbb{T})$ is κ -invariant whereas B is not. As one expects there are two cases and we first deal with the abelian one. The other will be more involved (but morally speaking less likely).

Suppose that H is abelian. Since $H \geq C_B^\circ(\mathbb{T}) \geq \mathbb{T}$ by abelianity of the latter, Step 5 (iii), and since H is supposed to be abelian too, $H = C_B^\circ(\mathbb{T}) \leq N_G^\circ(J_\kappa)$. We now consider $N_G^\circ(J_\kappa)$. It is not clear at all whether B contains $N_G^\circ(J_\kappa)$ but one may ask.

If (H is abelian and) B happens to be the only Borel subgroup of G containing $N_G^\circ(J_\kappa)$, then

$$U_\pi(N_{C_G^\circ(\mathbb{T})}^\circ(J_\kappa)) \leq U_\pi(N_{C_B^\circ(\mathbb{T})}^\circ(J_\kappa)) = U_\pi(C_B^\circ(\mathbb{T})) = J_\kappa$$

and $J_\kappa \leq C_G^\circ(\mathbb{T})$ is a Carter π -subgroup of $C_G^\circ(\mathbb{T})$. As the latter is κ -invariant, by normalisation principles κ has a $C_G^\circ(\mathbb{T})$ -conjugate λ normalising J_κ . But our current assumption that B is the only Borel subgroup of G containing $N_G^\circ(J_\kappa)$ forces λ to normalise B as well. By Step 2 (iii) and since λ like κ inverts the 2-divisible group \mathbb{T} , $\mathbb{T} = [\mathbb{T}, \lambda] \leq F^\circ(B)$ which certainly contradicts Step 4 (ii).

So (provided H is abelian) B is not the only Borel subgroup of G containing $N_G^\circ(J_\kappa)$: let $D \neq B$ one such. Then $C_B^\circ(\mathbb{T}) = H \leq N_B^\circ(J_\kappa) \leq (B \cap D)^\circ$ so by maximality of H , $H = (B \cap D)^\circ = N_B^\circ(J_\kappa)$ and $J_\kappa = U_\pi(C_B^\circ(\mathbb{T})) = U_\pi(H)$ is a Carter π -subgroup of B . By normalisation principles there is a B -conjugate \hat{S} of \hat{S}_0 normalising J_κ . For $\iota \in \hat{S} \cap I_B$ one has $C_G^\circ(\iota) \leq B$ and

$$C_{U_\pi(N_G^\circ(J_\kappa))}^\circ(\iota) \leq N_B^\circ(J_\kappa) = H \leq C_G^\circ(\mathbb{T})$$

It is not easy to say more as $N_G^\circ(J_\kappa)$ need not be nilpotent, but we are done with the proof in the abelian case.

We now suppose that H is not abelian. However $H \geq C_B^\circ(\mathbb{T})$ so if $D \neq B$ is a Borel subgroup of G containing H , one has by definition of the latter $H = (B \cap D)^\circ$. By [DJ12, 4.50(3) and (6),

our Fact 10], we are under the assumptions of Fact 11. Keep it at hand. Let $Q \leq H$ be a Carter subgroup of H . Let ρ' denote the parameter of the homogeneous group H' . Studying J_κ certainly means asking about ρ' and π .

Here is a useful principle: if σ is a set of parameters not containing ρ' , $V_\sigma \leq H$ is a nilpotent σ -subgroup of H , and $\hat{S} \leq N_{\hat{G}}(B) \cap N_{\hat{G}}(V_\sigma) \cap N_{\hat{G}}(C)$ is a 2-torus, then \hat{S} centralises V_σ . It is easily proved: let $\hat{B} = B \cdot d(\hat{S})$, a definable, connected, soluble subgroup of \hat{G} . Then by Zilber's indecomposibility theorem, $[B, \hat{S}] \leq (F^\circ(\hat{B}) \cap B)^\circ \leq F^\circ(B)$ and likewise in C . Hence $[V_\sigma, \hat{S}] \leq (F^\circ(B) \cap F^\circ(C))^\circ$ which is ρ' -homogeneous [DJ12, 4.52(3)]. As $\rho' \notin \sigma$, we have $[V_\sigma, \hat{S}] = 1$ by push-forward (Fact 7), and \hat{S} centralises V_σ .

The argument really starts here. First, $\rho' \in \pi$. Otherwise by lemma J, \mathbb{T} is included in a Carter subgroup of H ; we may assume $\mathbb{T} \leq Q$, and in particular by abelianity of Q (Fact 9) $Q \leq C_G^\circ(\mathbb{T})$. By Lemma A, $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B) \cup N_{\hat{G}}(C)$. So there are two cases (yes, this does work for groups).

- First suppose ($\rho' \notin \pi$ and) $N_{\hat{G}}(Q) \leq N_{\hat{G}}(C)$. In particular $N_B^\circ(Q) \leq N_H^\circ(Q) = Q$ and Q is a Carter subgroup of B . By normalisation principles, \hat{S}_0 has a B -conjugate \hat{S} in $N_{\hat{G}}(B) \cap N_{\hat{G}}(Q) \leq N_{\hat{G}}(B) \cap N_{\hat{G}}(U_\pi(Q)) \cap N_{\hat{G}}(C)$. As we noted \hat{S} must centralise $U_\pi(Q) \geq \mathbb{T}$. But there is an involution $\iota \in \hat{S} \cap I_B$, and this contradicts Step 5 (i).
- Hence (still assuming $\rho' \notin \pi$) one has $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B)$. Then $N_{C_G^\circ(\mathbb{T})}^\circ(Q) \leq N_{C_B^\circ(\mathbb{T})}^\circ(Q) \leq N_H^\circ(Q) = Q$ and $Q \leq C_G^\circ(\mathbb{T})$ is a Carter subgroup of $C_G^\circ(\mathbb{T})$. As the latter is κ -invariant, by normalisation principles κ has a $C_G^\circ(\mathbb{T})$ -conjugate λ normalising Q . Now since $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B)$, λ normalises B . Then \mathbb{T} is inverted by λ and 2-divisible, whence $\mathbb{T} = [\mathbb{T}, \lambda] \leq [B, \lambda] \leq F^\circ(B)$ by Step 2 (iii), contradicting Step 4 (ii).

So we have proved $\rho' \in \pi$. On the other hand $\rho_B = \rho \notin \pi$ as otherwise $C_G^\circ(U_\rho(\mathbb{T}))$ would contradict Step 5 (ii). Suppose for a second $\rho_C \succ \rho_B$; then since $\rho \neq \rho'$, one has $U_\rho(Z(F^\circ(B))) \leq Z(H) \leq C_G^\circ(\mathbb{T})$ [DJ12, 4.52(7)], against Step 4 (ii) again. Since parameters differ [DJ12, 4.50(6)] one has $\rho_B \succ \rho_C$. In particular [DJ12, 4.52(2)], Q is a Carter subgroup of B .

We now show that \mathbb{T} is ρ' -homogeneous, i.e. $\pi = \{\rho'\}$. Let $\sigma = \pi \setminus \{\rho'\}$. Since H' is ρ' -homogeneous, by Lemma J we may assume that $U_\sigma(\mathbb{T}) \leq Q$. Now $U_{\rho'}(H) = U_{\rho'}(F^\circ(H))$ is a Sylow ρ' -subgroup of B [DJ12, implicit but clear in 4.52(6)]. By normalisation principles \hat{S}_0 has a B -conjugate \hat{S} in $N_{\hat{G}}(B) \cap N_{\hat{G}}(U_{\rho'}(H)) \leq N_{\hat{G}}(B) \cap N_{\hat{G}}(C)$ [DJ12, 4.52(6)]. Hence \hat{S} normalises H . But Q is a Carter subgroup of H so by normalisation principles over H , \hat{S} has an H -conjugate \hat{S}_1 in $N_{\hat{G}}(B) \cap N_{\hat{G}}(C) \cap N_{\hat{G}}(Q)$. By our initial principle, \hat{S}_1 centralises $U_\sigma(Q) \geq U_\sigma(\mathbb{T})$. Since \hat{S}_1 contains an involution in I_B , $U_\sigma(\mathbb{T}) = 1$ by Step 5 (i), as desired. Hence \mathbb{T} is ρ' -homogeneous.

As a conclusion $\pi = \{\rho'\}$ and $J_\kappa = U_{\rho'}(C_B^\circ(\mathbb{T})) \leq U_{\rho'}(H)$. The latter is an abelian Sylow ρ' -subgroup of B [DJ12, implicit but clear in 4.52(6) and noted above]. Also, $\mathbb{T} \leq U_{\rho'}(H) \leq C_B^\circ(\mathbb{T})$ and $J_\kappa = U_{\rho'}(H)$. We constructed a maximal 2-torus $\hat{S} \leq N_{\hat{G}}(B) \cap N_{\hat{G}}(J_\kappa)$ a minute ago.

Finally fix $\iota \in \hat{S} \cap I_B$. We aim at showing that $C_{U_{\rho'}(N_{\hat{G}}(J_\kappa))}^\circ(\iota) \leq C_G^\circ(\mathbb{T})$. Recall that \hat{S} normalises C . By normalisation principles \hat{S} normalises some Sylow ρ' -subgroup $V_{\rho'}$ of C . Then with Lemma E under the action of ι , $V_{\rho'} = (V_{\rho'}^+)^{\circ} \cdot \{V_{\rho'}, \iota\}$. Now $(V_{\rho'}^+)^{\circ}$ is a ρ' -subgroup of $(B \cap C)^{\circ} = H$, so $(V_{\rho'}^+)^{\circ} \leq J_\kappa \leq F^\circ(C)$ [DJ12, 4.52(6)]. Letting $\hat{C} = C \cdot d(\hat{S})$ one easily sees (as we already did) that $\{V_{\rho'}, \iota\} \subseteq F^\circ(C)$. So $V_{\rho'} \leq F^\circ(C)$ and $V_{\rho'} \leq U_{\rho'}(F^\circ(C))$. Conjugating Sylow ρ' -subgroups in C this means that $U_{\rho'}(F^\circ(C))$ is actually the only Sylow ρ' -subgroup of C . But by [DJ12, 4.52(8)] any Sylow ρ' -subgroup of G containing $U_{\rho'}(H)$ is contained in C . This means that $U_{\rho'}(F^\circ(C))$ is the only Sylow ρ' -subgroup of G containing $U_{\rho'}(H) = J_\kappa$.

As a conclusion, any Sylow ρ' -subgroup of $N_G^\circ(J_\kappa)$ lies in $U_{\rho'}(F^\circ(C))$. Hence, paying attention to the fact that ι normalises the nilpotent ρ' -group $U_{\rho'}(F^\circ(C))$:

$$C_{U_\pi(N_G^\circ(J_\kappa))}^\circ(\iota) \leq C_{U_{\rho'}(F^\circ(C))}^\circ(\iota) \leq U_{\rho'}(H) = J_\kappa \leq C_G^\circ(\mathbb{T}) \quad \diamond$$

5.5 The Reaction

Notation 8.

- We now write \mathbb{T}_κ for $\mathbb{T}_B(\kappa)$, as the involution κ will vary in K_B .

- Let $Y = \{B, \iota_0\}$.

Step 9 (conjugacy). (i) Y is a normal subgroup of B ; (ii) $\text{rk } B = \text{rk } C_G(\iota_0) + \text{rk } Y$, and (iii) any element of $Y \setminus \{1\}$ lies in finitely many conjugates of Y . (iv) Moreover, \mathbb{T}_κ and Y are G -conjugate.

Proof of Step. As a matter of fact we let $Y_\iota = \{B, \iota\}$ for any $\iota \in I_B$. Since $I_B = \iota_0^{N_G(B)}$, such sets are $N_G(B)$ -conjugate to Y .

Let $\iota \in \hat{S} \cap I_B$; do not forget that there is such an involution alright. Under the action of ι we may write $J_\kappa = J_\kappa^+(\iota)[J_\kappa, \iota]$. By Step 5 (i), $\mathbb{T}_\kappa \cap J_\kappa^+ = 1$. So using the very definition of $\kappa \in K_B$ this yields the rank estimate:

$$\text{rk}[J_\kappa, \iota] = \text{rk } J_\kappa - \text{rk } J_\kappa^+ \geq \text{rk } \mathbb{T}_\kappa \geq \text{rk } B - \text{rk } C_G^\circ(\iota_0) = \text{rk } B - \text{rk } C_B^\circ(\iota) = \text{rk } \iota^B \geq \text{rk } \iota^{J_\kappa} = \text{rk}[J_\kappa, \iota]$$

Equality follows. In particular $[J_\kappa, \iota] \subseteq Y_\iota$ is generic in Y_ι . Since a definable set of degree 1 contains at most one definable, generic subgroup, one has $C_B(\iota) \leq N_G(Y_\iota) \leq N_B([J_\kappa, \iota])$. On the other hand since \hat{G} is W_2^\perp , $[J_\kappa, \iota]$ has no involutions; it is disjoint from $C_B(\iota)$. Hence $[J_\kappa, \iota] \cdot C_B(\iota) = [J_\kappa, \iota] \rtimes C_B(\iota)$ is a generic subgroup of B . It follows $B = [J_\kappa, \iota] \rtimes C_B(\iota)$. At this stage it is clear that $Y_\iota = [J_\kappa, \iota]$ is a normal subgroup of B contained in $F^\circ(B)$, and the same holds of Y by $N_G(B)$ -conjugacy. Moreover $\text{rk } Y_\iota = \text{rk } \mathbb{T}_\kappa$; we are not done.

Consider the definable function $f : \mathbb{T}_\kappa \rightarrow Y_\iota$ which maps t to $[t, \iota]$; as J_κ is abelian, it is a group homomorphism. Bearing in mind that $\mathbb{T}_\kappa \cap C_{J_\kappa}(\iota) = 1$ by Step 5 5 and by rank equality, f is actually a group isomorphism; we are not done.

We claim that $J_\kappa \leq C_G^\circ(\mathbb{T}_\kappa)$ is a Carter π -subgroup of $C_G^\circ(\mathbb{T}_\kappa)$. For let $N = U_\pi(N_G^\circ(J_\kappa))$ and $N_1 = U_\pi(N \cap C_G^\circ(\mathbb{T}_\kappa))$. We wish to decompose under the action of ι . Be very careful however that ι need not normalise N_1 . But since \hat{S} normalises J_κ it also normalises N . Then $\hat{N} = N \cdot d(\hat{S})$ is yet another definable, connected, soluble group, so $\{N, \iota\} \subseteq (\hat{N}' \cap N)^\circ \leq F^\circ(N)$, and Lemma F applies to N . Now take $n_1 \in N_1$ and write its decomposition $n_1 = pn$ in N , with $p \in (N^+)^\circ$ and $n \in \{N, \iota\}$. Then $p \in C_{U_\pi(N_G^\circ(J_\kappa))}^\circ(\iota) \leq C_G^\circ(\mathbb{T}_\kappa)$ by Step 7. So $n \in C_G^\circ(\mathbb{T}_\kappa)$. On the other hand, for any $t \in \mathbb{T}_\kappa$ one has using a famous identity:

$$\begin{aligned} 1 &= [[\iota, n^{-1}], t]^n \cdot [[n, t^{-1}], \iota]^t \cdot [[t, \iota], n]^t \\ &= [n^{-2}, t]^n \cdot [[t, \iota], n]^t \\ &= [[t, \iota], n]^t \end{aligned}$$

Hence n commutes with $[\mathbb{T}_\kappa, \iota] = Y_\iota$ and $n \in N_G(N_G^\circ(Y_\iota)) = N_G(B)$. So $n_1 = pn \in N_G(B)$, meaning $N_1 \leq N_G^\circ(B) = B$. Now $N_1 \leq U_\pi(N_B^\circ(J_\kappa))$ and since J_κ is a Carter π -subgroup of B , $N_1 = J_\kappa$. Therefore J_κ is a Carter π -subgroup of $C_G^\circ(\mathbb{T}_\kappa)$.

This extra rigidity has devastating consequences. By normalisation principles, κ has a $C_G^\circ(\mathbb{T}_\kappa)$ -conjugate λ normalising J_κ . If λ normalises B then $\mathbb{T}_\kappa \leq [J_\kappa, \lambda] \leq F^\circ(B)$ by Step 2 (iii), which contradicts $[U_\rho(Z(F^\circ(B))), \mathbb{T}_\kappa] \neq 1$ from Step 4 (ii). So λ does not normalise B . On the other hand $T_\lambda(B)$ contains \mathbb{T}_κ so $\lambda \in K_B$. In particular, everything we said so far of κ holds of λ : by rank equality, $\mathbb{T}_\lambda = \mathbb{T}_\kappa$.

By conjugacy of Sylow 2-subgroups, λ has an $N_{\hat{G}}(J_\kappa)$ -conjugate μ in \hat{S} . Remember that we took $\hat{G} = G \cdot d(\hat{S}^\circ)$, so $N_{\hat{G}}(J_\kappa) = N_G(J_\kappa) \cdot d(\hat{S})$ and $\mu = \lambda^n$ for some $n \in N_G(J_\kappa)$. Moreover $\mu \in \hat{S}$ commutes with the involution ι we fixed earlier in the proof. Let $X = C_{Y_\iota}^\circ(\mu) \leq F^\circ(B)$.

- Suppose $X = 1$. Then μ inverts Y_ι , so:

$$Y_\iota \leq [J_\kappa, \mu] = [J_\kappa, \lambda^n] = [J_\kappa, \lambda]^n \leq \mathbb{T}_\lambda^n = \mathbb{T}_\kappa^n$$

and equality follows from the equality of ranks.

- Suppose $X \neq 1$. We apply the Devil's Ladder, Proposition 4, to the action of $\langle \mu, \iota \rangle$ on X inside B_μ , the only Borel subgroup of G containing $C_G^\circ(\mu)$ by Uniqueness Step 2 (i). We find that B_μ is the only G -conjugate of B containing $C_G^\circ(X) \geq U_\rho(Z(F^\circ(B)))$. Uniqueness principles force $B_\mu = B$, which means $\mu \in I_B \cap \hat{S}$. In particular, everything we said so far of ι holds of μ , and

$$Y_\mu = [J_\kappa, \mu] = [J_\kappa, \lambda^n] = [J_\kappa, \lambda]^n \leq \mathbb{T}_\lambda^n = \mathbb{T}_\kappa^n$$

and equality follows from the equality of ranks.

In any case, \mathbb{T}_κ is G -conjugate to Y .

It remains to show that any non-trivial element of Y lies in finitely many conjugates. For if $a \in Y \setminus \{1\}$ then by the isomorphism $\mathbb{T}_\kappa \simeq Y$ and inductive torsion control, Proposition 5, a has infinite order: $C = C_G^\circ(a) \geq \langle U_\rho(Z(F^\circ(B))), Y \rangle$ is therefore soluble, and ι_0 -invariant. If $\rho_C \succ \rho_B$ then ι_0 inverts both $U_{\rho_C}(C)$ and Y , and commutation principles yield $[U_{\rho_C}(C), Y] = 1$ whence $U_{\rho_C}(C) \leq N_G^\circ(Y) = B$, a contradiction. Hence $\rho_C \preceq \rho_B$ and uniqueness principles show that B is the only Borel subgroup of G containing C . If $a \in Y^g$ then B^g is the only Borel subgroup of G containing C likewise. Since $B \leq N_G(Y) \leq N_G(B)$, this can happen only for a finite number of conjugates of Y . \diamond

5.6 Critical Mass

Step 10 (the collapse).

We first determine $\text{rk}\{\mathbb{T}_\kappa : \kappa \in K_B\}$. The set under consideration is definable alright as a subset of $\{Y^g : g \in G\} = G/N_G(Y)$ by Step 9 (iii). If $\mathbb{T}_\kappa = \mathbb{T}_\lambda$ then there is $g \in G$ with $\mathbb{T}_\kappa = Y^g$. In particular, κ and λ lie in $N_{\hat{G}}(N_G^\circ(Y^g)) = N_{\hat{G}}(B^g)$ by Step 9 (i). Since κ and λ are G -conjugate, $\kappa\lambda \in N_G(B^g)$. Now κ inverts $\kappa\lambda$ so by Step 2 (iv), $\kappa\lambda \in B^g$, and $\lambda \in \kappa T_{B^g}(\kappa)$. The latter has the same rank as Y by Proposition 5 and Step 9 (iii). It follows that $\text{rk}\{\mathbb{T}_\kappa : \kappa \in K_B\} \geq \text{rk } K_B - \text{rk } Y = \text{rk } G - \text{rk } C_G(\iota) - \text{rk } Y$.

We move to something else. Let \mathcal{F} be a definable family of conjugates of Y . Since an element in Y lies in only finitely many conjugates by Step 9 (iv), $\text{rk} \bigcup_{\mathcal{F}} = \text{rk } \mathcal{F} + \text{rk } Y$. We first apply this to $\mathcal{F}_1 = \{\mathbb{T}_\kappa : \kappa \in K_B\}$, finding:

$$\text{rk} \bigcup_{\kappa \in K_B} \mathbb{T}_\kappa \geq \text{rk } G - \text{rk } C_G(\iota_0) - \text{rk } Y + \text{rk } Y = \text{rk } G - \text{rk } C_G(\iota_0)$$

We now apply it to $\mathcal{F}_2 = \{Y^g : g \in G/N_G(Y)\}$, finding:

$$\text{rk } Y^G = \text{rk } G - \text{rk } N_G(B) + \text{rk } Y = \text{rk } G - \text{rk } B + \text{rk } Y$$

Both agree by Step 9 (ii), so $\bigcup_{\mathcal{F}_1}$ is generic in $\bigcup_{\mathcal{F}_2}$. However $\bigcup_{\mathcal{F}_1} \subseteq \bigcup_{\mathcal{F}_2} \cap B$, which contradicts [DJ12, Lemma 3.33].

This concludes the proof of Proposition 6. \square

6 The Proof – After the Maximality Proposition

6.1 The Dihedral Case

The following is a combination of two different lines of thought: the study of a pathological “ $W = 2$ ” configuration in [Del07b, Chapitre 4] (published as [Del08, §3]) and the final argument in [BCD09]. Since we can quickly focus on the 2^\perp case only a few details need be adapted in order to move from minimal connected simplicity to $*$ -local $^\circ$ solubility, so we feel that the resulting proposition owes much to Burdges and Cherlin. The final contradiction is by constructing two disjoint generic subsets of some definable subset of G .

Proposition 7 (dihedral case). *Let \hat{G} be a connected, U_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup. Suppose that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.*

Suppose that the Sylow 2-subgroup of \hat{G} is isomorphic to that of $\text{PSL}_2(\mathbb{C})$. Suppose in addition that for $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is included in a unique Borel subgroup of G .

Then \hat{G}/G is 2^\perp and $B_\iota = C_G^\circ(\iota)$ is a Borel subgroup of G inverted by any involution $\omega \in C_G(\iota) \setminus \{\iota\}$.

Proof. First observe that by torality principles, all involutions in \hat{G} are conjugate; it follows that G or \hat{G}/G is 2^\perp .

Notation 1.

- Let $V = \{1, \iota, \omega, \iota\omega\} \leq \hat{G}$ be a four-group.
- Let \hat{T}_ι be a 2-torus containing ι and inverted by ω , and \hat{T}_ω likewise.
- Let B_ι be the only Borel subgroup of G containing $C_G^\circ(\iota)$, and B_ω likewise (observe that by uniqueness of B_ι over $C_G^\circ(\iota)$, ω normalises B_ι and vice-versa).
- Let $\rho = \rho_{B_\iota}$.

Here is a small unipotence principle we shall use with no reference: if $L \leq G$ is a definable, connected, soluble, V -invariant subgroup, then $\rho_L \preceq \rho$. This is obvious by bigeneration, Fact 3, which will play a growing role.

Step 2. $B_\iota \neq B_\omega$.

Proof of Step. Suppose not. If G is 2^\perp , then it is W_2^\perp : by the Maximality Proposition 6, B_ι is a Borel subgroup of G . Hence $C_G^\circ(\iota) = B_\iota = B_\omega = C_G^\circ(\omega)$, and therefore $B_\iota = C_G^\circ(\omega)$ as well. Yet bigeneration, Fact 3, applies to the action of V on the 2^\perp group G : a contradiction.

If G is not 2^\perp then bigeneration might fail. But then all involutions are in G ; by totality principles $\iota \in C_G^\circ(\iota) \leq B_\iota = B_\omega$ so B_ω contains $\hat{T}_\omega \rtimes \langle \iota \rangle$, which contradicts the structure of torsion in connected, soluble groups. \diamond

Notation 3. Let $H = (B_\iota \cap B_\omega)^\circ$.

Since ω normalises B_ι and vice-versa, H is V -invariant.

Step 4. H is abelian and 2^\perp . Moreover ι centralises $U_\rho(B_\iota)$ and ω inverts it; V centralises H and $N_G^\circ(H) = C_G^\circ(H)$.

Proof of Step. If $H = 1$ then $C_{B_\iota}^\circ(\omega) = 1$ and ω inverts B_ι ; since ω inverts \hat{T}_ι which normalises B_ι , commutation principles yield $[\hat{T}_\iota, B_\iota] = 1$ and $B_\iota \leq C_G^\circ(\iota)$. So $B_\iota = C_G^\circ(\iota)$ is an abelian Borel subgroup inverted by ω and by $\iota\omega$. Hence all our claims hold if $H = 1$. We now suppose $H \neq 1$.

Suppose that H is not abelian and let $L = N_G^\circ(H)$, a definable, connected, soluble, V -invariant group. Then $\rho_L \preceq \rho$ but since L contains $U_\rho(Z(F^\circ(B_\iota)))$ and $U_\rho(Z(F^\circ(B_\omega)))$, equality holds. Hence $U_\rho(Z(F^\circ(B_\iota))) \leq U_\rho(L)$; by uniqueness principles $U_\rho(B_\iota)$ is the only Sylow ρ -subgroup of G containing $U_\rho(L)$. The same holds of $U_\rho(B_\omega)$, proving equality and $B_\iota = B_\omega$, against Step 2. So H is abelian.

Now suppose that $U_\rho(H) \neq 1$ and let $L = N_G^\circ(U_\rho(H))$. Same causes having the same effects, we reach a contradiction again. Hence $U_\rho(H) = 1$, and it follows that ω inverts $U_\rho(B_\iota)$. The same argument works for ω , so ι centralises $U_\rho(B_\iota)$.

We now claim that V centralises H . For let $K = [H, \iota]$; since H is abelian, using Zilber's indecomposibility theorem we see that K is a definable, connected, abelian group inverted by ι ; in particular it is 2-divisible. Since ι centralises $U_\rho(B_\iota)$ and inverts $U_\rho(B_\omega)$, commutation principles yield $\langle U_\rho(B_\iota), U_\rho(B_\omega) \rangle \leq C_G^\circ(K)$ and the latter is V -invariant. Uniqueness principles and Step 2 forbid solubility of $C_G^\circ(K)$: this means $K = 1$, and ι centralises H . The same holds of ω .

Suppose that H has involutions: since it is V -invariant, so is its Sylow 2-subgroup T (no need for normalisation principles here). If $\iota \in T$, then $\iota \in H \leq B_\iota$ and $\omega \in B_\omega$ by conjugacy; hence B_ω contains $\hat{T}_\omega \rtimes \langle \iota \rangle$, against the structure of torsion in connected, soluble groups. So $\iota \notin T$, and by assumption on the structure of the Sylow 2-subgroup of \hat{G} , ι inverts T ; the same holds of ω and $\iota\omega$, a contradiction.

It remains to show that $N_G^\circ(H) = C_G^\circ(H)$. Let $N = N_G^\circ(H)$. First assume that G is 2^\perp . Then using Lemma E under the action of ι we write $N = (N^{+\iota})^\circ \cdot \{N, \iota\}$ where $\{N, \iota\}$ is 2-divisible. Since ι centralises H , commutation principles applied pointwise force $\{N, \iota\} \subseteq C_G(H)$. We turn to the action of ω on $N_1 = (N^{+\iota})^\circ$; with Lemma E again $N_1 = (N_1^{+\omega})^\circ \cdot \{N_1, \omega\}$, and here again $\{N_1, \omega\} \subseteq C_G(H)$. Finally $(N_1^{+\omega})^\circ \leq C_G^\circ(\iota, \omega) \leq H \leq C_G(H)$ by abelianity, so $N \leq C_G(H)$ and we conclude by connectedness of N .

Now suppose that \hat{G}/G is 2^\perp : as a consequence $V \leq G$. It is not quite clear whether N has involutions and whether $\{N, \iota\}$ is 2-divisible, so we argue as follows. By normalisation principles, there is a V -invariant Carter subgroup Q of N . The previous argument applies to Q , so $Q \leq C_G^\circ(H)$; it also applies to $F^\circ(N)$, so $F^\circ(N) \leq C_G^\circ(H)$, and $N = F^\circ(N) \cdot Q \leq C_G^\circ(N)$. \diamond

Step 5. We may suppose that G is 2^\perp .

Proof of Step. Suppose that G contains involutions, i.e. $V \leq G$. We shall prove that $H = 1$. So suppose in addition that $H \neq 1$. For the consistency of notations, we $i = \iota \in G$ and $w = \omega \in G$, and $T_i = \hat{T}_i$, $T_w = \hat{T}_w$.

We claim that w does *not* invert $F^\circ(B_i)$. For if it does, then w inverts $T_i \leq B_i$ and $F^\circ(B_i)$, so by commutation principles $[T_i, F^\circ(B_i)] = 1$. Let $Q \leq B_i$ be a Carter subgroup of B_i containing T_i ; then $B_i = F^\circ(B_i) \cdot Q$ centralises T_i , and $T_w \leq Z(B_w)$ by conjugacy. Hence $T_i \rtimes \langle w \rangle \leq \langle T_i, T_w \rangle \leq C_G^\circ(H)$, against the structure of torsion in connected, soluble groups and $*$ -local $^\circ$ solubility.

Hence $Y_i = C_{F^\circ(B_i)}^\circ(w) \neq 1$. Since $U_\rho(B_i)$ is abelian by Step 4, $U_\rho(B_i) \leq C_G^\circ(Y_i)$; since Y_i is V -invariant, our small unipotence principle and general uniqueness principles force $C_G^\circ(Y_i) \leq B_i$. Hence by Step 4:

$$N_{B_w}^\circ(H) = C_{B_w}^\circ(H) \leq C_{B_w}^\circ(Y_i) \leq H$$

which proves that H is a Carter subgroup of B_w . It therefore contains involutions, against Step 4.

This contradiction shows that if G has involutions then $H = 1$. Hence, like in the beginning of Step 4, w inverts $B_i = C_G^\circ(i)$ and so does any other involution in $C_G^\circ(i) \setminus \{i\}$: we are done. \diamond

From now on, we suppose that G is 2^\perp ; we are after a contradiction. Since G is W_2^\perp , Maximality Proposition 6 applies and $C_G^\circ(\iota) = B_\iota$ is a Borel subgroup of G . Moreover since G is 2^\perp , it admits a decomposition $G = G^{+\iota} \cdot G^{-\iota}$ by Lemma E, and the fibers are trivial. By connectedness of G , $C_G(\iota) = G^+$ is connected. Finally, since the 2-torus \hat{T}_ι normalises B_ι , it centralises the finite quotient $N_G(B_\iota)/B_\iota$, and so does ι . Now $N = N_G(B_\iota)$ admits a decomposition $N = N^+ \cdot \{N, \iota\}$ as well; we just proved $N^+ \leq B$ and $\{N, \iota\} \subseteq B$. Hence $B_\iota = C_G(\iota)$ is a self-normalising Borel subgroup of G , which will be used with no reference.

Step 6. For any involution $\lambda \in C_{\hat{G}}(\iota) \setminus \{\iota\}$, $B_\iota^{-\lambda} = F^\circ(B_\iota)$.

Proof of Step. The claim is actually obvious if $H = 1$, an extreme case in which the below argument remains however valid. Let $X_\iota = C_{F^\circ(B_\iota)}^\circ(\omega)$ and $X_\omega = C_{F^\circ(B_\omega)}^\circ(\iota)$.

Suppose that $X_\iota \neq 1$ and $X_\omega \neq 1$. By abelianity of $U_\rho(B_\iota)$ from Step 4, $U_\rho(B_\iota) \leq C_G^\circ(X_\iota)$ which is V -invariant; unipotence and uniqueness principles show that B_ι is the only Borel subgroup of G containing $C_G^\circ(X_\iota)$, and likewise for B_ω over $C_G^\circ(X_\omega)$. It follows that $C_{B_\omega}^\circ(H) \leq (B_\iota \cap B_\omega)^\circ = H$ and H is a Carter subgroup of B_ω . The latter is $\hat{T}_\omega \rtimes \langle \iota \rangle$ -invariant, so by normalisation principles $N_{\hat{G}}(H)$ contains a Sylow 2-subgroup \hat{S} of \hat{G} . Since $V \leq C_{\hat{G}}(H)$ by Step 4, we may assume $V \leq \hat{S}$.

Still assuming that $X_\iota \neq 1$ and $X_\omega \neq 1$, we denote μ the involution of V which lies in $\hat{S} = \hat{T}_\mu$ and fix $\nu \in V \setminus \langle \mu \rangle$. Then by assumption on the structure of the Sylow 2-subgroup of \hat{G} , ν inverts \hat{T}_μ ; it also centralises H , so by commutation principles $\hat{T}_\mu \rtimes \langle \nu \rangle = \hat{S}$ centralises $H \geq \langle X_\iota, X_\omega \rangle$. Since B_ι is the only Borel subgroup of G containing $C_G^\circ(X_\iota)$ (and likewise for ω), \hat{S} normalises both B_ι and B_ω ; up to taking $\nu\mu$ instead of ν , we may suppose that \hat{S} normalises B_ν . Now ν inverts \hat{T}_μ and centralises B_ν , so by commutation principles $[\hat{T}_\mu, B_\nu] = 1$ and $B_\nu \leq C_G^\circ(\mu) = B_\mu$: a contradiction to Step 2.

All this shows that $X_\iota = 1$ or $X_\omega = 1$; we suppose the first. Then ω inverts $F^\circ(B_\iota)$. Using Lemma E we write $B_\iota = B_\iota^{+\omega} \cdot \{B_\iota, \omega\}$. Notice that since B_ι is 2^\perp , $B_\iota^- = \{B_\iota, \omega\}$ (the sign $-$ refers to the action of ω throughout the present paragraph). Since ω inverts the 2-divisible subgroup $F^\circ(B_\iota)$, one has $F^\circ(B_\iota) \subseteq B_\iota^-$. Since the set B_ι^- is 2-divisible, commutation principles applied pointwise show $F^\circ(B_\iota) \subseteq B_\iota^- \subseteq C_{B_\iota}(F^\circ(B_\iota))$. Hence B_ι^- is a union of translates of $F^\circ(B_\iota)$. Now $C_{B_\iota}(F^\circ(B_\iota))$ is normal in B_ι and nilpotent, so by definition of the Fitting subgroup $C_{B_\iota}(F^\circ(B_\iota)) \leq F(B_\iota)$. As a consequence $B_\iota^- \subseteq F(B_\iota)$ is a union of *finitely many* translates of $F^\circ(B_\iota)$. But $\deg B_\iota^- = \deg \{B_\iota, \omega\} = \deg \omega^{B_\iota} = 1$, so $F^\circ(B_\iota) = B_\iota^-$.

The previous paragraph shows that if $X_\iota = 1$, then our desired conclusion holds of $\lambda = \omega$; it then also holds of $\lambda = \iota\omega$. Now any involution $\lambda \in C_{\hat{G}}(\iota) \setminus \{\iota\}$ is a $C_{\hat{G}}(\iota)$ -conjugate of ω or $\iota\omega$, say $\lambda = \omega^n$ with $n \in C_{\hat{G}}(\iota) \leq N_{\hat{G}}(B_\iota) \leq N_{\hat{G}}(F^\circ(B_\iota))$, so:

$$B_\iota^{-\lambda} = B_\iota^{-\omega^n} = (B_\iota^{-\omega})^n = (F^\circ(B_\iota))^n = F^\circ(B_\iota)$$

Similarly, if $X_\omega = 1$, then for any $\lambda \in C_{\hat{G}}(\omega) \setminus \{\omega\}$, $B_\omega^{-\lambda} = F^\circ(B_\omega)$. We conjugate ω to ι and see that in this case we are done as well. \diamond

Step 7. $\text{rk } G^{-\iota} \leq 2 \text{rk } F^\circ(B_\iota)$.

Proof of Step. Let $\kappa = \iota\omega$ and $\check{G} = G \rtimes V$. Observe that in \check{G} the involutions ι, ω, κ are *not* conjugate; one has exactly three conjugacy classes, which also are G -classes. So for $(\omega_1, \kappa_1) \in \omega^G \times \kappa^G$, $d(\omega_1, \kappa_1)$ contains a unique involution which must be a conjugate ι_1 of ι .

Now consider the definable function from $\omega^G \times \kappa^G$ to ι^G which maps (ω_1, κ_1) to ι_1 ; we shall compute its fibers. If (ω_2, κ_2) also maps to ι_1 then $\omega_1\omega_2 \in C_G(\iota_1) = B_{\iota_1}$. Hence $\omega_1\omega_2 \in B_{\iota_1}^{-\omega_1} = F^\circ(B_{\iota_1})$ and fibers have rank at most $2 \text{rk } F^\circ(B_\iota)$. As the map is obviously onto, one has $2 \text{rk } F^\circ(B_\iota) \geq \text{rk } G - \text{rk } B = \text{rk } G^{-\iota}$. \diamond

Step 8. $(F^\circ(B_\omega))^{F^\circ(B_\iota)}$ and $(F^\circ(B_{\iota\omega}))^{F^\circ(B_\iota)}$ are generic subsets of $G^{-\iota}$.

Proof of Step. Recall from Step 6 that ι inverts $F^\circ(B_\omega)$ and centralises B_ι . In particular G is 2^\perp and $F^\circ(B_\omega) \cap B_\iota = 1$; moreover $(F^\circ(B_\omega))^{F^\circ(B_\iota)} \subseteq G^{-\iota}$. We now compute the rank. Consider the definable function from $F^\circ(B_\iota) \times F^\circ(B_\omega)$ to G which maps (a, x) to x^a . Let us prove that it has finite fibers.

Suppose $x^a = y^b$ with $b \in F^\circ(B_\iota)$ and $y \in F^\circ(B_\omega)$; then $x^{ab^{-1}} = y$, and $ab^{-1} \in F^\circ(B_\iota)$ which is abelian. Applying ω , $y = y^\omega = x^{ab^{-1}\omega} = x^{\omega a^{-1}b} = x^{a^{-1}b} = y^{a^{-2}b^2}$. Since G is 2^\perp , this results in $a^{-1}b \in C_G(y)$ and $x = y$.

So if $x^a = y^b$ with notations above, $x = y$. We shall determine $C_{F^\circ(B_\iota)}(x)$. Suppose $Y = C_{F^\circ(B_\iota)}^\circ(x)$ is infinite. Since Y is $\langle \iota, \omega \rangle$ -invariant, so is $C_G^\circ(Y)$, a definable, connected, soluble group containing $F^\circ(B_\iota)$. As we know $C_G^\circ(Y)$ has unipotence parameter at most ρ , so $C_G^\circ(Y)$ normalises $U_\rho(B_\iota)$ and $C_G^\circ(Y) \leq B_\iota$; as a matter of fact, by uniqueness principles B_ι is the only Borel subgroup of parameter ρ containing $C_G^\circ(Y)$. It follows $x \in N_G(B_\iota)$. Hence $x \in N_G(B_\iota) \cap F^\circ(B_\omega) = C_G(\iota) \cap F^\circ(B_\omega) = 1$.

As a result, fibers are finite; it follows $\text{rk}(F^\circ(B_\omega))^{F^\circ(B_\iota)} = 2 \text{rk } F^\circ(B_\iota) \geq \text{rk } G^{-\iota}$ by Step 7; inclusion forces equality. The same holds of $(F^\circ(B_{\iota\omega}))^{F^\circ(B_\iota)}$. \diamond

We now finish the proof of Proposition 7. By Step 8, both $(F^\circ(B_\omega))^{F^\circ(B_\iota)}$ and $(F^\circ(B_{\iota\omega}))^{F^\circ(B_\iota)}$ are generic in $G^{-\iota}$. So there is $t \in F^\circ(B_\omega) \cap F^\circ(B_{\iota\omega})^f \setminus \{1\}$ for some $f \in F^\circ(B_\iota)$. Then the involution $(\iota\omega)^f = f^{-1}\iota\omega f = f^\omega \iota\omega f = \iota\omega f^2$ centralises t , whereas $\iota\omega$ inverts it. So $f^2 \in G$ inverts t . This creates an involution in G : against Step 5. \square

6.2 Strong Embedding

Strong embedding is a classical topic in finite group theory [Ben71]. Recall that a subgroup M of a group G is said to be strongly embedded if M contains an involution but $M \cap M^g$ does not for any $g \notin M$. The reader should also keep in mind a few basic facts about strongly embedded configurations [BN94b, Theorem 10.19 (checking the apparently missing assumptions would be almost immediate here)]:

- involutions in M are M -conjugate;
- a Sylow 2-subgroup of M is a Sylow 2-subgroup of G ;
- M contains the centraliser of each of its involutions.

We need no more. The study of a minimal connected simple group with a strongly embedded subgroup was carried in [BCJ07, Theorem 1].

Proposition 8. *Let G be a connected, U_2^\perp , non-soluble, $*$ -locally $^\circ$ soluble group of finite Morley rank. Suppose that G has a definable, soluble, strongly embedded subgroup. Then $\text{Pr}_2(G) \leq 1$.*

Our proof will be considerably shorter than [BCJ07]: thanks to the Maximality Proposition 6 we need only handle the case of central involutions [BCJ07, §4]. Apart from this, our argument is a subset of the one in [BCJ07, §4]: we construct two disjoint generic sets. We only hope to have helped clarify matters in Step 8 below.

(Incidentally, an alternative proof of [BCJ07, Theorem 1] was suggested using state-of-the-art genericity arguments in minimal connected simple groups [ABF13, Theorem 6.1]. Yet this new proof reproduces [BCJ07, §4] and affects only the case we need not consider by Maximality.)

Proof. We let G be a minimal counterexample, i.e. G satisfies the assumptions but $\text{Pr}_2(G) \geq 2$. By the 2-structure Proposition 1, the Sylow 2-subgroups of G are connected.

Notation 1. Let $M \leq G$ be a definable, soluble, strongly embedded subgroup. Let $S \leq M$ be a Sylow 2-subgroup of G and $A = \Omega_2(S^\circ)$.

Step 2. For all $i \in I(G)$, $C_G^\circ(i)$ is soluble.

Proof of Step. First observe that $Z(G)$ has no involutions by strong embedding, as they would lie in $S \leq M$ and in any conjugate.

Suppose that there is $i \in A \setminus \{1\}$ with non-soluble $C_G^\circ(i)$. Fix some 2-torus $\tau_i \leq S$ of Prüfer rank 1 containing i ; since $C_G^\circ(\tau_i)$ is soluble by $*$ -local $^\circ$ solubility, there exists by descending chain condition some $\alpha \in \tau_i$ with $C_G^\circ(\alpha)$ soluble. We take α with minimal order; then $C_G^\circ(\alpha^2)$ is not soluble, and $\alpha^2 \neq 1$ since $\alpha \neq i$.

Let $H = C_G^\circ(\alpha^2)$ and $N = M \cap H$. Since $\alpha^2 \neq 1$ and $Z(G)$ has no 2-elements, $H < G$. Observe how $\alpha \in \tau_i \leq S \leq N$. Let $\bar{H} = H/\langle \alpha^2 \rangle$ and $\bar{N} = N/\langle \alpha^2 \rangle$. Then \bar{N} is definable, soluble, and strongly embedded in \bar{H} which still has Prüfer rank ≥ 2 : against minimality of G as a counterexample. \diamond

Notation 3. Let $B = M^\circ$.

Step 4. B is a Borel subgroup of G and $A \leq Z(B)$; the group M/B is non-trivial and has odd order. Moreover (i) strongly real elements of G which lie in B actually lie in A ; (ii) if $i \in I(B)$ inverts $n \in N_G(B)$ then $n \in B$. (iii) For any $g \in G$, $BgI(G)$ is generic in G . (iv) Finally $(B \cap B^g)^\circ = 1$ for $g \notin N_G(B)$.

Proof of Step. By Step 2, connectedness of the Sylow 2-subgroup, and the maximality Proposition 6, $C_G^\circ(i)$ is a Borel subgroup of G for any $i \in I(G)$. But for $i \in A \setminus \{1\}$, $C_G^\circ(i) \leq M$ by strong embedding of the latter, so $C_G^\circ(i) \leq B$ and equality follows. In particular, $A \leq Z(B)$.

By structure of the Sylow 2-subgroup, $N_G(B)/B$ has odd order, and so has its subgroup M/B . But M being strongly embedded conjugates its (more than one) involutions, which are central in B : this shows $B < M$.

If $b \in B$ is inverted by some $k \in I(G)$ then k normalises $C_G(b) \geq A$; by normalisation principles and structure of the Sylow 2-subgroup, one has $k \in C_G(b)$, so b has order at most 2; this is claim (i). If $i \in I(B)$ inverts $n \in N_G(B)$ then computing modulo B : $n^{-1}B = n^iB = nB$, and $n^2 \in B$. Since $N_G(B)/B$ has odd order, $n \in B$, proving (ii).

We move to (iii). Consider the definable function $B \times I(G)$ which maps (b, k) to bk . If $b_1k_1 = b_2k_2$ with obvious notations, then $b_2^{-1}b_1$ is a strongly real element of G lying in B , hence has order at most 2 by claim (i): this happens only finitely many times, so fibers are finite and $\text{rk}(B \cdot I(G)) = \text{rk } B + \text{rk } I(G) = \text{rk } B + \text{rk } G - \text{rk } B = \text{rk } G$. Then for any $g \in G$:

$$\text{rk}(BgI(G)) = \text{rk}(gB^gI(G)^g) = \text{rk}(g(BI(G))^g) = \text{rk}(BI(G)) = \text{rk } G$$

It remains to control intersections of conjugates of B , claim (iv). Suppose that $H = (B \cap B^g)^\circ$ is infinite. Let $Q \leq H$ be a Carter subgroup of H ; since A^g centralises $B^g \geq H \geq Q$, A^g normalises the definable, connected, soluble group $N_G^\circ(Q)$. By bigeneration, Fact 3, $N_G^\circ(Q) \leq \langle C_G^\circ(a^g) : a \in A \setminus \{1\} \rangle \leq B^g$, so $N_B^\circ(Q) \leq N_H^\circ(Q) = Q$ and Q is actually a Carter subgroup of B . By solubility of B , Q contains a Sylow 2-subgroup of B : hence $A \leq Q \leq B^g$, and strong embedding guarantees $g \in N_G(B)$. \diamond

Notation 5. Let $w \in M \setminus B$ (denoted σ in [BCJ07, Notation 4.1(2)]).

Step 6. We may assume that w is strongly real, in which case the following applies: (i) $C_G(w)$ has no involutions; (ii) if some involution $k \in I(G)$ inverts w , then k inverts $C_G^\circ(w)$; (iii) finally $C_B^\circ(w) = 1$.

Proof of Step. By Step 4 (iii), both $BI(G)$ and $BwI(G)$ are generic in G , so they intersect. Hence up to translating by an element of B , we may suppose that w is a strongly real element.

Suppose that there is an involution $\ell \in C_G(w)$. Then $w \in C_G(\ell) = C_G^\circ(\ell)$ by Steinberg's torsion theorem and connectedness of the Sylow 2-subgroup; $C_G^\circ(\ell)$ is a conjugate of B (toral principles suffice here; no need to invoke strong embedding). But w is strongly real, so by Step 4 (i) it is an involution, against the fact that M/B has odd order.

Let k be an involution inverting w (this exists as noted). Then $C_G^\circ(k)$ is a conjugate B_k of B , and $k \in B_k$ by torality principles. Observe how $w \notin N_G(B_k)$ by Step 4 (ii). So $C_G^\circ(k, w) \leq (B_k \cap B_k^w)^\circ$ is trivial by Step 4 (iv), and k inverts $C_G^\circ(w)$.

Finally let $H = C_B^\circ(w)$ and suppose $H \neq 1$. Bear in mind that A centralises H , so it normalises the definable, soluble group $N_G^\circ(H)$. By bigeneration, Fact 3, $N_G^\circ(H) \leq B$. But k inverts H , so it normalises $N_G^\circ(H)$ as well. Hence $N_G^\circ(H) \leq B \cap B^k$, and Step 4 (iv) forces $k \in N_G(B)$. Now $k \in B$ inverts $w \in N_G(B) \setminus B$, a contradiction to Step 4 (ii). This shows that $C_B^\circ(w) = 1$. \diamond

Notation 7. Let $\check{C} = C_G^\circ(w) \setminus N_G(B)$.

\check{C} is obviously generic in $C_G^\circ(w)$, as $C_{N_G(B)}^\circ(w) \leq C_B^\circ(w) = 1$ by Step 6.

Step 8. $B\check{C}B$ is generic in G .

Proof of Step. This is the only part where we slightly rewrite the argument in [BCJ07]. Let $\mathcal{F} = \{(m, \ell) \in Bw \times I(G) : m^\ell = m^{-1}\}$.

Let $m \in Bw$. If m is inverted by some involution in G , then by Step 6 (iii) $C_B^\circ(m) = 1$ and $m^B \subseteq Bm$ is generic in Bm . So is w^B , and m is therefore B -conjugate with w . So let us count involutions inverting w . First, there is such an involution k by Step 6. If ℓ is yet another such, then $k\ell \in C_G(w)$ and $\ell \in kC_G(w)$. Moreover, since k inverts $C_G^\circ(w)$ by Step 6 (ii), any element in $kC_G^\circ(w)$ is an involution inverting w . This together shows:

$$\text{rk } \mathcal{F} = \text{rk } w^B + \text{rk } C_G^\circ(w) = \text{rk } B + \text{rk } C_G^\circ(w)$$

Since $BwI(G)$ and $BI(G)$ are generic in G by Step 4 (iii), a generic $\ell \in I(G)$ inverts some element in Bw . Hence $\text{rk } \mathcal{F} \geq \text{rk } I(G) = \text{rk } G - \text{rk } B$.

Finally consider the definable function which maps $(b_1, c, b_2) \in B \times \check{C} \times B$ to b_1cb_2 . We claim that all fibers are finite. Since the fiber over $b_1c_0b_2$ has same rank as the fiber over c_0 , we compute the latter. Suppose $b_1cb_2 = c_0$ with obvious notations. Then applying w :

$$c_0 = c_0^w = b_1^w c b_2^w = [w, b_1^{-1}] b_1 c b_2 [b_2, w] = [w, b_1^{-1}] c_0 [b_2, w]$$

In particular, $[w, b_1^{-1}]^{c_0} = [b_2, w]^{-1} \in B \cap B^{c_0}$ which is finite by Step 4 (iv). Since $C_B^\circ(w) = 1$ by Step 6 (iii), there are finitely many possibilities for b_1 and b_2 , and c is then determined. So the function has finite fibers, and therefore:

$$\text{rk } (B\check{C}B) = 2 \text{rk } B + \text{rk } C_G^\circ(w) = \text{rk } \mathcal{F} + \text{rk } B \geq \text{rk } G \quad \diamond$$

We now finish the proof of Proposition 8. By Steps 4 (iii) and 8, both $BI(G)$ and $B\check{C}B$ are generic in G . So they intersect; there is an involution $k = b_1cb_2 \in B\check{C}B$. Conjugating by b_1 , there is an involution $\ell = cb \in \check{C}B$. Now applying w one finds:

$$\ell^w = cb^w = cb[b, w] = \ell[b, w]$$

which means that $[b, w] \in B$ is a strongly real element. There are two possibilities. If $[b, w] \neq 1$ then by Step 4 (i) $[b, w] \in A \setminus \{1\}$ and $\ell \in C_G([b, w])$, so ℓ and c lie in B : a contradiction. If $[b, w] = 1$ then w centralises b and $cb = \ell$: against Step 6 (i). \square

6.3 November

Theorem. Let \hat{G} be a connected, U_2^\perp group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, non-soluble, $*$ -locally $^\circ$ soluble subgroup.

Then the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$, isomorphic to that of $\mathrm{SL}_2(\mathbb{C})$, or is a 2-torus of Prüfer 2-rank at most 2.

Suppose in addition that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.

Then $m_2(\hat{G}) \leq 2$, G or \hat{G}/G is 2^\perp , and involutions are conjugate in \hat{G} . Moreover one of the following cases occurs:

- **PSL₂**: $G \simeq \mathrm{PSL}_2(\mathbb{K})$ in characteristic not 2; \hat{G}/G is 2^\perp ;
- **CiBo₀**: G is 2^\perp ; $m_2(\hat{G}) \leq 1$; for $\iota \in I(\hat{G})$, $C_G(\iota) = C_G^\circ(\iota)$ is a self-normalising Borel subgroup of G ;
- **CiBo₁**: $m_2(G) = m_2(\hat{G}) = 1$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G ;
- **CiBo₂**: $\mathrm{Pr}_2(G) = 1$ and $m_2(G) = m_2(\hat{G}) = 2$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G^\circ(i)$ is an abelian Borel subgroup of G inverted by any involution in $C_G(i) \setminus \{i\}$ and satisfies $\mathrm{rk} G = 3 \mathrm{rk} C_G^\circ(i)$;
- **CiBo₃**: $\mathrm{Pr}_2(G) = m_2(G) = m_2(\hat{G}) = 2$; \hat{G}/G is 2^\perp ; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G .

Proof.

Step 1. We may suppose that for all $\iota \in I(\hat{G})$, $C_G^\circ(\iota)$ is soluble.

Proof of Step. All we must do for the moment is determine the structure of the Sylow 2-subgroup of G so we may take $\hat{G} = G$; by the 2-Structure Proposition 1 it suffices to bound the Prüfer 2-rank of G . Suppose that G has Prüfer rank at least 3, and has minimal rank among such counterexamples. Note that $G/Z(G)$ has the same properties but is now centreless.

For the current Step we also suppose that there is some involution $i \in G$ with $C_G^\circ(i)$ non-soluble. Then as in Step 2 of Proposition 8 we take a 2-torus of rank 1 τ_i containing i and $\alpha \in \tau_i$ of minimal order with $C_G^\circ(\alpha)$ soluble; $\alpha^2 \neq 1$. Let $H = C_G^\circ(\alpha^2)$: by torality principles, it has the same Prüfer 2-rank as G , hence by minimality of G as a counterexample $H = G$ and $\alpha^2 \in Z(G)$, a contradiction. \diamond

Step 2. We may suppose that G is W_2^\perp .

Proof of Step. Suppose G is not. By the 2-Structure Proposition 1 and since centralisers in G of involutions are quasi-soluble, the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$, that is $\mathrm{Pr}_2(G) = 1$ and $m_2(G) = 2$. Fix $i \in I(G)$ an involution of G .

If $C_G^\circ(i)$ is contained in at least two Borel subgroups of G , then by the Algebraicity Proposition 3, $G \simeq \mathrm{PSL}_2(\mathbb{K})$ for some algebraically closed field of characteristic not 2. The latter has no outer automorphisms [BN94b, Theorem 8.4]; by assumption on centralisers of involutions, \hat{G}/G is 2^\perp and we are in case **PSL₂**.

So we may assume that $C_G^\circ(i)$ is contained in a unique Borel subgroup of G . We then apply the Dihedral Proposition 7 inside $\check{G} = G$ to find that $C_G^\circ(i)$ is an abelian Borel subgroup of G inverted by any involution in $C_G(i) \setminus \{i\}$. By torality principles in G there exist a Sylow 2-subgroup $S_i = S_i^\circ \rtimes \langle w \rangle$ with $i \in S_i^\circ$ and another Sylow 2-subgroup $S_w = S_w^\circ \rtimes \langle i \rangle$ likewise. In order to reach case **CiBo₂** one first shows that \hat{G}/G is 2^\perp ; only the rank estimate will remain to prove.

If \hat{G}/G is not 2^\perp then S_i is no Sylow 2-subgroup of \hat{G} . Let $\hat{S} \leq \hat{G}$ be a Sylow 2-subgroup containing S_i properly; it is folklore that $\mathrm{Pr}_2(\hat{S}) \geq 2$. Since \hat{S}° is 2-divisible and invariant under $\omega \in \hat{S}$, we may apply Maschke's Theorem (see for instance [Del12, Fact 2]) to find a quasi-complement, i.e. a w -invariant 2-torus \hat{T} with $\hat{S}^\circ = S_i^\circ(+)\hat{T}$. Then using Zilber's indecomposibility theorem, $[\hat{T}, w] \leq (\hat{T} \cap G)^\circ = 1$, that is, w centralises \hat{T} . It follows that \hat{T} normalises both $C_G^\circ(i)$ and $C_G^\circ(w)$; by the rigidity of tori, it centralises therefore both S_i° and S_w° . Hence $S_i^\circ \rtimes \langle w \rangle \leq \langle S_i^\circ, S_w^\circ \rangle \leq C_G^\circ(\hat{T})$,

so by the structure of torsion in connected, soluble groups, $C_G^\circ(\hat{T})$ may not be soluble. This contradicts the fact that $\hat{T} \neq 1$ contains an involution of \hat{G} , which has soluble centraliser^o by assumption.

Hence \hat{G}/G is 2^\perp ; we finally show $\text{rk } G = 3 \text{ rk } C_G^\circ(i)$. This exactly follows [Del08, Proposition 3.26 and Corollaire 3.27]: since $C_G(i)$ is not connected for $i \in I(G)$, using the Borovik-Cartan decomposition one sees that generic, independent $j, k \in I(G)$ are such that $d(jk)$ is not 2-divisible, and we let ℓ be the only involution in $d(jk)$. Then $(j, k) \mapsto \ell$ is a well-(generically) defined, definable function; obvious rank computations yield $\text{rk } G = 3 \text{ rk } C_G^\circ(i)$. \diamond

Notation 3. For $\iota \in I(\hat{G})$ let $B_\iota = C_G^\circ(\iota)$.

By Steps 1 and 2 and the Maximality Proposition 6, B_ι is a Borel subgroup of G .

Step 4. $\text{Pr}_2(\hat{G}) \leq 2$.

Proof of Step. Suppose $\text{Pr}_2(\hat{G}) \geq 3$. We may assume that $\hat{G} = G \cdot d(\hat{S})$ for some 2-torus \hat{S} of \hat{G} . In particular \hat{G}/G is W_2^\perp . But so is G by Step 2; by Lemma K, so is \hat{G} , i.e. \hat{S} is actually a Sylow 2-subgroup of \hat{G} . Let $A = \Omega_2(\hat{S})$ be the group generated by the involutions of \hat{S} ; $A \leq \hat{G}$ is an elementary abelian 2-group with 2-rank $\text{Pr}_2(\hat{G}) \geq 3$. Let $\rho = \max\{\rho_{B_\iota} : \iota \in A \setminus \{1\}\}$ and $\iota \in A \setminus \{1\}$ be such that $\rho_{B_\iota} = \rho$.

We show that for any involution $\lambda \in A \setminus \{1\}$, $B_\lambda = B_\iota$. Let $\kappa \in A \setminus \langle \iota \rangle$ be such that $C_{U_\rho(Z(F^\circ(B_\iota)))}(\kappa) \neq 1$; this certainly exists as A has rank at least 3. Then $X \leq C_G^\circ(\kappa) = B_\kappa$, so $\rho_\kappa = \rho$ and $X \leq U_\rho(B_\kappa)$. Let as always $\hat{B}_\iota = B_\iota \cdot d(\hat{S})$; one has $\{B_\iota, \kappa\} \subseteq (\hat{B}_\iota \cap B)^\circ \leq F^\circ(B_\iota)$ so we may apply Lemma F and write $B_\iota = B_\iota^{+\kappa} \cdot \{B_\iota, \kappa\} \subseteq B_\iota^+ \cdot F^\circ(B_\iota)$. Now both B_ι^+ and $F^\circ(B_\iota)$ normalise X , hence X is normal in B_ι . Uniqueness principles imply that $U_\rho(B_\iota)$ is the only Sylow ρ -subgroup of G containing X . In particular $U_\rho(B_\iota) = U_\rho(B_\kappa)$. Hence $C_G^\circ(\iota) = B_\iota = B_\kappa = C_G^\circ(\kappa) = C_G^\circ(\iota\kappa)$. Turning to an arbitrary $\lambda \in A \setminus \{1\}$ we apply bigeneration, Fact 3, to the action of $V = \langle \iota, \kappa \rangle$ on the soluble group B_λ , and find $B_\lambda \leq \langle C_{B_\lambda}^\circ(\mu) : \mu \in V \setminus \{1\} \rangle \leq B_\iota$. So $B_\lambda = B_\iota$ for any $\lambda \in A \setminus \{1\}$.

We claim that $\text{Pr}_2(G) = 1$. First, if G is 2^\perp we contradict bigeneration, i.e. the fact that $G = \langle C_G^\circ(\mu) : \mu \in V \setminus \{1\} \rangle$. Therefore G has involutions. In order to bound its Prüfer 2-rank we shall use the Strong Embedding Proposition 8. We argue that $M = N_G(B_\iota)$ is strongly embedded in G . For let j be an involution in $S = \hat{S} \cap G$, which is a Sylow 2-subgroup of G ; then $j \in N_G(B_\iota)$. But G is W_2^\perp and B_ι contains a maximal 2-torus of G , so $j \in B_\iota$. Let $V = \langle \iota, \kappa \rangle$; recall that V centralises B_ι . In particular V centralises j , and normalises B_j . As the latter is soluble we apply bigeneration, Fact 3, and find $B_j = \langle C_{B_j}^\circ(\lambda) : \lambda \in V \setminus \{1\} \rangle \leq B_\iota$. Now if $j \in M^x$ with $x \in G$, then we argue likewise: $j \in B_\iota^x$, V^x centralises j , V^x normalises B_j , and $B_j = B_\iota^x$. Hence $x \in N_G(B_\iota)$, and $M = N_G(B_\iota)$ is strongly embedded in G . By the Strong Embedding Proposition 8, $\text{Pr}_2(G) = 1$, as desired.

Observe that any two commuting involutions of \hat{G} centralise the same Borel subgroup of G : for if $\langle \mu, \nu \rangle$ is a four-subgroup of \hat{G} then up to conjugacy $\langle \mu, \nu \rangle \leq A$, so $B_\mu = B_\nu$. Now any two non-conjugate involutions of \hat{G} commute to a third involution, so they centralise the same Borel subgroup of G . But there are at least two conjugacy classes of involutions in \hat{G} , since $\text{Pr}_2(G) = 1$ and $\text{Pr}_2(\hat{G}) \geq 3$. So actually any two involutions of \hat{G} centralise the same Borel subgroup of G . This is to mean: for any $g \in G$, $B_\iota^g = B$; B_ι is normal in G , which contradicts \ast -local^o solubility. \diamond

Step 5. If $i \in I(G)$ then B_i is self-normalising.

Proof of Step. We claim that i is the only involution in $Z(B_i)$. If $\text{Pr}_2(G) = 1$ this is clear by the structure of torsion in connected, soluble groups. If $\text{Pr}_2(G) \geq 2$ (and one has equality by Step 4), then let $k \in I(B_i) \setminus \{i\}$: if $k \in Z(B_i)$ then $B_k = B_i = B_{ik}$ is clearly strongly embedded, against Proposition 8.

In particular, $N_G(B_i) \leq B_i \cdot C_G(i) \leq C_G(i) = C_G^\circ(i) = B_i$ by Steinberg's torsion theorem and connectedness of the Sylow 2-subgroup of G (Step 2). \diamond

Notation 6. For $\kappa, \lambda \in I(\hat{G})$ let $T_\kappa(\lambda) = T_{B_\kappa}(\lambda)$.

Before reading the following be very careful that Inductive Torsion Control, Proposition 5, requires \hat{G} to be W_2^\perp ; for the moment only G need be by Step 2.

Step 7 (Antalya). *If \hat{G} is W_2^\perp and $\lambda \notin N_{\hat{G}}(B_\kappa)$ then $T_\kappa(\lambda)$ is finite.*

If in addition $\hat{G} = G \cdot d(\hat{S}^\circ)$ for some maximal 2-torus $\hat{S}^\circ \leq \hat{G}$, then $\text{rk } C_{\hat{G}}^\circ(\kappa) = \text{rk } C_{\hat{G}}^\circ(\lambda)$ and the generic left translate $\hat{g}C_{\hat{G}}^\circ(\lambda)$ contains a conjugate of κ .

Proof of Step. Suppose that \hat{G} is W_2^\perp and $T_\kappa(\lambda)$ is infinite. Then by Inductive Torsion Control, Proposition 5, $\mathbb{T}_\kappa(\lambda)$ is infinite and contains no torsion elements. Then λ inverts $\mathbb{T}_\kappa(\lambda)$ pointwise, and normalises $C_{\hat{G}}(\mathbb{T}_\kappa(\lambda))$; the latter contains κ . By the structure of the Sylow 2-subgroup of \hat{G} and normalisation principles, λ has a $C_{\hat{G}}(\mathbb{T}_\kappa(\lambda))$ -conjugate μ commuting with κ . Now μ inverts $\mathbb{T}_\kappa(\lambda)$ and normalises B_κ . Since $N_{\hat{G}}(B_\kappa)$ already contains a Sylow 2-subgroup of \hat{G} which is a 2-torus, μ is toral in $N_{\hat{G}}(B_\kappa)$ by torality principles. Hence $\mathbb{T}_\kappa(\lambda) \subseteq \{B, \mu\} \subseteq F^\circ(B)$. We now take any $t \in \mathbb{T}_\kappa(\lambda) \setminus \{1\}$ and $X = d(t)$, and we climb the Devil's Ladder, Proposition 4: B_κ is the only Borel subgroup of G containing $C_G^\circ(X)$. In particular, λ normalises B_κ , a contradiction.

For the rest of the argument we assume in addition that $\hat{G} = G \cdot d(\hat{S}^\circ)$ for some maximal 2-torus $\hat{S}^\circ \leq \hat{G}$; in particular \hat{G} is W_2^\perp by Lemma K and Step 2.

Let us introduce the following definable maps:

$$\begin{aligned} \pi_{\kappa, \lambda} : \quad \kappa^{\hat{G}} \setminus N_{\hat{G}}(B_\lambda) &\rightarrow \hat{G}/C_{\hat{G}}^\circ(\lambda) \\ \kappa_1 &\mapsto \kappa_1 C_{\hat{G}}^\circ(\lambda) \end{aligned}$$

We shall compute fibers.

Suppose that $\pi_{\kappa, \lambda}(\kappa_1) = \pi_{\kappa, \lambda}(\kappa_2)$. Then using the assumption that $\hat{G} = G \cdot d(\hat{S}^\circ)$, G controls \hat{G} -conjugacy of involutions. Hence $\kappa_1 \kappa_2 \in C_{\hat{G}}^\circ(\lambda) \cap G \leq C_G(\lambda)$. Be very careful that we do not a priori have connectedness of the latter, insofar as there is no ‘‘outer’’ version of Steinberg’s torsion theorem; as a matter of fact connectedness is immediate only when G is 2^\perp or $\lambda \in G$, not in general.

But if $c \in C_G(\lambda)$ is inverted by κ , then κ normalises $C_{\hat{G}}(c)$ which contains λ ; since \hat{G} is W_2^\perp and by normalisation principles, κ as a $C_{\hat{G}}(c)$ -conjugate μ commuting with λ . Now $\mu \in N_{\hat{G}}(C_G(\lambda))$ which contains a maximal 2-torus by torality principles; torality principles again provide some maximal 2-torus $T_\mu \leq N_{\hat{G}}(C_G(\lambda))$ containing μ . Then by Zilber’s indecomposibility theorem, $[c, T_\mu] \leq C_G^\circ(\lambda)$, that is, $c^2 \in C_G^\circ(\lambda)$. If G is 2^\perp the conclusion comes easily; if G contains involutions, then by torality principles $C_G^\circ(\lambda)$ contains a Sylow 2-subgroup of G which is connected by Step 2, so $c \in C_G^\circ(\lambda)$.

Turning back to our fiber computation, we do have $\kappa_1 \kappa_2 \in C_G^\circ(\lambda)$, that is, $\kappa_1 \kappa_2 \in T_\lambda(\kappa)$. The latter is finite as first proved. Hence $\pi_{\kappa, \lambda}$ has finite fibers, and it follows, keeping the Genericity Proposition 2 in mind:

$$\text{rk } \kappa^{\hat{G}} \leq \text{rk } \hat{G} - \text{rk } C_{\hat{G}}^\circ(\lambda)$$

that is, $\text{rk } C_{\hat{G}}^\circ(\lambda) \leq \text{rk } C_{\hat{G}}^\circ(\kappa)$, and vice-versa. So equality holds. By a degree argument, $\pi_{\kappa, \lambda}$ is now generically onto. \diamond

Step 8. *We may suppose that $\text{Pr}_2(\hat{G}) = 1$.*

Proof of Step. Suppose that $\text{Pr}_2(\hat{G}) \geq 2$; equality follows from Step 4 and we aim at finding case **CiBo**₃. There seem to be three cases depending on the values of $\text{Pr}_2(G)$ and $\text{Pr}_2(\hat{G}/G) = 2 - \text{Pr}_2(G)$. We however give a common argument.

Let $\hat{S}^\circ \leq \hat{G}$ be a maximal 2-torus of \hat{G} and $\check{G} = G \cdot d(\hat{S}^\circ)$. Bear in mind that \check{G} is W_2^\perp by Step 2 and Lemma K. In particular, \hat{S}° is a Sylow 2-subgroup of \hat{G} . Let κ, λ, μ be the three involutions in \hat{S}° .

If κ, λ and μ are not pairwise G -conjugate, then they are not \check{G} -conjugate either. So \check{G} has at least (hence exactly) three conjugacy classes of involutions by Lemma M: κ, λ and μ are pairwise not G -conjugate. We now apply Step 7 in \check{G} . The generic left-translate $\check{g}C_{\check{G}}^\circ(\lambda)$ contains both a conjugate κ_1 of κ and a conjugate μ_1 of μ . Now κ_1 and μ_1 are not \check{G} -conjugate so $d(\kappa_1 \mu_1)$ contains an involution ν . By the structure of the Sylow 2-subgroup of \check{G} , ν must be a conjugate λ_1 of λ . Now $\lambda_1 \in d(\kappa_1 \mu_1) \leq C_{\check{G}}^\circ(\lambda)$. By the structure of the Sylow 2-subgroup of \check{G} again, λ is the only

conjugate of λ in its centraliser. Hence $\lambda_1 = \lambda$. It follows that $\kappa_1, \mu_1 \in C_{\check{G}}(\lambda)$, and $\check{g} \in C_{\check{G}}(\lambda)$: a contradiction to genericity of $\check{g}C_{\check{G}}^\circ(\lambda)$ in $\check{G}/C_{\check{G}}^\circ(\lambda)$.

So involutions in \check{G} are G -conjugate. This certainly rules out the case where $\text{Pr}_2(G) = 1 = \text{Pr}_2(\check{G}/G)$. Actually this also eliminates the case where $\text{Pr}_2(G) = 0$ and $\text{Pr}_2(\check{G}/G) = 2$. For in that case, κ, λ, μ remain distinct in the quotient \check{G}/G : so G cannot conjugate them in \check{G} .

Hence $\text{Pr}_2(G) = 2$ and \check{G}/G is 2^\perp ; the latter means $\hat{S}^\circ \leq G$ so \hat{G}/G is 2^\perp as well. We have proved that G conjugates its involutions: with a look at Step 5 we recognize case **CiBo**₃. \diamond

We shall now finish the proof. If G has involutions then by Steps 2 and 8, $m_2(G) = \text{Pr}_2(G) = 1$ and $\text{Pr}_2(\hat{G}/G) = 0 = m_2(\hat{G}/G)$: with a look at Step 5 this is case **CiBo**₁. So we may suppose that G is 2^\perp . Since $\text{Pr}_2(\hat{G}) = 1$, by Proposition 7 $m_2(\hat{G}) = 1$. This is case **CiBo**₀: only self-normalisation of B_ι in G remains to be proved (although it was covered in Proposition 7, between Steps 5 and 6). Since $N = N_G(B_\iota) \leq G$ is 2^\perp it admits a decomposition $N = N^{+\iota} \cdot N^{-\iota}$ under the action of ι . But on the one hand so does G : hence $G = C_G(\iota) \cdot G^{-\iota}$ with trivial fibers and by a degree argument $C_G(\iota)$ is connected, so $N^+ \leq B_\iota$. On the other hand, by torality principles there exists a 2-torus \hat{S}° of \hat{G} containing ι ; \hat{S}° normalises B_ι and $N_G(B_\iota)$. By connectedness, \hat{S}° centralises the finite group $N_G(B_\iota)/B_\iota$, and so does ι . So $N^- \subseteq B_\iota$ and therefore $N = B_\iota$.

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