DENSE CODENSE PREDICATES AND NTP_2

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Abstract. We show that if T is any geometric theory having NTP_2 then the corresponding theories of lovely pairs of models of T and of H-structures associated to T also have NTP_2. We also prove that if T is strong then the same two expansions of T are also strong.

1. Introduction

The family of NTP_2 theories has attracted a lot of attention recently. On the one hand it provides a setting that includes both simple and dependent theories. On the other hand the family is small enough to have some very nice model-theoretic properties, such as the equivalence of forking and dividing for types over models [8]. There are several recent results on natural theories that are NTP_2 but are not simple nor dependent, they include ultraproducts of p-adics [6] and the theory of the non-standard Frobenius automorphism acting on an algebraically closed valued field of equicharacteristic 0 [7].

In this paper we study some expansions of geometric NTP_2 theories and prove that NTP_2 also holds in those expansions. In technical terms, we prove that expansions of any geometric NTP_2 theory with a dense/co-dense predicate are also NTP_2, provided that the predicate defines either an algebraically independent subset or an elementary substructure.

There are several papers on the preservation of stability and NIP under adding predicates. For example, there are the work of Casanovas and Ziegler on stability [5] and a similar result by Chernikov and Simon on NIP theories [9], [10]. A key fact in both papers is the following result: inside a highly saturated model of T, the family of stable (or NIP) formulas is closed under boolean combination and it suffices to check (under some technical assumptions) that the induced structure on a predicate is stable (respectively NIP).

On the other hand, for NTP_2 and simple theories, there are no results along the lines of Casanovas-Ziegler, Chernikov-Simon and we require different tools for analyzing expansions of these theories. In the case of simple theories, one can study independence relations that characterize forking and simplicity. For NTP_2 theories, the main tool is that the burden is sub-multiplicative [6] and thus it suffices to look at formulas in a single variable. This is the path followed in [6] to show that any geometric NTP_2 theory with a random predicate is again NTP_2. In this paper we use a similar approach, we concentrate on formulas with a single variable and apply tools about indiscernible arrays and burden to prove our results.

Date: May 2014.

2000 *Mathematics Subject Classification.* 03C45, 03C64.

Key words and phrases. NTP_2, geometric structures, strong theories.

Strong theories and burden were first defined by Adler in [1] and they provide notions analogous to those like superstability and U-rank but in the setting of NTP$_2$ theories. Examples include henselian valued field of equicharacteristic 0 where the residue field and the valued group are both strong [6], and also the ultraproduct of p-adics. In this paper we also prove a result on the preservation of strongness: if T is strong then the expansion by a dense/co-dense predicate is again strong.

Our work is organized as follows. In section 2 we review the notions of H-structures and lovely pairs and improve some results about the definable sets of these expansion from [3, 4]. In section 3 we review some key results on NTP$_2$ theories, burden and indiscernible arrays. Finally in section 4 we prove our main results.

2. H-structures and lovely pairs

In this section, we review the notions of *H-structures* and *lovely pairs* of geometric theories. All the essential definitions and results in this section are due to Berenstein and Vassiliev ([3], [4]), but we also refine some of their results on definable sets (Propositions 2.10 and 2.11 below).

Recall that a theory is called *geometric* if (1) it eliminates the quantifier $\exists^\infty$ and (2) the algebraic closure satisfies the exchange property. By 'independence' we shall mean algebraic independence and use the symbol $\mathop{\smile\!\!\!\!\!\!|}$ to denote this independence relation. Whenever $M \models T$ and $\bar{a} \in M$, we write $\mathrm{acl}_T(\bar{a})$ for the algebraic closure of $\bar{a}$ inside M and $\mathrm{tp}_T(\bar{a})$ for the type of $\bar{a}$ inside M.

Definition 2.1. Given a geometric complete theory T in a language $\mathcal{L}$ and a model $M \vDash T$, add a new unary predicate symbol H to form an extended language $\mathcal{L}_H := \mathcal{L} \cup \{H\}$. Let $(M, H(M))$ denote an expansion of M to $\mathcal{L}_H$, where $H(M) := \{x \in M \mid H(x)\}$.

1. (1) $(M, H(M))$ is called a *dense/co-dense* expansion if, for any non-algebraic $\mathcal{L}$-type $p(x) \in S_1(A)$ where $A \subseteq M$ has a finite dimension, $p(x)$ has realizations both in $H(M)$ and in $M \setminus \mathrm{acl}_T(A \cup H(M))$.
2. (2) A dense/co-dense expansion $(M, H(M))$ is called a *lovely pair* if $H(M)$ is an elementary substructure of M.
3. (3) A dense/co-dense expansion $(M, H(M))$ is called an *H-structure* if $H(M)$ is an $\mathcal{L}$-algebraically independent subset of M.

Theorem 2.2 ([3], [4])**.** *Given any geometric complete theory T, all the lovely pairs (resp. H-structures) associated with T are elementarily equivalent to one another.*

Notation 2.3. *T_P and T^{ind} denote the common complete theories of the lovely pairs and the H-structures, respectively, associated with T. By T^* we shall mean either T_P or T^{ind}.*

Remark 2.4. *Not every model of T^* may be an H-structure (resp. lovely pair), but all the sufficiently saturated ones are. (See* [4, Examples 2.11, 2.12]*.)*

Notation/convention

1. (1) Throughout the rest of the paper, we shall fix a geometric complete theory T and work inside some fixed, $\bar{\kappa}$-saturated model $(M, H(M)) \vDash T^*$ for some sufficiently large cardinal $\bar{\kappa}$. When we talk about subsets of M, we shall mean subsets of cardinality $< \bar{\kappa}$ unless stated otherwise.

(2) For a subset $A \subseteq M$, $H(A) := \{x \in A \mid H(x)\}$.
(3) We shall use H-subscripts to distinguish those operations in $\mathcal{L}_H$ from those in $\mathcal{L}$. (e.g., $\mathrm{acl}_H(\bar{a})$, $\mathrm{tp}_H(\bar{a})$.)

Definition 2.5. A subset $A \subseteq M$ is called *H-independent* if $A \mathop{\smile\!\!\!\!\!\!|}_{H(A)} H(M)$.

The following proposition is easy to verify.

Proposition 2.6. (1) *For any tuple $\bar{a}$, there exists some finite tuple $\bar{h}$ in $H(M)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\bar{h}} H(M)$. And, for such $\bar{h}$, $\bar{a}\bar{h}$ is H-independent.*
(2) *If $\bar{a}$ is any H-independent tuple, then for any finite tuple $\bar{h}$ in $H(M)$, $\bar{a}\bar{h}$ is also H-independent.*

Lemma 2.7 ([3], [4])**.** *For any H-independent tuples $\bar{a}$ and $\bar{b}$,*

$$\mathrm{tp}_H(\bar{a}) = \mathrm{tp}_H(\bar{b}) \;\Leftrightarrow\; \mathrm{tp}_T(\bar{a}H(\bar{a})) = \mathrm{tp}_T(\bar{b}H(\bar{b}))$$

Definition 2.8. For any subset $A \subseteq M$,

$$\mathrm{scl}(A) := \mathrm{acl}_T(A \cup H(M))$$

is called the *small closure* of A. Any subset $B \subseteq \mathrm{scl}(A)$ is called *A-small.*

Remark 2.9. *It is easy to check that $M \setminus \mathrm{scl}(A)$ is type-definable (in $\mathcal{L}_H$) over A.*

In the remainder of the section, we prove Propositions 2.10 and 2.11 which are refined versions of the original theorems by Berenstein and Vassiliev [4], [3]. (The original versions do not make explicit references to parameters.) We need these stronger versions to prove our main results later, so we include their full proofs.

Proposition 2.10. *If $\varphi(\bar{x}, \bar{a})$ is any $\mathcal{L}_H$-formula where $\bar{a}$ is H-independent, then there exists some $\mathcal{L}$-formula $\psi(\bar{x}, \bar{a})$ such that*

$$\vDash\; \varphi(\bar{x}, \bar{a}) \wedge H(\bar{x}) \;\leftrightarrow\; \psi(\bar{x}, \bar{a}) \wedge H(\bar{x})$$

Proof. Let $X \subseteq M^n$ be the set defined by $\varphi(\bar{x}, \bar{a})$. We may assume that $H(M)^n \cap X$ and $H(M)^n \setminus X$ are both nonempty.

Claim. For any $\bar{h}_1 \in H(M)^n \cap X$ and any $\bar{h}_2 \in H(M)^n \setminus X$, there exists some $\mathcal{L}$-formula $\theta_{\bar{h}_1\bar{h}_2}(\bar{x}, \bar{a})$ such that $\bar{h}_1 \vDash \theta(\bar{x}, \bar{a})$ and $\bar{h}_2 \vDash \neg\theta(\bar{x}, \bar{a})$.

Proof of Claim. Let $\bar{h}_1 \in H(M)^n \cap X$ and $\bar{h}_2 \in H(M)^n \setminus X$. Then $\mathrm{tp}_H(\bar{h}_1\bar{a}) \neq \mathrm{tp}_H(\bar{h}_2\bar{a})$ since $\varphi(\bar{x}, \bar{y}) \in \mathrm{tp}_H(\bar{h}_1\bar{a}) \setminus \mathrm{tp}_H(\bar{h}_2\bar{a})$. Moreover, $\bar{h}_1\bar{a}$ and $\bar{h}_2\bar{a}$ are both H-independent by Proposition 2.6(2). Hence $\mathrm{tp}_T(\bar{h}_1\bar{a}) \neq \mathrm{tp}_T(\bar{h}_2\bar{a})$ by Lemma 2.7. This completes the proof of Claim.

For each $\bar{h}_2 \in H(M)^n \setminus X$, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma_{\bar{h}_2}(\bar{x}) := \{H(\bar{x}) \wedge \varphi(\bar{x}, \bar{a})\} \cup \{\neg\theta_{\bar{h}_1\bar{h}_2}(\bar{x}, \bar{a}) \mid \bar{h}_1 \in H(M)^n \cap X\}$$

which is clearly inconsistent. Hence, since M is saturated, there exist finitely many tuples $\bar{h}_1^1, \cdots, \bar{h}_1^k$ in $H(M)^n \cap X$ such that the $\mathcal{L}$-formula

$$\psi_{\bar{h}_2}(\bar{x}, \bar{a}) := \bigvee_{i=1}^{k} \theta_{\bar{h}_1^i\bar{h}_2}(\bar{x}, \bar{a})$$

is satisfied by every tuple in $H(M)^n \cap X$. Note $\bar{h}_2 \nvDash \psi_{\bar{h}_2}(\bar{x}, \bar{a})$.

Next, consider the following $\mathcal{L}_H$-type $\Sigma(\bar{x})$ over $\bar{a}$:

$$\Sigma(\bar{x}) := \{H(\bar{x}) \wedge \neg\varphi(\bar{x}, \bar{a})\} \cup \{\psi_{\bar{h}_2}(\bar{x}, \bar{a}) \mid \bar{h}_2 \in H(M)^n \setminus X\}$$

which is clearly inconsistent. Hence, since the structure M is saturated, there exist finitely many tuples $\bar{h}_2^1, \cdots, \bar{h}_2^m$ in $H(M)^n \setminus X$ such that the $\mathcal{L}$-formula

$$\psi(\bar{x}, \bar{a}) := \bigwedge_{i=1}^{m} \psi_{\bar{h}_2^i}(\bar{x}, \bar{a})$$

is *not* satisfied by any tuple in $H(M)^n \setminus X$. But $\psi(\bar{x}, \bar{a})$ is satisfied by every tuple in $H(M)^n \cap X$, and hence $\psi(\bar{x}, \bar{a})$ is a desired $\mathcal{L}$-formula. □

Proposition 2.11. *If $\varphi(x, \bar{a})$ is any $\mathcal{L}_H$-formula where x is a single variable and $\bar{a}$ is an H-independent tuple, then there exists some $\mathcal{L}$-formula $\psi(x, \bar{a})$ such that the symmetric difference $\varphi(x, \bar{a}) \triangle \psi(x, \bar{a})$ defines an $\bar{a}$-small set.*

Proof. Let $X \subseteq M$ be the set defined by $\varphi(x, \bar{a})$. And let

$$Y_1 := \{x \in X \mid x \notin \mathrm{scl}(\bar{a})\} \quad \text{and} \quad Y_2 := \{x \in M \setminus X \mid x \notin \mathrm{scl}(\bar{a})\}$$

We may assume that Y_1 and Y_2 are both nonempty.

Claim. For any $c_1 \in Y_1$ and any $c_2 \in Y_2$, there exists some $\mathcal{L}$-formula $\theta_{c_1 c_2}(x, \bar{a})$ such that $c_1 \models \theta_{c_1 c_2}(x, \bar{a})$ and $c_2 \models \neg\theta_{c_1 c_2}(x, \bar{a})$.

Proof of Claim. Let $c_1 \in Y_1$ and $c_2 \in Y_2$. Then $\mathrm{tp}_H(c_1\bar{a}) \neq \mathrm{tp}_H(c_2\bar{a})$ since $\varphi(x, \bar{y}) \in \mathrm{tp}_H(c_1\bar{a}) \setminus \mathrm{tp}_H(c_2\bar{a})$. Moreover, note that $c_1\bar{a}$ and $c_2\bar{a}$ are both H-independent. Hence, $\mathrm{tp}_T(c_1\bar{a}) \neq \mathrm{tp}_T(c_2\bar{a})$ by Lemma 2.7. This completes the proof of Claim.

As was observed in Remark 2.9, both Y_1 and Y_2 are type-definable (in $\mathcal{L}_H$) over $\bar{a}$. So let $\Sigma_1(x)$ and $\Sigma_2(x)$ be $\mathcal{L}_H$-types over $\bar{a}$ defining Y_1 and Y_2, respectively.

For each $c_2 \in Y_2$, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma_1(x) \cup \{\neg\theta_{c_1 c_2}(x, \bar{a}) \mid c_1 \in Y_1\}$$

which is clearly inconsistent. Since M is saturated, there exist some finitely many $c_1^1, \cdots, c_1^k \in Y_1$ such that the $\mathcal{L}$-formula

$$\psi_{c_2}(x, \bar{a}) := \bigvee_{i=1}^{k} \theta_{c_1^i c_2}(x, \bar{a})$$

is satisfied by every element in Y_1. Note $c_2 \not\models \psi_{c_2}(x, \bar{a})$.

Next, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma_2(x) \cup \{\psi_{c_2}(x, \bar{a}) \mid c_2 \in Y_2\}$$

which is clearly inconsistent. Again, using that M is saturated, there exist some finitely many $c_2^1, \cdots, c_2^m \in Y_2$ such that the $\mathcal{L}$-formula

$$\psi(x, \bar{a}) := \bigwedge_{i=1}^{m} \psi_{c_2^i}(x, \bar{a})$$

is *not* satisfied by any element of Y_2. But $\psi(x, \bar{a})$ is satisfied by every element of Y_1, and hence $\varphi(x, \bar{a}) \triangle \psi(x, \bar{a})$ defines an $\bar{a}$-small set. □

3. TP$_2$, BURDEN AND INDISCERNIBLE ARRAYS

In this section, we review some key results on NTP$_2$ theories and burden. All the definitions and results in this section are in a general setting, and we do not assume that T is geometric.

Definition 3.1. (1) A theory T has *k-TP*$_2$ (for some integer $k \geq 2$) if there exist a formula $\varphi(\bar{x}, \bar{y})$ and a set of tuples $\{\bar{a}_{i,j} \mid i, j < \omega\}$ (in some model of T) such that $\{\varphi(\bar{x}, \bar{a}_{i,f(i)}) \mid i < \omega\}$ is consistent for every function $f\colon \omega \to \omega$ and $\{\varphi(\bar{x}, \bar{a}_{i,j}) \mid j < \omega\}$ is k-inconsistent for every $i < \omega$.
(2) TP$_2$ means 2-TP$_2$.
(3) A theory has NTP$_2$ if it does not have TP$_2$.

Remark 3.2. *Notice that, if a formula $\varphi(\bar{x}, \bar{y})$ witnesses k-TP$_2$ with some array $\{\bar{a}_{i,j} \mid i, j < \omega\}$ then $\bigwedge_{i<\omega} \varphi(\bar{x}, \bar{a}_{i,f(i)})$ must have infinitely many realizations for every function $f\colon \omega \to \omega$.*

The following definitions of *indiscernible array* and *array-basedness* are due to [2].

Definition 3.3. Consider the Cartesian product $\omega \times \omega$ as a model in the language $\mathcal{L}_{\mathrm{ar}} := \{<_1, <_2\}$ where $<_1$ and $<_2$ are binary relation symbols interpreted in $\omega \times \omega$ as follows:

$$(a, b) <_1 (c, d) \iff a < c$$
$$(a, b) <_2 (c, d) \iff (a = c) \wedge (b < d)$$

Let M be any model in some language $\mathcal{L}$.

(1) A set of parameters $\{\bar{a}_\mu \mid \mu \in \omega \times \omega\}$ in M is called an *indiscernible array* if the $\mathcal{L}$-type of any finite tuple $(\bar{a}_{\mu_1}, \cdots, \bar{a}_{\mu_n})$ is determined by the quantifier-free $\mathcal{L}_{\mathrm{ar}}$-type of the tuple $(\mu_1, \cdots, \mu_n)$.
(2) Let $\mathcal{A} := \{\bar{a}_\mu \mid \mu \in \omega \times \omega\}$ and $\mathcal{B} := \{\bar{b}_\mu \mid \mu \in \omega \times \omega\}$ be any sets of parameters in M. $\mathcal{A}$ is *array-based* (or simply *based*) on $\mathcal{B}$ if for any $\mathcal{L}$-formula $\varphi(\bar{x}_1, \cdots, \bar{x}_n)$ and any tuple $(\mu_1, \cdots, \mu_n)$ in $\omega \times \omega$, there exists some tuple $(\nu_1, \cdots, \nu_n)$ in $\omega \times \omega$ such that
   (a) $\mathrm{qftp}_{\mathcal{L}_{\mathrm{ar}}}(\mu_1, \cdots, \mu_n) = \mathrm{qftp}_{\mathcal{L}_{\mathrm{ar}}}(\nu_1, \cdots, \nu_n)$
   (b) $M \vDash \varphi(\bar{a}_{\mu_1}, \cdots, \bar{a}_{\mu_n}) \leftrightarrow \varphi(\bar{b}_{\nu_1}, \cdots, \bar{b}_{\nu_n})$

Theorem 3.4 (Array-modeling [2])**.** *Given any set of tuples $\mathcal{A} := \{\bar{a}_\eta \mid \eta \in \omega \times \omega\}$ (in some sufficiently saturated model), there exists an indiscernible array $\{\bar{b}_\eta \mid \eta \in \omega \times \omega\}$ based on $\mathcal{A}$.*

Corollary 3.5. *If a formula $\varphi(\bar{x}, \bar{y})$ witnesses k-TP$_2$ then it may do so with an indiscernible array.*

Corollary 3.6. *If a formula $\varphi(\bar{x}, \bar{y})$ witnesses k-TP$_2$ then some finite conjunction $\psi(\bar{x}, \bar{y}_1, \cdots, \bar{y}_n) := \bigwedge_{i=1}^{n} \varphi(\bar{x}, \bar{y}_i)$ witnesses TP$_2$. Hence, if a theory T does not have TP$_2$ then it does not have k-TP$_2$ for any $k \geq 2$.*

Proof. An easy application of the array-modeling theorem. See [2] or [**?**] for details. □

Proposition 3.7. *A theory T has TP$_2$ if there exist a formula $\varphi(x, \bar{y})$ and an indiscernible array $\{\bar{a}_{i,j} \mid i, j < \omega\}$ such that*

(1) $\bigwedge_{i<\omega} \varphi(x, \bar{a}_{i,0})$ *has infinitely many realizations,*
(2) $\bigwedge_{j<\omega} \varphi(x, \bar{a}_{0,j})$ *has at most finitely many realizations.*

Proof. Suppose that there exist such a formula $\varphi(x, \bar{y})$ and an indiscernible array $\mathcal{A} := \{\bar{a}_{i,j} \mid i, j < \omega\}$ in some sufficiently saturated model $M \models T$. Then clearly there exist some integer $N \geq 1$ and some algebraic formula $\theta(x, \bar{b}_0)$ such that, for any indices $i_1 < \cdots < i_N$,

$$M \models \left(\bigwedge_{k=1}^{N} \varphi(x, \bar{a}_{0,i_k}) \right) \to \theta(x, \bar{b}_0)$$

Then it's easy to check that the formula $\psi(x, \bar{y}\bar{z}) := \varphi(x, \bar{y}) \wedge \neg\theta(x, \bar{z})$ witnesses N-TP$_2$. Hence T has TP$_2$ by Corollary 3.6. □

Theorem 3.8 (Chernikov [6])**.** *If a theory has TP$_2$, then there exists some formula $\varphi(x, \bar{y})$ (where x is a single variable) witnessing TP$_2$.*

Next, we recall the following definition from [?].

Definition 3.9. Given a (partial) type $p(\bar{x})$, an *inp-pattern in* $p(\bar{x})$ consists of a set of formulas $\{\varphi_i(\bar{x}, \bar{y}_i) \mid i < \kappa\}$ (for some cardinal κ) and a set of tuples $\{\bar{a}_{i,j} \mid i < \kappa, \, j < \omega\}$ satisfying the following properties:

(1) For each $i < \kappa$, $\{\varphi_i(\bar{x}, \bar{a}_{i,j}) \mid j < \omega\}$ is k_i-inconsistent for some integer $k_i \geq 2$.
(2) For every function $f : \kappa \to \omega$, $p(\bar{x}) \cup \{\varphi_i(\bar{x}, \bar{a}_{i,f(i)}) \mid i < \kappa\}$ is consistent.

The cardinal κ is called the *depth* of the inp-pattern. The *burden* of a type $p(\bar{x})$, denoted by $\mathrm{bdn}(p(\bar{x}))$, is the supremum of the depths of all inp-patterns in $p(\bar{x})$. $\mathrm{bdn}(\bar{a}/B)$ denotes $\mathrm{bdn}(\mathrm{tp}(\bar{a}/B))$.

Proposition 3.10.

(1) $\mathrm{bdn}(p(\bar{x})) = 0 \;\Leftrightarrow\; p(\bar{x})$ *is an algebraic type.*
(2) $\mathrm{bdn}(\bar{a}/\bar{b}) \leq \mathrm{bdn}(\bar{a}) \leq \mathrm{bdn}(\bar{a}\bar{b})$.
(3) *If* $\models p(\bar{x}) \to q(\bar{x})$ *then* $\mathrm{bdn}(p(\bar{x})) \leq \mathrm{bdn}(q(\bar{x}))$.

Proof. Immediate from the definition of burden. □

Proposition 3.11 ([?], [6])**.** *If there exists an inp-pattern $\{\varphi_i(\bar{x}, \bar{a}_{i,j}) \mid i < \kappa, \, j < \omega\}$ in a type $p(\bar{x})$, then we may assume that the array $\{\bar{a}_{i,j} \mid i < \kappa, \, j < \omega\}$ is* mutually indiscernible*, that is, for each $i_0 < \kappa$, the sequence $\{\bar{a}_{i_0,j} \mid j < \omega\}$ is indiscernible over the parameters $\{\bar{a}_{i,j} \mid i \neq i_0, \, j < \omega\}$.*

Remark 3.12 (Indiscernible array vs. mutually indiscernible array)**.** *Any indiscernible array is necessarily a mutually indiscernible array, but the converse is not true in general. (Also note that, all the tuples in an indiscernible array have the same arity by definition, but it is not necessarily the case for mutually indiscernible array.)*

Theorem 3.13 (Sub-multiplicativity of burden, [6])**.** *If there is an inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\mathrm{tp}(\bar{a}\bar{b})$ then either there is an inp-pattern of depth κ_1 in $\mathrm{tp}(\bar{a})$ or there is an inp-pattern of depth κ_2 in $\mathrm{tp}(\bar{b}/\bar{a})$. In particular:*

(1) *For any finitely many tuples $\bar{a}_1, \cdots, \bar{a}_n$ and any infinite cardinal κ, if $\mathrm{bdn}(\bar{a}_i) < \kappa$ for all i then $\mathrm{bdn}(\bar{a}_1 \cdots \bar{a}_n) < \kappa$.*

(2) *For any finitely many tuples $\bar{a}_1, \cdots, \bar{a}_n$, if all the inp-patterns in* $\mathrm{tp}(\bar{a}_i)$ *have finite depths, for all i, then all the inp-patterns in* $\mathrm{tp}(\bar{a}_1 \cdots \bar{a}_n)$ *have finite depths.*

Proposition 3.14 ([6])**.** *Let T be a complete theory. Then the following are equivalent:*

(1) *There does not exist any formula $\varphi(x, \bar{y})$ witnessing TP$_2$.*
(2) *$bdn(b/C) < |T|^+$ for all $b \in M$ and all $C \subset M$ in some saturated model M of T.*

4. Main Results

Throughout this section, we work inside some fixed, sufficiently saturated model $(M, H(M)) \models T^*$

Proposition 4.1. *Let $\varphi(\bar{x}, \bar{y})$ be any $\mathcal{L}_H$-formula witnessing k-TP$_2$ for some $k \geq 2$. Then for some dummy variables $\bar{z}$, the formula $\varphi(\bar{x}, \bar{y}\bar{z})$ witnesses k-TP$_2$ with some indiscernible array $\{\bar{c}_{i,j} \mid i, j < \omega\}$ where each $\bar{c}_{i,j}$ is H-independent.*

Proof. By Corollary 3.5, $\varphi(\bar{x}, \bar{y})$ witnesses k-TP$_2$ with some indiscernible array $\{\bar{a}_{i,j} \mid i, j < \omega\}$. (In particular, all $\bar{a}_{i,j}$ have the same $\mathcal{L}_H$-type.) Moreover, by Proposition 2.6(1), there exists some finite tuple $\bar{h}_{0,0}$ in $H(M)$ such that $\bar{a}_{0,0}\bar{h}_{0,0}$ is H-independent. For each $(i, j) \neq (0, 0)$, define $\bar{h}_{i,j}$ to be the image of $\bar{h}_{0,0}$ under some $\mathcal{L}_H$-automorphism sending $\bar{a}_{0,0} \mapsto \bar{a}_{i,j}$. So, in particular, each $\bar{a}_{i,j}\bar{h}_{i,j}$ is H-independent. It's also clear that, for any choice of dummy variables $\bar{z}$ having the same arity as $\bar{h}_{i,j}$, the formula $\varphi(\bar{x}, \bar{y}\bar{z})$ still witnesses k-TP$_2$ with the array $\{\bar{b}_{i,j} := \bar{a}_{i,j}\bar{h}_{i,j} \mid i, j < \omega\}$. Now, by Theorem 3.4, there exists some indiscernible array $\{\bar{c}_{i,j} \mid i, j < \omega\}$ based on $\{\bar{b}_{i,j} \mid i, j < \omega\}$. Then it is straightforward to check that each $\bar{c}_{i,j}$ is H-independent and $\varphi(\bar{x}, \bar{y}\bar{z})$ witnesses k-TP$_2$ with $\{\bar{c}_{i,j} \mid i, j < \omega\}$. □

Proposition 4.2. *T has TP$_2$ if there exists some $\mathcal{L}_H$-formula $\varphi(x, \bar{y})$ (where x is a single variable) such that $\varphi(x, \bar{y}) \wedge H(x)$ witnesses k-TP$_2$ for some $k \geq 2$.*

Proof. Assume that there exists such an $\mathcal{L}_H$-formula $\varphi(x, \bar{y})$. We may assume that $\varphi(x, \bar{y}) \wedge H(x)$ witnesses TP$_2$ with some indiscernible array $\mathcal{A} := \{\bar{a}_{i,j} \mid i, j < \omega\}$ where each $\bar{a}_{i,j}$ is H-independent (by Proposition 4.1 and Corollary 3.6). Then, by Proposition 2.10, there exists some $\mathcal{L}$-formula $\psi(x, \bar{y})$ such that for all $i, j < \omega$,

$$\models \varphi(x, \bar{a}_{i,j}) \wedge H(x) \leftrightarrow \psi(x, \bar{a}_{i,j}) \wedge H(x).$$

Since $\bigwedge_{i<\omega} \varphi(x, \bar{a}_{i,0}) \wedge H(x)$ has infinitely many realizations (by Remark 3.2), $\bigwedge_{i<\omega} \psi(x, \bar{a}_{i,0})$ also has infinitely many realizations. Moreover, $\bigwedge_{j<\omega} \psi(x, \bar{a}_{0,j})$ has at most finitely many realizations. (Otherwise, the density condition for H (Definition 2.1(1)) implies that $\varphi(x, \bar{a}_{0,0}) \wedge \varphi(x, \bar{a}_{0,1}) \wedge H(x)$ is consistent, contradiction.) Hence, T has TP$_2$ by Proposition 3.7. □

Proposition 4.3. *If there exists $h \in H(M)$ such that* $\mathrm{bdn}_H(h) \geq |T|^+$ *then T has TP$_2$.*

Proof. This is a local version of Lemma 3.14. The existence of such h implies that there exists some $\mathcal{L}_H$-formula $\varphi(x, \bar{y})$ such that $\varphi(x, \bar{y}) \wedge H(x)$ witnesses k-TP$_2$ for some $k \geq 2$. This follows easily from the pigeon hole principle (i.e., the regularity of the cardinal $|T|^+$). Hence, T has TP$_2$ by Proposition 4.2. □

Corollary 4.4. *If there exists some $\mathcal{L}_H$-formula $\varphi(\bar{x},\bar{y})$ such that $\varphi(\bar{x},\bar{y}) \wedge H(\bar{x})$ witnesses k-TP$_2$ for some $k \geq 2$ then T has TP$_2$.*

Proof. The existence of such an $\mathcal{L}_H$-formula $\varphi(\bar{x},\bar{y})$ implies that $\mathrm{bdn}_H(\bar{h}) \geq |T|^+$ for some tuple $\bar{h}$ in $H(M)$. This follows easily from compactness and the fact that we can always choose an indiscernible array (Corollary 3.5). Then, by the sub-multiplicativity of burden (Theorem 3.13), there exists some $h \in H(M)$ such that $\mathrm{bdn}_H(h) \geq |T|^+$. Hence T has TP$_2$ by Proposition 4.3 □

Theorem 4.5 (Main Theorem)**.** *If T^* has TP$_2$ then so does T.*

Proof. Assume T^* has TP$_2$. So there exists some $\mathcal{L}_H$-formula $\varphi(x,\bar{y})$ (where x is a single variable due to Theorem 3.8) witnessing TP$_2$ with some indiscernible array $\mathcal{A} := \{\bar{a}_{i,j} \mid i,j < \omega\}$ where each $\bar{a}_{i,j}$ is H-independent (by Proposition 4.1). There are two possible cases:

Case 1. $\bigwedge_{i<\omega} \varphi(x,\bar{a}_{i,0})$ is realized by some $b \in \mathrm{scl}(\mathcal{A})$.

Such b is realized by some algebraic $\mathcal{L}$-formula $\theta(x,\bar{c},\bar{h})$ where $\bar{c}$ and $\bar{h}$ are some tuples in $\mathcal{A}$ and $H(M)$, respectively. Clearly we may assume that, for any parameters $\bar{c}'$ and $\bar{h}'$, the formula $\theta(x,\bar{c}',\bar{h}')$ always has at most k realizations for some fixed integer k. Choose any $N < \omega$ such that $\bar{c}$ appears in the sub-array $\{\bar{a}_{i,j} \mid i < N, j < \omega\}$ and let $\bar{d}_{i,j} := \bar{a}_{N+i,j}$ for each $i,j < \omega$. Then $\varphi(x,\bar{y})$ still witnesses TP$_2$ with the array $\{\bar{d}_{i,j} \mid i,j < \omega\}$ but this array is now *indiscernible over $\bar{c}$*. Then it's easy to check that the $\mathcal{L}_H$-formulas

$$\mu(\bar{z},\bar{c},\bar{d}_{i,j}) := H(\bar{z}) \wedge \exists x(\theta(x,\bar{c},\bar{z}) \wedge \varphi(x,\bar{d}_{i,j}))$$

for $i,j < \omega$, witness $(k+1)$-TP$_2$. Hence T has TP$_2$ by Corollary 4.4.

Case 2. All the realizations of $\bigwedge_{i<\omega} \varphi(x,\bar{a}_{i,0})$ are in $M \setminus \mathrm{scl}(\mathcal{A})$.

By Proposition 2.11, there exists some $\mathcal{L}$-formula $\psi(x,\bar{y})$ such that, for each $i,j < \omega$, $\varphi(x,\bar{a}_{i,j}) \triangle \psi(x,\bar{a}_{i,j})$ defines an $\bar{a}_{i,j}$-small set. Then every realization of $\bigwedge_{i<\omega} \varphi(x,\bar{a}_{i,0})$ must also be a realization of $\bigwedge_{i<\omega} \psi(x,\bar{a}_{i,0})$. In particular, $\bigwedge_{i<\omega} \psi(x,\bar{a}_{i,0})$ has infinitely many realizations.

Moreover, $\bigwedge_{j<\omega} \psi(x,\bar{a}_{0,j})$ has at most finitely many realizations. (Otherwise, the co-density condition for H (Definition 2.1(1)) implies that $\varphi(x,\bar{a}_{0,0}) \wedge \varphi(x,\bar{a}_{0,1})$ is consistent, contradiction.) Hence, T has TP$_2$ by Proposition 3.7. □

Now we study how burden behaves in T^*.

Theorem 4.6. *Let $p(x)$ be any partial $\mathcal{L}$-type in a single variable over any subset. Then for any inp-pattern of $\mathcal{L}_H$-formulas in $p(x) \wedge H(x)$, there exists an inp-pattern of $\mathcal{L}$-formulas in $p(x)$ having the same depth. In particular,* $\mathrm{bdn}_H(p(x) \wedge H(x)) \leq \mathrm{bdn}_T(p(x))$.

Proof. Let $\{\varphi_i(x,\bar{a}_{i,j}) \mid i < \kappa,\ j < \omega\}$ be any inp-pattern of $\mathcal{L}_H$-formulas in $p(x) \wedge H(x)$ (where κ is some cardinal). Clearly we may assume that $\models \varphi_i(x,\bar{a}_{i,j}) \to H(x)$ for all $i < \kappa$ and $j < \omega$. We may further assume that the array $\{\bar{a}_{i,j} \mid i < \kappa,\ j < \omega\}$ is mutually indiscernible (by Proposition 3.11) and that each $\bar{a}_{i,j}$ is H-independent. Then, by Proposition 2.10, there exist $\mathcal{L}$-formulas $\psi_i(x,\bar{y}_i)$ such that

$$\models \varphi_i(x,\bar{a}_{i,j}) \leftrightarrow \psi_i(x,\bar{a}_{i,j}) \wedge H(x)$$

for all $i < \kappa$ and $j < \omega$. Notice that, for each $i < \kappa$, $\bigwedge_{j<\omega} \psi_i(x, \bar{a}_{i,j})$ has at most finitely many realizations. (Otherwise, the density property of H and compactness imply that $\bigwedge_{j<\omega} \varphi_i(x, \bar{a}_{i,j})$ is consistent, contradiction.) Let $\bar{e}_0 = \{e_1, \cdots, e_\ell\}$ be the set of realizations of $\bigwedge_{j<\omega} \psi_0(x, \bar{a}_{0,j})$. Note $\bar{e}_0 \cap H(M) = \emptyset$. Moreover, by the indiscernibility of $\{\bar{a}_{0,j} \mid j < \omega\}$, there exists some $N_0 < \omega$ such that $\bigwedge_{j<\omega} \psi_0(x, \bar{a}_{0,j}) = \bigwedge_{j\in I} \psi_0(x, \bar{a}_{0,j})$ whenever $I \subset \omega$ has the size N_0. Let $\bar{z}_0 = (z_1, \cdots, z_\ell)$ be a tuple of new variables and consider the $\mathcal{L}$-formula

$$\psi'_0(x, \bar{y}_0, \bar{z}_0) := \psi_0(x, \bar{y}_0) \wedge (\wedge_{k\le\ell} x \neq z_k).$$

Then $\{\psi'_0(x, \bar{a}_{0,j}, \bar{e}_0) \mid j < \omega\}$ is N_0-inconsistent. Repeating the same process for each $i < \kappa$, we obtain an array of $\mathcal{L}$-formulas

$$\{\psi'_i(x, \bar{a}_{i,j}, \bar{e}_i) \mid i < \kappa, \, j < \omega\}$$

which is an inp-pattern of depth κ in $p(x)$. This completes the proof. □

Before stating the next result (Theorem 4.8), let us quickly review the notion of $\kappa_{inp}(T)$ defined by Shelah [11, Section III.7] and the notion of *strong theory* defined by Adler [**?**].

Given an arbitrary theory T (not necessarily geometric) and any $n < \omega$, $\kappa^n_{inp}(T)$ denotes the least cardinal τ such that there is no inp-pattern of depth τ in the type $\{\bar{x} = \bar{x}\}$ where $\bar{x}$ is a tuple of n variables. And $\kappa_{inp}(T) := \sup_{n<\omega} \kappa^n_{inp}(T)$. We list below some basic properties (all of which follow immediately from the definition).

(1) $n < \kappa^n_{inp}(T)$ (due to the equality symbol in every language).
(2) $n \le m \Rightarrow \kappa^n_{inp}(T) \le \kappa^m_{inp}(T)$.
(3) $\kappa_{inp}(T) \ge \aleph_0$.
(4) $\kappa_{inp}(T) = \aleph_0 \Leftrightarrow \kappa^n_{inp}(T) \le \aleph_0$ for all $n < \omega$.
(5) $\kappa^n_{inp}(T) \le \aleph_0 \Leftrightarrow$ Every inp-pattern in $\bar{x} = \bar{x}$ (where $|\bar{x}| = n$) has a finite depth.

In [**?**], Adler defines a theory T to be *strong* if every inp-pattern in $\bar{x} = \bar{x}$ has a finite depth, for every finite tuple of variables $\bar{x}$. i.e., T is strong iff $\kappa_{inp}(T) = \aleph_0$.

Theorem 4.7 (Chernikov [6])**.** *In any theory T, either $\kappa^n_{inp}(T) < \aleph_0$ for all $n < \omega$, or there exists some infinite cardinal τ such that $\kappa^n_{inp}(T) = \tau$ for all $n < \omega$. In particular, T is strong if and only if $\kappa^1_{inp}(T) \le \aleph_0$.*

Theorem 4.8. *For any geometric complete theory T, if T is strong then so is T^*.*

Proof. Assume that T is strong. Then, by Theorem 4.6 and the sub-multiplicativity of burden (Theorem 3.13), every inp-pattern in $\text{tp}_H(\bar{h})$ has a finite depth, for all finite tuples $\bar{h}$ in $H(M)$.

Claim. Let $p(x)$ be any partial $\mathcal{L}_H$-type in a single variable over any subset. Suppose that all the realizations of $p(x)$ belong to $\text{scl}(\bar{b})$ for some tuple $\bar{b}$. Then every inp-pattern of $\mathcal{L}_H$-formulas in $p(x)$ has a finite depth.

Proof of claim. Suppose not, i.e., there exists some inp-pattern $\{\varphi_i(x, \bar{a}_{i,j}) \mid i, j < \omega\}$ of depth $\aleph_0$ in $p(x)$. We may assume that the array $\{\bar{a}_{i,j} \mid i, j < \omega\}$ is mutually indiscernible over $\bar{b}$. Since M is saturated, we can find some $\mathcal{L}$-formula $\theta(x, \bar{y}, \bar{b})$ such that

(1) $\theta(x, \bar{c}, \bar{b})$ is algebraic for all parameters $\bar{c}$,

(2) $\models p(x) \to \exists \bar{y}(H(\bar{y}) \wedge \theta(x,\bar{y},\bar{b}))$.

In particular, there exists some tuple $\bar{h}$ in $H(M)$ such that

$$p(x) \cup \{\theta(x,\bar{h},\bar{b}) \wedge \varphi_i(x,\bar{a}_{i,0}) \mid i<\omega\}$$

is consistent. Then the $\mathcal{L}_H$-formulas

$$\mu_i(\bar{y},\bar{b},\bar{a}_{i,j}) := H(\bar{y}) \wedge \exists x(\theta(x,\bar{y},\bar{b}) \wedge \varphi_i(x,\bar{a}_{i,j}))$$

for $i,j<\omega$ form an inp-pattern of depth $\aleph_0$ in $\mathrm{tp}_H(\bar{h})$, contradicting that every inp-pattern in $\mathrm{tp}_H(\bar{h})$ has a finite depth. This completes the proof of Claim.

Now, to prove that T^* is strong, it suffices to show that every inp-pattern of $\mathcal{L}_H$-formulas in $x=x$ (where x is a single variable) has a finite depth. Suppose not, i.e., there exists some inp-pattern $\{\varphi_i(x,\bar{a}_{i,j}) \mid i,j<\omega\}$ of depth $\aleph_0$ in $x=x$. We may assume that the array $\{\bar{a}_{i,j} \mid i,j<\omega\}$ is mutually indiscernible and that each $\bar{a}_{i,j}$ is H-independent. Then, by Proposition 2.11, there exist $\mathcal{L}$-formulas $\psi_i(x,\bar{y}_i)$ such that

$$\varphi_i(x,\bar{a}_{i,j}) \triangle \psi_i(x,\bar{a}_{i,j})$$

defines an $\bar{a}_{i,j}$-small set, for each $i,j<\omega$. Note that, for each $n<\omega$ and any function $f\colon n\to\omega$, the formulas

$$\{\varphi_i(x,\bar{a}_{i,j}) \mid n\le i<\omega,\ j<\omega\}$$

form an inp-pattern of depth $\aleph_0$ in the type $\{\varphi_i(x,\bar{a}_{i,f(i)}) \mid i<n\}$. Hence, Claim above implies that, for each $n<\omega$ and any function $f\colon n\to\omega$, the formula

$$\bigwedge_{i<n} \psi_i(x,\bar{a}_{i,f(i)})$$

is realized by some infinitely many elements in $M \setminus \mathrm{scl}\{\bar{a}_{i,f(i)} \mid i<n\}$.

Furthermore, for each $i<\omega$, $\bigwedge_{j<\omega} \psi_i(x,\bar{a}_{i,j})$ has at most finitely many realizations. (Otherwise, the co-density property of H and compactness imply that $\{\varphi_i(x,\bar{a}_{i,j}) \mid j<\omega\}$ is consistent, contradiction.) Hence, we may repeat the same argument in the latter part of the proof for Theorem 4.6 to obtain an array of $\mathcal{L}$-formulas

$$\{\psi'_i(x,\bar{a}_{i,j},\bar{e}_i) \mid i<\omega,\ j<\omega\}$$

which forms an inp-pattern of depth $\aleph_0$, contradicting that T is strong. We conclude that every inp-pattern of $\mathcal{L}_H$-formulas in $x=x$ has a finite depth, i.e., T^* is strong. □

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