COMBINATORIAL GEOMETRIES OF THE FIELD EXTENSIONS

JAKUB GISMATULLIN

Abstract. We classify projective planes in algebraic combinatorial geometries in arbitrary fields of characteristic zero. We investigate the first-order theories of such geometries and pregeometries. Then we classify the algebraic combinatorial geometries of arbitrary field extensions of the transcendence degree ≥ 5 and describe their groups of automorphisms. Our results and proofs extend similar results and proofs by Evans and Hrushovski in the case of algebraically closed fields.

Introduction

Let $K \subset L$ be an arbitrary field extension. We investigate the algebraic combinatorial geometry $\mathbb{G}(L/K)$ and pregeometry $\mathrm{G}(L/K)$ in L obtained from algebraic dependence relation over K. Such a geometry is sometimes called a full algebraic matroid.

In [1] the authors classify projective planes in $\mathbb{G}(L/K)$ for algebraically closed K and L. Using their results, we give such a classification for arbitrary fields K and L of characteristic zero. We prove a theorem about formulas with one quantifier of the first-order theory of $\mathbb{G}(L/K)$. Assume that the transcendence degree of L over K is at least 5. When considering $\mathbb{G}(L/K)$ we may also assume that L is a perfect field and K is relatively algebraically closed in L. One of the main results of [2] is the reconstruction of the algebraically closed field L from $\mathbb{G}(L/K)$. We generalize this reconstruction to arbitrary field extension $K \subset L$ (of transcendence degree ≥ 5), and thus we obtain full classification of combinatorial geometries of fields: $\mathbb{G}(L_1/K_1)$ and $\mathbb{G}(L_2/K_2)$ are isomorphic if and only if field extensions

$$K_1 \subset L_1 \quad \text{and} \quad K_2 \subset L_2$$

are isomorphic (here we assume that L_1 and L_2 are perfect and K_1, K_2 are relatively algebraically closed). We also give a description of $\mathrm{Aut}(\mathbb{G}(L/K))$.

By $\widehat{F}$ and $\widehat{F}^r$ we denote algebraic and purely inseparable closure of F. Throughout this paper we assume that $K \subset L$ is an arbitrary field extension and the transcendence degree of L over K is at least 3. We take basic definitions of algebraic combinatorial geometry and pregeometry from [1, 2]. For $X \subseteq L$, let $\mathrm{acl}_K(X)$ be $\widehat{K(X)}$. We denote by $\mathrm{G}(L/K)$ *the pregeometry* (L, acl_K). *The geometry* $\mathbb{G}(L/K)$ is obtained from $L \setminus \widehat{K}$ by factoring out the equivalence relation:

$$x \sim y \iff \widehat{K(x)} = \widehat{K(y)}.$$

We can also transfer the closure operation acl_K from $\mathrm{G}(L/K)$ to $\mathbb{G}(L/K)$:

$$\mathrm{acl}_K(Y/\sim) = \mathrm{acl}_K(Y)/\sim.$$

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Therefore we can regard the points of $\mathbb{G}(L/K)$ as sets $\mathrm{acl}_K(x)$, where $x \in L \setminus \widehat{K}$, and acl_K as the usual algebraic closure. When considering $\mathbb{G}(L/K)$ we assume that L is a perfect field and K is relatively algebraically closed in L (because $\mathbb{G}(L/K) = \mathbb{G}(\widehat{L}^r/K) = \mathbb{G}(\widehat{L}^r/\widehat{L}^r \cap \widehat{K})$). Subsets of $\mathbb{G}(L/K)$ of the form $\mathrm{acl}_K(X)$, $X \subseteq \mathbb{G}(L/K)$, are called *closed*. *The rank* of a subset of $\mathbb{G}(L/K)$ or $\mathrm{G}(L/K)$ is its transcendence degree. We also have notions of *independent set* (for each $x \in X$, $x \notin \mathrm{acl}_K(X \setminus \{x\})$) and a *basis* of a closed subset as a maximal independent set (transcendence basis). Note that the closure operation acl_K satisfies *the exchange condition*:

$$x \in \mathrm{acl}_K(A \cup \{y\}) \setminus \mathrm{acl}_K(A) \quad \Longrightarrow \quad y \in \mathrm{acl}_K(A \cup \{x\}).$$

Closed subset of rank 1 (respectively 2, 3) is a *point* (respectively *line* and *plane*). If X is a closed subset of $\mathbb{G}(L/K)$, and a tuple $\overline{x} \subset L$ satisfies $X = \mathrm{acl}_K(\overline{x})$, then we say that $\overline{x}$ is *generic* in X.

Let F be a skew field (division ring). We will denote by $\mathbb{P}(F)$ the projective plane over F. It is simply the set $F^3 \setminus \{0\}$ factored out by the relation:

$$(x_1, x_2, x_3) \simeq (y_1, y_2, y_3) \quad \Longleftrightarrow \quad \big(\exists\ 0 \neq \lambda \in F\big)\ (x_1, x_2, x_3) = \lambda(y_1, y_2, y_3).$$

The paper is organized as follows. The first section is devoted to give some preliminary definitions and results from [1]. In the second section we classify the projective planes arising in $\mathbb{G}(L/K)$. Section 3 contains a theorem about first-order theory of $\mathrm{G}(L/K)$ and formulas with one quantifier. In Section 4 we transfer theorems from [2] to geometries of arbitrary field extensions and prove a general classification theorem for them.

The reader is referred to [6] for the model-theoretic background and notation, and to [8] for general background on pregeometries and matroids.

1. Preliminaries

For definitions and proofs in this section we refer the reader to [1]. Throughout this section we assume that K and L are algebraically closed. Let X be a subset of $\mathbb{G}(L/K)$ and let ${\mathrm{acl}_K}^X$ be the relative closure operation: ${\mathrm{acl}_K}^X(Y) = \mathrm{acl}_K(Y) \cap X$ for $Y \subseteq X$. We say that X is a *projective plane of* $\mathbb{G}(L/K)$ if the geometry $(X, {\mathrm{acl}_K}^X)$ is itself a projective plane, meaning that:

1) the geometry $(X, {\mathrm{acl}_K}^X)$ has rank 3;
2) there are three noncollinear points in X;
3) any line has at least three different points;
4) any two lines intersect.

If a projective plane X is isomorphic to $\mathbb{P}(F)$, for some skew field F, then we say that X is *coordinatised* by F. It is well known ([3, Chapter 7]) that if the Desargues theorem is true in X, then X is coordinatised by a unique skew field. The converse is also true. If X_1 and X_2 are Desarguesian projective planes coordinatised by F_1 and F_2 respectively, and $X_1 \subseteq X_2$, then F_1 is a subskewfield of F_2. It is proved in [5] that any projective plane in $\mathbb{G}(L/K)$ is Desarguesian. The aim of the next section is to find all skew fields coordinatising some projective planes in $\mathbb{G}(L/K)$ for arbitrary $K \subset L$ of characteristic zero. The paper [1] describes all such skew fields in the case when L and K are algebraically closed.

Let $(G, *)$ be a one-dimensional irreducible K-definable algebraic group in L. Then G is isomorphic over K ([1, Section 3.1]), as an algebraic group, to one of the following commutative groups: $(L, +)$, $(L^*, \cdot)$ or an elliptic curve. Since G is commutative, the group $\mathrm{End}_K(G) = \mathrm{Hom}_K(G, G)$ of definable over K morphisms of G (as an algebraic group) may be given a ring structure $(\mathrm{End}_K(G), +, \circ)$ and is embeddable into a skew field of quotients $\mathrm{End}_K(G)_0$. If $char(L) > 0$, then $\mathrm{End}_K(L, +)$ is the ring of p-polynomials over K and we donote by $\mathcal{O}_{\widehat{K}}$ the skew field $\mathrm{End}_K(L, +)_0$. Let $\overline{x}, \overline{y}, \overline{z} \in G$ be an independent generics over K. We may consider G as an $\mathrm{End}_K(G)$ module and define

$$\mathbb{P}((G, *)\colon \overline{x}, \overline{y}, \overline{z}) = \{\mathrm{acl}_K(a(\overline{x}) * b(\overline{y}) * c(\overline{z}))\colon (a, b, c) \in \mathrm{End}_K(G)^3 \setminus \{\mathbf{0}\}\}.$$

This is a projective plane in $\mathbb{G}(L/K)$, coordinatised by $\mathrm{End}_K(G)_0$ i.e. elements of $\mathbb{P}((G, *)\colon \overline{x}, \overline{y}, \overline{z})$ are dependent with respect to $\mathrm{End}_K(G)$ exactly if they are acl_K-dependent.

Lemma 1.1. *Let x_1, x_2, x_3, x'_1, x'_2, $x'_3 \in L$.*

(i) If each triple $\{x_1, x_2, x_3\}$ and $\{x'_1, x'_2, x'_3\}$ is algebraically independent over K, and

$$\mathrm{acl}_K(x_1 + x_2) = \mathrm{acl}_K(x'_1 + x'_2), \quad \mathrm{acl}_K(x_i) = \mathrm{acl}_K(x'_i), \text{ for } i = 1, 2, 3,$$
$$\mathrm{acl}_K(x_1 + x_3) = \mathrm{acl}_K(x'_1 + x'_3),$$

then there exist $0 \neq c, c' \in \mathrm{End}_K(L, +)$ and $d_1, d_2, d_3 \in K$ such that

$$c'(x'_1) = c(x_1) + d_1,\ c'(x'_2) = c(x_2) + d_2,\ c'(x'_3) = c(x_3) + d_3.$$

(ii) If each pair $\{x_1, x_2\}$ and $\{x'_1, x'_2\}$ is algebraically independent over K, and

$$\mathrm{acl}_K(x_1) = \mathrm{acl}_K(x'_1), \quad \mathrm{acl}_K(x_2) = \mathrm{acl}_K(x'_2), \quad \mathrm{acl}_K(x_1 \cdot x'_1) = \mathrm{acl}_K(x_2 \cdot x'_2),$$

then there exist $0 \neq n, m \in \mathbb{Z}$ and $0 \neq a, b \in K$ such that $x_1^n = a x_2^m$, $\quad y_1^n = b y_2^m$.

Proof. First statement follows from [1, Theorem 2.2.2] and the second from [2, Theorem 1.1]. □

2. Projective planes in $\mathbb{G}(L/K)$

Throughout this section we assume that $K \subset L$ is an arbitrary field extension and $\mathrm{tr\,deg}_K(L) \geq 3$. The geometry $\mathbb{G}(L/K)$ naturally embeds into $\mathbb{G}(\widehat{L}/\widehat{K})$. Therefore we can use theorems about $\mathbb{G}(\widehat{L}/\widehat{K})$ to investigate $\mathbb{G}(L/K)$.

From the proof of [1, Theorem 3.3.1] we obtain some maximal projective planes in $\mathbb{G}(\widehat{L}/\widehat{K})$ in the following way. Suppose $x, y, z \in \widehat{L}$ are algebraically independent over K. Then the projective plane

$$\mathbb{P}((\widehat{L}, +)\colon x, y, z)$$

is the largest projective plane in $\mathbb{G}(\widehat{L}/\widehat{K})$ containing the tuple $\big(\mathrm{acl}_K(x), \mathrm{acl}_K(y), \mathrm{acl}_K(z), \mathrm{acl}_K(x+y), \mathrm{acl}_K(x+z)\big)$, i.e. if a projective plane $\mathbb{P} \subset \mathbb{G}(\widehat{L}/\widehat{K})$ contains points $\big(\mathrm{acl}_K(x), \mathrm{acl}_K(y), \mathrm{acl}_K(z), \mathrm{acl}_K(x + y), \mathrm{acl}_K(x + z)\big)$, then $\mathbb{P} \subseteq \mathbb{P}((\widehat{L}, +)\colon x, y, z)$.

The next theorem generalizes above remark to the geometry $\mathbb{G}(L/K)$ in characteristic zero. The case of positive characteristic requires detailed knowledge of the structure of $\mathcal{O}_{\widehat{K}}$.

Theorem 2.1. *($char(L) = 0$) Suppose that $x, y, z \in \widehat{L}$ are independent over $\widehat{K}$ and the tuple $\big(\mathrm{acl}_K(x), \mathrm{acl}_K(y), \mathrm{acl}_K(z), \mathrm{acl}_K(x + y), \mathrm{acl}_K(x + z)\big)$ is in $\mathbb{G}(L/K)$ (x, y and z do not need to be in L). Then*

$$\mathbb{P}((\widehat{L}, +)\colon x, y, z) \cap \mathbb{G}(L/K) = \{\mathrm{acl}_K(ax + by + cz)\colon (a, b, c) \in (\widehat{K} \cap L)^3 \setminus \{\mathbf{0}\}\},$$

is the projective plane in $\mathbb{G}(L/K)$, coordinatised by $\widehat{K}\cap L$. Moreover the above plane is the largest projective plane in $\mathbb{G}(L/K)$ containing the tuple $\big(\mathrm{acl}_K(x),\mathrm{acl}_K(y),\mathrm{acl}_K(z),$ $\mathrm{acl}_K(x+y),\mathrm{acl}_K(x+z)\big)$.

Proof. Let $f\in \mathrm{Aut}(\widehat{L}/L)$ be arbitrary. By assumption we have

$$\begin{aligned}&\mathrm{acl}_K(x)=f[\mathrm{acl}_K(x)]=\mathrm{acl}_K(f(x)),\\&\mathrm{acl}_K(y)=\mathrm{acl}_K(f(y)),\quad \mathrm{acl}_K(x+y)=\mathrm{acl}_K(f(x)+f(y)),\\&\mathrm{acl}_K(z)=\mathrm{acl}_K(f(z)),\quad \mathrm{acl}_K(x+z)=\mathrm{acl}_K(f(x)+f(z)).\end{aligned}$$

Therefore by Lemma 1.1 we obtain $f(x)=c'\cdot x+d_1, f(y)=c'\cdot y+d_2, f(z)=c'\cdot z+d_3$, for some $d_1,d_2,d_3\in\widehat{K}$ and $0\neq c'\in\widehat{K}$.

$\subseteq$: Let $v\in\mathbb{P}((\widehat{L},+)\colon x,y,z)\cap\mathbb{G}(L/K)$. We have $v=\mathrm{acl}_K(ax+by+cz)=\mathrm{acl}_K(l)$, where $a,b,c\in\widehat{K}$ and $l\in L$. It follows $f[v]=v$, so

$$\begin{aligned}&\mathrm{acl}_K(ax+by+cz)=\mathrm{acl}_K(f(a)f(x)+f(b)f(y)+f(c)f(z))\\&=\mathrm{acl}_K(c'\cdot(f(a)x+f(b)y+f(c)z)+d')=\mathrm{acl}_K(f(a)x+f(b)y+f(c)z),\end{aligned}$$

for $c', d'=f(a)d_1+f(b)d_2+f(c)d_3\in\widehat{K}$ (because $f[\widehat{K}]=\widehat{K}$). By [1, Example 2, Section 3.3] there is a nonzero $\lambda\in\widehat{K}$ such that $(f(a),f(b),f(c))=\lambda(a,b,c)$. If e.g. $a\neq 0$, then $f(\frac{b}{a})=\frac{b}{a}$ and $f(\frac{c}{a})=\frac{c}{a}$. But f has been arbitrary, so $\frac{b}{a},\frac{c}{a}\in L$. Finally $v=\mathrm{acl}_K(ax+by+cz)=\mathrm{acl}_K(x+\frac{b}{a}y+\frac{c}{a}z)$, where $\frac{b}{a},\frac{c}{a}\in\widehat{K}\cap L$.

$\supseteq$: Let $a,b,c\in\widehat{K}\cap L$ and consider $v=\mathrm{acl}_K(ax+by+cz)$. It remains to prove that $v\in\mathbb{G}(L/K)$. We have $f[v]=v$, because

$$\begin{aligned}&f[\mathrm{acl}_K(ax+by+cz)]=\mathrm{acl}_K(af(x)+bf(y)+cf(z))\\&=\mathrm{acl}_K(c'\cdot(ax+by+cz)+d')=\mathrm{acl}_K(ax+by+cz).\end{aligned}$$

Let $w(x)$ be a minimal monic polynomial for $ax+by+cz$ over L. Then $v=\mathrm{acl}_K(ax+by+cz)=\mathrm{acl}_K(\text{roots of } w)=\mathrm{acl}_K(\text{coefficients of } w)\in\mathbb{G}(L/K)$.

The last part of the theorem follows from the first part and from remarks at the begining of this section. □

From the above we have that the geometries $\mathbb{G}(\mathbb{C}/\mathbb{Q})$ and $\mathbb{G}(\mathbb{R}/\mathbb{Q})$ are not isomorphic, because in $\mathbb{G}(\mathbb{R}/\mathbb{Q})$ there is a maximal projective plane, coordinatised by $\widehat{\mathbb{Q}}\cap\mathbb{R}$ and in $\mathbb{G}(\mathbb{C}/\mathbb{Q})$ there is no such plane.

The next result generalizes [1, Corollary 3.3.2] and follows from Theorem 2.1.

Corollary 2.2. *(char$(L)=0$) If $\mathbb{P}\subset\mathbb{G}(L/K)$ is a projective plane, then $\mathbb{P}$ is coordinatised by a subfield of one of the following fields: $\mathbb{Q}(\sqrt{-d})$, $d\in\omega$ and $\widehat{K}\cap L$.*

3. The first-order theory of $\mathrm{G}(L/K)$

We can regard $\mathrm{G}(L/K)$ (and thus $\mathbb{G}(L/K)$) as a model in the countable first-order language $\mathcal{L}=\{\mathrm{acl}_n\colon n<\omega\}$. Namely let

$$\mathrm{acl}_n(a_0,\dots,a_n)\iff a_0\in\mathrm{acl}_K(a_1,\dots,a_n).$$

We obtain a structure $(L,\mathcal{L})$. The following Theorem 3.2 describes a small part of the first-order theory of $(L,\mathcal{L})$.

Proposition 3.1. *Let F be an arbitrary field. If $F = F_1 \cup \dots \cup F_n$, for some subfields $F_1, \dots, F_n$ of F, then $F = F_i$ for some $1 \leq i \leq n$.*

Proof. It follows from a well known result of B. H. Neumann [7]: if there is a covering of an abelian group by finitely many cosets of subgroups, then one of these subgroup has finite index. We leave the proof to the reader. □

Theorem 3.2. *Let $K \subseteq L_1 \subseteq L_2$ be arbitrary field extensions. Assume that* $\operatorname{tr}\deg_K L_1 = \operatorname{tr}\deg_K L_2 < \aleph_0$ *or* $\operatorname{tr}\deg_K L_1, \operatorname{tr}\deg_K L_2 \geq \aleph_0$. *Then*

$$(L_1, \mathcal{L}) \prec_1 (L_2, \mathcal{L}),$$

i.e. for every $\mathcal{L}$-statement $\psi \in \mathcal{L}(L_1)$ with one quantifier and parameters from L_1 we have $(L_1, \mathcal{L}) \models \psi \iff (L_2, \mathcal{L}) \models \psi$.

It is easy to check that without the condition on transcendence degree, the theorem will not be true.

Proof. We can assume that $\psi = \exists x \varphi(x, \bar{c})$, where $\bar{c} \subseteq L_1$ and φ is a quantifier free formula. Using the exchange property for acl_K we may assume that $\varphi(x, \bar{c})$ is of the form

$$\bigvee_{k<l} \Bigg(\Big(x \in \bigcap_{i<n_k} \operatorname{acl}_K(A_{k,i}) \setminus \bigcup_{j<m_k} \operatorname{acl}_K(B_{k,j}) \Big) \wedge \big(\text{(in)equality about } x,\, \bar{c} \big) \Bigg),$$

where $A_{k,i}, B_{k,j} \subseteq \bar{c} \subseteq L_1$.

Let $\{p, q\} = \{1, 2\}$ and assume that $L_p \models \exists x \varphi(x, \bar{c})$, then there exists $a \in L_p$ with $L_p \models \varphi(a, \bar{c})$. Without loss of generality we may assume that $a \notin \bar{c}$, so

$$a \in \bigcap_{i<n} \operatorname{acl}_K(A_i) \setminus \bigcup_{j<m} \operatorname{acl}_K(B_j).$$

If $n = 0$, then by assumptions we have $a' \in L_q$ such that $L_q \models \varphi(a', \bar{c})$. Let $n \neq 0$. Note that by Proposition 3.1, $L_p \models \exists x \varphi(x, \bar{c})$ is equivalent to:

$$\big(\forall j < m\big) \operatorname{acl}_K(B_j) \cap \bigcap_{i<n} \operatorname{acl}_K(A_i) \subsetneq \bigcap_{i<n} \operatorname{acl}_K(A_i),$$

i.e. for $j < m$, $\widehat{K(B_j)} \cap \bigcap_{i<n} \widehat{K(A_i)} \cap L_p \subsetneq \bigcap_{i<n} \widehat{K(A_i)} \cap L_p$. The next lemma will be useful in the proof.

Lemma 3.3. *Suppose that A and B are finite subsets of L_1. Then there exists a finite subset $C \subseteq L_1$ satisfying*

$$\widehat{K(A)} \cap \widehat{K(B)} = \widehat{K(C)}.$$

Proof. Let C' be a transcendence basis of a $\widehat{K(A)} \cap \widehat{K(B)}$ over K. Write $C' = \{c_1, \dots, c_k\} \subset \widehat{L_1}$. Then $\widehat{K(A)} \cap \widehat{K(B)} = \widehat{K(C')}$. Take a minimal monic polynomial $w_i \in L_1[X]$ for c_i over L_1 and let $C = \bigcup_{1 \leq i \leq k} (\text{coefficients of } w_i) \subseteq L_1$. We shall show that $\widehat{K(C')} = \widehat{K(C)}$. By definition $C' \subset \widehat{K(C)}$, hence $\subseteq$. By symmetric polynomials we obtain

$$C \subseteq K\Bigg(\bigcup_{1 \leq i \leq k} \text{roots of } w_i \Bigg) \subseteq \widehat{K(C')}.$$

We explain the last inclusion: if $w_i(a) = 0$, then there exist $f \in \mathrm{Aut}(\widehat{L_1}/L_1)$, $f(c_i) = a$, and thus $c_i \in \widehat{K(C')}$. Finally $a = f(c_i) \in f[\widehat{K(C')}] = f[\widehat{K(A)} \cap \widehat{K(B)}] \overset{A,B\subseteq L_1}{=} \widehat{K(A)} \cap \widehat{K(B)} = \widehat{K(C')}$. □

By Lemma 3.3, to finish the proof it remains to show the following lemma.

Lemma 3.4. *For* $A, B \subseteq L_1$

$$\widehat{K(A)} \cap L_1 \subsetneq \widehat{K(B)} \cap L_1 \iff \widehat{K(A)} \cap L_2 \subsetneq \widehat{K(B)} \cap L_2.$$

Proof. Implication $\Rightarrow$ is obvious. $\Leftarrow$: Suppose, contrary to our claim, that $\widehat{K(A)} \cap L_1 = \widehat{K(B)} \cap L_1$. Take $a \in (\widehat{K(B)} \setminus \widehat{K(A)}) \cap L_2$ and the minimal monic polynomial $w(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in K(B)[X]$ for a over $K(B)$. Then for $i < n$ we have $a_i \in K(B) \subseteq L_1$, so by assumption $a_i \in \widehat{K(A)} \cap L_1$, hence $w \in (\widehat{K(A)} \cap L_1)[X]$ and thus $a \in (\widehat{K(A)} \cap L_1)\widehat{\ } \subseteq \widehat{K(A)}$. But by assumption $a \notin \widehat{K(A)} \cap L_2$ and $a \in L_2$, which is impossible. □

□

4. The reconstruction L from $\mathbb{G}(L/K)$ and corollaries

In this section we generalize some theorems of [2] from the case of algebraically closed fields to the case of arbitrary field extensions. Throughout this section $K \subset L$ will be an arbitrary field extension, $char(L) = p$ and $\operatorname{tr}\deg_K(L) \geq 5$.

We begin with important definitions (see [2] Definitions 2.1, 2.3 and 2.6). Let $(L\backslash K)^{(2)}$ denote the set of pairs $(x, y) \in L^2$ such that x and y are algebraically independent over K. We define the following subsets of $\mathbb{G}(L/K)^4$:

$$\begin{aligned}
\mathscr{Q} &= \{(\mathrm{acl}_K(x), \mathrm{acl}_K(y), \mathrm{acl}_K(x+y), \mathrm{acl}_K(x/y))\colon (x, y) \in (L \setminus K)^{(2)}\}\\
&= \{(\mathrm{acl}_K(x), \mathrm{acl}_K(xz), \mathrm{acl}_K(xz+x), \mathrm{acl}_K(z))\colon (x, z) \in (L \setminus K)^{(2)}\},\\
\mathscr{Q}' &= \{(\mathrm{acl}_K(x), \mathrm{acl}_K(y), \mathrm{acl}_K(x+y), \mathrm{acl}_K(x \cdot y))\colon (x, y) \in (L \setminus K)^{(2)}\},\\
\mathscr{J} &= \mathrm{Im}(j),
\end{aligned}$$

where $j\colon (L \setminus K)^{(2)} \to \mathbb{G}(L/K)^5$ is the function

$$j(x, a) = (\mathrm{acl}_K(x), \mathrm{acl}_K(x + a), \mathrm{acl}_K(xa), \mathrm{acl}_K(x + xa), \mathrm{acl}_K(a)).$$

Let $\psi(A_1, A_2, B_1, B_2, C_1, C_2, D, E, F, G, H, I, P, Q, R, S, T, U, X, Y, Z)$ be an $\mathcal{L}$-formula (see Section 3), standing for the assumptions from [2, Lemma 3.1 (1), (2)] and [2, Corollary 3.4] (where $A = \mathrm{acl}_K(A_1, A_2)$, etc.).

Theorem 4.1. *The sets* $\mathscr{Q}$, $\mathscr{Q}'$ *and* $\mathscr{J}$ *are definable without parameters in* $\mathbb{G}(L/K)$.

Proof. When K and L are algebraically closed then the proof of this theorem can be found in [2, Section 3]. We sketch it in this case.

Let $\psi_{\mathscr{Q}}(P, D, Y, I) = (\exists A_1, \ldots, Z)\ \psi(A_1, \ldots, Z)$ where the quantifier is free from P, D, Y, I. Then the formula $\psi_{\mathscr{Q}}$ defines $\mathscr{Q}$. Now we find a formula for $\mathscr{Q}'$. Lemma 2.2 in [2] gives us configuration for multiplication: if $(x, y) \in (L \setminus K)^{(2)}$ and if the points A', $B', C', D', E' \in \mathbb{G}(L/K)$ are such that the configuration of points and lines in $\mathbb{G}(L/K)$ holds as in Figure 4.1, then

$$E' = \mathrm{acl}_K(x \cdot y),$$

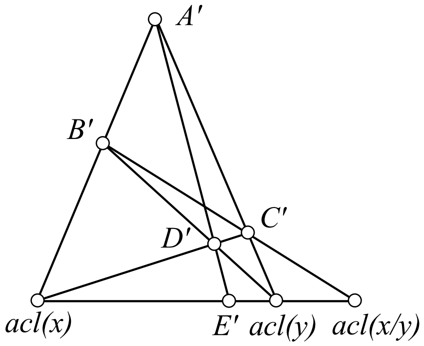

FIGURE 4.1. Configuration for the multiplication

and there exist $a \in A'$ such that $A' = \mathrm{acl}_K(a)$, $B' = \mathrm{acl}_K(ax)$, $C' = \mathrm{acl}_K(ay)$ and $D' = \mathrm{acl}_K(axy)$. Thus if we know $\mathrm{acl}_K(x)$, $\mathrm{acl}_K(y)$ and $\mathrm{acl}_K(x/y)$, then in $\mathbb{G}(L/K)$ we can construct $\mathrm{acl}_K(xy)$. Let $\psi_{\mathscr{Q}'}(A,B,C,D)$ be

$$\big(\exists A',B',C',D',V\big)\ \psi_{\mathscr{Q}}(A,B,C,V) \wedge (\text{the configuration in Figure 4.1 holds}),$$

where in Figure 4.1 we put A instead of $\mathrm{acl}_K(x)$, D instead of E', B instead of $\mathrm{acl}_K(y)$ and V instead of $\mathrm{acl}_K(x/y)$. Therefore $\psi_{\mathscr{Q}'}$ defines $\mathscr{Q}'$. To find a formula for $\mathscr{J}$ we recall [2, Proposition 2.4]: let X,P,Q,R,A be in $\mathbb{G}(L/K)$. Then

$$(X,P,Q,R,A) \in \mathscr{J} \quad \Longleftrightarrow \quad (X,Q,R,A) \in \mathscr{Q} \wedge \big((X,A,P,Q),(X,A,P,R) \in \mathscr{Q}'\big).$$

Hence, the formula

$$\psi_{\mathscr{J}}(X,P,Q,R,A) = \psi_{\mathscr{Q}}(X,Q,R,A) \wedge \psi_{\mathscr{Q}'}(X,A,P,Q) \wedge \psi_{\mathscr{Q}'}(X,A,P,R)$$

defines $\mathscr{J}$ in algebraically closed case.

Now we turn to the general case, i.e. when K and L are arbitrary fields. It is sufficient to prove the next Claim, because we have for instance $(X,Q,R,A) \in \mathscr{Q} \Leftrightarrow \big(\exists P\big)(X,P,Q,R,A) \in \mathscr{J}$.

Claim. The formula $\psi_{\mathscr{J}}$ defines $\mathscr{J}$ in $\mathbb{G}(L/K)$.

We will prove that the following conditions are equivalent:

(1) $(X,P,Q,R,A) \in \mathscr{J}^{\mathbb{G}(L/K)}$,
(2) $(X,P,Q,R,A) \in \mathscr{J}^{\mathbb{G}(\widehat{L}/\widehat{K})} \wedge (X,P,Q,R,A) \subset \mathbb{G}(L/K)$,
(3) $\mathbb{G}(\widehat{L}/\widehat{K}) \models \psi_{\mathscr{J}}(X,P,Q,R,A) \wedge (X,P,Q,R,A) \subset \mathbb{G}(L/K)$,
(4) $\mathbb{G}(L/K) \models \psi_{\mathscr{J}}(X,P,Q,R,A)$.

Implications (i)⇒(ii) and (ii)⇔(iii) are obvious. For (ii)⇒(i) take an arbitrary $f \in \mathrm{Aut}(\widehat{L}/L)$ and write $(X,P,Q,R,A) = j(x,a)$ for some $(x,a) \in (\widehat{L}\backslash\widehat{K})^{(2)}$. Since $j(x,a) \subset \mathbb{G}(L/K)$, we have $j(x,a) = f(j(x,a)) = j(f(x),f(a))$, so by [2, Lemma 2.5] there exist $n \in \mathbb{Z}$ such that $f(x) = x^{p^n}$, $f(a) = a^{p^n}$. We show that $f(x) = x$ and $f(a) = a$. On the contrary, suppose that $p \neq 0$ and $n \neq 0$. Then $f(f(x)) = f(x^{p^n}) = f(x)^{p^n} = x^{p^{2n}}$ and in general $f^m(x) = x^{p^{m\cdot n}}$. However $x \in \widehat{L}$, so the set $\{f^m(x) : m < \omega\}$ is finite. Hence, there is $k < \omega$ such that $x^k = 1$, which implies $x \in \widehat{K}$, a contradiction.

(iv)⇒(iii): It is sufficient to show that for $(A,B,C,D) \subset \mathbb{G}(L/K)$

$$\mathbb{G}(L/K) \models \psi_{\mathscr{Q}}(A,B,C,D) \quad \Longrightarrow \quad \mathbb{G}(\widehat{L}/\widehat{K}) \models \psi_{\mathscr{Q}}(A,B,C,D),$$

$$\mathbb{G}(L/K) \models \psi_{\mathscr{Q}'}(A,B,C,D) \quad \Longrightarrow \quad \mathbb{G}(\widehat{L}/\widehat{K}) \models \psi_{\mathscr{Q}'}(A,B,C,D).$$

It is immediately seen that we must only prove the following: for X, Y, A_1, $A_2 \in \mathbb{G}(L/K)$, if $\mathbb{G}(L/K) \models \big(\forall A' \in \mathrm{acl}_K(A_1, A_2)\big)\ X \notin \mathrm{acl}_K(A'Y)$ then $\mathbb{G}(\widehat{L}/\widehat{K}) \models \big(\forall A' \in \mathrm{acl}_K(A_1, A_2)\big)\ X \notin \mathrm{acl}_K(A'Y)$. Since $\mathbb{G}(\widehat{L}/\widehat{K}) = \mathbb{G}(\widehat{L}/K)$, and the above formula has one quantifier, our statement follows from Theorem 3.2.

(i)$\wedge$(iii)$\Rightarrow$(iv): Take an element $(x', a') \in (L \setminus K)^{(2)}$ such that $(X, P, Q, R, A) = j(x', a')$. We must show the following (remember that $\mathrm{acl}_K(a'+1) = \mathrm{acl}_K(a')$, etc.)

- $\mathbb{G}(L/K) \models \psi_{\mathscr{Q}}(\mathrm{acl}_K(x'), \mathrm{acl}_K(x'a'), \mathrm{acl}_K(x'+x'a'), \mathrm{acl}_K(x'a'/x'))$,
- $\mathbb{G}(L/K) \models \psi_{\mathscr{Q}'}(\mathrm{acl}_K(x'), \mathrm{acl}_K(a'), \mathrm{acl}_K(x'+a'), \mathrm{acl}_K(x'a'))$,
- $\mathbb{G}(L/K) \models \psi_{\mathscr{Q}'}(\mathrm{acl}_K(x'), \mathrm{acl}_K(a'+1), \mathrm{acl}_K(x'+(a'+1)), \mathrm{acl}_K(x'(a'+1)))$.

It is an easy consequence of [2, Corollary 3.4]. We give the proof only for the first case, the other cases are left to the reader. Let

$$(P, D, Y, I) = (\mathrm{acl}_K(x'), \mathrm{acl}_K(x'a'), \mathrm{acl}_K(x'+x'a'), \mathrm{acl}_K(x'a'/x') = \mathrm{acl}_K(a')).$$

We can find an algebraically independent (over $\widehat{K}$) set $\{a, b, c, d, x\} \in L$ such that

$$(P, D, Y, I) = (\mathrm{acl}_K(b), \mathrm{acl}_K(ax), \mathrm{acl}_K(ax+b), \mathrm{acl}_K(ax/b)),$$

and define points as in thesis of from [2, Corollary 3.4] i.e. $A = \mathrm{acl}_K(a, b), \ldots, R = \mathrm{acl}_K(cb+d)$. Finally points $A, \ldots, Z \in \mathbb{G}(L/K)$ satisfy the assumption of [2, Corollary 3.4], and thus they fulfil the formula ψ, so P, D, Y and I fulfil $\psi_{\mathscr{Q}}$ in $\mathbb{G}(L/K)$. □

Now we prove the main classification theorem. We recall that $\widehat{F}^r = \bigcup_{n\in\omega} F^{p^{-n}}$ is purely inseparable closure of F.

Theorem 4.2. *Suppose that $K \subset L$ and $K' \subset L'$ are field extensions and* $\mathrm{tr\,deg}_K(L)$, $\mathrm{tr\,deg}_{K'}(L') \geq 5$.

(i) *The field $\widehat{L}^r$ is uniformly interptable in $\mathbb{G}(L/K)$, using a formula with one (arbitary) parameter from $L \setminus \widehat{K}$.*

(ii) *Every isomorphism $F\colon \mathbb{G}(L/K) \xrightarrow{\cong} \mathbb{G}(L'/K')$ is induced by some isomorphism $\widetilde{F}\colon \widehat{L}^r \xrightarrow{\cong} \widehat{L'}^r$ such that $\widetilde{F}\big[\widehat{L}^r \cap \widehat{K}\big] = \widehat{L'}^r \cap \widehat{K'}$, and for each $x \in \widehat{L}^r \setminus \widehat{K}$, $F(\mathrm{acl}_K(x)) = \mathrm{acl}_K(\widetilde{F}(x))$. In particular $\mathbb{G}(L/K) \cong \mathbb{G}(L'/K')$ if and only if field extensions $\widehat{L}^r \cap \widehat{K} \subset \widehat{L}^r$ and $\widehat{L'}^r \cap \widehat{K'} \subset \widehat{L'}^r$ are isomorphic.*

(iii) *The natural mapping*

$$H\colon\ \mathrm{Aut}(\widehat{L}^r/\{\widehat{L}^r \cap \widehat{K}\}) \longrightarrow \mathrm{Aut}(\mathbb{G}(L/K)),$$

is an epimorphism. If $char(L) = 0$, then H is an isomorphism of groups and if $char(L) \neq 0$ then $\ker H \cong \mathbb{Z}$ *is generated by the Frobenius automorphism.*

Proof. Let $\cong$ be the following equivalence relation on $\mathscr{J}$ ([2, Definition 2.9])

$$j(x, a) \cong j(x', a') \quad \Longleftrightarrow \quad \big(\exists n \in \mathbb{Z}\big)\ a' = a^{p^n}.$$

Using [2, Lemma 2.8] we obtain that $\cong$ is a definable (without parameters) equivalence relation on $\mathscr{J}$. When x, x' and a are algebraically indepedent (over K), then

$$j(x, a) \cong j(x', a') \ \Leftrightarrow\ (\exists P) \text{ configuration from Fig. 3 in [2, Lem. 2.8] holds.}$$

Implication $\Leftarrow$ follows from [2, Lemma 2.8]. For $\Rightarrow$ assume that $a' = a^{p^n}$ for some $0 \leq n \in \mathbb{Z}$. We must find a suitable P from $\mathbb{G}(L/K)$. We have $\mathrm{acl}_K(ax) = \mathrm{acl}_K(a^{p^n}x^{p^n}) = \mathrm{acl}_K(a'x^{p^n})$, $\mathrm{acl}_K((a+1)x) = \mathrm{acl}_K((a^{p^n}+1)x^{p^n}) = \mathrm{acl}_K((a'+1)x^{p^n})$ and $\mathrm{acl}_K(x) = \mathrm{acl}_K(x^{p^n})$. Hence $P = \mathrm{acl}_K(x^{p^n}/x') \in \mathbb{G}(L/K)$.

When x, x' and a are collinear, then we put ([2, Definition 2.9])

$$j(x,a) \cong j(x',a') \;\Leftrightarrow \text{there exist } j(z,a'') \text{ with } z \notin \mathrm{acl}_K(a,x) \text{ and the configuration in Fig. 3 holds between } j(x,a), j(z,a'') \text{ and } j(z,a''), j(x',a').$$

Take an arbitrary $a \in L \setminus \widehat{K}$ and let

$$\mathscr{I}_1 = [j(x,a)]_{\cong} = \{j(x',a) : x' \in \widehat{L}^r \setminus \mathrm{acl}_K(a)\}$$

be one of the classes of $\cong$ (here we use the equality $j(x', a^{p^n}) = j(x'^{p^{-n}}, a)$ and properties of $\widehat{L}^r$). We repeat the Proof [2, Theorem C]. Let

$$\mu\colon \mathscr{I}_1 \to \widehat{L}^r \setminus \mathrm{acl}_K(a), \quad \mu(j(x,a)) = x$$

be a bijective map. It follows from [2, Lemma 2.11] that there are generic definable over a operations $\oplus$ and $\odot$ on $\mathscr{I}_1 \times \mathscr{I}_1$ which satisfy: if $j(x,a), j(y,a) \in (L \setminus K)^{(2)}$ and x, x', a are independent, then

$$j(x,a) \oplus j(x',a) = j(x+x',a)$$
$$j(x,a) \odot j(x',a) = j(x \cdot x',a).$$

(the same definition works for non-algebraically closed case). Note that the map μ respects these operations (when defined). We now interpret the field $\widehat{L}^r$ in $\mathbb{G}(L/K)$. Define a relation $\equiv$ on $\mathscr{I}_1^2$ by

$$(j(x_1,a), j(x_2,a)) \equiv (j(y_1,a), j(y_2,a)) \quad \Longleftrightarrow \quad \frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

It is a definable over a equivalence relation. We moreover define the product and the sum of two classes $[j(x_1,a), j(x_2,a)]_{\equiv}$ and $[j(x_1,a), j(x_2,a)]_{\equiv}$ as in [2], i.e.

$$[j(x',a), j(x,a)]_{\equiv} \;\cdot\; [j(y',a), j(y,a)]_{\equiv} \;=\; [j(x'',a), j(y'',a)]_{\equiv},$$

for suitable x'' and y'' such that $\frac{x'}{x}\frac{y'}{y} = \frac{x''}{y''}$. We need a new class $0_{\equiv}$ to define the sum of classes in a standard fashion. Finally we extend μ to the isomorphism of fields:

$$\mu\colon (\mathscr{I}_1^2/\equiv) \cup \{0_{\equiv}\} \xrightarrow{\cong} \widehat{L}^r, \quad \mu(j(x,a), j(x',a)) = \frac{x}{x'},\ \mu(0_{\equiv}) = 0,$$

which establishes (i).

(ii): Let $F\colon \mathbb{G}(L/K) \xrightarrow{\cong} \mathbb{G}(L'/K')$. Then

$$F\colon \mathscr{I}_1^{\mathbb{G}(L/K)} = [j(x,a)]_{\cong} \;\xrightarrow{\cong}\; \mathscr{I}_1^{\mathbb{G}(L'/K')} = [j(y,b)]_{\cong},$$

$$F\colon \left(\mathscr{I}_1^{\mathbb{G}(L/K)}\right)^2/\equiv\; \cup \{0_{\equiv}\} \;\xrightarrow{\cong}\; \left(\mathscr{I}_1^{\mathbb{G}(L'/K')}\right)^2/\equiv\; \cup \{0_{\equiv}\},$$

for some $y, b \in L'$. Hence F induces an isomorphism of fields $\widetilde{F}\colon \widehat{L}^r \xrightarrow{\cong} \widehat{L'}^r$.

First, we show that $\widetilde{F}[\widehat{L}^r \cap \widehat{K}] = \widehat{L'}^r \cap \widehat{K'}$. Let $c \in \widehat{L}^r \cap \widehat{K}$ and $x \in \widehat{L}^r \setminus \mathrm{acl}_K(a)$. Then we may write

(1) $\quad F(j(cx,a)) = j(y_1,b),$

(2) $\quad F(j(x,a)) = j(y_2,b),$

(3) $\quad \widetilde{F}(c) = \mu(F([j(cx,a), j(x,a)]_{\equiv})) = \mu([j(y_1,b), j(y_2,b)]_{\equiv}) = \frac{y_1}{y_2},$

for some $y_1, y_2 \in \widehat{L'}^r$. However $c \in \widehat{K}$ yields that

$$\begin{aligned} j(cx, a) &= (\mathrm{acl}_K(cx), \mathrm{acl}_K(cx+a), \mathrm{acl}_K(cxa), \mathrm{acl}_K(cxa+cx), \mathrm{acl}_K(a)) \\ &= (\mathrm{acl}_K(x), \mathrm{acl}_K(cx+a), \mathrm{acl}_K(xa), \mathrm{acl}_K(xa+x), \mathrm{acl}_K(a)). \end{aligned}$$

Thus from (1) and (2) above, we have

$$\begin{aligned} F(\mathrm{acl}_K(x)) &= \mathrm{acl}_K(y_1) = \mathrm{acl}_K(y_2), \\ F(\mathrm{acl}_K(xa)) &= \mathrm{acl}_K(y_1 b) = \mathrm{acl}_K(y_2 b), \\ F(\mathrm{acl}_K(xa+x)) &= \mathrm{acl}_K(y_1 b + y_1) = \mathrm{acl}_K(y_2 b + y_2). \end{aligned}$$

Hence from Lemma 1.1 (ii) we obtain $n, m \in \mathbb{Z}$ and $d_1, d_2 \in \widehat{K}$ satisfying $y_1^n = d_1 y_2^m$ and $(y_1 b)^n = d_2 (y_2 b)^m$. It gives that $n = m$, and $y_1 = c' y_2$ for some $c' \in \widehat{K}$. Finally $\widetilde{F}(c) = \frac{y_1}{y_2} = c' \in \widehat{L'}^r \cap \widehat{K'}$.

Now we show the following

$$\big(\forall x \in \widehat{L}^r \setminus \widehat{K}\big) \quad F(\mathrm{acl}_K(x)) = \mathrm{acl}_K(\widetilde{F}(x)).$$

It follows from the preceding results that for $x_1, x_2 \in \widehat{L}^r \setminus \mathrm{acl}_K(a)$

$$F([j(x_1, a), j(x_2, a)]_\equiv) \;=\; [j(y\widetilde{F}\Big(\frac{x_1}{x_2}\Big), b), j(y, b)]_\equiv,$$

for some $b \in \widehat{L'}^r$ and $y \in \widehat{L'}^r \setminus \mathrm{acl}_K(b)$. Let $t = \dfrac{y}{\widetilde{F}(x_2)}$. We obtain

$$\big(\forall x \in \widehat{L}^r \setminus \mathrm{acl}_K(a)\big) \quad F(j(x, a)) = j(\widetilde{F}(x)t, b).$$

Let $x_1, x_2 \in \widehat{L}^r$ be algebraically independent over $\mathrm{acl}_K(a)$. Then

$$\begin{aligned} j(\widetilde{F}(x_1 x_2)t, b) &= F(j(x_1 x_2, a)) = F(j(x_1, a) \odot j(x_2, a)) = F(j(x_1, a)) \odot F(j(x_2, a)) \\ &= j(\widetilde{F}(x_1)t, b) \odot j(\widetilde{F}(x_2)t, b) = j(\widetilde{F}(x_1)\widetilde{F}(x_1)t^2, b). \end{aligned}$$

Hence $t = 1$ and from the above

$$\big(\forall x \in \widehat{L}^r \setminus \mathrm{acl}_K(a)\big) \quad F(\mathrm{acl}_K(x)) = \mathrm{acl}_K(\widetilde{F}(x)).$$

What is left is to show our claim for points from $\mathrm{acl}_K(a) \setminus \widehat{K}$. Let $a' \in \mathrm{acl}_K(a) \setminus \widehat{K}$. Take independent points $t, s \in \widehat{L}^r \setminus \mathrm{acl}_K(a)$, then

$$\mathrm{acl}_K(a') = \mathrm{acl}_K(t, ta') \cap \mathrm{acl}_K(s, sa'),$$

so as $ta', sa' \in \widehat{L}^r \setminus \mathrm{acl}_K(a)$ from the preceding result we have

$$F(\mathrm{acl}_K(a')) = \mathrm{acl}_K(\widetilde{F}(t)), \widetilde{F}(ta')) \cap \mathrm{acl}_K(\widetilde{F}(s), \widetilde{F}(sa')) = \mathrm{acl}_K(\widetilde{F}(a')).$$

The observation that $\mathbb{G}(L/K) = \mathbb{G}(\widehat{L}^r/K)$ finishes the proof of (ii).

(iii) It follows immediately from (ii) that H is an epimorphism. Let $f \in \ker H$. Then $j(x, a) = f(j(x, a)) = j(f(x), f(a))$, so from [2, Lemma 2.5] there is $n \in \mathbb{Z}$ such that $f(x) = x^{p^n}$ and $f(a) = a^{p^n}$. But x and a were arbitrary (independent), so $f = \mathrm{Frob}^n$. $\square$

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Mathematical Institute, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland
E-mail address: gismat@math.uni.wroc.pl