ALGEBRAIC $\overline{\mathbb{Q}}$-GROUPS AS ABSTRACT GROUPS

OLIVIER FRÉCON

ABSTRACT. We analyze the abstract structure of algebraic groups over an algebraically closed field K, using techniques from the theory of groups of finite Morley rank.

For K of characteristic zero and G a given connected affine algebraic $\overline{\mathbb{Q}}$-group, the main theorem describes the algebraic structure of all the groups $H(K)$ isomorphic as abstract groups to $G(K)$, with H an affine algebraic $\overline{\mathbb{Q}}$-group. As a corollary, for any two connected algebraic $\overline{\mathbb{Q}}$-groups G and H, the abstract isomorphy of the groups $G(K)$ and $H(K)$ implies the algebraic isomorphy of their quotients by the center. Furthermore, a model theoretical consequence is that the elementary equivalence of the pure groups $G(K)$ and $H(K)$ implies the abstract isomorphy.

Along the way, we consider the classical problem of abstract isomorphisms. As a new result, we characterize the connected algebraic groups all of whose abstract automorphisms are standard, when K is either $\overline{\mathbb{Q}}$ or of positive characteristic. In characteristic zero, a sufficient algebraic condition is exhibited.

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1. INTRODUCTION

1.1. **General context.** This paper studies *non necessarily simple* algebraic groups as abstract groups. We will tackle classical subjects as the analysis of their abstract isomorphisms [4, 36, 29, 32, 13], and their elementary equivalence [26, 6, 9]. The

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latter is closed for Chevalley groups defined over an algebraically closed field of characteristic not equal to two [9, Theorem 6.10]. Moreover, for simple algebraic groups over a perfect field, the first of these topics is closed too, since Steinberg demonstrated in [34, Theorem 30 p.158] (see also [35]) that, in this case, any abstract automorphism is *standard*, *i.e.* it is the composition of a field isomorphism $\varphi : K \to L$ and an L-isogeny (*i.e.* L-rational homomorphism, surjective with finite kernel). However these results do not hold for non-simple groups, since Remark 8.11 gives two connected solvable algebraic groups over $\overline{\mathbb{Q}}$ with a finite center which are abstractly isomorphic but not algebraically isomorphic. Furthermore Example 3.1 (3) provides a centerless perfect algebraic group over an algebraically closed field of characteristic zero with an abstract automorphism which is not even continuous for the Zariski topology. In particular, the isomorphism is not standard. Thus, in general, an abstract automorphism does not necessarily preserve the geometric structure of the algebraic group. Based on these observations, the analyzes of abstract automorphisms and of the elementary equivalence in the general case appear less relevant than in the simple case.

In the present paper, we will focus on a closely related question, namely the abstract isomorphy. In other words we will look for the implications of the existence of an abstract isomorphism between two algebraic groups. For instance, if α is an abstract isomorphism between two connected affine algebraic groups G and H, then α may not be standard, but the existence of α may imply the existence of a standard isomorphism from G to H. This new issue is an intermediate topic between the previous ones, and it seems to us a more natural subject than abstract isomorphisms for the general case. Another feature of the present work is that no structural assumption on the groups is made. Thus the algebraic groups are not supposed simple or solvable for instance.

The main result of this article is Theorem 1.1. For K an algebraically closed field of characteristic zero and G a given connected affine algebraic $\overline{\mathbb{Q}}$-group, it describes the algebraic structure of all the groups $H(K)$ isomorphic as abstract groups to $G(K)$, with H an affine algebraic $\overline{\mathbb{Q}}$-group. Its proof lies heavily on the theory of *groups of finite Morley rank* and *model theory*. One of its leading consequences is the following result.

Corollary 1.2. – Let K be an algebraically closed field of characteristic zero, and G and H two connected affine algebraic $\overline{\mathbb{Q}}$-groups. If $G(K)$ and $H(K)$ are not the central product of two infinite closed subgroups with a finite intersection, then:

$$G(K) \simeq H(K) \text{ (as abstract groups)} \iff G(K) \simeq H(K) \text{ (as algebraic groups)}$$

As shown in Remarque 8.11, Corollary 1.2 does not hold when G and H are the central product of two infinite closed subgroups with finite intersection. Moreover, we emphasize that the proof of our main theorem would be easier if it would consider only the groups which are not the central product of two infinite closed subgroups with a finite intersection. Indeed, the main difficulty of §8 is precisely to consider no such a restrictive hypothesis.

On the other side, we need the complete statement of Theorem 1.1 in order to obtain the following very strong property about the elementary equivalence of algebraic groups. *Actually, the following result could have been announced as the main theorem of this paper. In particular since both results are proven simultaneously.*

However, the statement of Theorem 1.1 is more general, since Theorem 1.6 could be obtained as a corollary of it by using §9.

Theorem 1.6. – Let K be an algebraically closed field of characteristic zero, and let G and H be two connected algebraic $\overline{\mathbb{Q}}$-groups. Then in the language of pure groups:

$$G(K) \simeq H(K) \iff G(K) \equiv H(K)$$

We note that, thanks to Lemma 1.7, the groups G and H are not necessarily affine in the previous theorem.

We do not know if our main results stated for groups of the form $G(K)$, where G denotes an algebraic $\overline{\mathbb{Q}}$-group and K an algebraically closed field of characteristic zero, could be generalized to all algebraic groups over K. Actually, we expect that a model theoretical argument allows this from our results. Indeed, in §9, a model theoretical argument enables the transition from $G(\overline{\mathbb{Q}})$ to $G(K)$, but this one does not work in a more general context.

Our analysis of the pure groups associated with an algebraic group is entirely based on the theory of *groups of finite Morley rank*. Their main properties are presented in §2. This method imposes an important restriction on us: since any infinite field of finite Morley rank is algebraically closed (Fact 2.3), this paper concerns only the algebraic groups over such a field. In comparison to the major results on the same subject, this constraint is very strong, since the Borel-Tits Theorem [4] on abstract homomorphisms concerns the algebraic groups over any commutative field. In the same vein, the study of the elementary equivalence of Chevalley groups by Bunina has no restriction on the ground field [7, 8]. But when one consider the history of these subjects, one sees that the first steps invariably involved algebraic groups over perfect fields [34, Theorem 30 p.158] or over algebraically closed fields [6]. Furthermore, as soon as we focus on non-simple groups, such a restriction seems mandatory. Indeed, there are few theorems on this subject initiated by Tits [36], mainly because the techniques used for simple groups are not effective. The main results known to us are the ones by Tits concerning perfect algebraic groups over $\mathbb{R}$ [36], and those by Sharomet and Ponomarëv about minimal solvable algebraic groups over fields of characteristic zero [33, 29]. More recently, other generalizations of the Borel-Tits Theorem have been studied, such as the homomorphisms from a simple algebraic group over a field *of characteristic zero* to a non-reductive algebraic group [25], and above all the analysis of the abstract isomorphisms of Kac-Moody groups by Caprace and Mühlherr, whose main result was first stated for Kac-Moody groups *over an algebraically closed field* [13], before being generalized [14, 12].

We emphasize that the main part of this paper concerns the algebraic groups over an algebraically closed field *of characteristic zero*. Although we obtain various results in positive characteristic as well. Indeed, it is possible to develop an analysis of algebraic groups in the presence of a sufficiently rich structure, for example for perfect groups and for centerless groups (cf. Theorem 10.1 below). But if such a rich structure is not present, for example when we are dealing with a group of nilpotency class two, such an analysis seems out of reach, as this is exemplified by Baudisch [3] that constructs a connected nilpotent group of class two and of finite Morley rank in which no infinite field can not interpreted. The Baudisch example does not present an unsurmontable obstacle to extending our analysis to nonzero characteristic since this group is not algebraic. Moreover, the theorems of Rabinovich [30] and of Hrushovski and Zil'ber [23] allow to interpret an infinite field of positive characteristic in some algebraic groups with rather poor structure. However, these do not cover all connected non-abelian groups of finite Morley rank, and the complexity of the arguments involved [30] and of [23] make it clear that this is a very difficult question.

Nevertheless, one is able to obtain strong results on abstract automorphisms of algebraic groups even in positive characteristic since nilpotent groups are not necessarily involved. Thus, our main result on abstract automorphisms is the following.

Theorem (Special case of Theorems 10.1 and 10.7). – Let G be a connected algebraic group over an algebraically closed field K of *positive* characteristic. Then the following three conditions are equivalent:

- any automorphism of G is standard;
- the algebraic group G and the pure group G are biinterpretable;
- (1) the center $Z(G)$ has no nontrivial torus, and either G/G' is a torus, or $Z(G)$ has no nontrivial unipotent element;
  (2) and the group G is not central product of two proper closed subgroups U and V with $U \cap V$ finite.

As in the rest of this paper, for any group G, we denote by $Z(G)$ the center of G and by G' its derived subgroup.

It should be noted that the statement of Theorem 10.1 is more general than the one above, because it can be extended to the case where $K = \overline{\mathbb{Q}}$ as well, provided that another technical condition than (1) above is satisfied (see Remark 10.2). However, Theorem 10.1 does not hold when K is of positive transcendance degree over $\overline{\mathbb{Q}}$ (Example 3.1 (3)). In this last case, we prove the following statement.

Theorem 10.9. – Let G be a connected algebraic group over an algebraically closed field K of characteristic zero. Then all of its automorphisms are standard if it satisfies the following three conditions:

(1) either G is perfect or $Z(G)$ is finite;
(2) the group G is not a central product of two proper closed subgroups U and V with $U \cap V$ finite;
(3) for each characteristic abelian closed closed subgroup A of G, and each maximal torus T of G, the centralizer $C_A(T)$ is central in G.

Furthermore, under these conditions, the algebraic group G and the pure group G are biinterpretable.

1.2. **Main theorem.** Some situations prevent the equivalence of the abstract isomorphy and isomorphy as algebraic groups. For example, when K is any algebraically closed field, for all the positive integers m and n, the additive groups K_+^m and K_+^n are abstractly isomorphic, but they are isomorphic as algebraic groups only if $m = n$. Moreover, Remark 8.11 describes another problematic situation, where the group has finite center and is a central product of two infinite closed subgroups with finite intersection. The main theorem of this paper, Theorem 1.1 below, shows that these are the only pathological case when $K = \overline{\mathbb{Q}}$. More precisely, for K an algebraically closed field of characteristic zero, when we fix a connected affine algebraic $\overline{\mathbb{Q}}$-group G, it determines all the other connected affine algebraic $\overline{\mathbb{Q}}$-groups H, such $G(K)$ is abstractly isomorphic with $H(K)$, up to isomorphism of algebraic groups. In this paper, we denote by G° the identity component of any algebraic group G defined over an algebraically closed field of characteristic zero. Moreover, we denote by $Z(G)$ the center of any group G, and by G' its derived subgroup.

Main Theorem 1.1. – *For any connected affine algebraic $\overline{\mathbb{Q}}$-group G and any algebraically closed field K of characteristic zero, there exists a connected affine algebraic $\overline{\mathbb{Q}}$-group D and a finite central subgroup F of $D(K)$ such that $G(K)$ has the form $D(K)/F \times K_+^r$ for $r \in \mathbb{N}$, and such that for each affine algebraic $\overline{\mathbb{Q}}$-group H, the followings conditions are equivalent:*

- *$H(K)$ is isomorphic as abstract group to $G(K)$;*
- *$H(K)$ is isomorphic as algebraic group to $D(K)/\alpha(F) \times K_+^s$ for a* quasi-standard *automorphism α of $D(K)$, and an integer s which is zero if $G(K)'$ contains $Z(G(K))^\circ$ and any natural number otherwise.*

An isomorphism $\alpha : G \to H$ between two algebraic groups over algebraically closed fields K and L is said to be *standard* if it is the composition of a field isomorphism $\varphi : K \to L$ and an L-isogeny (*i.e.* L-rational homomorphism, surjective with finite kernel). *We point out that, as throughout this paper, we identify the field isomorphism φ to the corresponding map $\varphi^\circ : G \to {}^\varphi G$, where ${}^\varphi G$ denotes the algebraic group over L obtained by transfer of base field.* Then an isomorphism $\alpha : G \to H$ is said to be *quasi-standard* if $G = G_1 \times \cdots \times G_n$ and $H = H_1 \times \cdots \times H_n$ for $G_1, \ldots, G_n, H_1, \ldots, H_n$ some algebraic subgroups such that, for each i, we have $\alpha(G_i) = H_i$, and $\alpha_{|G_i} : G_i \to H_i$ is a standard isomorphism.

We notice that, since any field automorphism of a field K and any K-isogeny commute, the standard automorphisms of an algebraic group form a group.

We will mention several consequences of Theorem 1.1. In particular, they are worth mentioning Theorem 1.6, whose the proof is not direct, and the following two results, which reformulate Theorem 1.1 in some interesting special cases.

Corollary 1.2. – *Let G and H be two connected affine algebraic $\overline{\mathbb{Q}}$-groups and let K be any algebraically closed field of characteristic zero. If $G(K)$ and $H(K)$ are not a central product of two infinite closed subgroups with a finite intersection, then $G(K)$ and $H(K)$ are isomorphic as algebraic groups.*

Proof – Theorem 1.1 provides a connected affine algebraic $\overline{\mathbb{Q}}$-group D, a finite central subgroup F of $D(K)$, and a quasi-standard automorphism α of $D(K)$, such that $G(K)$ and $D(K)/F$ (resp. $H(K)$ and $D(K)/\alpha(F)$) are isomorphic as algebraic groups. Our hypotheses over $G(K)$ and $H(K)$ imply that $D(K)$ has no decomposition as a direct product of two infinite closed subgroups, so α is standard. Now the map $\overline{\alpha} : D(K)/F \to D(K)/\alpha(F)$ defined by $\overline{\alpha}(xF) = \alpha(x)\alpha(F)$ for each $x \in D(K)$ is a standard isomorphism, hence the algebraic groups $D(K)/F$ and $D(K)/\alpha(F)$ are isomorphic. □

Corollary 1.3. – *Let G and H be two connected (nonnecessarily affine) algebraic $\overline{\mathbb{Q}}$-groups and let K be any algebraically closed field of characteristic zero. If $G(K)$ and $H(K)$ are abstractly isomorphic, then they have a common algebraic central extension.*

Moreover, $G(K)/Z(G(K))$ and $H(K)/Z(H(K))$ are isomorphic as algebraic groups.

Proof – By Lemma 1.7 below, there is a connected affine algebraic $\overline{\mathbb{Q}}$-group G_A (resp. H_A) such that $G_A(K)$ and $G(K)$ (resp. $H_A(K)$ and $H(K)$) are abstractly isomorphic, and such that $G_A(K)/Z(G_A(K))$ and $G(K)/Z(G(K))$ (resp.

$H_A(K)/Z(H_A(K))$ and $H(K)/Z(H(K)))$ are isomorphic as algebraic groups. Since $G_A(K)$ and $H_A(K)$ are abstractly isomorphic, Theorem 1.1 provides a connected algebraic $\overline{\mathbb{Q}}$-group D, a finite central subgroup F of $D(K)$, a quasi-standard automorphism α of $D(K)$, and two integers r and s such that the algebraic groups $G_A(K)$ and $D(K)/F \times K_+^r$ (resp. $H_A(K)$ and $D(K)/\alpha(F) \times K_+^s$) are isomorphic. Since F and $\alpha(F)$ are finite, and since D is connected, we have $Z(D(K)/F \times K_+^r) = Z(D(K))/F \times K_+^r$ and $Z(D(K)/\alpha(F) \times K_+^s) = Z(D(K))/\alpha(F) \times K_+^s$, so $G_A(K)/Z(G_A(K))$ and $H_A(K)/Z(H_A(K))$ are isomorphic to $D(K)/Z(D(K))$ as algebraic groups. Consequently, $G(K)/Z(G(K))$ and $H(K)/Z(H(K))$ are isomorphic as algebraic groups.

Finally, the preimage in $G(K) \times H(K)$ of the graph of any isomorphism of algebraic groups from $G(K)/Z(G(K))$ to $H(K)/Z(H(K))$ is a common algebraic central extension for $G(K)$ and $H(K)$. □

We should point that, thanks to the following lemma, it would be easy to state similar results to above ones by considering two abstractly isomorphic groups $G(K)$ and $H(L)$, where G and H would be as above and K and L would be *distinct* algebraically closed fields of characteristic zero. The proof of Lemma 1.4 is based on the fact that the pure group G has finite Morley rank.

Lemma 1.4. – *Let G and H be two abstractly isomorphic connected (nonnecessarily affine) algebraic groups over algebraically closed fields K and L of characteristic zero. If G is either nonabelian or uncountable, there is a field isomorphism between K and L.*

Otherwise, either G is trivial and there is no condition over L, or G is infinite and L is any countable algebraically closed field of characteristic zero.

Proof – We may assume that G is nontrivial. If G is nonabelian, its Borel subgroups, *i.e.* its maximal solvable connected closed subgroups, are nonabelian too. Let B be such a subgroup of G. Then B is a maximal solvable subgroup of G, so it is definable in the pure group G [5, Corollary 5.38]. Moreover B is connected in the pure field K, and since all the definable sets of the pure group G are definable in K too, B is connected in the pure group G. We consider the *Fitting subgroup* $F(B)$ of B (see §2.5), which is a normal nilpotent definable subgroup of B (Fact 2.9). Then Fact 2.21 says that $F(B)/Z(B)$ is a $\widetilde{U}$*-group* (see Definition 2.13). In particular, $F(B)/Z(B)$ is connected in the pure group G. Since B is a solvable connected algebraic group, then B' is unipotent, so $F(B)$ contains B', and $B/F(B)$ is abelian. Thus, if $F(B) = Z(B)$, then B' is central in B and B is nilpotent, so we obtain $B = F(B) = Z(B)$, contradicting that B is nonabelian. Consequently $F(B)/Z(B)$ is infinite. But K has characteristic zero and B is algebraic over K, so $B/Z(B)$ has no infinite subgroup of bounded exponent. Hence it follows from Definition 2.13 that there is a field K_0 of characteristic zero, interpretable in the pure group G. By Fact 2.6, the fields K and K_0 are definably isomorphic in the pure field K. Since G and H are abstractly isomorphic, there is a field L_0 interpretable in the pure group H, with L_0 isomorphic to K_0. In the same way, L_0 and L are isomorphic, so there is a field isomorphism between K and L. Hence we may assume that G is abelian.

We note that G, K, H and L have the same cardinality. Moreover, K and L are elementarily equivalent by completeness of ACF_0 [27, Corollary 3.2.3]. So, if G is uncountable, then K and L are two uncountable algebraically closed fields of characteristic zero with the same cardinality, hence they are isomorphic by the κ-categoricity of ACF_0 [27, Proposition 2.2.5].

If G is abelian and countable, then K and L are necessarily countable. By Lemma 1.7, there is a connected affine algebraic group G_A over K such that G_A and G are abstractly isomorphic. Then we find two integers r and s with $rs \neq 0$ such that the algebraic groups G_A and $(K^*)^r \times K_+^s$ are isomorphic. But, for any countable algebraically closed field F of characteristic zero, $(K^*)^r \times K_+^s$ is abstractly isomorphic to $(F^*)^r \times F_+^s$, so we obtain the result. □

1.3. **Methods.** The methods and tools that will be used in this paper comes from the theory of *groups of finite Morley rank*. Indeed, the geometric and algebraic classical methods seem not appropriate for the study of the abstract structure of nonnecessarily simple groups, while the bulk of the present work concerns nonnecessary simple groups. By cons, in the 80's, B.I. Zil'ber [37], then B. Poizat [28], showed that model theory is potentially effective for our subject, since they dealt with a model-theoretic proof of the theorem of the abstract isomorphisms of simple algebraic groups in the algebraically closed context. Furthermore, the main part of the arguments does not deal with simple groups, and the last part of the proof of [28, Corollary 4.17] provides the following rather general piece of information about the abstract isomorphisms. Actually, a slightly more elaborate version of this result will be fundamental for us (see Fact 3.6).

Fact 1.5. – [28, Proof of Corollaire 4.17] *Let α be an abstract isomorphism between two algebraic groups G and H over algebraically closed field. If, in the pure group G, there is an interpretable field K and a definable isomorphism from G to an algebraic group over K, then α is a standard isomorphism.*

A brief introduction to the theory of groups of finite Morley rank is given in §2, and for more details about these groups, we may refer to [1, 5, 27, 28]. Roughly speaking, *Morley rank* is an ordinal-valued abstract dimension notion that arose in model theory. Model theory studies structure using a certain formalism. A structure is an underlying set together with the graphs of some distinguished relations and functions. Thus an "abstract" group, or a *pure group* as model theorists would say, is a set G with the group law, inversion and the constant 1, as the distinguished binary, unary and 0-ary operations (one writes $(G, \cdot, ^{-1}, 1)$). Any subset of G^n ($n \in \mathbb{N}$) described using first-order logic and symbols defining the three functions, the pure group language, is a *definable set*. More generally, one defines a *interpretable set* as a quotient of a definable set by a definable equivalence relation.

Using this line, a *(pure) field* is a structure of the form $(K, +, -, \cdot, 0, 1)$, where $+$, $-$ and $\cdot$ are binary functions and 0 and 1 are constants. In this structure, the definable sets coincide with the constructible ones. This structural equivalence between definability and constructibility in the case of algebraically closed fields has a consequence at the level of model-theoretic and geometric dimensions:

$$\text{Morley rank} \; = \; \text{Zariski dimension.}$$

One question in the present paper is to compare the definable sets in an affine algebraic group when this group is seen as a pure group structure and when it is seen as a constructible set in an algebraically closed field. Since affine algebraic groups are examples of groups of finite Morley rank, it is natural that model theoretic techniques are relevant for questions of algebro-geometric nature such as the ones addressed in this paper.

Usually, the two most natural ways for a model theorist to view an algebraic group is as a pure group, or as a group, interpretable in a pure field. In this paper, we introduce an intermediate notion: the *ACF-expansion of a pure group* (Definition 3.2). Indeed, on the one hand, the pure group is not sufficiently rich for our subject (see Example 3.1 (2) and (3)), and on the other hand abstract isomorphisms do not preserve all the sets interpretable in the pure field. However, the ACF-expansion of a pure group is effective just if the ground field is isomorphic to the algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$ (see Lemma 3.10 for instance). Thus, most of our analysis is concerned with algebraic groups over $\overline{\mathbb{Q}}$. In the final argument (§9), a model-theoretical argument will generalize the result over $\overline{\mathbb{Q}}$ to any algebraically closed field of characteristic zero.

As in [28], in order to analyze the abstract structure of algebraic groups over $\overline{\mathbb{Q}}$, and more generally the one of algebraic groups over an algebraically closed field, we remark that their associated pure groups are of finite Morley rank. In fact, we introduce in §4 the notion of a *ACF-group*, including the one of a pure group and the one of the ACF-expansion of a pure group. Then a very substantial part of the article is to prove Theorem 7.13. This one says that, if G is a connected algebraic group over $\overline{\mathbb{Q}}$ then, in the ACF-expansion of pure group G, the quotient $G/Z(G)$ is definably linear, in the sense of Definition 4.2. The core of the proof relies on the Burdges' unipotence theory [10], its improvements in [17] and [18, §3.2], and the theory of solvable groups of finite Morley rank. Another key point is the notion of a *pseudo-torus* (Definition 2.8), a concept derived from the one of a *decent torus* of a group of finite Morley rank, and of a torus of an algebraic group. Indeed, they are the basis of the analysis of *$\widetilde{U}$-groups*, and thus they make possible the use of the Burdges' unipotence for algebraic groups.

1.4. **About elementary equivalence.** It is noticeable that the final argument of the proof of Theorem 1.1 provides another strong result. Indeed, Theorem 1.6 below is proven in §9 for affine groups, simultaneously with the main theorem. Thus, in this paper, if G denotes a connected affine algebraic $\overline{\mathbb{Q}}$-group and K an algebraically closed field of characteristic zero, we describe all the affine algebraic $\overline{\mathbb{Q}}$-groups H such that $H(K)$ is elementarily equivalent to $G(K)$, and not only abstractly isomorphic to $G(K)$.

Theorem 1.6. – *Let G and H be two connected (nonnecessarily affine) algebraic $\overline{\mathbb{Q}}$-groups and let K be an algebraically closed field of characteristic zero. Then $G(K)$ and $H(K)$ are abstractly isomorphic if and only if their pure groups are elementarily equivalent.*

Furthermore, thanks to Lemma 1.7 below, we note that the groups are not necessarily affine in Theorem 1.6.

Lemma 1.7. – *Let G be any connected algebraic group over an algebraically closed field K of characteristic zero. Then there is a connected affine algebraic group G_A over K such that G_A and G are abstractly isomorphic.*

Moreover, we may choose G_A such that $G_A/Z(G_A)$ and $G/Z(G)$ are isomorphic as algebraic groups.

Proof – Let N be the smallest connected normal algebraic subgroup of G such that G/N is affine [31, §5, Corollary 3 p.431], and let A be the largest connected affine algebraic subgroup of G [31, §5, Theorem 16]. Then, by [31, §5, Corollary 1 p.433], we have $N \leq Z(G)$, and by [31, §5, Corollary 5 p.440], we have $G = AN$ and N contains only a finite number of elements of any given finite order. So $(A \cap N)^\circ$ and N are abelian and divisible, and $(A \cap N)^\circ$ has a (nonnecessarily algebraic) divisible complement R in N.

By the choice of A, the group $(A \cap N)^\circ$ is the largest connected affine algebraic subgroup of N, and $N/(A \cap N)^\circ$ is an abelian variety by [31, §5, Theorem 16]. Consequently, for each integer n, the group $R \simeq N/(A \cap N)^\circ$ has n^{2g} elements of order dividing n, where g denotes the dimension of $N/(A \cap N)^\circ$. This implies that R is abstractly isomorphic to the torus $(K^*)^{2g}$, so the affine algebraic group $A \times (K^*)^{2g}$ is abstractly isomorphic to $A \times R$. But we have $G = AN = AR$ and $A \cap R \leq (A \cap N) \cap R$ is finite, so there is an abstract surjective homomorphism $\tau : A \times (K^*)^{2g} \to G$ such that $\tau(x, 1) = x$ for each $x \in A$, and with finite kernel E. Hence there is an abstract isomorphism σ from G to the connected affine algebraic group $G_A := (A \times (K^*)^{2g})/E$, such that its restriction $\sigma_{|A} : A \to (A \times \{1\})E/E$ is an isomorphism of algebraic groups. Thus, since we have $N \leq Z(G)$ and $G = AN$, we obtain $G = AZ(G)$, and the map $\overline{\sigma} : G/Z(G) \to G_A/Z(G_A)$ induced by σ is an isomorphism of algebraic groups. □

1.5. **Plan of the article.** After reminders of known facts in §2, we analyze the following very special group in §3:

$$G = \left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^*,\ (a, u, v) \in \overline{\mathbb{Q}}^3 \right\}.$$

For that, we introduce the notion of a *ACF-expansion of a pure group* (Definition 3.2), and we prove that the maximal tori of G are definable in the ACF-expansion of the pure group G (Proposition 3.11).

In §4, we study the unipotent groups over $\overline{\mathbb{Q}}$, and we prove that, for each connected nilpotent algebraic group G over $\overline{\mathbb{Q}}$, the quotient $G/Z(G)$ is *definably linear* (in the sense of Definition 4.2) in the ACF-expansion of the pure group G (Theorem 4.3). In order to study the minimal configuration, we need Proposition 3.11. Moreover, a key point of this section is the Hoschild-Mostow Theorem (Fact 4.5), of which we provide a model-theoretic proof for the special case used here (Fact 4.6).

Section §5 is devoted to the study of the *ACF-groups* and *ACF_0-groups* (see the beginning of §4 for the definitions). These concepts encompass different ways of seeing an algebraic group for the model-theorist: as a pure group, as the ACF-expansion of a pure group, or as a group interpretable in a pure field for instance. Moreover, it appears that the notion of a definably linear group is not sufficient to analyze the ACF-groups (see Example 5.1). Therefore we introduce a slightly thinness concept: the *definably affine groups* (Definition 5.2). In particular, each ACF-group G has a largest connected definably affine subgroup (Corollary 5.10), and a smallest normal definable subgroup $W(G)$ such that $G/W(G)$ is definably affine (Corollary 5.18). It is noticeable that the proofs are somewhat more formal in this section, and most results are valid for the algebraic groups over any algebraically closed field, not just over $\overline{\mathbb{Q}}$.

When the ground field of an algebraic group G is algebraically closed of characteristic zero, the definability of tori becomes a problem. We approach this problem using *decent tori* and *pseudo-tori* (Definition 2.8). However, even if G is a centerless connected algebraic group over $\overline{\mathbb{Q}}$, the maximal tori of the pure group G may not be definable (Example 3.1 (2)). In §6, we prove that, when G is a connected solvable algebraic group over $\overline{\mathbb{Q}}$, then $T \cap F(G)$ is central in G for each pseudo-torus T of the ACF-expansion of the pure group G (Theorem 6.3), where $F(G)$ denotes the *Fitting subgroup* of G (see §2.5). Moreover, it is noticeable that, in order to study the minimal configuration, we need Proposition 3.11 again.

From this result, from the analysis of the unipotent groups in §4, and from the one of the ACF-groups in §5, we deduce Theorem 7.13, which is fundamental for us. Indeed, it says that when G is a connected algebraic group over $\overline{\mathbb{Q}}$, then $G/Z(G)$ is definably linear in the ACF-expansion of the pure group G. It is the base of the construction of the group D in the main theorem. Actually, this last group is built in §8. Its construction is rather natural, but above all, we have a serious problem with some central finite subgroups of it (see the proof of Theorem 8.14).

The transition from $\overline{\mathbb{Q}}$ to any algebraically closed field of characteristic zero is made in §9, essentially by a classical model-theoretic argument. Furthermore, it is noticeable that the end of the proof of the main result provides simultaneously the one of Theorem 1.6 in the affine case (the general case of Theorem 1.6 follows from the affine case and from Lemma 1.7).

In §10, we consider the abstract isomorphisms between algebraic groups over an algebraically closed field of any characteristic, and the bi-interpretability of algebraic groups with their pure groups associated. The proofs depends mainly from §5 and §7.

2. Facts

The notations will be as in [5], which is also our main reference for groups of finite Morley rank. In this section we recall some definitions and known results.

Here is a list of the main notations and definitions used in this paper, with where they are defined.

$\cdot^{\circ}$	§2.3
$d(\,\cdot\,)$	§2.3
$F(\,\cdot\,)$	§2.5
$J(\,\cdot\,)$	§2.6
$U_K(\,\cdot\,)$	Not. 2.12
$\widetilde{U}(\,\cdot\,)$	Def. 2.13
$V(\,\cdot\,)$	Not. 4.10
$A(\,\cdot\,)$	Cor. 5.10
$W(\,\cdot\,)$	Cor. 5.18
$\Phi(\,\cdot\,)$	§6
$\mathcal{S}(\,\cdot\,)$	§7
$\mathcal{T}(\,\cdot\,)$	§7
$Q(\,\cdot\,)$	§7
$T(\,\cdot\,)$	§8
$D(\,\cdot\,)$	§8

S-minimal	§2.3
Decent torus	Def. 2.8
Pseudo-torus	Def. 2.8
Groups of finite Morley rank Fitting subgroup	§2.5
Indecomposable group	§2.6
U_K-group	Def. 2.13
$\widetilde{U}$-group	Def. 2.13
$\mathscr{T}$-expansion	Def. 3.2
ACF-group	§4
Definably linear group	Def. 4.2
Definably affine group	Def. 5.2
Carter subgroup	§6
Frattini subgroup	§6
Quasiunipotent radical	§7
Centrally indecomposable group	Def. 8.3

2.1. **Definable sets.** For the convenience of readers notfamiliar with groups of finite Morley rank, we devote this part to the introduction of these groups. We refer to [1, 5, 27, 28] for details and many other information.

A *structure* $\mathcal{M}$ is an underlying set M equipped with

- a possibly empty family $\{c_i \mid i \in I_C\}$ of distinguished elements of M, called *constants*;
- a possibly empty family $\{f_i \mid i \in I_F\}$ of *functions* with $f_i : M^{n_i} \to M$ for each $i \in I_F$, where $n_i \in \mathbb{N}^*$ and depends only on f_i;
- a family $\{R_i \mid i \in I_R\}$ of *relations* on M^{k_i} for each $i \in I_R$, where $k_i \in \mathbb{N}^*$ and depends only on R_i.

The three sets I_C, I_F, I_R are the families of indices. It is worth noting that the equality is always part of the relations, the reason why the family of relations is never empty. Also, constants are nothing but 0-ary functions.

To concretize this formalism, a group can be regarded as the following structure

$$\mathcal{G} \;=\; (G;\ \cdot\ ,\ ^{-1}, 1, =)$$

where G is the underlying non-empty set, $\cdot$ is the binary group operation, the unary function $^{-1}$ is the groupfirst order formula inversion, 1 is the identity element of the group $\mathcal{G}$ and $=$ is the only relation. It is common practice to exclude the equality from the notation. A group structure considered with no other constant, function or relation is called a *pure group*. *Pure fields* are other examples of structures. Such a structure is of the form $(K, +, -, \cdot, 0, 1)$, where $+$, $-$ and $\cdot$ are binary functions and 0 and 1 are constants.

The set of *terms* is inductively defined using constants, functions and *variables symbols*. An *atomic formula* is any expression of the form $R(t_1, \cdots, t_n)$, where $t_1, \cdots, t_n$ are terms. The set of *(first order) formulas* is inductively defiNevertheлessned :

- any atomic formula is a formula;
- if α and β are two formulas, then $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$ and $\alpha \leftrightarrow \beta$ are formulas;
- if α is a formula and if x is a variable symbol, then $\exists x\alpha$ and $\forall x\alpha$ are formulas.

A subset of a cartesian power of the underlying set M is said to be *definable* in the structure $\mathcal{M}$ if it can be described using a formula. For example, the center $Z(G)$ of a group G is definable in the pure group $(G; \cdot, {}^{-1}, 1)$ as the set of the elements $z \in G$ satisfying the formula $\forall x\ x \cdot z = z \cdot x$. By cons, in general, a set defined as the generated set by a definable subset is not definable. Thus the derived subgroup G' of G is not necessarily definable.

A function or relation is said be *definable* if its graph is a definable set. Using these notions, one extends the notion of definability, and introduces a structure that is definable in another structure. Intuitively speaking, a structure $\mathcal{M}$ is said to be definable in a structure $\mathcal{M}'$ if its underlying set and signature are definable in $\mathcal{M}'$. This definition is extended further by allowing "quotients", in other words definable sets modulo *definable equivalence relations*. Some call these structures *interpretable*. We will keep using the word "definable" since in a suitable model-theoretic setup everything interpretable becomes definable.

A relevant group-theoretic example is an algebraic group over a field. By its very definition, the underlying set of such a group, its group operations and identity element are all definable using field operations. On the other hand, whether one can recover up to a reasonable isomorphism, the underlying field and its geometry using the bare group structure is a less obvious question. Indeed, the answer may even be negative, and the quest for such an answer is a major activity in model theory that lies among the aims of this paper.

2.2. **Groups of finite Morley rank.** For the model-theoretical definition of the *Morley rank* and of the groups of finite Morley rank, we refer to [27, §6-7]. For simplicity, we prefer to introduce the groups of finite Morley rank from the *Borovik-Poizat axiomatization.*

First A.V. Borovik introduced a rank for any structure $\mathcal{M}$. Namely, to any non-empty interpretable set A, he associates a *rank* denoted by $\mathrm{rk}(A)$, belonging to $\mathbb{N} \cup \{+\infty\}$ and defined by the following assertion

- for all $n \in \mathbb{N}$, we have $rk(A) \geq n + 1$ if and only if there are infinitely many pairwise disjoint, interpretable, non-empty subsets of A each of rank at least n.

In particular, an interpretable non-empty set A is of rank zero if and only if it is finite.

Moreover, A.V. Borovik introduced several axioms, and B. Poizat provided the following characterization of groups of finite Morley rank.

Fact 2.1. – [28] *We consider a structure of group* $(G;\ \cdot\ ,\ ^{-1}, 1, \cdots)$ *not necessarily assumed pure, and the function* rk *defined above.*

The group G is of finite Morley rank *is and only if, for each definable application f between two definable sets A and B, the following three axioms hold:*

- *the rank* $\mathrm{rk}(A)$ *is an integer;*
- *the set* $\{b \in B \mid rk(f^{-1}(b)) = n\}$ *is interpretable for each* $n \in \mathbb{N}$*;*
- *there is an integer m such that, for any $b \in B$, the set $f^{-1}(b)$ is infinite whenever it contains at least m elements.*

If the group G is of finite Morley rank, then for each non-empty definable set A, the integer $\mathrm{rk}(A)$ is the *Morley rank* of A.

The main class of examples of groups of finite Morley rank is algebraic groups defined over an algebraically closed field (see [28] for more details). Indeed, if we consider an algebraically closed field structure $\mathcal{K} = (K; +, ., \ ^{-1}, -, 0, 1)$, then one can show that the underlying set K has Morley rank one. Moreover, one can that any structure definable in a structure of finite Morley rank is of finite Morley rank too. Thus algebraic groups over algebraically closed fields are groups of finite Morley rank. In this case, the definable sets are the constructible ones, and the Morley rank is the Zariski dimension. The main conjecture on groups of finite Morley rank concerns their links with algebraic groups in the simple case. It is the following one.

Cherlin-Zil'ber Conjecture 2.2. – Any infinite simple group of finite Morley rank is isomorphic as abstract group to a linear algebraic group defined over an algebraically closed field.

At this stage, we have to note that, by the following result, the fields of finite Morley rank are the finite ones, and the algebraically closed ones. Thus, in this paper, *all the infinite fields are algebraically closed*.

Fact 2.3. – [5, Theorem 8.1] *A field definable in a structure of finite Morley rank is either finite or algebraically closed.*

2.3. **Descending chain condition on definable subgroups.** The class of groups of finite Morley rank is a subclass of *stable groups* (see [27]). These last groups satisfy the *descending chain condition on definable subgroups*. In other words, in a group of finite Morley rank, there is no infinite descending chain of definable subgroups. This leads several natural notions, and very useful in this paper.

Indeed, by descending chain condition on definable subgroups, any definable subgroup H of a group G of finite Morley rank has a smallest definable subgroup of finite index. The latter is the *connected component* of H, and it is denoted by H°. The link between the connected component of a group of finite Morley rank and the *identity component* of an algebraic group is clarified by Lemma 4.1.

Naturally, a group G of finite Morley rank is said be *connected* if it is equal to its connected component.

Again by descending chain condition on definable subgroups, for any subset X of a group G of finite Morley rank, the intersection of all the definable subgroups of G which contain X is a definable subgroup of G. The latter is the *definable hull* of X, and it is denoted by $d(X)$. This notion offers an analogue to the Zariski closure in algebraic geometry.

An infinite normal definable subgroup A of a group G of finite Morley rank is said to be S-*minimal*, where S is any subset of G, if A is infinite, definable, normalized by S and minimal for these conditions. By descending chain condition on definable subgroups, for each subset S of G, any infinite normal definable subgroup B of G contains an S-minimal subgroup. It is noticeable that, when G is solvable and connected, then any S-minimal subgroup of G is abelian [5, Proposition 7.7].

2.4. **Some fundamental theorems.** The Zil'ber's Indecomposability Theorem is one of fundamental result of the theory of groups of finite Morley rank. It is rather technical, and we refer to [28, 5] for its complete statement, which is not directly used in this paper. However, its important corollaries are away used.

Fact 2.4. – *Let G be a group of finite Morley rank. Then the following assertions are true.*

- *(Special case of* [5, Theorem 5.26]*) The subgroup generated by a set of definable connected subgroups of G is definable and connected.*
- [5, Corollary 5.29] *If H is a definable connected subgroup of G, the subgroup $[H, X]$ is definable and connected for any subset X of G.*
- [5, Corollary 5.38] *If H is a solvable (resp. nilpotent) subgroup of class n of G, then $d(H)$ is also solvable (resp. nilpotent) of class n.*

The following result due also to B. Zil'ber confers a central importance on fields of finite Morley rank. We recall that, by Fact 2.3, an infinite field of finite Morley rank is always algebraically closed.

Fact 2.5. – [5, Theorem 9.1] *Let $G = A \rtimes H$ be a group of finite Morley rank where A and H are two infinite definable abelian subgroups, A is H-minimal and $C_H(A) = 1$. Then G interprets an algebraically closed field K such that $A \simeq K_+$ definably, and H is definably isomorphic to a subgroup of K^*.*

B. Poizat proved the following two results, which will be very useful in our context.

Fact 2.6. – [28, Théorème 4.15] *Let F be an algebraically closed field. Then, in the pure field F, every infinite definable field K is definably isomorphic to F.*

Fact 2.7. – [28, Corollaire 3.3] *Let K be a field of finite Morley rank of characteristic zero. Then K_+ has no nontrivial proper definable subgroup.*

2.5. **Decent tori and pseudo-tori.** In [15], G. Cherlin defines *decent tori* as an analogue to algebraic tori. In [20], we introduce *pseudo-tori*, as a more general notion, independant of torsion. Here we relate just the more general results used in this paper concerning pseudo-tori, but their proofs are very often similar to their analogues in [15] concerning decent tori. Moreover, pseudo-tori are the basis of $\widetilde{U}$*-groups* defined below.

Definition 2.8. – *Let T be a radicable abelian group of finite Morley rank.*

- *We say that T is a* decent torus *if T is the definable hull of its torsion.*

- *We say that T is a* pseudo-torus *if no definable quotient of T is definably isomorphic to the additive group K_+ of an interpretable field K.*

We note that any decent torus is a pseudo-torus, and any pseudo-torus is connected. In Fact 2.10, we summarize the main properties of pseudo-tori. Before stating it, we have to recall that the *Fitting subgroup* $F(G)$ of an arbitrary group G is the subgroup generated by all the normal nilpotent subgroups of G. In groups of finite Morley rank, its main property is its definability.

Fact 2.9. – [5, Theorem 7.3]. *In any group G of finite Morley rank, the Fitting subgroup is nilpotent and definable.*

Fact 2.10. – *Let G be a group of finite Morley rank. Then,*

(i) [20, Theorem 1.7] *the maximal pseudo-tori of G are conjugate;*
(ii) [20, Proposition 2.7] *for any pseudo-torus T of G, $N_G(T)^\circ$ centralizes T;*
(iii) [20, Corollaries 2.8 and 2.9] *$F(G)$ has a unique maximal pseudo-torus, and this one is central in G°;*
(iν) [20, Corollary 2.13] *if N is a normal definable subgroup of G, the maximal pseudo-tori of G/N are the images of the maximal pseudo-tori of G.*

2.6. **Unipotence.** In [10], J. Burdges introduced some analogues of the algebraic unipotence for groups of finite Morley rank. In [17, 18], we continued the analysis of these concepts. Here are considered *$\widetilde{U}$-groups* introduced in [18]. This notion heavily depends on pseudo-tori [20]. We summarize the general results needed, and we refer to [10, 17, 18] for a more complete introduction to these unipotence notions.

To obtain a notion analogous to the unipotence in algebraic groups, Burdges [10] first introduces the notion of *indecomposable group*.

An abelian connected group A of finite Morley rank is *indecomposable* if it is not the sum of two proper definable subgroups. If $A \neq 1$, then A has a unique maximal proper definable connected subgroup $J(A)$, and if $A = 1$, let $J(1) = 1$.

The first result is nontrivial and is consequence of Fact 2.10 (iν) above.

Fact 2.11. – [20, Lemma 2.2] *Let G be a group of finite Morley rank and H a definable normal subgroup of G. If $\overline{B}$ is a radicable indecomposable definable abelian subgroup of G/H, then there is an indecomposable definable abelian subgroup A of G such that $\overline{B} = AH/H$.*

In particular, Fact 2.11 says that any (nonnecessarily abelian) radicable group of finite Morley rank is generated by its indecomposable definable abelian subgroups.

We recall the definitions and properties of $\widetilde{U}$-groups.

Notation 2.12. – *For any group G of finite Morley rank and any interpretable field K of characteristic zero, we denote by $U_K(G)$ the subgroup of G generated by its indecomposable definable abelian subgroups A such that $A/J(A)$ is definably isomorphic to K_+.*

Definition 2.13. –

- *A group G of finite Morley rank is said to be a* U_K-group*, where K is an interpretable field of characteristic zero, if $G = U_K(G)$. We say that a U_K-group G is* homogeneous *if each definable connected subgroup of G is a U_K-subgroup.*

- *For every group G of finite Morley rank, we denote by $\widetilde{U}(G)$ the subgroup of G generated by its normal homogeneous U_K-subgroups, for the interpretable fields K of characteristic zero, and by its normal definable connected subgroups of bounded exponent.*
- *A $\widetilde{U}$-group is a group G of finite Morley rank such that $G = \widetilde{U}(G)$.*

Remark 2.14. –

- By Fact 2.7, any radicable indecomposable definable abelian subgroup is either a pseudo-torus or a U_K-group for an interpretable field K of characteristic 0.
- By Fact 2.10 (iν) and Fact 2.17 below, in any $\widetilde{U}$-group, each pseudo-torus is trivial.

The below facts 2.15-2.21 comes from [18], but their proofs are not given in [18]. Indeed, they are mainly obtained by using [20] in place of [15], and their proofs are similar to the ones of [10, 11, 17].

Fact 2.15. – (see [18, Fact 3.10] and [11, Theorem 3.4]) *Let G be a radicable nilpotent group of finite Morley rank, and let T be its maximal pseudo-torus. Then G interprets some fields $K_1, \cdots, K_n$ of characteristic zero such that*

$$G = T * U_{K_1}(G) * U_{K_2}(G) * \cdots * U_{K_n}(G),$$

where $*$ denotes the central product.

Fact 2.16. – (see [18, Fact 3.11] and [17, Theorem 4.11]) *Let G be a connected group of finite Morley rank, and K an interpretable field of characteristic zero. Assume that G acts definably by conjugation on H, a nilpotent U_K-group. Then $[G, H]$ is a homogeneous U_K-subgroup.*

Fact 2.17. – (see [18, Fact 3.13] and [17, Theorem 5.4]) *Let G be a nilpotent $\widetilde{U}$-group. Then G interprets some algebraically closed fields $K_1, \cdots, K_n$ of characteristic zero such that the following decomposition holds:*

$$G = B \times U_{K_1}(G) \times U_{K_2}(G) \times \cdots \times U_{K_n}(G)$$

where B is a definable connected characteristic subgroup of bounded exponent, and $U_{K_s}(G)$ a homogeneous U_{K_s}-subgroup (for $s \in \{1, 2, \ldots, n\}$).

Fact 2.18. – (see [18, Fact 3.15] and [17, Proposition 5.7]) *Let G be a torsion-free group of finite Morley rank containing no nontrivial pseudo-torus. Then G is a $\widetilde{U}$-group if and only if, for each interpretable field K of characteristic zero, $U_K(G)$ is a homogeneous U_K-subgroup.*

Fact 2.19. – (see [18, Fact 3.16] and [17, Corollary 5.8])

- *Every definable quotient of a $\widetilde{U}$-group is a $\widetilde{U}$-group.*
- *Every definable connected subgroup of a $\widetilde{U}$-group is a $\widetilde{U}$-group.*

Fact 2.20. – (see [18, Fact 3.18] and [10, Lemma 2.11]) *Let G be a group of finite Morley rank, U and V be two definable subgroups with V normal in G, and K be an interpretable field of characteristic zero. Then $U_K(UV/V) = U_K(U)V/V$.*

Fact 2.21. – (see [18, Fact 3.25] and [17, Results 5.8, 6.12 and 6.20]) *Let G be a solvable connected group of finite Morley rank. Then $F(G)/Z(G)$ is a $\widetilde{U}$-group.*

Furthermore, the proof of [17, Theorem 6.10] applied with $\widetilde{U}$-groups provides the following result.

Fact 2.22. – (see [17, Theorem 6.10]) *Let G be a connected group of finite Morley rank. Assume that G acts definably by conjugation on H a solvable connected group of finite Morley rank. Then $[G, H]$ is a $\widetilde{U}$-group.*

3. The ACF-expansion of a pure group

First we focus on a very particular algebraic group. We consider the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and the following subgroup of $\mathrm{GL}(3, \overline{\mathbb{Q}})$:

$$\left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^*,\ (a,u,v) \in \overline{\mathbb{Q}}^3 \right\}.$$

This group will play a key role for us. However, the following examples show that the notion of a pure group is not sufficiently rich to analyze it. In order to remedy to this problem, we will define the *ACF-expansion of a pure group* (Definition 3.2).

Example 3.1. –

(1) Let $\tilde{\mathbb{Q}}$ denote a proper elementary extension of the pure field $\overline{\mathbb{Q}}$. We consider the following solvable centerless connected algebraic group

$$\tilde{G} = \left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \tilde{\mathbb{Q}}^*,\ (a,u,v) \in \tilde{\mathbb{Q}}^3 \right\}.$$

Then $\tilde{G}$ has an abstract automorphism α such that $\alpha(T)$ is not Zariski closed for any maximal torus T of $\tilde{G}$. Thus the maximal tori of $\tilde{G}$ are not definable in the pure group $\tilde{G}$.

Furthermore, the latter implies that the automorphism α is not continuous for the Zariski topology of $\tilde{G}$. In particular, α is not standard.

Indeed, since $\tilde{\mathbb{Q}}$ is a proper extension of $\overline{\mathbb{Q}}$, it has a nonzero derivation δ. Then we consider the map $\alpha : \tilde{G} \to \tilde{G}$ defined by

$$\alpha \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t & a+\delta(t) & u+\delta(v) \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix},$$

for each $t \in \tilde{\mathbb{Q}}^*$ and each $(a,u,v) \in \tilde{\mathbb{Q}}^3$. It is an abstract automorphism of $\tilde{G}$, so for each subset X of $\tilde{G}$, the set X is definable in the pure group $\tilde{G}$ if and only if $\alpha(X)$ is definable in the pure group $\tilde{G}$.

We verify $\alpha(T)$ is not Zariski closed for any maximal torus T of $\tilde{G}$. By conjugacy of maximal tori in $\tilde{G}$, we may assume that T is the diagonal subgroup of $\tilde{G}$. Then we have

$$T \cap \alpha(T) = \{diag(t,t,1) \mid t \in \tilde{\mathbb{Q}}^*,\ \delta(t) = 0\},$$

so neither $T \cap \alpha(T)$, nor $\alpha(T)$ is Zariski closed.

(2) Now we consider the pure group

$$G = \left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^*,\ (a,u,v) \in \overline{\mathbb{Q}}^3 \right\}.$$

By Lemma 3.10 below, all of its abstract automorphisms are standard. Nevertheless, we cannot apply Fact 1.5 to G, since G does not verify its hypotheses.

Indeed, otherwise, in the pure group G, there would be an interpretable field K and a definable isomorphism ν from G to an algebraic group H over K. In particular, K and ν would be definable in the pure field $\overline{\mathbb{Q}}$, so $\nu(T)$ would be a maximal torus of H for each maximal torus T of G. Then, by the definability of K in G, the subgroups $\nu(T)$ and T would be definable in G. But $\tilde{\mathbb{Q}}$ is a proper elementary extension of the pure field $\overline{\mathbb{Q}}$, so the pure group $\tilde{G}$ is an elementary extension of the pure group G, and the extension $\tilde{T}$ of T to $\tilde{G}$ is a maximal torus of $\tilde{G}$. Hence $\tilde{T}$ would be definable in the pure group $\tilde{G}$, contradicting the previous example.

We note that the argument above shows that, in the pure group G,

- there is no definable isomorphism from G to an algebraic group over an interpretable field.
- the maximal tori of G are not definable.

(3) The algebraic groups $\tilde{G}$ and G above are solvable of class two, centerless and connected. However, we obtain the same conclusions by considering the algebraic groups $\tilde{H}$ and H below instead of $\tilde{G}$ and G respectively, whereas $\tilde{H}$ and H are perfect, centerless and connected.

Indeed, consider the following perfect connected affine algebraic group

$$R = \left\{ \begin{pmatrix} a & b & r & s \\ c & d & t & u \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \mid (a,b,c,d,r,s,t,u) \in \overline{\mathbb{Q}}^8,\ ad-bc=1 \right\},$$

and the group $H := R/Z(R)$. Then H is a perfect centreless connected algebraic group over $\overline{\mathbb{Q}}$.

Now let $\tilde{\mathbb{Q}}$ be as above, and let $\tilde{R}$ and $\tilde{H}$ be the elementary extensions of R and H respectively to $\tilde{\mathbb{Q}}$. Then $\tilde{H}$ is a perfect centreless connected algebraic group over $\tilde{\mathbb{Q}}$.

If we consider a nonzero derivation δ of $\tilde{\mathbb{Q}}$, and the abstract automorphism β of $\tilde{R}$ defined by

$$\beta \begin{pmatrix} a & b & r & s \\ c & d & t & u \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} = \begin{pmatrix} a & b & r+\delta(a) & s+\delta(b) \\ c & d & t+\delta(c) & u+\delta(d) \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix},$$

for each $(a,b,c,d,r,s,t,u) \in \tilde{\mathbb{Q}}^8$ satisfying $ad-bc=1$. Then β induces an automorphism $\overline{\beta}$ of $\tilde{H}$. Arguing as for the automorphism α above, we show that $\overline{\beta}$ is not continuous for the Zariski topology of $\tilde{G}$. In particular, $\overline{\beta}$ is not standard.

Thus, as in (2), H does not verify the hypotheses of Fact 1.5. Even so, since H is not a direct product of two proper subgroups, it follows from Fact 3.6, Lemma 5.6 and Theorem 7.13 below, that all the abstract automorphisms of H are standard.

The three examples in 3.1 above show that the pure group structure of a connected affine algebraic group may not be as rich as its definable geometric structure, even when the group is centerless. This motivates the following definition. *We note that, in this paper, the word* definable *signifies* definable with parameters.

Definition 3.2. – *Let $\mathscr{T}$ be a theory in a language $\mathscr{L}$, and let $\mathscr{M} = (M; \cdots)$ be a structure in a language $\mathscr{L}'$. We assume that $\mathscr{M}$ is interpretable in a model of $\mathscr{T}$.*

- *Let $n \in \mathbb{N}$ and $A \subseteq M^n$. Then A is said to be $\mathscr{T}$-*definable *if, for each isomorphism f from $\mathscr{M}$ to another $\mathscr{L}'$-structure $\mathscr{M}_1$, and for each model $\mathscr{N}$ of $\mathscr{T}$ interpreting $\mathscr{M}_1$, the set $f(A)$ is definable in $\mathscr{N}$.*
- *For each $n \in \mathbb{N}$, let $\mathscr{A}_n$ be the family of the $\mathscr{T}$-definable subsets of M^n and, for each $A \in \mathscr{A}_n$, let $\mathscr{R}_A$ be a symbol of an n-ary relation. We consider $\mathscr{A} = \cup_{n\in\mathbb{N}}\mathscr{A}_n$ and $\mathscr{L}^* = \mathscr{L}' \cup (\bigcup_{A\in\mathscr{A}} \mathscr{R}_A)$.*
- *The $\mathscr{T}$-*expansion of *$\mathscr{M} = (M; \cdots)$ is the $\mathscr{L}^*$-expansion $\mathscr{M}^* = (M; \cdots)$ of $\mathscr{M}$, where $\mathscr{R}_A^{\mathscr{M}^*} = A$ for each $A \in \mathscr{A}$.*

By the following results, if a structure $\mathscr{M}$ is interpretable in a model of a theory $\mathscr{T}$, then any automorphism of $\mathscr{M}$ preserves the definable sets of its $\mathscr{T}$-expansion $\mathscr{M}^*$ (Corollary 3.5). Thus, in our context where we consider some group isomorphisms, and since $\mathscr{M}^*$ seems to be richer than $\mathscr{M}$, it appears more interesting to work in $\mathscr{M}^*$ than in $\mathscr{M}$.

Lemma 3.3. – *Let $\mathscr{T}$ be a theory in a language $\mathscr{L}$, and let $\mathscr{M} = (M; \cdots)$ be a structure interpretable in a model of $\mathscr{T}$. Then, for each $n \in \mathbb{N}$ and each $A \subseteq M^n$, the set A is definable in the $\mathscr{T}$-expansion $\mathscr{M}^*$ of $\mathscr{M}$ if and only if it is $\mathscr{T}$-definable.*

Proof – We may assume that A definable in $\mathscr{M}^*$, and we have just to prove that A is $\mathscr{T}$-definable. We assume toward a contradiction that A is not $\mathscr{T}$-definable. In particular, it is not definable in $\mathscr{M}$, its complement is not $\mathscr{T}$-definable, and it is neither a union nor an intersection of two $\mathscr{T}$-definable sets. Moreover, it is not of the form $M \times X$ for a $\mathscr{T}$-definable set X, and there is no $\mathscr{T}$-definable subset X of M^{n+1} such that $A = \pi(X)$ where $\pi : M^{n+1} \to M^n$ denotes the projection map defined for any $(x_1, \ldots, x_{n+1}) \in M^{n+1}$ by $\pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$. Hence [27, Proposition 1.3.4] says that we may assume that there exists $m \in \mathbb{N}$, a $\mathscr{T}$-definable subset X of M^{n+m} and $b \in M^m$ such that $A = \{a \in M^n \mid (a, b) \in X\}$. But again in this last case, it follows from the definition that A is $\mathscr{T}$-definable, as claimed. □

Corollary 3.4. – *Let $\mathscr{T}$ be a theory in a language $\mathscr{L}$, and let $\mathscr{M} = (M; \cdots)$ and $\mathscr{M}' = (M'; \cdots)$ be two $\mathscr{L}'$-structures interpretable in some models of $\mathscr{T}$, where $\mathscr{L}'$ is another language. Let $f : \mathscr{M} \to \mathscr{M}'$ be an isomorphism, and let $\mathscr{M}^*$ and $\mathscr{M}'^*$ be the $\mathscr{T}$-expansions of $\mathscr{M}$ and $\mathscr{M}'$ respectively. Then, for each $n \in \mathbb{N}$ and each $A \subseteq M^n$ definable in $\mathscr{M}^*$, the set $f(A)$ is definable in $\mathscr{M}'^*$.*

Proof – Let g be an isomorphism from $\mathscr{M}'$ to another $\mathscr{L}'$-structure $\mathscr{M}_1$. We assume that $\mathscr{M}_1$ is interpretable in a model $\mathscr{N}$ of $\mathscr{T}$. Then $g \circ f$ is an isomorphism from $\mathscr{M}$ to $\mathscr{M}_1$. Now, since A is $\mathscr{T}$-definable by Lemma 3.3, the set $g(f(A))$ is definable in $\mathscr{N}$. This proves that $f(A)$ is definable in $\mathscr{M}'^*$. □

Corollary 3.5. – *Let $\mathscr{T}$ be a theory in a language $\mathscr{L}$, and let $\mathscr{M}$ be a structure interpretable in a model of $\mathscr{T}$. Then any automorphism of $\mathscr{M}$ preserves the definable sets of its $\mathscr{T}$-expansion $\mathscr{M}^*$.*

It is noticeable that, by the previous result, the proof of [28, Corollaire 4.17] provides the very general following fact, which is fundamental for us.

Fact 3.6. – [28, Proof of Corollaire 4.17] *Let α be an abstract isomorphism between two algebraic groups G and H over algebraically closed fields. If, in the ACF-expansion of the pure group G, there are an interpretable field K and a definable isomorphism from a definable section U/V to an algebraic group over K, then the isomorphism $\alpha_{|U/V} : U/V \to \alpha(U)/\alpha(V)$ is standard.*

Furthermore, we notice that the definition of the $\mathscr{T}$-expansion of a structure $\mathscr{M}$ for a theory $\mathscr{T}$ is rather robust.

Lemma 3.7. – *Let $\mathscr{T}$ be a theory in a language $\mathscr{L}$, and let $\mathscr{M}$ be a structure in a language $\mathscr{L}'$. We assume that $\mathscr{M}$ is interpretable in a model of $\mathscr{T}$. Let $\mathscr{M}_0$ be another structure, interpretable in the $\mathscr{T}$-expansion of $\mathscr{M}$. If A is a set, definable in the $\mathscr{T}$-expansion of $\mathscr{M}_0$, then A is definable in $\mathscr{T}$-expansion of $\mathscr{M}$ too.*

Proof – Let M (resp. M_0) be the domain of $\mathscr{M}$ (resp. $\mathscr{M}_0$). Let A be a subset of M_0^n for $n \in \mathbb{N}$, with A definable in the $\mathscr{T}$-expansion of $\mathscr{M}_0$. Since $\mathscr{M}_0$ is interpretable in the $\mathscr{T}$-expansion of $\mathscr{M}$, there are $m \in \mathbb{N}$, a subset B of M^m and an equivalence relation R over B, with B and R definable in the $\mathscr{T}$-expansion of $\mathscr{M}$, such that $M_0 = B/R$. We denote by A_M the preimage of A in $(M^m)^n$. Then we have just to prove that A_M is definable in the $\mathscr{T}$-expansion of $\mathscr{M}$.

Let f be an isomorphism from $\mathscr{M}$ to another $\mathscr{L}'$-structure $\mathscr{M}_1$, and let $\mathscr{N}$ be a model of $\mathscr{T}$ interpreting $\mathscr{M}_1$. Then f induces an isomorphism $\overline{f}$ from $\mathscr{M}_0$ to a structure $\mathscr{M}_B$ with domain $f(B)/f(R)$. Moreover, since the structure $\mathscr{M}_0$ is interpretable in the $\mathscr{T}$-expansion of $\mathscr{M}$, then $\mathscr{M}_B$ is interpretable in $\mathscr{N}$. Now, since $\overline{f}$ is an isomorphism from $\mathscr{M}_0$ to $\mathscr{M}_B$, then $\overline{f}(A)$ is definable in $\mathscr{N}$. Thus $f(A_M)$ is definable in $\mathscr{N}$, and A is definable in the $\mathscr{T}$-expansion of $\mathscr{M}$. □

Nevertheless, by the following remark, this notion is not preserved in elementary extensions.

Remark 3.8. – Let $\tilde{G}$ and G be as in Example 3.1 (1) and (2) respectively. Then the pure group $\tilde{G}$ is an elementary extension of G. Moreover, if T denotes any maximal torus of G, then its extension $\tilde{T}$ to $\tilde{G}$ is a maximal torus of $\tilde{G}$. But T is definable in the ACF-expansion of the pure group G by Lemma 3.11 below, while that $\tilde{T}$ is not definable in the ACF-expansion of the pure group $\tilde{G}$. Indeed, if α is as in Example 3.1 (1), then $\alpha(\tilde{T})$ is not Zariski closed (see Example 3.1 (1)), so neither $\alpha(\tilde{T})$, nor $\tilde{T}$ by Corollary 3.5, is definable in the ACF-expansion of the pure group $\tilde{G}$.

From now on, we study the abstract automorphisms of some algebraic groups.

Lemma 3.9. – *Let K be an algebraically closed field of characteristic zero. Then each abstract automorphism α of the group*

$$G = \left\{ \begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix} \mid t \in K^*,\ a \in K \right\}$$

has the form $\alpha = \beta \circ \mu$ for μ a field automorphism of K, and β an inner automorphism of G.

Proof – Let $u \in K \setminus \{-1, 0, 1\}$, $x = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. Then we have $G = A \rtimes H$ for $A = C_G(x)$ and $H = C_G(y)$. In particular, A and H are definable in the pure group G, and Fact 2.5 says that G interprets an algebraically closed field L such that A is definably isomorphic to L_+ and such that H is definably isomorphic to a subgroup of L^*, acting by multiplication on A. But, in the pure field K, the fields K and L are definably isomorphic by Fact 2.6, so L^* has no proper infinite definable subgroup, and we obtain $H \simeq L^*$ definably. Hence the groups $G = A \rtimes H$ and $L_+ \rtimes L^* \simeq \left\{ \begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix} \mid t \in L^*,\ a \in L \right\}$ are isomorphic, definably in the pure group G. Thus, by Fact 3.6 (or Fact 1.5), we have $\alpha = \beta \circ \mu$ for μ a field automorphism of K, and β an algebraic automorphism of G.

We show that β is an inner automorphism of G. Since H is abelian and self-normalizing in G, it is a Cartan subgroup, and $\beta(H)$ is a Cartan subgroup too. Hence H and $\beta(H)$ are conjugate, and we may assume that β stabilizes H. The action of H on $A \setminus \{0\}$ is transitive, so there exists $h \in H$ such that $\beta(x) = x^h$, and we may assume that β fixes x. Therefore, since $A \simeq K_+$ is torsion-free of dimension one, and since x is nontrivial, β centralizes A. Thus we have $\beta(x^y) = x^y$. But if β inverts H, then we have $\beta(x^y) = x^{y^{-1}}$, and y^2 centralizes x, contradicting $u \notin \{-1, 0, 1\}$. Hence β does not invert H. Since H is a torus of dimension one, its algebraic automorphisms are the identity map and the inversion map, so we obtain $\beta(h) = h$ for each $h \in H$, and β centralizes $G = AH$, proving the result. □

Lemma 3.10. – *Let K be an algebraically closed field of characteristic zero. Then each abstract automorphism α of the group*

$$G = \left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in K^*,\ (a, u, v) \in K^3 \right\}$$

has the form $\alpha = \beta_{r,\delta} \circ \gamma \circ \mu$ for μ a field automorphism of K, γ an inner automorphism of G and $\beta_{r,\delta}$ an abstract automorphism of G such that, for each $t \in K^*$ and each $(a, u, v) \in K^3$, we have

$$\beta_{r,\delta}(\begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} t & ra + \delta(t) & ru + \delta(v) \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix},$$

where δ is a derivation of K and r an element of K.

Proof – The unipotent part of G is its Fitting subgroup, so it is characteristic and definable in the pure group G, and its center $Z = \left\{ \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid u \in K \right\}$ is characteristic and definable too. Also the preimage in G of the center of G/Z is $U = \left\{ \begin{pmatrix} 1 & a & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid (a, u) \in K^2 \right\}$, so U is characteristic and definable in the pure group G. Moreover, there is an isomorphism of algebraic groups between the quotient group G/U and $\left\{ \begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix} \mid t \in K^*,\ a \in K \right\}$, and Lemma 3.9 shows that the automorphism $\overline{\alpha}$ of G/U induced by α has the form $\overline{\alpha} = \overline{\gamma} \circ \mu$ for μ a field automorphism of K, and $\overline{\gamma}$ an inner automorphism of G/U. Hence we may assume that α centralizes G/U, and we have just to prove that α has the form $\alpha = \beta_{r,\delta} \circ \gamma$ for γ an inner automorphism of G induced by an element of U.

We consider

$$C = \left\{ \begin{pmatrix} t & a & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid t \in K^*,\ a \in K \right\}.$$

This subgroup is abelian and self-normalizing in G, so it is definable in the pure group G, and it is a Cartan subgroup of G. In particular, C and $\alpha(C)$ are two Cartan subgroups of $V = UC = Z \rtimes C$, and they are conjugate by an element of $Z \leq U$. Hence we may assume that α normalizes C and we have just to prove that α has the form $\alpha = \beta_{r,\delta} \circ \gamma$ for γ an inner automorphism of G induced by an element of $C \cap U$.

The center of V is $C \cap U$, and there is an isomorphism of algebraic groups between $V/(C \cap U)$ and

$$\left\{ \begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix} \mid t \in K^*,\ a \in K \right\}.$$

Then Lemma 3.9 shows that the automorphism $\overline{\alpha}$ of $V/(C \cap U)$ induced by α has the form $\overline{\alpha} = \overline{\gamma} \circ \mu$ for μ a field automorphism of K, and $\overline{\gamma}$ an inner automorphism of $V/(C \cap U)$. Since $U/(C \cap U)$ is the unique nontrivial normal abelian subgroup of $V/(C \cap U)$, it is characteristic in $V/(C \cap U)$, and $\overline{\gamma}$ and μ normalize it. But V/U is abelian, so $\overline{\gamma}$ centralizes it, and since α centralizes G/U, the field automorphism μ centralizes V/U too. Furthermore, V/U is a torus of dimension one, hence μ is a field automorphism centralizing K^* and, consequently, μ is the identity map. Now $\overline{\alpha} = \overline{\gamma}$ is an inner automorphism of $V/(C \cap U)$, and there exists $r \in K$ such that, for each $t \in K^*$ and each $(a, u) \in K^2$, there is $b \in K$ such that $\alpha(\begin{pmatrix} t & a & u \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} t & b & ru \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Moreover, since α normalizes C, the element b depends just on t and a, and there is a map $\nu_0 : K^* \times K \to K$ such that for each $t \in K^*$ and each $a \in K$, we have:

$$\alpha(\begin{pmatrix} t & a & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} t & \nu_0(t,a) & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In addition, we have $G' = \left\{ \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \mid (u,v) \in K^2 \right\}$, and α centralizes G/U, so there is a map $\mu_0 : K \times K \to K$ such that, for each $(u, v) \in K^2$, we have

$$\alpha(\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & \mu_0(u,v) \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix},$$

and μ_0 satisfies $\mu_0(u,0) = ru$ for each $u \in K$. Thus, for each $t \in K^*$ and each $(a,u,v) \in K^3$, we have

$$\alpha(\begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix}) = \alpha(\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}) \cdot \alpha(\begin{pmatrix} t & a & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} t & \nu_0(t,a) & \mu_0(u,v) \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix}.$$

By considering $t \in K^*$, $(a,u,v) \in K^3$ and the equality $\alpha(xy) = \alpha(x)\alpha(y)$ for $x = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (resp. $x = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$) and $y = \begin{pmatrix} 1 & t^{-1}a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (resp. $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$), we obtain $\nu_0(t,a) = \nu_0(t,0) + t\nu_0(1,t^{-1}a)$ (resp. $\mu_0(u,v) = ru + \mu_0(0,v)$). Moreover, by considering $a \in K$, and the equality $\alpha(xy) = \alpha(x)\alpha(y)$ for $x = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, we obtain $\mu_0(a,1) = \mu_0(0,1) + \nu_0(1,a)$. Since we have $\mu_0(a,1) = ra + \mu_0(0,1)$, this yields $\nu_0(1,a) = ra$ for each $a \in K$, and $\nu_0(t,a) = \nu_0(t,0) + ra$ for each $(t,a) \in K^* \times K$.

Now we consider the maps $\nu : K \to K$ and $\mu : K \to K$ defined by $\nu(0) = 0$ and, for each $t \in K^*$ and each $v \in K$, $\nu(t) = \nu_0(t,0)$ and $\mu(v) = \mu_0(0,v)$. Then, for each $t \in K^*$, $(a,u,v) \in K^3$, we have

$$\alpha(\begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} t & ra+\nu(t) & ru+\mu(v) \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that we have $\nu(1) = 0$ and $\mu(0) = 0$. Moreover, if γ denotes the conjugation by $\begin{pmatrix} 1 & -r^{-1}\mu(1) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in C \cap U$, we obtain

$$(\alpha \circ \gamma^{-1})(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

so we may assume $\mu(1) = 0$, and we have just to prove that $\alpha = \beta_{r,\delta}$ for a derivation δ of K.

We consider $t \in K^*$ and the equality $\alpha(xy) = \alpha(x)\alpha(y)$ for $x = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Since $\mu(1) = 0$, we obtain $\nu(t) = \mu(t)$, and since $\nu(0) = 0 = \mu(0)$, we find $\nu = \mu$. From now on, we have just to prove that ν is a derivation of K.

Firstly, for each $(v,v') \in K^2$, by considering the equality $\alpha(xy) = \alpha(x)\alpha(y)$ for $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v' \\ 0 & 0 & 1 \end{pmatrix}$, we find $\nu(v+v') = \nu(v) + \nu(v')$.

Secondly, for each $(t,t') \in (K^*)^2$, by considering the equality $\alpha(xy) = \alpha(x)\alpha(y)$ for $x = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} t' & 0 & 0 \\ 0 & t' & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we find $\nu(tt') = t\nu(t') + t'\nu(t)$. Since we have $\nu(tt') = 0 = t\nu(t') + t'\nu(t)$ as soon as $t = 0$ or $t' = 0$, the map ν is indeed a derivation, and this finishes the proof. $\square$

Proposition 3.11. – *We consider the group*

$$G = \left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^*,\ (a,u,v) \in \overline{\mathbb{Q}}^3 \right\}.$$

Then its maximal tori are definable in the ACF-expansion of the pure group G.

Proof – We consider another algebraically closed field L, a pure group G_L interpretable in L, and an abstract group isomorphism $\mu : G \to G_L$. We remark that G has no proper subgroup of finite index, so the groups G and G_L are connected as pure groups as well as algebraic groups. We consider the subgroup H of G formed by the matrices $\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $u \in \overline{\mathbb{Q}}$. Then H is a normal definable subgroup of the pure group G, since it is the centralizer in G of the subgroup M formed by the matrices $\begin{pmatrix} 1 & a & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$ for $(a,u,v) \in \overline{\mathbb{Q}}^3$. We may consider the groups $H \rtimes G/M$ and $G'/H \rtimes G/M$ where G/M acts by conjugation on H and G'/H: they are definable in the pure group G because $M = C_G(H)$ is definable. Moreover, the actions are transitive on $H \setminus \{0\}$ and on $G'/H \setminus \{0\}$ respectively, and faithfull, so in the pure group G, there are two interpretable fields K_1 and K_2 such that H (resp. G'/H) is definably isomorphic to $(K_1)_+$ (resp. $(K_2)_+$) (Fact 2.5). We note also that, in the pure field $\overline{\mathbb{Q}}$, the fields K_1 and K_2 are definably isomorphic to $\overline{\mathbb{Q}}$ (Fact 2.6).

Now there are two fields L_1 and L_2, isomorphic to K_1 and K_2 respectively, and interpretable in the pure group G_L, such that $\mu(H)$ (resp. $\mu(G')/\mu(H)$) is definably isomorphic to $(L_1)_+$ (resp. $(L_2)_+$). So Fact 2.6 says that, in the pure field L, the fields L_1 and L_2 are definably isomorphic to L. In particular, this proves that the fields $\overline{\mathbb{Q}}$ and L are isomorphic. Moreover, $\mu(G')$ is an abelian group and, in the pure field L, the groups $\mu(H)$ and $\mu(G')/\mu(H)$ are definably isomorphic to L_+, so $\mu(G')$ is isomorphic to $L_+ \times L_+$, definably in the pure field L.

We consider a maximal torus T of G_L and its centralizer C. Then we have $G_L = CG'_L$, so $Z(C) \cap G'_L$ is central in G_L. Since G_L is centerless, $Z(C) \cap G'_L$ is trivial, and since C is nilpotent, we obtain $G_L = G'_L \rtimes C$. Since G'_L is a maximal abelian subgroup of G_L, the group $C \simeq G_L/G'_L$ acts faithfully by conjugation on G'_L, and C is isomorphic to an abelian definable subgroup of $\mathrm{GL}(2,L)$, definably in the pure field L. Consequently, in the pure field L, there is a definable isomorphism

f from G_L to a subgroup of

$$\left\{ \begin{pmatrix} t_1 & a & u \\ b & t_2 & v \\ 0 & 0 & 1 \end{pmatrix} \mid (t_1, t_2, a, b, u, v) \in L^6,\ t_1t_2 - ab \neq 0 \right\},$$

such that

- $f(\mu(H))$ is formed by the matrices where $t_1 = t_2 = 1$ and $a = b = v = 0$;
- $f(G'_L) = f(\mu(G'))$ by the matrices where $t_1 = t_2 = 1$ and $a = b = 0$;
- $f(C)$ by some matrices where $u = v = 0$.

In addition, since $\mu(H)$ is normal in G_L, we obtain $b = 0$ for each element of $f(G_L)$. Also, $\mu(M)/G'_L$ is an infinite abelian torsion-free subgroup of G_L/G'_L, definable in the pure group G_L, so $C \simeq G_L/G'_L$ contains a closed infinite torsion-free subgroup, that is a nontrivial unipotent subgroup. Since the unipotent part of

$$\left\{ \begin{pmatrix} t_1 & a & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid (t_1, t_2) \in (L^*)^2,\ a \in L \right\}$$

has dimension one, it is contained in $f(C)$, and we obtain

$$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in L \right\} \subseteq f(C) \subseteq \left\{ \begin{pmatrix} t_1 & a & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid (t_1, t_2) \in (L^*)^2,\ a \in L \right\}.$$

Since $f(C)$ is abelian, has torsion, and is connected because it is isomorphic to G/G' which is radicable, this forces $f(C)$ to be equal to

$$\left\{ \begin{pmatrix} t & a & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid t \in L^*,\ a \in L \right\},$$

and $f(G_L)$ is equal to

$$\left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in L^*,\ (a, u, v) \in L^3 \right\}.$$

Since the fields $\overline{\mathbb{Q}}$ and L are isomorphic, we may consider a field isomorphism $\delta : L \to \overline{\mathbb{Q}}$. Then $\delta \circ f \circ \mu$ is an abstract automorphism of G, and Lemma 3.10 shows that $\delta \circ f \circ \mu$ is a standard automorphism. So the image of any maximal torus of G by $\delta \circ f \circ \mu$ is a maximal torus of G. But δ is a field isomorphism, and the isomorphism f is definable in the pure field L, so the preimage by $\delta \circ f$ of any maximal torus of G is a maximal torus of G_L. Hence the image of any maximal torus of G by μ is a maximal torus of G_L. In particular, the image of any maximal torus of G is definable in L. Thus, the maximal tori of G are definable in the ACF-expansion of the pure group G. □

4. Unipotent groups over $\overline{\mathbb{Q}}$

In this paper, we study the structures $\mathcal{M} = (G, \cdot, ^{-1}, 1, \cdots)$ interpretable in a pure algebraically closed field K, where $(G, \cdot, ^{-1}, 1)$ is a group. Such a structure is said to be an *ACF-group* and, if p denotes the characteristic of K, $\mathcal{M}$ is said to be an *ACF_p-group*.

In particular, any ACF-group is a group of finite Morley rank, and each interpretable group in an ACF-group is an ACF-group too. Also, in the pure field K, any ACF-group G is constructible [28, §4.a], and there is an isomorphism definable in the pure field K, between G and an algebraic group over K [28, Théorème 4.13].

The connected component G° of an ACF-group G as a group of finite Morley rank is not necessarily equal to the one $G^{\circ\circ}$ of G as a constructible group. Indeed, if p is a prime integer, the pure group $K_+ \oplus \mathbb{F}_p$ is connected as a group of finite Morley rank, and it is not connected as a constructible group. However, by Lemma 4.1, the situation is different for ACF_0-groups. Then, in the rest of this paper, for any ACF_0-group G, we will denote by G° its connected component as a group of finite Morley rank, as well as a constructible group.

Lemma 4.1. – *Let G be an ACF_p-group for p a prime or zero. If $G^\circ \neq G^{\circ\circ}$, then p is a prime and $G^\circ/G^{\circ\circ}$ is an abelian p-group.*

Moreover, G has a normal definable connected subgroup I contained in $G^{\circ\circ}$ such that G°/I is an abelian p-group.

Proof – We notice that, if p is a prime and if G is an ACF_p-group such that $G^\circ/G^{\circ\circ}$ is a nontrivial abelian p-group, then $G^{\circ\circ}$ contains $(G^\circ)'$. Let $I/(G^\circ)'$ denote the image of the endomorphism f of $G^\circ/(G^\circ)'$ defined by $f(x) = x^n$ where n is the index of $G^{\circ\circ}$ in G°. Then I is a normal definable connected subgroup of G, contained in $G^{\circ\circ}$ and such that G°/I is an abelian p-group. Hence we have just to prove that, if $G^\circ \neq G^{\circ\circ}$, then p is a prime and $G^\circ/G^{\circ\circ}$ is an abelian p-group.

We may assume $G = G^\circ$ and we proceed by induction on the Morley rank of G. Let A be a G-normal definable connected subgroup of $G^{\circ\circ}$. If $A \neq 1$, the induction hypothesis applied with G/A provides the result. Hence we may assume that $G^{\circ\circ}$ contains no nontrivial G-normal definable connected subgroup. Then the Zil'ber's Indecomposability Theorem [5, Results 5.26 and 5.29] yields $[G^{\circ\circ}, G] = 1$. Now $Z(G)$ contains $G^{\circ\circ}$ and it is a definable subgroup of finite index in the connected group G, so $G = Z(G)$ is abelian.

If $p = 0$, we consider $N = \{g^n \mid g \in G\}$ where $n = |G/G^{\circ\circ}|$. It is a connected definable subgroup of the pure group G contained in $G^{\circ\circ}$, and the previous paragraph gives $N = 1$. Thus G has bounded exponent, and since G is a constructible group defined over an algebraically closed field of characteristic zero, it is finite. But G is connected, so $G = 1$, contradicting $G \neq G^{\circ\circ}$.

From now on, p is a prime. We denote by R the smallest subgroup of G containing $G^{\circ\circ}$ such that p does not divide the index $[G : R]$ of R in G. We show that $G = R$. We consider $N = \{g^n \mid g \in G\}$ where $n = |G/R|$. It is a connected definable subgroup of the pure group G contained in R. Thus, if $N \neq 1$, the induction hypothesis applied with G/N shows that $G/G^{\circ\circ}N$ is a p-group. Since G/N has no nontrivial p-element, we obtain $G = G^{\circ\circ}N = R$ as desired. If $N = 1$, then G has bounded exponent m, and p does not divide m. Since G is an ACF_p-group, it is constructible, and for any integer r, any subgroup of exponent r is finite, unless p divides r. But p does not divide m, so G is finite, and it is trivial since it is connected, contradicting $G \neq G^{\circ\circ}$. Thus we have proved that $G = R$, and since G is abelian, this implies that $G/G^{\circ\circ}$ is a p-group. □

The following notion is central for our paper.

Definition 4.2. – *A group of finite Morley rank is said to be* definably linear *(over finitely many interpretable fields $K_1, \dots, K_n$), if G has an interpretable faithful linear representation over the ring $K_1 \oplus \cdots \oplus K_n$.*

In other words, G definably embeds in $H_1 \times \cdots \times H_n$, where H_i is an affine algebraic group over K_i for each $i = 1, \dots, n$.

In this section, we prove our first theorem. Its proof is based on the previous section and on a Hochschild-Mostow's theorem (Facts 4.5 and 4.6).

Theorem 4.3. – *Let G be a connected nilpotent algebraic group over $\overline{\mathbb{Q}}$. Then, in the ACF-expansion of the pure group G, the quotient $G/Z(G)$ is definably linear.*

Remark 4.4. – This result fails when G is a nilpotent algebraic group over an algebraically closed field K of characteristic zero not isomorphic to $\overline{\mathbb{Q}}$.

Indeed, consider a nonzero derivation δ of K, the group

$$G = \left\{ \begin{pmatrix} 1 & a & b & x \\ 0 & 1 & a & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid (a, b, x, y, z) \in K^5 \right\},$$

and the group automorphism $\alpha : G \to G$ defined for each $(a, b, x, y, z) \in K^5$ by:

$$\alpha(\begin{pmatrix} 1 & a & b & x \\ 0 & 1 & a & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} 1 & a & b+\delta(a) & x+\delta(y) \\ 0 & 1 & a & y+\delta(z) \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then suppose toward a contradiction that $G/Z(G)$ is definably linear in the ACF-expansion of the pure group G. That is $G/Z(G)$ is definably isomorphic to $H_1 \times \cdots \times H_n$, where H_i is an affine algebraic group over an interpretable field K_i for each i (Lemma 5.6). Moreover, we may assume that, for each $i \neq j$, the fields K_i and K_j are not definably isomorphic, and that H_i is not trivial. Since H_i is a U_{K_i}-group for each i, and since G' contains $Z(G)$, Facts 2.15 and 2.20 provide $G = U_{K_1}(G) * \cdots * U_{K_n}(G)$, where $*$ denotes the central product, and Fact 2.16 gives $G' = U_{K_1}(G)' \times \cdots \times U_{K_n}(G)'$. If, for some i, we have $U_{K_i}(G)' = 1$, then $U_{K_i}(G)$ is central in G, contradicting that H_i is nontrivial. Since G is nilpotent, this implies that $Z(G) \cap U_{K_i}(G)'$ is infinite for each i. Since $Z(G)$ is torsion-free and of dimension one over K, its Morley rank is one in the ACF-expansion of the pure group G too, and we obtain $Z(G) \leq U_{K_i}(G)'$ for each i. Thus we find $n = 1$, and $G/Z(G)$ is definably linear over K_1.

From now on, since K_1 and K are definably isomorphic in the pure field K (Fact 2.6), the definable subsets of $G/Z(G)$ are precisely its constructible subsets. In particular, for each constructible subset $X/Z(G)$ of $G/Z(G)$, the image of X by any group automorphism of G is constructible too. But, if we set

$$X = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid (x, z) \in K^2 \right\},$$

then X is constructible and $X \cap \alpha(X)$ is not constructible, contradicting that $G/Z(G)$ is definably linear.

Our study starts with the analysis of some automorphisms of algebraic groups. The following theorem by Hochschild-Mostow, together with Lemma 4.7, has several consequences very useful for us.

An affine algebraic group is said to be *conservative* if the action of its algebraic automorphism group on its algebra of polynomial functions is locally finite. Actually, by [22], a group G is conservative if and only if the holomorph of G inherit an affine algebraic group structure with which it is the semidirect product, in the sense of affine algebraic groups, of G and its algebraic automorphism group.

Fact 4.5. – [22, Theorem 3.2] *Let G be a connected affine algebraic group over the algebraic closed field F of characteristic zero. Then G is conservative if and only if one of the following two conditions is satisfied:*

(1) *the center of G is finite over its unipotent part;*
(2) *the center of a maximal reductive subgroup of G has dimension at most 1.*

Actually, only a special case of this result will be used: when the group is generated by its unipotent elements. Fortunately, for this particular case, we can provide below a model theoretic proof for Fact 4.5.

Fact 4.6. – (Special case of Fact 4.5) *Let K be an algebraically closed field of characteristic zero, and let G be an affine algebraic group over K.* If G is generated by its unipotent elements*, then there is an affine algebraic group of the form $H = G \rtimes A$, where A is an algebraic subgroup acting faithfully on G by conjugation and such that, for each algebraic group automorphism φ of G, there exists $a \in A$ satisfying $\varphi(g) = g^a$ for each $g \in G$.*

Proof – We may assume $G \neq 1$. We consider the pure language of fields. Then G is an interpretable group and it has a finite Morley rank. Moreover, $G \times G$ definably embeds in $L = \mathrm{GL}_m(K)$ for a positive integer m. Since there is finitely many Jordan decompositions for unipotent elements in L, there is a finite subset X of nontrivial unipotent elements of $G \times G$ such that each element of the set $\mathscr{U}$ of the nontrivial unipotent elements of $G \times G$ is conjugate in L with a unique element of X. For each $x \in X$, let $V_x = d(x)$. Since each $x \in X$ is nontrivial and unipotent, V_x is definably isomorphic to K_+. Thus $\mathscr{F} = \{V_x^g \mid (x, g) \in X \times L, x^g \in G \times G\}$ is a uniformly definable family of subgroups of $G \times G$ such that $V_1 \cap V_2 = 1$ for each distinct elements V_1 and V_2 of $\mathscr{F}$. Moreover, for each $x \in X$, the subgroup V_x is unipotent, so all the elements of $\cup\mathscr{F}$ are unipotent and, by choice of X, we have $\cup\mathscr{F} = \mathscr{U} \cup \{1\}$.

We consider n nontrivial unipotent elements $u_1, \cdots, u_n$ of G, such that $u_{i+1} \notin d(u_1, \cdots, u_i)$ for each $i < n$, and such that $G = d(u_1, \cdots, u_n)$. The existence of n is ensured by the Zil'ber Indecomposability Theorem ([28, Theorem 2.9] or [5, Theorem 5.26]), and because, for each nontrivial unipotent element u of G, the subgroup $d(u)$ is definably isomorphic to K_+, so it is connected. In particular, we notice that G is connected.

From now on, by the Zil'ber Indecomposability Theorem, there is an integer k such that, for any elements $F_1, \cdots, F_n$ of $\mathscr{F}$, the product $(F_1 \cdots F_n)^k$ is a subgroup of $G \times G$. In particular, the family $\mathscr{H}_0$ of these products is a uniformly definable family of subgroups of $G \times G$. Let

$$\mathscr{H} = \{F \in \mathscr{H}_0 \mid rk(F) = rk(G),\, F \cap (G \times \{1\}) = 1,\, F \cap (\{1\} \times G) = 1\}.$$

Then $\mathscr{H}$ is a uniformly definable family of subgroups of $G \times G$, and since $rk(F) = rk(G)$ for each $F \in \mathscr{H}$, the connectedness of G implies that each element of $\mathscr{H}$ is the graph of an algebraic group automorphism of G. Conversely, if α is an algebraic group automorphism of G, then for each i, the element $(u_i, \alpha(u_i))$ is nontrivial and unipotent, so $d((u_i, \alpha(u_i))) \in \mathscr{F}$. Moreover, the graph Δ of α contains the subgroup $H \in \mathscr{H}_0$ generated by the subgroups $d((u_i, \alpha(u_i)))$ for $i = 1, \cdots, n$. But we have $G = d(u_1, \cdots, u_n)$, so there is a definable surjection from H to G, and $rk(H) \geq rk(G)$. Since Δ contains H, we obtain $rk(H) = rk(\Delta) = rk(G)$ and $H \in \mathscr{H}$. Now, since Δ is the graph of an automorphism of G, it is connected, so $\Delta = H$ belongs to $\mathscr{F}$. This proves that $\mathscr{H}$ is the set of graphs of the algebraic group automorphisms of G. In particular the group $\mathrm{Aut}_{alg}(G)$ of all the algebraic group automorphisms of G is interpretable in K.

Furthermore, our argument proves that each $\varphi \in \mathrm{Aut}_{alg}(G)$ is characterized by $\varphi(u_1), \cdots, \varphi(u_n)$, so the multiplication in $\mathrm{Aut}_{alg}(G)$ is interpretable too, as well as its action over G. Thus, there is an algebraic group of the form $G \rtimes A$, where A is an algebraic subgroup acting faithfully on G by conjugation and such that, for each $\varphi \in \mathrm{Aut}_{alg}(G)$, there exists $a \in A$ satisfying $\varphi(g) = g^a$ for each $g \in G$.

From now on, we have to show that the algebraic group $G \rtimes A$ is affine. Since $G^* = G \rtimes A^\circ$ is connected and since $A^\circ \cap Z(G^*) = 1$, the group A° is affine by the Rosenlicht's Theorem [31, §5, Theorem 13]. Then G^* is affine by [31, §5, Theorem 16 p.439], and $G \rtimes A$ is affine too by [31, §5, Corollary 1 p.430], as desired. □

Lemma 4.7. – *Let K be a field of finite Morley rank of characteristic zero, and let G be an affine algebraic group over K. If G is generated by its unipotent elements, then any definable automorphism of G is an algebraic automorphism.*

Proof – We may assume $G \neq 1$. For any nontrivial unipotent element u of G, the subgroup $d(u)$ is contained in the Zariski closure $\overline{\langle u \rangle}$ of $\langle u \rangle$, which is definably isomorphic to K_+. By Fact 2.7, we obtain $d(u) = \overline{\langle u \rangle}$, and $d(u)$ is closed and connected (as an algebraic group, as well as a group of finite Morley rank). Let $u_1, \cdots, u_n$ be n nontrivial unipotent elements of G such that $u_{i+1} \notin d(u_1, \cdots, u_i)$ for each $i < n$, and such that $G = d(u_1, \cdots, u_n)$. The existence of n is ensured since $d(u)$ is connected for each unipotent element u, and by the finiteness of the Morley rank of G.

Let α be a definable automorphism of G. We show that α is an algebraic group automorphism of G. Let Δ_α be the graph of α. Then Δ_α is a definable subgroup of $G \times G$, and it is definably isomorphic to G. In particular Δ_α is connected (as a group of finite Morley rank). But $G = d(u_1, \cdots, u_n)$, so we have $\Delta_\alpha = d(\langle (u_i, \alpha(u_i)) \mid i = 1, \cdots, n \rangle)$. For each $i = 1, \ldots, n$, the subgroup $d(u_i, \alpha(u_i))$ is contained in $\overline{\langle (u_i, \alpha(u_i)) \rangle}$, which is definably isomorphic to K_+. So we obtain $d(u_i, \alpha(u_i)) = \overline{\langle (u_i, \alpha(u_i)) \rangle}$ (Fact 2.7), and Δ_α contains $\Delta_0 = \langle \overline{\langle (u_i, \alpha(u_i)) \rangle} \mid i = 1, \cdots, n \rangle$. Since $\overline{\langle (u_i, \alpha(u_i)) \rangle}$ is a connected closed subgroup for each i, the group Δ_0 is connected and closed too. In particular, it is definable. But it is contained in Δ_α, and it contains $(u_i, \alpha(u_i))$ for each i, hence it is equal to Δ_α, and Δ_α is closed. Thus, the automorphism α is interpretable in the pure field K, and α is algebraic. □

Corollary 4.8. – *Let G be a group of finite Morley rank interpreting a field K of characteristic zero. We assume that G acts faithfully and definably on an affine* *algebraic group U over K. If U is generated by its unipotent elements, then $U \rtimes G$ is definably isomorphic to a definable subgroup of an affine algebraic group over K.*

Proof – By Fact 4.5 or 4.6, there is an affine algebraic group of the form $H = U \rtimes A$ for a closed subgroup A, such that H is isomorphic to the holomorph of U. By Lemma 4.7, each element of G acts on U by algebraic group automorphism, so there is a definable isomorphism from G to a definable subgroup of A. □

Proposition 4.9. – *Let G be a nilpotent group of finite Morley rank interpreting a field K of characteristic zero. Then G has a largest definable subgroup definably isomorphic to a unipotent algebraic group over K.*

Proof – We may assume that G is connected. We proceed by induction on the Morley rank of G. Let N be a definable subgroup, maximal among the ones definably isomorphic to a unipotent algebraic group over K. We may assume $N < G$ and we consider a maximal proper definable subgroup M of G containing N. By induction hypothesis, N is definably characteristic in M, so N is normal in G.

Let U be a definable subgroup of G definably isomorphic to a unipotent algebraic group over K. We may assume that N does not contain U and that U is minimal for these conditions. Then, by induction hypothesis, we have $G = NU$. Also, U has a normal definable subgroup V such that V is definably isomorphic to a unipotent algebraic group over K, and such that $U = V \rtimes A$ for a definable subgroup A definably isomorphic to K_+. By minimality of U, N contains V and, finally, we have $U = A \simeq K_+$. Hence, by Fact 2.7, we have $G = N \rtimes U$, and either $C_U(N) = 1$ or $C_U(N) = U$.

In the first case, Corollary 4.8 gives the result. In the second case, G is definably isomorphic to $N \times K_+$, so we have the result too. □

In the pure language of fields, if A is an indecomposable unipotent algebraic group over a field K of characteristic zero, then $J(A) = 1$. This remark induces the following notion.

Notation 4.10. – *In each group G of finite Morley rank, we consider*

$$V(G) = \langle J(A) \mid A \text{ is a radicable indecomposable definable abelian subgroup of } G\rangle.$$

Remark 4.11. –

- By the Zil'ber Indecomposability Theorem, for each group G of finite Morley rank, $V(G)$ is definable and connected.
- Moreover, by Fact 2.11, $V(G/H) = V(G)H/H$ for every normal definable subgroup H of G.

Lemma 4.12. – *Let G be a nilpotent group of finite Morley rank, and let K be an interpretable field of characteristic zero such that G is a U_K-group. Then G is definably isomorphic to a unipotent algebraic group over K if and only if $V(G) = 1$.*

Moreover, in this case, G is a homogeneous U_K-group.

Proof – First we assume that G is definably isomorphic to a unipotent algebraic group $\tilde{G}$ over K. Then each nontrivial indecomposable definable abelian subgroup A of $\tilde{G}$ is contained in a closed subgroup B definably isomorphic to K_+. So Fact 2.7 yields $A = B$ and $J(A) = 1$, and we have $V(\tilde{G}) = 1$ and $V(G) = 1$. Now each indecomposable subgroup of G is a U_K-group, so G is a homogeneous U_K-group.

If $V(G) = 1$, we proceed by induction on the rank of G. We may assume $G \neq 1$. Let M be a maximal proper connected definable subgroup of G. Then we have

$V(M) \leq V(G) = 1$ and, by induction hypothesis, M is definably isomorphic to a unipotent algebraic group over K. Since G is a U_K-group, there is an indecomposable definable abelian subgroup A of G not contained in M such that $A/J(A)$ is definably isomorphic to K_+. In particular, by the maximality of M, we have $G = MA$. On the other hand, since $J(A) \leq V(G)$ is trivial, A is definably isomorphic to K_+. Now Proposition 4.9 says that $G = MA$ is definably isomorphic to a unipotent algebraic group over K. □

Corollary 4.13. – *Let G be a nilpotent U_K-group of finite Morley rank for an interpretable field K of characteristic zero. If there are finitely many algebraically closed fields $K_1, \ldots, K_n$ such that G is definably isomorphic to a direct product of unipotent groups $U_1, \ldots, U_n$ over $K_1, \ldots, K_n$ respectively, then $V(G) = 1$.*

Proof – In this case, each nontrivial indecomposable definable abelian subgroup A of G is contained in a definable subgroup B which is definably isomorphic to $(K_1)_+ \times \cdots \times (K_n)_+$. So Fact 2.7 says that A is definably isomorphic to $(K_i)_+$ for some i and that $J(A) = 1$. In particular, this proves that $V(G) = 1$. □

Now, by a relatively technical argument, we can prove Theorem 4.3, as a consequence of the previous study.

Proof of Theorem 4.3 – We proceed by induction on the nilpotence class of G and on the Morley rank of G. By Fact 2.15, if T_0 is the maximal pseudo-torus of G, then G interprets some fields $K_1, \cdots, K_n$ of characteristic zero such that $G = T_0 * U_{K_1}(G) * \cdots * U_{K_n}(G)$, where $*$ denotes the central product, so $G/Z(G)$ is definably isomorphic to $U_{K_1}(G)/Z(U_{K_1}(G)) \times \cdots \times U_{K_n}(G)/Z(U_{K_n}(G))$. But, if G is not a U_{K_i}-group for some $i = 1, \ldots, n$, then the induction hypothesis shows that $U_{K_i}(G)/Z(U_{K_i}(G))$ satisfies the result for each i, so G satisfies the result. Hence we may assume that G is a U_L-group for an interpretable field L. In particular, by Lemma 4.12, we have just to prove that $V(G)$ is central in G, and we may assume $V(G) \not\leq Z(G)$. We note that, by Facts 2.18 and 2.21, $G/Z(G)$ is a homogeneous U_L-group.

If G has two distinct G-minimal subgroups A_1 and A_2, we consider $Z_i/A_i = Z(G/A_i)$ for $i = 1, 2$. By induction hypothesis and Corollary 4.13, we have $V(G) \leq Z_1 \cap Z_2$. Now $V(G)$ is central in G, contradicting $V(G) \not\leq Z(G)$. Hence G has a unique G-minimal subgroup A.

We denote by $\mathscr{M}$ the set of the maximal proper connected definable subgroups of G. If there are two distinct elements M_1 and M_2 of $\mathscr{M}$ such that $V(M_1)$ and $V(M_2)$ are contained in $Z(G)$, then Proposition 4.9 and Lemma 4.12 show that $G/Z(G) = M_1M_2/Z(G)$ satisfies the theorem. Hence there is at most one element M_V of $\mathscr{M}$ such that $Z(G)$ contains $V(M_V)$. We denote by $\mathscr{M}^*$ the set of the elements M of $\mathscr{M}$ such that $Z(G)$ does not contain $V(M)$.

For each $M \in \mathscr{M}$, by induction hypothesis, $Z(M)$ contains $V(M)$. If $Z(M)$ does not contain $Z(G)$, then M does not contain $Z(G)$. Moreover, since G is nilpotent and connected, its torsion part is contained in its maximal torus, which is central in G, so $Z(G)$ is connected, and we have $G = MZ(G)$ by maximality of M. But this implies that $Z(M) = Z(G) \cap M$, so $G/Z(G)$ is definably isomorphic to $M/Z(M)$, and we obtain $V(G) \leq Z(G)$, contradicting $V(G) \not\leq Z(G)$. Hence $Z(M)$ contains $Z(G)$ for each $M \in \mathscr{M}$.

Let $M_1 \in \mathscr{M}$. Since G is a U_L-group, G/M_1 is definably isomorphic to L_+ (Facts 2.7 and 2.20). By Fact 2.11, G has an indecomposable subgroup I_0 covering

G/M_1. Since G is nonabelian, we have $G \neq I_0$, and there exists $M_2 \in \mathscr{M}$ containing I_0. Therefore G/M_2 is definably isomorphic to L_+, and $G/(M_1 \cap M_2)$ is definably isomorphic to $L_+ \times L_+$. Let $N = M_1 \cap M_2$. Since $Z(M_1) \cap Z(M_2)$ centralizes $M_1M_2 = G$, the previous paragraph yields $Z(G) = Z(M_1) \cap Z(M_2)$. In particular, N contains $Z(G)$ and we have $V(N) \leq V(M_1) \cap V(M_2) \leq Z(M_1) \cap Z(M_2) = Z(G)$, so $N/Z(G) \leq G/Z(G)$ is definably isomorphic to a unipotent group over L (Lemma 4.12). Moreover, since $G/N \simeq L_+ \times L_+$, we have $V(G) \leq N$.

If there exists $M_3 \in \mathscr{M}$ does not containing N, then $G/(N \cap M_3)$ is definably isomorphic to $L_+ \times L_+ \times L_+$. Now we find N_1, N_2 and N_3 such that, for $i = 1, 2, 3$, we have $N_i = M_1^i \cap M_2^i$ for some maximal definable subgroups M_1^i and M_2^i, and such that $G = \langle N_1, N_2, N_3 \rangle$. By the previous paragraph, N_i contains $Z(G)$ for $i = 1, 2, 3$, the quotient $N_i/Z(G)$ is definably isomorphic to a unipotent group over L, and Proposition 4.9 shows that $G/Z(G)$ is definably isomorphic to a unipotent group over L too. Now Lemma 4.12 contradicts $V(G) \nleq Z(G)$. Hence each element of $\mathscr{M}$ contains N. Moreover, we have $G'V(G) \leq N$ and $G/N \simeq L_+ \times L_+$, and $G/G'V(G)$ is definably isomorphic to a direct product of copies of L_+ by Lemma 4.12, so the intersection of the elements of $\mathscr{M}$ is contained in $G'V(G)$. Hence we obtain $N = G'V(G)$.

We consider the subgroup Z defined by $Z/A = Z(G/A)$. By induction hypothesis, Z contains $V(G)$, and G/Z is definably isomorphic to a unipotent group over L (Lemma 4.12). For each $z \in Z \setminus Z(G)$, the groups $G/C_G(z)$ and A are definably isomorphic, so we have $C_G(z) \in \mathscr{M}$, and $N \leq C_G(z)$. Thus N centralizes Z. Moreover, $C_G(z) \in \mathscr{M}$ implies that $A \simeq G/C_G(z)$ is definably isomorphic to L_+. We show that N contains Z. If $G = Z$, then we have $N \leq C_G(Z) = Z(G)$, contradicting $V(G) \leq N$ and $V(G) \nleq Z(G)$. Therefore Z is contained in an element M_Z of $\mathscr{M}$. Since $M_Z \in \mathscr{M}$ contains N, and since G/N is definably isomorphic to $L_+ \times L_+$, either N contains Z or $NZ = M_Z$. In the second case, let I_1 be an indecomposable subgroup of G such that $J(I_1)$ is not contained in $Z(G)$. Let $x \in J(I_1) \setminus Z(G)$. Then x belongs to $V(G) \leq Z$, so $C_G(x)$ belongs to $\mathscr{M}$ and $C_G(x)$ contains N. Moreover, we have $x \in V(G) \leq N$, so x centralizes Z, and we obtain $C_G(x) = NZ$. But x centralizes I_1, hence I_1 is contained in NZ. This shows that each indecomposable subgroup I_1 satisfying $J(I_1) \nleq Z(G)$ is contained in NZ. Since G/N is definably isomorphic to $L_+ \times L_+$, there exists $M^* \in \mathscr{M}^* \setminus \{NZ\}$, and we find an indecomposable subgroup I_2 in M^* such that $J(I_2) \nleq Z(G)$. Consequently, we have $I_2 \leq NZ \cap M^* = N$, contradicting $V(N) \leq Z(G)$. This proves that N contains Z. In particular, since Z contains $V(G)$, we obtain $N = G'Z$.

We show that $Z/Z(G)$ is definably isomorphic $L_+ \times L_+$. Since we have $Z(G) < Z \leq Z(N)$, we have either $C_G(Z) = N$, or $C_G(Z) \in \mathscr{M}$. We consider two distinct elements H_1 and H_2 of $\mathscr{M} \setminus \{C_G(Z)\}$. For $i = 1, 2$, we fix $h_i \in H_i \setminus C_G(Z)$, and let $\gamma_i : Z \to A$ be the homomorphism defined by $\gamma_i(z) = [z, h_i]$. Then $Z/C_Z(h_i)$ is definably isomorphic to $A \simeq L_+$. Thus, since we have $G = Nd(h_1, h_2)$, we obtain $C_Z(h_1, h_2) = Z(G)$, and $Z/Z(G)$ is either definably isomorphic to L_+, or definably isomorphic to $L_+ \times L_+$. In the first case, for $i = 1, 2$, we have $C_Z(h_i) = Z(G)$, so $V(H_i)$ is contained in $Z \cap Z(H_i) = Z(G)$, and H_i belongs to $\mathscr{M} \setminus \mathscr{M}^*$. This contradicts that $\mathscr{M} \setminus \mathscr{M}^*$ has at most one element. Hence $Z/Z(G)$ is definably isomorphic $L_+ \times L_+$. Furthermore, if $C_G(Z)$ belongs to $\mathscr{M}$, we consider $g \in G \setminus C_G(Z)$. Then $C_Z(g)$ centralizes $d(C_G(Z), g) = G$, and we have $Z(G) = C_Z(g)$.

But the homorphism $\gamma_g : Z \to A$ defined by $\gamma_g(z) = [g, z]$ is surjective by G-minimality of A, hence $Z/C_Z(g)$ is definably isomorphic to $A \simeq L_+$, contradicting that $Z/Z(G)$ is definably isomorphic $L_+ \times L_+$. Thus $C_G(Z)$ does not belong to $\mathscr{M}$, and we have $N = C_G(Z)$.

By Fact 2.6, the fields $\overline{\mathbb{Q}}$ and L are isomorphic, definably in the pure field $\overline{\mathbb{Q}}$. Hence there is an isomorphism f, definable in the pure field $\overline{\mathbb{Q}}$, from G to a connected nilpotent algebraic group G_L over L. We consider the induced isomorphism $\overline{f} : G/Z \to G_L/f(Z)$, and its graph $\overline{\Delta}$. In the ACF-expansion of the pure group G, there is a definable isomorphism i between $G/Z \times G_L/f(Z)$ and a unipotent group U_1 over L. Since, in the pure field $\overline{\mathbb{Q}}$, the fields $\overline{\mathbb{Q}}$ and L are definably isomorphic, there is an isomorphism $j : U_1 \to U_2$, definable in $\overline{\mathbb{Q}}$, from U_1 to a unipotent group U_2 over $\overline{\mathbb{Q}}$, such that the preimage of each closed subgroup of U_2 is a subgroup of U_1, definable in the ACF-expansion of the pure group G. But $(j \circ i)(\overline{\Delta})$ is a subgroup of U_2, definable in $\overline{\mathbb{Q}}$, so it is a closed subgroup of U_2, and $\overline{\Delta}$ is a subgroup of $G/Z \times G_L/f(Z)$, definable in the ACF-expansion of the pure group G.

We consider the preimage Δ of $\overline{\Delta}$ in $G \times G_L$. It is a definable subgroup of $G \times G_L$, and if Δ^* is the graph of f, then we have $\Delta = (Z \times \{1\})\Delta^*$. Since G centralizes Z/A and does not centralize Z, we have $\Delta' = (A \times \{1\})(\Delta^*)'$. In the same way, since G centralizes A, if Δ^2 denotes $[\Delta, \Delta']$, we have $\Delta^2 = (\Delta^*)^2$. In particular, $\Delta^2 \cap (G \times \{1\})$ is trivial. We consider Δ/Δ^2, and let Z_Δ/Δ^2 be its center. Then $[\Delta, Z_\Delta \cap (G \times \{1\})]$ is contained in $\Delta^2 \cap (G \times \{1\}) = 1$, so $Z_\Delta \cap (G \times \{1\}) \leq Z(G) \times \{1\}$. If the nilpotence class of G is not 2, then the induction hypothesis applied with Δ/Δ^2 shows that $V(\Delta)$ is contained in Z_Δ. But G_L is a unipotent group over L, so we have $V(G_L) = 1$ and $V(G \times G_L)$ is contained in $G \times \{1\}$. Hence $V(\Delta)$ is contained in $Z_\Delta \cap (G \times \{1\}) \leq Z(G) \times \{1\}$, and $V(G)$ is contained in $Z(G)$. This contradiction implies that the nilpotence class of G is 2. In particular, G' is contained in $Z(G) \leq Z$ and we have $N = G'Z = Z$.

We show that $\mathscr{M} = \mathscr{M}^*$. Indeed, if M_V exists, we consider $M \in \mathscr{M}^*$. Then Proposition 4.9 and Lemma 4.12 show that $V(G) = V(MM_V)$ is contained in $V(M)Z(G)$. But we have $Z = N = G'V(G) = Z(G)V(G)$, so we obtain $N = Z(G)V(M) \leq Z(M)$, and $M \leq C_G(N) = Z$, contradicting our choice of M. Hence we have $V(M) \not\leq Z(G)$ for each $M \in \mathscr{M}$.

Now, for each $g \in G \setminus N$, we consider the map $\gamma_g : G \to Z(G)$ defined by $\gamma_g(x) = [g, x]$. Since $g \notin Z$ does not centralize $N = Z$ and centralizes Z/A, we have $\gamma_g(N) = A \simeq L_+$ and, since $Z/Z(G)$ is definably isomorphic to $L_+ \times L_+$, the group $C_Z(g)/Z(G)$ is definably isomorphic to L_+. Moreover, since $g \notin Z(G)$, we have $C_G(g) < G$ and, since $N \leq M$ for each $M \in \mathscr{M}$, we have $C_G(g)N < G$. Since $g \notin N$ and since G/N is definably isomorphic to $L_+ \times L_+$, this implies that $C_G(g)N/N$ is definably isomorphic to L_+. Now, if we consider $h \in G \setminus C_G(g)N$, then we have $C_G(h)N/N \simeq L_+$ too, and since $h \in C_G(h) \setminus C_G(g)N$, Fact 2.7 gives $G = d(g)d(h)N$. Thus, for each $z \in C_G(g, h)$, we have $G = C_G(z)N$, Therefore, since N is contained in each element of $\mathscr{M}$, we have $G = C_G(z)$ and $z \in Z(G)$, so $C_G(g)/Z(G) \cap C_G(h)/Z(G)$ is trivial. On the other hand, again since N is contained in each element of $\mathscr{M}$, we obtain $G = C_G(g)C_G(h)$, and $G/Z(G)$ is the direct product of $C_G(g)/Z(G)$ and $C_G(h)/Z(G)$.

We fix $g \in G \setminus N$ and $h \in G \setminus C_G(g)N$. The previous paragraph shows that, for any definable isomorphism $\alpha : C_G(g)/Z(G) \to C_G(h)/Z(G)$, there is a definable subgroup $U/Z(G)$ of $G/Z(G)$ representing the graph of α. We remark that, since

G/N is definably isomorphic to $L_+ \times L_+$, we have either $U \leq N$, or $UN/N \simeq L_+$ definably, or $UN/N = G/N$. Since $G = C_G(g)U$ and since $G \neq C_G(g)N$, we have not $U \leq N$. Moreover, since N is contained in each element of $\mathscr{M}$ and since $U < G$, we do not have $UN/N = G/N$, and we obtain $UN/N \simeq L_+$ and $UN \in \mathscr{M}$. In particular, we have $V(UN) \not\leq Z(G)$ and $V(UN) \leq Z(UN)$. We consider $u \in U \setminus N$. Since $UN/N \simeq L_+$, the subgroup $C_G(u)N$ contains UN, and since $UN \in \mathscr{M}$, we have either $G = C_G(u)N$ or $C_G(u) \leq UN$. In the first case, since each element of $\mathscr{M}$ contains N, the group $C_G(u)$ is contained in no element of $\mathscr{M}$, and we obtain $G = C_G(u)$ and $u \in Z(G)$, contradicting $Z(G) \leq N$. Hence UN contains $C_G(u)$. Then $C_G(u)$ centralizes $V(UN)$, and since the previous paragraph gives $C_Z(u)/Z(G) \simeq L_+$, we obtain $C_Z(u) = V(UN)Z(G) = V(U)V(N)Z(G)$. Also, since $Z(G)$ contains $V(N)$, we have $C_Z(u) = V(U)Z(G) \leq U$. But we have $Z/Z(G) \simeq L_+ \times L_+$, so either $U \cap Z = C_Z(u)$ or U contains Z. Since we have $U/Z(G) \cap C_G(g)/Z(G) = 1$ and $C_Z(g)/Z(G) \simeq L_+$, U does not contain Z, and we obtain $U \cap Z = C_Z(u)$. Moreover, since we have $u \in U \setminus N$, we have $C_U(u)N/N \neq 1$, and since the previous paragraph gives $C_G(u)N/N \simeq L_+ \simeq UN/N$, we obtain $C_U(u)N/N = C_G(u)N/N = UN/N$. In particular, we have $C_G(u) = C_U(u)N \cap C_G(u) = C_U(u)C_N(u)$, and since U contains $C_N(u)$, we obtain $C_G(u) \leq U$. Furthermore, since $U \cap Z = C_Z(u)$, we obtain $U = C_G(u)(N \cap U) \leq C_G(u)$ and $U = C_G(u)$.

We note that, if $u \in C_G(g)N$, then we have $UN = C_G(g)N$ because $UN/N \simeq L_+ \simeq C_G(g)N/N$, contradicting $G = C_G(g)U$ and $C_(g)N < G$. Thus we have $u \notin C_G(g)N$ and, in the same way, $u \notin C_G(h)N$. In particular, the following uniformly definable family $\mathscr{F}$ of subgroups of $G/Z(G)$ contains the graph of each definable isomorphism from $C_G(g)/Z(G)$ to $C_G(h)/Z(G)$:

$$\mathscr{F} = \{C_G(u)/Z(G) \mid u \in G \setminus (C_G(g)N \cup C_G(h)N)\}.$$

Conversely, for each $u \in G \setminus (C_G(g)N \cup C_G(h)N)$, the quotient group $G/Z(G)$ is the direct product of $C_G(g)/Z(G)$ and $C_G(u)/Z(G)$, and also of $C_G(u)/Z(G)$ and $C_G(h)/Z(G)$, so $C_G(u)/Z(G)$ is the graph of an isomorphism from $C_G(g)/Z(G)$ to $C_G(h)/Z(G)$. Now the set $\mathscr{I}$ of the definable isomorphisms from $C_G(g)/Z(G)$ to $C_G(h)/Z(G)$ is a (nonempty) uniformly definable family. But, if we fix $r \in \mathscr{I}$, then the set of the definable automorphisms of $C_G(g)/Z(G)$ is $\mathscr{A} = \{r^{-1} \circ s \mid s \in \mathscr{I}\}$, and it is a uniformly definable family.

We show that the Morley rank of $\mathscr{A}$ is 2. We note that, in the pure field $\overline{\mathbb{Q}}$, we have $rk(L_+) = 1$ (Fact 2.6), so we have $rk(L_+) = 1$ in the ACF-expansion of the pure group G too. Since there are some definable bijections between $\mathscr{F}$, $\mathscr{I}$ and $\mathscr{A}$, we have to prove that $\mathscr{F}$ has Morley rank 2. First we show that, if $C_G(u)/Z(G)$ and $C_G(v)/Z(G)$ are two distinct elements of $\mathscr{F}$, then $C_G(u) \cap C_G(v)$ is contained in N, that is the elements of $\mathscr{F}$ are generically disjoint. The previous analysis shows that, if we have $v \notin C_G(u)N$, then $C_G(u) \cap C_G(v) = Z(G)$, so we may assume $v \in C_G(u)N$. Therefore $C_G(u)N/N \cap C_G(v)N/N$ is not trivial and, since $C_G(u)N/N$ and $C_G(v)N/N$ are definably isomorphic to L_+, we obtain $C_G(u)N = C_G(v)N$. Also, the previous study gives $C_Z(u) = V(C_G(u)N)Z(G) = V(C_G(v)N)Z(G) = C_Z(v)$. Since $C_G(u)/C_Z(u)$ and $C_G(v)/C_Z(v)$ are definably isomorphic to $C_G(u)Z/Z = C_G(v)Z/Z \simeq L_+$, we have either $C_G(u) = C_G(v)$ or $C_G(u) \cap C_G(v) = C_Z(u) = C_Z(v) \leq N$. Hence the elements of $\mathscr{F}$ are generically disjoint. Moreover, each element $C_G(u)/Z(G)$ of $\mathscr{F}$ has Morley rank

$$rk(C_G(u)/C_Z(u)) + rk(C_Z(u)/Z(G)) = 2rk(L_+) = 2,$$

and $\cup\mathscr{F}$ contains $G/Z(G) \setminus (C_G(g)N/Z(G) \cup C_G(h)N/Z(G))$ which is a generic subset of $G/Z(G)$. So the Morley rank of $\cup\mathscr{F}$ is $rk(G/Z(G)) = rk(G/N) + rk(N/Z(G)) = 4rk(L_+) = 4$, and the one of $\mathscr{A}$ is $rk(\cup\mathscr{F}) - rk(C_G(u)/Z(G)) = 2$, as claimed.

The same reasoning applied in the pure field $\overline{\mathbb{Q}}$ shows that the group $\mathscr{A}$ has dimension 2 over $\overline{\mathbb{Q}}$ too.

In the pure field $\overline{\mathbb{Q}}$, the group $C_G(g)/Z(G)$ is definably isomorphic to $\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+$, since $C_G(g)/C_Z(g)$ and $C_Z(g)/Z(G)$ are definably isomorphic to $L_+ \simeq \overline{\mathbb{Q}}_+$. Hence there is a definable embedding μ from the canonical semidirect product $C_G(g)/Z(G) \rtimes \mathscr{A}$ to $(\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+) \rtimes \mathrm{GL}(2,\overline{\mathbb{Q}})$, where $\mathrm{GL}(2,\overline{\mathbb{Q}})$ acts by multiplication on $\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+$, and such that $\mu(C_G(g)/Z(G)) = \overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+$ and $\mu(\mathscr{A})$ is a closed subgroup of $\mathrm{GL}(2,\overline{\mathbb{Q}})$. By Lemma 4.1, the group $\mu(\mathscr{A}^\circ)$ is connected, and since $\mathscr{A}$ has dimension 2 over $\overline{\mathbb{Q}}$, the group $\mu(\mathscr{A}^\circ)$ has dimension two. Let T be a maximal torus of $\mu(\mathscr{A}^\circ)$. It is nontrivial since $\mu(\mathscr{A}^\circ)$ is a closed subgroup of dimension two of $\mathrm{GL}(2,\overline{\mathbb{Q}})$. First we assume that $\mu^{-1}(T)$ is definable in the ACF-expansion of the pure group G. Since T is nontrivial, there exists $e_1 \in \overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+$ such that $T \cdot e_1 = \overline{\mathbb{Q}}^* \cdot e_1$, and the subset $V_1 := (T \cdot e_1) \cup \{(0,0)\} = \overline{\mathbb{Q}} \cdot e_1$ is an infinite subgroup of $\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+$. Now either T centralizes $(\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+)/V_1$ and $V_2 := C_{\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+}(T)$ is a complement of V_1 in $\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+$, or T does not centralize $(\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+)/V_1$ and we find $e_2 \in (\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+) \setminus V_1$ such that $T \cdot e_2 = \overline{\mathbb{Q}}^* \cdot e_2$. In this last case, the subset $V_2 := (T \cdot e_2) \cup \{(0,0)\} = \overline{\mathbb{Q}} \cdot e_2$ is an infinite complement of V_1 in $\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+$. Thus, for $i = 1, 2$, the preimage $\mu^{-1}(V_i) = (\mu^{-1}(T) \cdot \mu^{-1}(e_i)) \cup \{\mu^{-1}(0,0)\}$ of V_i is definable in the ACF-expansion of the pure group G. Since V_1 and V_2 are infinite and satisfy $\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+ = V_1 \oplus V_2$, and since $C_G(g)/Z(G)$ has Morley rank two then, for $i = 1, 2$, the subgroup $\mu^{-1}(V_i)$ is torsion-free of Morley rank one and we have $C_G(g)/Z(G) = \mu^{-1}(V_1) \oplus \mu^{-1}(V_2)$. Since $C_G(g)/Z(G)$ is an homogeneous U_L-group, this implies that $\mu^{-1}(V_i)$ is definably isomorphic to L_+ for $i = 1, 2$, and that $C_G(g)/Z(G)$ is definably isomorphic to $L_+ \times L_+$ in the ACF-expansion of the pure group G. But $C_G(h)/Z(G)$ is definably isomorphic to $C_G(g)/Z(G)$, and $G/Z(G)$ is the direct product of $C_G(g)/Z(G)$ and $C_G(h)/Z(G)$, so $G/Z(G)$ is isomorphic to $(L_+)^4$, definably in the ACF-expansion of the pure group G. This contradicts $V(G) \not\leq Z(G)$. Hence $\mu^{-1}(T)$ is not definable in the ACF-expansion of the pure group G.

Since $\mu^{-1}(T)$ is not definable in G, the group $\mu(\mathscr{A}^\circ)$ is not a torus, and since it has dimension two while the maximal unipotent subgroups of $\mathrm{GL}(2,\overline{\mathbb{Q}})$ have dimension one, we have $\mu(\mathscr{A}^\circ) = P \rtimes T$ for P a maximal unipotent subgroup of $\mathrm{GL}(2,\overline{\mathbb{Q}})$. Moreover, since $\mu^{-1}(T)$ is not definable in G, we have $\mu^{-1}(T) < C_{\mathscr{A}^\circ}(\mu^{-1}(T))$ and $T < C_{\mu(\mathscr{A}^\circ)}(T)$, and since P is a torsion-free group of dimension one, we obtain $\mu(\mathscr{A}^\circ) = C_{\mu(\mathscr{A}^\circ)}(T)$ and $\mu(\mathscr{A}^\circ)$ is abelian. Now T centralizes a nontrivial unipotent subgroup of $\mathrm{GL}(2,\overline{\mathbb{Q}})$, hence T is central in $\mathrm{GL}(2,\overline{\mathbb{Q}})$ and it is conjugate with the following subgroup of $\mathrm{GL}(2,\overline{\mathbb{Q}})$:

$$R = \left\{ \begin{pmatrix} t & a \\ 0 & t \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^*,\ a \in \overline{\mathbb{Q}} \right\}.$$

Thus there is an isomorphism ν, definable in $\overline{\mathbb{Q}}$, from $(\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+) \rtimes \mu(\mathscr{A}^\circ)$ to $H_1 = (\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+) \rtimes R$, where R acts on $\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}_+$ by multiplication, and such that $\nu(T)$ is the maximal torus of R. But there is an isomorphism γ, definable in $\overline{\mathbb{Q}}$, from H_1 to the following group

$$H_2 = \left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^*,\ (a,u,v) \in \overline{\mathbb{Q}}^3 \right\},$$

and satisfying $(\gamma \circ \nu)(T) = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^* \right\}$. Since $\mu^{-1}(T)$ is not definable in G, this implies that, in the ACF-expansion of the pure group H_2, the maximal torus $(\gamma \circ \nu)(T)$ is not definable. This contradicts Lemma 3.7 and Proposition 3.11, and finishes the proof. □

5. Definably affine groups

Generally a quotient of a definably linear group of finite Morley rank by a normal definable subgroup, even finite, is not definably linear (Example 5.1). In order to overcome this obstacle, we introduce *definably affine groups* (Definition 5.2). Then we obtain a structural result about the definably affine ACF-groups (Proposition 5.11). Moreover, we introduce two new subgroups $A(G)$ and $W(G)$ for any ACF-group G (Corollaries 5.10 and 5.18). We notice that the subsections 5.1 and 5.2 concern all the ACF-groups, not just the ACF_0-groups, since we do not need Lemma 4.1 in them. Furthermore, the following results will be used in §10 for ACF_p-groups when p is a prime.

Example 5.1. – We consider the pure group $G = H_1 \times H_2$, where H_1 and H_2 are two copies of $\mathrm{SL}_2(K)$ for an algebraically closed field K of characteristic zero. On the one hand, for $i = 1, 2$, the quotient $H_i/Z(H_i)$ is definably linear [28, Corollaire 4.16], and H_i is definably linear too (Proposition 5.15). Hence G is definably linear. On the other hand, if i and j denote the involutions of $Z(H_1)$ and $Z(H_2)$ respectively, then $\overline{G} = G/\langle (i,j) \rangle$ is not definably linear.

Indeed, let μ be a nontrivial field automorphism of K, and let μ^* be the automorphism of H_1 induced by μ. We consider the automorphism α of $\overline{G}$ defined for each $\overline{(h_1, h_2)} \in \overline{G}$ by $\alpha(\overline{(h_1, h_2)}) = \overline{(\mu^*(h_1), h_2)}$. Then, if Δ is the graph in G of an algebraic group isomorphism from H_1 to H_2, and if $\overline{\Delta} := \Delta/\langle (i,j) \rangle$ denotes its image in $\overline{G}$, the intersection of $\overline{\Delta}$ and of $\alpha(\overline{\Delta})$ is not definable. So $\alpha(\overline{\Delta})$ is not definable and α is not a standard isomorphism. Consequently, by Fact 3.6 (or Fact 1.5), the pure group $\overline{G}$ is not definably linear over one interpretable field. But if $\overline{G}$ is definably linear over several interpretable fields, then Lemma 5.6 shows that $\overline{G}$ is a direct product of two proper definable subgroups. This contradicts that each nontrivial normal subgroup of $\overline{G}$ contains the central involution of $\overline{G}$. Hence $\overline{G}$ is not definably linear.

5.1. **Definition and generalities.**

Definition 5.2. – *A group G of finite Morley rank is said to be* definably affine *(over finitely many fields $K_1, \ldots, K_n$, interpretable in G) if G is definably isomorphic to a definable section of $H_1 \times \cdots \times H_n$, where H_i is an affine algebraic group over K_i for each $i = 1, \ldots, n$.*

Remark 5.3. –

(1) If G is a definably affine ACF-group, interpretable in the pure algebraically closed field K, then G is affine by [28, §4.e (2)] and Fact 2.6.

(2) Every definable quotient and every definable subgroup of a definably affine group is definably affine too.
(3) Consider an ACF-group G definably affine over *one* algebraically closed field L. By [28, §4.e (2)] and Fact 2.6, there is a definable isomorphism $\rho : G \to H$ for an affine algebraic group H over L, and any subgroup of H is definable in G if and only if it is closed. In particular, G is definably linear over L.

Our first result is a remark concerning some definable fields isomorphisms. For the fields of finite Morley rank and of characteristic zero, a result of the same vein is known [18, Corollary 2.8].

Lemma 5.4. – *Let K and L be two fields, definable in an ACF-group G. If one of the following two conditions is satisfied, then K and L are definably isomorphic:*

- *K_+ and L_+ are definably isomorphic;*
- *K^* and L^* are definably isomorphic.*

Proof – Let F be an algebraically closed field such that G is interpretable in the pure field F. We may assume that K and L are infinite. Thus there is a field isomorphism $\alpha : K \to L$, definable in the pure field F (Fact 2.6). If $\delta : K_+ \to L_+$ is a group isomorphism, definable in G, then $\mu = \delta^{-1} \circ \alpha$ is a group automorphism of K_+, definable in F. By Fact 2.6, the map μ is definable in the pure field K, so $\alpha = \delta \circ \mu$ is definable in G.

If $\gamma : K^* \to L^*$ is a group isomorphism, definable in G, we consider the map $\beta : K \to L$ defined by $\beta(x) = \gamma(x)$ for each $x \in K^*$ and by $\beta(0) = 0$. Then $\nu = \beta^{-1} \circ \alpha$ is a permutation of K definable in F, and ν is definable in the pure field K (Fact 2.6). Thus $\alpha = \beta \circ \nu$ is definable in G. □

Corollary 5.5. – *Let G be an infinite ACF-group. Suppose that K and L are two interpretable fields such that G is definably affine over K (resp. L). Then K and L are definably isomorphic.*

Proof – We may assume that G has no proper infinite definable subgroup. Let H_K (resp. H_L) be an affine algebraic group over K (resp. L) such that G is definably isomorphic to H_K (resp. H_L). By the minimality of G, the group H_K (resp. H_L) is definably isomorphic to either K^* (resp. L^*), or K_+ (resp. L_+). Now Lemma 5.4 provides the result. □

Lemma 5.6. – *Let G be an ACF-group, and let $K_1, \ldots, K_n$ be n infinite definable fields, such that K_i is not definably isomorphic to K_j for $i \neq j$. For $i = 1, \ldots, n$, we consider an affine algebraic group H_i. Then, for each connected definable subgroup U of $H = H_1 \times \cdots \times H_n$, we have $U = (U \cap H_1) \times \cdots \times (U \cap H_n)$.*

Proof – We assume toward a contradiction that U is a counterexample of minimal Morley rank. Then, for each proper connected definable subgroup M_0 of U, we have $M_0 = (M_0 \cap H_1) \times \cdots \times (M_0 \cap H_n)$, so U has a unique maximal proper connected definable subgroup M. For each i, we denote by $\rho_i : U \to H_i$ the projection map from U to H_i. We may assume $\rho_i(U) = H_i \neq 1$ for each i. From now on, $M = (M \cap H_1) \times \cdots \times (M \cap H_n)$ is normal in H. Thus, if $M \neq 1$, we have $H_i \cap M \neq 1$ for some i, and $H/H_i \cap M$ is equal to $H_1(H_i \cap M)/(H_i \cap M) \times \cdots \times H_n(H_i \cap M)/(H_i \cap M)$. Then, in $H/(H_i \cap M)$, the minimality of the Morley rank of U gives

$$U/(H_i \cap M) = (U \cap H_1)(H_i \cap M)/(H_i \cap M) \times \cdots \times (U \cap H_n)(H_i \cap M)/(H_i \cap M),$$

and $U = (U \cap H_1) \times \cdots \times (U \cap H_n)$, contradicting the choice of U, so $M = 1$.

For each i, since ρ_i is definable, the triviality of M implies that each proper closed subgroup of H_i is finite, so H_i is definably isomorphic either to $(K_i)^*$, or to $(K_i)_+$. Moreover, for each i, since $\rho_i(U) = H_i \neq 1$, the kernel R_i of ρ_i is finite, and U/R_i is definably isomorphic either to $(K_i)^*$, or to $(K_i)_+$. Now $R = \langle R_i \mid i = 1, \ldots, n\rangle$ is finite and, for each i, U/R is definably isomorphic either to $(K_i)^*$, or to $(K_i)_+$. Hence Lemma 5.4 yields a contradiction. □

Corollary 5.7. – *Let G be an ACF-group, definably affine over interpretable fields $K_1, \ldots, K_n$. Then, for each $i = 1, \ldots, n$, there is an affine algebraic group H_i over K_i, such that G is definably isomorphic to U/F, for a definable subgroup U of $H_1 \times \cdots \times H_n$, and a finite normal subgroup F of U.*

Proof – We may assume that K_i is not definably isomorphic to K_j for $i \neq j$. Since G is definably affine, it is definably isomorphic to a definable section U/F of $H_1 \times \cdots \times H_n$, where H_i is an affine algebraic group over K_i for each $i = 1, \ldots, n$. We may assume $F \cap H_i = 1$ for each i. But Lemma 5.6 gives $F^\circ = (F^\circ \cap H_1) \times \cdots \times (F^\circ \cap H_n) = 1$, so F is finite. □

From now on, we connect the notions of definably linear groups and of definably affine groups.

At first we note that, if G is both an ACF-group and an affine algebraic group over an algebraically closed field K, then since any quotient of an affine algebraic group by a normal closed subgroup is affine, it follows that the ACF-group G is definably affine over one interpretable field if and only if it is definably linear over one interpretable field.

Corollary 5.8. – *Let G be an ACF-group, definably affine over interpretable fields $K_1, \ldots, K_n$. Then G has a finite normal subgroup E such that G/E is definably linear over $K_1, \ldots, K_n$.*

Proof – Let U/F and $H_1, \ldots, H_n$ be as in the previous result. We may assume $G = U/F$. For each i, let F_i be the projection of F on H_i. Then $E = (F_1 \times \cdots \times F_n)/F$ is convenient. □

5.2. **The subgroup $A(G)$.**

Theorem 5.9. – *Let G be an ACF-group. If G is generated by its connected definably affine subgroups over interpretable fields $K_1, \ldots, K_n$, then G is definably affine over $K_1, \ldots, K_n$.*

Proof – Let L be an algebraically closed field such that G is interpretable in the pure field L. First we assume that G is generated by its connected definably affine subgroups over an interpretable field K. We may assume that K is infinite, therefore K is isomorphic to L, definably in the pure field L (Fact 2.6). Let φ be an isomorphism from G to an algebraic group G_K over K, definable in the pure field L, and let Δ be its graph. By the Zilber's Indecomposability Theorem, G is connected and there exist finitely many connected definably affine subgroups $R_1, \ldots, R_m$ of G over K such that $G = \langle R_i \mid i = 1, \ldots, m\rangle$. Since K and L are definably isomorphic in L, for each i, the subgroup $\varphi(R_i)$ of G_K is affine, and any subgroup of $R_i \times \varphi(R_i)$ is definable in G if and only if it is definable in L. In particular, the subgroup $S_i = \{(x, \varphi(x)) \mid x \in R_i\}$ of Δ is affine and definable in G for each i. Moreover, for each i, since R_i is connected (in G), this implies that

S_i and $\varphi(R_i)$ are connected in G. Consequently, the Zilber's Indecomposability Theorem says that $\Delta = \langle S_i \mid i = 1, \dots, n\rangle$ is definable in G, so φ is definable in G. Moreover, for each i, since $\varphi(R_i)$ is connected in G, it is connected in K too, and $G_K = \langle \varphi(R_i) \mid i = 1, \dots, n\rangle$ is an affine algebraic group. Hence G is definably linear over K.

For the proof of the result, we may assume that K_i is not definably isomorphic to K_j for each $i \neq j$. For each i, we denote by U_i the (normal) subgroup of G generated by the connected definable subgroups of G, definably affine over K_i. By the Zilber's Indecomposability Theorem, U_i is definable and connected for each i, and the previous paragraph says that U_i is definably linear over K_i for each i. Now, for each $i \neq j$, by the Zilber's Indecomposability Theorem, $[U_i, U_j]$ is a connected definable subgroup of $U_i \cap U_j$, and it is definably affine over K_i (resp. K_j). Hence Corollary 5.5 yields $[U_i, U_j] = 1$.

We show that G is the central product of $U_1, \dots, U_n$. By the Zilber's Indecomposability Theorem, G is connected and there exist finitely many connected definably affine subgroups $R_1, \dots, R_m$ of G such that $G = \langle R_i \mid i = 1, \dots, m\rangle$. By Lemma 5.6 and Corollary 5.7, for each $i = 1, \dots, n$ and each $j = 1, \dots, m$, there is an affine algebraic group H_{ij} over K_i, such that R_j is definably isomorphic to V_j/F_j, for a definable subgroup V_j of $H_{1j} \times \cdots \times H_{nj}$, and a finite normal subgroup F_j of V_j. Since R_j is connected for each j, we may assume that V_j is connected for each j. Now, for each $j = 1, \dots, m$, Lemma 5.6 shows that R_j has some connected definable subgroups $S_{1j}, \dots, S_{nj}$ such that $R_j = \langle S_{ij} \mid i = 1, \dots, n\rangle$ and such that S_{ij} is definably affine over K_i for each i. Since for each i, j, the subgroup S_{ij} is contained in U_i, this shows that G is generated by $U_1, \dots, U_n$. Since $[U_i, U_j] = 1$ for each $i \neq j$, the group G is the central product of $U_1, \dots, U_n$. Hence we find a definable epimorphism from $U_1 \times \cdots \times U_n$ to G. Since U_i is definably linear over K_i for each i, this finishes the proof. $\square$

Corollary 5.10. – *Any ACF-group G has a largest connected definably affine subgroup denoted $A(G)$.*

From now on, we can describe the structure of any connected definably affine ACF-group.

Proposition 5.11. – *Let G be a connected ACF-group. Suppose that G is definably affine over the fields $K_1, \dots, K_n$, and that K_i is not definably isomorphic to K_j for each $i \neq j$. For each i, let G_i be the largest connected subgroup of G definably linear over K_i. Then the following conditions hold:*

- *G is the central product of $G_1, \dots, G_n$;*
- *G has a nontrivial finite normal subgroup E such that G/E is the direct product of $G_1E/E, \dots, G_nE/E$; in particular $G_i \cap G_j$ is finite for each $i \neq j$;*
- *$\{G_1, \cdots, G_n\}$ is stable by each automorphism of the ACF-group G.*

Proof – By Corollary 5.7, for each i, there is an affine algebraic group H_i over K_i such that G is definably isomorphic to U/F, for U a definable subgroup of $H_1 \times \cdots \times H_n$ and a finite normal subgroup F of U. We may assume $G = U/F$. Since G is connected, we may assume that U is connected, and Lemma 5.6 gives $U = (U \cap H_1) \times \cdots \times (U \cap H_n)$. In particular, $U \cap H_i$ is connected for each i, and we may assume $H_i \leq U$ for each i. Now G is the central product of $H_1F/F, \dots, H_nF/F$, and $H_iF/F \cap H_jF/F$ is finite for each $i \neq j$.

For each i, we consider V_i such that $G_i = V_i/F$. By Lemma 5.6, we have $V_i^\circ = (V_i^\circ \cap H_1) \times \cdots \times (V_i^\circ \cap H_n)$ for each i. In particular, $V_i^\circ \cap H_j$ is connected for each i, j. Thus, for each i, j, the subgroup $(V_i^\circ \cap H_j)F/F \leq G_i \cap H_jF/F$ is connected and definably affine over K_i (resp. K_j). Hence Corollary 5.5 gives $V_i^\circ \cap H_j = 1$ for each $i \neq j$, so V_i° is contained in H_i for each i. Since G_i contains H_iF/F for each i, we obtain $G_i = H_iF/F$ for each i. This proves that G is the central product of $G_1, \ldots, G_n$, and that $G_i \cap G_j$ is finite for each $i \neq j$.

Moreover, Corollary 5.8 provides a finite normal subgroup E such that G/E is definably linear over $K_1, \ldots, K_n$, and Lemma 5.6 says that G/E is definably isomorphic to $A_1 \times \cdots \times A_n$ for some affine algebraic groups $A_1, \ldots, A_n$ over $K_1, \ldots, K_n$ respectively. Then Corollary 5.5 and Lemma 5.6 imply that G_iE/E is definably isomorphic to A_i for each i, and that G/E is the direct product of $G_1E/E, \ldots, G_nE/E$.

Let φ be an automorphism of the ACF-group G. We fix $i \in \{1, \ldots, n\}$, and we show that $\varphi(G_i) \in \{G_1, \ldots, G_n\}$. Then, as for G_i in G, there is an infinite interpretable field L such that $\varphi(G_i)$ is the largest connected definable subgroup of G, definably linear over L. If L is not definably isomorphic to K_j for each j, then G is definably affine over $K_1, \ldots, K_n, L$, and the previous paragraphs applied with this situation give $dim(G) = dim(G_1) + \cdots + dim(G_n) + dim(\varphi(G_i))$. On the other hand, by the previous paragraph, we have $dim(G) = dim(G_1) + \cdots + dim(G_n)$, so $\varphi(G_i)$ is finite. Since $\varphi(G_i)$ is connected, we obtain $\varphi(G_i) = 1$, so $G_i = 1 = \varphi(G_i)$. Hence we may assume that L is definably isomorphic to K_j for some j. Since $\varphi(G_i)$ is the largest connected definable subgroup of G, definably linear over L, we obtain $\varphi(G_i) = G_j$. This finishes the proof. □

Corollary 5.12. – *Let G be an ACF-group, and let K be an algebraically closed field interpreting G. Let T be a maximal torus of G viewed as an algebraic group over K. Then T is definable in the ACF-group G.*

Proof – We may assume that G is connected as ACF-group. Let $G_1, \cdots, G_n$ be as in Proposition 5.11. Then G is the central product of $G_1, \ldots, G_n$, so there are maximal tori $T_1, \ldots, T_n$ in $G_1, \ldots, G_n$ respectively such that $T = T_1 \cdots T_n$. Since, for each i, the subgroup G_i is definably linear over one interpretable field, then T_i is definable, consequently T is definable too. □

5.3. **The subgroup $W(G)$.** We provide some crucial criterions for the ACF-groups to be definably affine (Theorems 5.16 and 5.9). Moreover, similarly to the existence of the subgroup $A(G)$ in any ACF-group G (Corollary 5.10), we show that any such a group G has a smallest normal definable subgroup $W(G)$ such that $G/W(G)$ is definably affine, and that $W(G)$ is connected when G is an ACF_0-group (Corollary 5.18).

In the proof of Corollary 5.14, we use the following result, due to A.V. Borovik and G. Cherlin.

Fact 5.13. – [19, Proof of proposition 4.3] *Let H be a normal definable subgroup of finite index of a group G of finite Morley rank. Then G definably embeds in the wreath product of H by G/H.*

Corollary 5.14. – *Let G be a group of finite Morley rank. If G° is definably affine over interpretable fields $K_1, \ldots, K_n$, then G is definably affine over $K_1, \ldots, K_n$ too.*

Proof – For each $i = 1, \ldots, n$, let H_i be an affine algebraic group over K_i such that G° is definably isomorphic to the definable section U/F of $H = H_1 \times \cdots \times H_n$.

Then the wreath product W of $G^\circ \simeq U/F$ by G/G° is definably isomorphic to a definable section of the wreath product of H by G/G°, which is definably linear over $K_1, \ldots, K_n$. Since Fact 5.13 says that G definably embeds in W, we obtain the result. □

Proposition 5.15. – *Let G be an ACF_p-group for p a prime or zero, and let E be a finite normal subgroup of G. If p does not divide $|E|$ and if G/E is definably linear over an interpretable field L, then G is definably linear over L too.*

Proof – Let K be an algebraically closed field of characteristic p such that the ACF-group G is interpretable in the pure field K. By Corollary 5.14, we may assume that G is connected and infinite. Moreover, G/E is definably linear over L, so G/E and G are affine over K by Fact 2.6, and we may assume that G is an infinite subgroup of $\mathrm{GL}_n(K)$ for an integer n. Since G/E is definably linear over L, we find an algebraic group A_0 over L definably isomorphic to G/E. Since G is infinite, L is infinite too and L is isomorphic to K, definably in K by Fact 2.6. Let α be a field automorphism from K to L, definable in the pure field K. We consider $A = \alpha^*(G)$ and $B = \alpha^*(E)$, where α^* is the isomorphism from $\mathrm{GL}_n(K)$ to $\mathrm{GL}_n(L)$ induced by α. Then A/B is isomorphic to A_0, definably in the pure field K. In the pure field K, for each i, the definable subsets of $\mathrm{GL}_i(L)$ are the images of the constructible subsets of $\mathrm{GL}_i(K)$ by the isomorphisms α_i^* induced by α. In particular A is definable in L and, for any isomorphism from A/B to A_0, definable in K, the graph Δ_0 is definable in L too.

Let $Z = B \times E$, and let Δ be the graph of the isomorphism u from A to G induced by α^*. Let v be an isomorphism from G/E to A_0 definable in G. Then u induces an isomorphism $\overline{u}$ from A/B to G/E, definable in K, and $v \circ \overline{u}$ is an isomorphism from A/B to A_0 definable in K. By the previous paragraph, $v \circ \overline{u}$ is definable in L too. Thus, since v is definable in G, $\overline{u}$ is definable in G. This shows that ΔZ is a subgroup of $A \times G$, definable in G. Moreover, since Z is abelian and finite of order $|E|^2$, the quotient $\Delta Z/\Delta$ is abelian, finite, and p does not divide its order. Now, by Lemma 4.1, we obtain $(\Delta Z)^\circ \leq \Delta$, and Δ is definable in G. This proves that u is definable in G, so G is definably linear over L. □

Theorem 5.16. – *Let G be an ACF_p-group for p a prime or zero. Suppose that G has a finite normal subgroup F such that G/F is definably affine over interpretable fields $K_1, \ldots, K_n$. If p does not divide $n = |F|$, then G is definably affine over $K_1, \ldots, K_n$*

Proof – By Corollary 5.14, we may assume that G is connected. By Corollary 5.8, G has a finite normal subgroup E containing F such that there exists a definable isomorphism ρ from G/E to a definable subgroup U of $H = H_1 \times \cdots \times H_n$, where H_i is an affine algebraic group over K_i for each i. By Lemma 5.6, we have $U = (U \cap H_1) \times \cdots \times (U \cap H_n)$.

For each i, we consider $V_i/E = \rho^{-1}(U \cap H_i)$. Then we have $G/E = V_1/E \times \cdots \times V_n/E$ and, for each $i \neq j$, we have $[V_i, V_j] \leq E$. Since V_i/E is connected for each i and since E is finite, the Zil'ber's Indecomposability Theorem [5, Corollary 5.29] shows that $[V_i, V_j] = 1$ for each $i \neq j$. Thus G is the central product of $V_1, \ldots, V_n$, and there is a definable epimorphism f from $V_1 \times \cdots \times V_n$ to G with a finite kernel. Hence, by Corollary 5.14, we have just to prove that V_i° is definably affine over K_i for each i. But, for each i, the quotient $V_i^\circ/(V_i^\circ \cap E)$ is definably affine over K_i, so

Corollary 5.5 and Lemma 5.6 show that $V_i^\circ/(V_i^\circ \cap F)$ is definably affine over K_i. Now Proposition 5.15 provides the result. □

Lemma 5.17. – *Let G be a group of finite Morley rank. If G is residually definably affine over interpretable fields $K_1, \dots, K_n$, then G is definably affine over $K_1, \dots, K_n$.*

Proof – We find finitely many normal definable subgroups $S_1, \dots, S_m$ of G such that $\cap_{i=1}^m S_i = 1$ and such that G/S_i is definably affine over $K_1, \dots, K_n$ for each i. Then G definably embeds in $(G/S_1) \times \cdots \times (G/S_m)$, and the result follows. □

Corollary 5.18. – *Let G be an ACF_p-group for p a prime or zero. Then G has a smallest normal definable subgroup $W(G)$ such that $G/W(G)$ is definably affine.*

Furthermore, $W(G)$ is contained in G° and, either $p = 0$ and $W(G)$ is connected, or p is a prime and $W(G)/W(G)^\circ$ is a p-group.

Proof – The existence of $W(G)$ follows from Lemma 5.17. Since G/G° is definably linear over any finite field, we have $W(G) \leq G^\circ$. In particular, $W(G)/W(G)^\circ$ is a central subgroup of $G^\circ/W(G)^\circ$, and it is abelian.

If $p = 0$, let $H = W(G)^\circ$, and if p is a prime, let $H/W(G)^\circ$ be the largest p-subgroup of $W(G)/W(G)^\circ$. Then Corollary 5.14 and Theorem 5.16 show that G/H is definably affine, so we obtain $H = W(G)$, and this equality finishes the proof. □

6. More on pseudo-tori

In this section, we consider an algebraic group G over $\overline{\mathbb{Q}}$, and its pseudo-tori in the ACF-expansion of the pure group G. Ideally, these subgroups are the tori of G. However, even if G is centerless and connected, the maximal tori of the pure group G may be not definable (Example 3.1 (2)).

Our first result concerns the ACF-expansion of the pure group G, when G is a connected centerless algebraic group over $\overline{\mathbb{Q}}$, with G solvable of class two (Proposition 6.2). We should note that, by the main theorem of [16], such a group interprets finitely many connected, solvable of class two and centerless algebraic groups $G_1, \dots, G_n$ over algebraically closed fields $K_1, \dots, K_n$ respectively, in such a way that G imbeds in $G_1 \times \cdots \times G_n$. However, contrary to what is announced, the embedding in [16] is not necessarily definable. *Indeed, otherwise the group G in Example 3.1 (2) would be definably linear, and since it is not a direct product of two proper subgroups, it would be definably linear over one interpretable field (Lemma 5.6), contradicting Example 3.1 (2).*

For the proof of Proposition 6.2 and of the main result of this section, that is Theorem 6.3, we need *Carter subgroups*. These subgroups are defined in any group of finite Morley rank as being definable, connected, nilpotent, and of finite index in their normalizers. They have turned out to be increasingly useful in the analysis of groups of finite Morley rank. In the algebraic groups over an algebraically closed field, and when the language is the one of the pure fields, these subgroups are precisely *Cartan subgroups*, namely the connected component of centralizers of the maximal tori. The following fact is a summarize, in the solvable context, of their properties useful for us. We refer to [21] for more details on Carter subgroups.

Fact 6.1. – *Let G be a connected solvable group of finite Morley rank, and let N be a normal definable subgroup of G. Then the following conditions are satisfied:*

(i) [20, Corollary 2.10] *any pseudo-torus of G lies in a Carter subgroup of G;*
(ii) [21, Theorem 3.11] *its Carter subgroups are self-normalizing;*
(iii) [21, Corollary 3.3] *the Carter subgroups of G/N are exactly of the form QN/N, with Q a Carter subgroup of G;*
(iν) [21, Proposition 3.19] *if G is 2-solvable, then G has a definable connected characteristic abelian subgroup A of G such that $G = A \rtimes C$ for every Carter subgroup C of G.*

Proposition 6.2. – *Let G be an algebraic group over $\overline{\mathbb{Q}}$. If G is connected, centerless, and solvable of class two, then the ACF-expansion of the pure group G is definably linear.*

Proof – We notice that, since G is connected and centerless, $F(G)$ is torsion-free. We suppose toward a contradiction that there exists a counterexample G of minimal Morley rank. For $i = 1, 2$, let A_i be a G-minimal subgroup, let Z_i/A_i be the center of G/A_i, and let W_i be the hypercenter of G/A_i. Since G is connected, W_i is definable for $i = 1, 2$ [28, Corollaire 3.15]. Moreover, since $F(G)$ is torsion-free, A_1 and A_2 are torsion-free too. Thus, if we have $A_1 \neq A_2$, then $A_1 \cap A_2$ is trivial, and $Z_1 \cap Z_2 \leq Z(G)$ is trivial too, so $W_1 \cap W_2$ is trivial. In this case, since G/W_1 and G/W_2 are definably affine by induction hypothesis, G is definably affine (Lemma 5.17). Hence G has a unique G-minimal subgroup A.

Let C be a Carter subgroup of G. By Fact 6.1 (iii), we have $G = G'C$. Then $Z(C) \cap C_C(G')$ is central in G, so $C_C(G')$ is trivial. Moreover, since G is 2-solvable, C' centralizes G', so C is abelian. Hence $C \cap G'$ is central in G, and we obtain $G = G' \rtimes C$.

We show that, if $A(G')$ denotes the largest connected definably affine subgroup of G' (Corollary 5.10), we have $A \leq A(G') < G'$, and that $C/C_C(A(G'))$ is definably linear over one interpretable field. By Facts 2.5, we have $A \leq A(G')$. Moreover, since $G' \leq F(G)$ is torsion-free, $A(G')$ is definably linear by Corollary 5.8. So the uniqueness of A and Lemma 5.6 show that $A(G')$ is definably linear over one interpretable field K. Moreover, since G' is torsion-free, K is of characteristic zero and $A(G')$ is definably isomorphic to a unipotent algebraic group over K. Consequently, Corollary 4.8 says that $C/C_C(A(G'))$ is definaly linear over K. Therefore, if $A(G') = G'$, since $C_C(G')$ is trivial, then C is definably linear over K. But, since G is solvable, we have $G'H < G$ for each normal proper subgroup H of G, so the conjugates of C generate G. Hence the condition $A(G') = G'$ implies that G is definably affine over K by Theorem 5.9. Thus we obtain $A(G') < G'$.

We claim that G has a unique maximal proper normal connected definable subgroup. Indeed, suppose that N_1 and N_2 are two distinct maximal proper normal connected definable subgroups of G. In particular, N_1 and N_2 contain G', and G is generated by N_1 and N_2. We may assume that A is noncentral in N_2. Since $Z(N_2) \leq F(G)$ is torsion-free, and since it is normal in G and does not contains A, it is trivial. Hence N_2 is definably linear by induction hypothesis. Moreover, by the maximality of N_2 and since G is solvable, N_2 contains G', therefore G' is definably linear. This contradicts the previous paragraph, so G has a unique maximal proper normal connected definable subgroup N.

We show that $N = F(G)$. Indeed, we have $F(G) \leq N$ by the uniqueness of N. Let $\overline{B}$ be a G-minimal section of G'. If C centralizes $\overline{B}$, then C covers $\overline{B}$ by Fact 6.1 (iii), contradicting $G' \cap C = 1$. So Fact 2.5 says that $G/C_G(\overline{B}) \simeq C/C_C(\overline{B})$ is definably isomorphic to a subgroup of L^* for an interpretable algebraically closed field L. But L and $\overline{\mathbb{Q}}$ are definably isomorphic in the pure field $\overline{\mathbb{Q}}$ (Fact 2.6), hence L^* has no proper connected definable subgroup, and $C_G(\overline{B})^\circ$ is a maximal proper connected definable subgroup of G. Then, by the uniqueness of N, we obtain $N = C_G(\overline{B})^\circ$, and N centralizes each G-minimal section of G'. This implies that N is nilpotent, so $N = F(G)$.

We show that $A = A(G')$. We note that, since G is centerless and since $A(G')$ contains A, we have $C_G(A(G')) < G$. So, by the uniqueness of N, the quotient $NC_G(A(G'))/C_G(A(G'))$ is the unique maximal proper normal connected definable subgroup of $G/C_G(A(G'))$. But $G/C_G(A(G'))$ is abelian since G' is abelian, and $G/C_G(A(G')) \simeq C/C_C(A(G'))$ is definably linear over K. Hence, since K and $\overline{\mathbb{Q}}$ are definably isomorphic in the pure field $\overline{\mathbb{Q}}$ (Fact 2.6), the quotient $G/C_G(A(G'))$ has dimension one over $\overline{\mathbb{Q}}$, and the uniqueness of N gives $N = C_G(A(G'))^\circ$. In particular, since the unipotent part of G is contained in $F(G) = N$, it centralizes $A(G')$, and $G/C_G(A(G'))$ is a torus. Moreover, $A(G')$ is a closed torsion-free abelian subgroup of G, so it is a $\overline{\mathbb{Q}}$-vector space, and there is a base $(a_1, \ldots, a_n)$ of $A(G')$ such that G normalizes $\overline{\mathbb{Q}} \cdot a_i$ for each i. Then, since $A(G')$ is definably linear over K and since K and $\overline{\mathbb{Q}}$ are definably isomorphic in the pure field $\overline{\mathbb{Q}}$, the subgroups $\overline{\mathbb{Q}} \cdot a_i$ are definable and normal in G for each i. So we obtain $n = 1$ by the uniqueness of A, and $A(G')$ is definably isomorphic to K_+. Now Fact 2.7 gives $A(G') = A$, as desired.

We show that AC is the unique maximal proper connected definable subgroup of G containing C. Indeed, if $G = AC$, then we have $G' = A = A(G')$, contradicting $A(G') < G'$. Let M be a maximal proper connected definable subgroup M of G containing C. Since AC is proper in G, we have $C < M$. Then $M \cap G'$ is a normal infinite definable subgroup of G, and we have $A \leq M$ by the uniqueness of A. Let Z be the hypercenter of M. It is definable by [28, Corollaire 3.15], and it is contained in C by Fact 6.1 (ii). In particular, $Z \cap G'$ is trivial. Moreover, by induction hypothesis, M/Z is definably linear, so $M \cap G'$ is definably linear, and the previous paragraph gives $M \cap G' \leq A(G') = A$. Now we have $M = AC$, as claimed.

From now on, we can prove that the maximal tori of G are definable. Indeed, by the maximality of AC and since $A < G'$, the quotient G'/A is C-minimal. Since $G = G' \rtimes C$, Fact 6.1 (iii) says that C does not centralize G'/A, and Fact 2.5 provides an infinite interpretable field K_1 such that G'/A is definably isomorphic to $(K_1)_+$. Then Fact 2.6 shows that G'/A is isomorphic to $\overline{\mathbb{Q}}_+$, definably in $\overline{\mathbb{Q}}$. Since $A \simeq \overline{\mathbb{Q}} \cdot a_1$ is isomorphic to $\overline{\mathbb{Q}}_+$, definably in $\overline{\mathbb{Q}}$ too, G' is isomorphic to a $\overline{\mathbb{Q}}$-vector space of dimension two, definably in $\overline{\mathbb{Q}}$. Thus, since $C_C(G')$ is trivial, C is isomorphic, definably in $\overline{\mathbb{Q}}$, to an abelian connected closed subgroup of

$$\left\{ \begin{pmatrix} s & a \\ 0 & t \end{pmatrix} \mid (s,t) \in (\overline{\mathbb{Q}}^*)^2,\ a \in \overline{\mathbb{Q}} \right\}.$$

In particular, C has dimension at most two over $\overline{\mathbb{Q}}$. Since $C/C_C(A)$ is definably linear over K, Proposition 5.15 says that $C/C_C(A)^\circ$ is definably linear over K too. Thus, since the conjugates of C generate G, and since G is not definably affine over K, Theorem 5.9 shows that $C_C(A)^\circ$ is nontrivial, so C has dimension exactly two over $\overline{\mathbb{Q}}$. Moreover, $C_C(A)^\circ \leq N = F(G)$ is torsion-free, so it is a unipotent subgroup, and C is not a torus. This implies that C is isomorphic, definably in $\overline{\mathbb{Q}}$, to

$$\left\{ \begin{pmatrix} t & a \\ 0 & t \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^*,\ a \in \overline{\mathbb{Q}} \right\}.$$

Consequently, $G = G' \rtimes C$ is isomorphic, definably in $\overline{\mathbb{Q}}$, to

$$\left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \overline{\mathbb{Q}}^*,\ (a, u, v) \in \overline{\mathbb{Q}}^3 \right\},$$

and Lemma 3.7 and Proposition 3.11 say that the maximal tori of G are definable.

Let T be the maximal torus of C. Since $C \cap N \leq F(G)$ is torsion-free, $C \cap N$ is a unipotent subgroup. On the other hand, the unipotent part of C is contained in the one of G, which is contained in $F(G) = N$, so we obtain $C = (C \cap N) \times T$. Actually, since $N = F(G)$ is torsion-free, it is the unipotent part of G, and since $G = G'C = NC = N \rtimes T$, the torus T is maximal in G. Consequently, by the previous paragraph, it is definable, and $G'T$ is definable too. But, since $G = G' \rtimes C$ and since $C \cap N = C_C(A)^\circ$ is nontrivial, $G'T$ is a proper normal connected definable subgroup of G. Hence $G'T$ is contained in N by the uniqueness of N. This contradicts $G = N \rtimes T$, and finishes the proof. □

From now on, we are going to prove the following fundamental result.

Theorem 6.3. – *Let G be a connected solvable algebraic group over $\overline{\mathbb{Q}}$. Then $T \cap F(G)$ is central in G for each pseudo-torus T of the ACF-expansion of the pure group G.*

First, we have to prove Lemma 6.5. Its proof uses the *Frattini subgroup*, which is defined for any group G of finite Morley rank, as being the intersection of all the maximal proper definable connected subgroups of G. It is denoted by $\Phi(G)$.

Fact 6.4. – *Let G be a connected group of finite Morley rank. Then,*

(i) [21, Lemma 2.14.b] *if H is a definable subgroup of G such that $G = \Phi(G)H$, we have $G = H$;*
(ii) [21, Lemma 2.14.a] *$\Phi(G/\Phi(G))$ is trivial;*
(iii) [21, Proposition 3.18] *if G is solvable, $\Phi(G)$ is nilpotent and the quotient $F(G)/\Phi(G) = F(G/\Phi(G))$ is abelian. In particular, $G/\Phi(G)$ is 2-solvable, and G is nilpotent if and only if $G/\Phi(G)$ is abelian;*
(iν) [21, Proposition 6.5] *if G is solvable, $F(G) \cap \Phi(C) \leq \Phi(G)$ for each Carter subgroup C of G;*
(ν) [19, Lemma 5.4] *if G is nilpotent and if $\Phi(G)$ is finite, $\Phi(G/A)$ is finite for each normal definable subgroup A of G.*

Lemma 6.5. – *Let C be a nilpotent group of finite Morley rank, and T a pseudo-torus of C. Then, for each connected definable subgroup A of T, the quotient $A\Phi(C)/\Phi(C)$ is a pseudo-torus.*

Proof – By Facts 2.10 (iν) and 6.4 (ii), we may assume $\Phi(C) = 1$. In particular C is abelian (Fact 6.4 (iii)). We assume toward a contradiction that A is not a pseudo-torus. Since T is abelian and radicable, then A is abelian and radicable too, and it has a definable normal subgroup B such that A/B is definably isomorphic to K_+ for an interpretable field K. Moreover, since A is radicable, the characteristic of K is zero, and A/B has no nontrivial proper definable subgroup (Fact 2.7). Since $\Phi(C/B)$ is finite (Fact 6.4 (ν)), there is a maximal proper connected definable subgroup M/B of C/B does not containing A/B. In particular, since A/B has no nontrivial proper definable subgroup, we have $A \cap M = B$. Moreover, by maximality of M/B, we have $C = MA = MT$, and $K_+ \simeq A/B = A/(A \cap M)$ is definably isomorphic to $T/(T \cap M)$, contradicting that T is a pseudo-torus. This finishes the proof. □

From now on, we can prove Theorem 6.3.

Proof – [Proof of Theorem 6.3] We assume toward a contradiction that G is a counterexample of minimal Morley rank and with $Z(G)$ of minimal Morley degree. We may assume that T is a maximal pseudo-torus of G. We show that no proper normal definable subgroup of G contains T. Indeed, suppose toward a contradiction that M is a proper normal definable subgroup of G containing T. Then M° contains T since T is connected, and $T \cap F(M^\circ)$ is central in M° by the minimality of $rk(G)$. But we have $G = MN_G(T)$ by Fact 2.10 (i) and a Frattini Argument, therefore by Fact 2.10 (ii) and since G is connected, we find $G = M^\circ C_G(T)$. Hence we obtain $T \cap F(M^\circ) \leq Z(G)$. Since M is normal in G, we have $F(M^\circ) = M^\circ \cap F(G)$, so $T \cap F(G)$ is central in G, contradicting the choice of G. This proves that no proper normal definable subgroup of G contains T. In particular, we have $G = G'T$.

We show that $Z(G) = 1$. By Fact 2.10 (iν), $TZ(G)/Z(G)$ is a pseudo-torus of $G/Z(G)$. Consequently, if $Z(G)$ is infinite, the minimality of $rk(G)$ implies that $(T \cap F(G))Z(G)/Z(G)$ is central in $G/Z(G)$, so $C_G(t)$ is a normal subgroup of G for each $t \in T \cap F(G)$. Now the previous paragraph gives $G = C_G(t)$ for each $t \in T \cap F(G)$, that is $T \cap F(G) \leq Z(G)$, which contradicts our hypothesis. Thus $Z(G)$ is finite and, since G is connected, $G/Z(G)$ is centerless. If $Z(G) \neq 1$, the minimality of the Morley degree of $Z(G)$ yields $(T \cap F(G))Z(G)/Z(G) = 1$ and $T \cap F(G) \leq Z(G)$, contradicting our hypothesis on G, so G is centerless.

We show that $F(G)$ is torsion-free. Indeed, for each prime p, since $\overline{\mathbb{Q}}$ has characteristic zero and since $F(G)$ is nilpotent, the subset F_p of the elements of order p in $F(G)$ is finite. Then, since G is connected and normalizes F_p, it centralizes F_p, and we obtain $F_p = \emptyset$ because $Z(G) = 1$. This proves that $F(G)$ is torsion-free. Furthermore, this implies that $F(G)$ is a unipotent subgroup of G.

By Fact 6.1 (i), T is contained in a Carter subgroup C of G. We show that $\Phi(C)$ contains $T \cap F(G)$. Since $F(G)$ is torsion-free, $T \cap F(G)$ is connected. By Lemma 6.5, the group $(T \cap F(G))\Phi(C)/\Phi(C)$ is a pseudo-torus, and Fact 2.10 (iν) yields a pseudo-torus T_0 of $T \cap F(G)$ such that $T_0\Phi(C) = (T \cap F(G))\Phi(C)$. By Fact 2.10 (iii), we have $T_0 \leq Z(G) = 1$, so $(T \cap F(G))\Phi(C)/\Phi(C) = 1$ and $T \cap F(G)$ is contained in $\Phi(C)$. In particular, $T \cap F(G)$ is contained in $\Phi(G)$ (Fact 6.4 (iν)).

We show that T is contained in a unique maximal proper definable connected subgroup M of G. Indeed, since $T < G$, the subgroup M exists. Moreover, we have $T \cap F(M) \leq Z(M)$ by the minimality of $rk(G)$. Since M contains $\Phi(G)$ by the definition of $\Phi(G)$, and since $\Phi(G)$ is nilpotent (Fact 6.4 (iii)), we have $\Phi(G) \leq F(M)$ and the previous paragraph yields $T \cap F(G) \leq T \cap F(M) \leq Z(M)$. Thus, if T is contained in two distinct maximal proper definable connected subgroups M_1 and M_2 of G, then $T \cap F(G)$ centralizes $\langle M_1, M_2\rangle = G$, contradicting $T \cap F(G) \not\leq Z(G)$. Hence we obtain the uniqueness of M.

We consider $Z/G'' = Z(G/G'')$, $\overline{G} = G/Z$ and $\overline{T} = TZ/Z$. We show that $\overline{G} = \overline{G}' \rtimes \overline{T}$ and that $\overline{G}$ is centerless. If $G'' = 1$ then, since G is centerless, we have $Z = 1$ and $Z(\overline{G}) = 1$. Moreover G' and T are abelian and, since the first paragraph gives $G = G'T$, we obtain $G' \cap T \leq Z(G) = 1$, so $G = G' \rtimes T$ and $\overline{G} = \overline{G}' \rtimes \overline{T}$. Hence we may assume $G'' \neq 1$. In particular, the minimality of $rk(G)$ yields $TG''/G'' \cap F(G/G'') \leq Z/G''$. But Fact 6.4 (iii) gives $G'' \leq \Phi(G) \leq F(G)$ and $F(G/\Phi(G)) = F(G)/\Phi(G)$, hence we have $F(G/G'') = F(G)/G''$ and we obtain $T \cap F(G) \leq Z$. Also, the equalities $Z/G'' = Z(G/G'')$ and $F(G/G'') = F(G)/G''$ imply $Z \leq F(G)$ and $F(\overline{G}) = F(G)/Z$, so we have $\overline{T} \cap F(\overline{G}) = 1$. In particular, since $F(\overline{G})$ contains the abelian subgroup $\overline{G}'$, we have $\overline{G}' \cap \overline{T} = 1$, and since the first paragraph gives $G = G'T$, we obtain $\overline{G} = \overline{G}' \rtimes \overline{T}$. Now, since $\overline{G}' \leq F(\overline{G})$ and since $\overline{T} \cap F(\overline{G}) = 1$, we find $\overline{G}' = F(\overline{G}) = F(G)/Z$. Moreover, $\overline{C} = CZ/Z$ is a Carter subgroup of $\overline{G}$ (Fact 6.1 (iii)), and Fact 6.1 (iν) provides a definable connected characteristic abelian subgroup $\overline{A} = A/Z$ in $\overline{G}$ such that $\overline{G} = \overline{A} \rtimes \overline{C}$. Since Fact 2.10 (iii) says that $\overline{T}$ is central in $\overline{C}$, the definable subgroup AT is normal in G and the first paragraph gives $G = AT$. Thus we obtain $\overline{C} = \overline{T}$. Since G' is torsion-free, $Z(\overline{G}) \leq F(\overline{G}) = \overline{G}'$ is torsion-free too. In particular, since $Z(\overline{G})$ normalizes the Carter subgroup $\overline{C}$, we obtain $Z(\overline{G}) \leq \overline{C} \cap F(\overline{G}) = \overline{T} \cap F(\overline{G}) = 1$. This proves that $\overline{G}$ is a centerless 2-solvable group.

We show that $\overline{G}'$ has no nontrivial proper definable subgroup. Fact 6.4 (iii) gives $G'' \leq \Phi(G) \leq M$, so M is normal in MZ, and $\overline{M} = MZ/Z$ is proper in $\overline{G}$ by the first paragraph. Then $\overline{M}$ is the unique maximal proper definable connected subgroup of $\overline{G}$ containing $\overline{T}$. In particular $\overline{G}'$ has no decomposition of the form $\overline{G}' = \overline{A_1}\,\overline{A_2}$ for two proper definable connected subgroups $\overline{A_1}$ and $\overline{A_2}$ of $\overline{G}'$, with $\overline{A_1}$ and $\overline{A_2}$ normal in $\overline{G}$, otherwise $\overline{M}$ would contain $\overline{A_i}\,\overline{T}$ for $i = 1, 2$. Then Lemma 5.6 and Proposition 6.2 show that $\overline{G}$ is definably linear over an interpretable field K. Thus, since K and $\overline{\mathbb{Q}}$ are isomorphic, definably in $\overline{\mathbb{Q}}$ (Fact 2.6), each closed subgroup of $\overline{G}$ is definable, and the pseudo-tori of $\overline{G}$ are tori. Moreover, in the pure field $\overline{\mathbb{Q}}$, since G' is torsion-free, $\overline{G}'$ is a $\overline{\mathbb{Q}}$-vector space. Thus, in $\overline{\mathbb{Q}}$, since $\overline{T}$ is a torus acting on $\overline{G}'$, there is a basis $(v_1, \cdots, v_n)$ of $\overline{G}'$, where n is the dimension of $\overline{G}'$ over $\overline{\mathbb{Q}}$, such that $\overline{T}$ normalizes $\overline{\mathbb{Q}} \cdot v_i$ for each i. Then, by the uniqueness of $\overline{M}$, we obtain $n = 1$ and $\overline{G}' = \overline{\mathbb{Q}} \cdot v_1$. Now Fact 2.7 says that $\overline{G}'$ has no nontrivial proper definable subgroup.

We show that G'/G'' has no nontrivial proper definable subgroup. By Fact 6.1 (iii) and (iν), G/G'' has a definable connected characteristic subgroup B/G'' such that $G/G'' = B/G'' \rtimes CG''/G''$. Since T is central in C (Fact 2.10 (iii)), the subgroup BT is normal in G and the first paragraph gives $G = BT$, so $CG''/G'' = TG''/G''$. In particular B contains G'. Since Fact 6.1 (iii) says that $G = G'C$, we obtain $B = G'$. Moreover, $Z/G'' = Z(G/G'')$ normalizes CG''/G'' therefore, since CG''/G'' is a Carter subgroup of G/G'' (Fact 6.1 (iii)), we obtain $Z \leq CG''$ (Fact 6.1 (ii)). Now $Z/G'' \cap G'/G''$ is trivial, and G'/G'' is definably isomorphic to $\overline{G}'$. Thus G'/G'' has no nontrivial proper definable subgroup. Since G is a solvable connected algebraic group, G' is nilpotent, and we have $\Phi(G') = G''$ (Fact 6.4 (iii)). Let $x \in G' \setminus G''$ and let $X := d(x)$ be the smallest definable subgroup of G' containing x. Then X is abelian and, since G'/G'' has no nontrivial proper definable subgroup, we have $G' = XG'' = X\Phi(G')$. Thus Fact 6.4 (i) gives $G' = X$. In particular, G' is abelian and $G'' = 1$. Now G' has no nontrivial proper definable subgroup. Since $G = G'T$ and since $F(G)$ contains G', we have $F(G) = G'(T \cap F(G))$. Consequently, either $F(G) = T \cap F(G)$, or $T \cap F(G)$ is a maximal proper definable subgroup of $F(G)$. In the first case, we obtain $G' \leq T$ and, since $G = G'T$, we have $G = T$ and

G is abelian, contradicting our choice of G. In the second case, $T \cap F(G)$ is normal in $F(G)$. But, since $T \cap F(G) \nleq Z(G)$, we have $T \cap F(G) \neq 1$ and $T \cap Z(F(G))$ is nontrivial. Hence, since we have $G = G'T = F(G)T$, the subgroup $T \cap Z(F(G))$ is central in G, contradicting that G is centerless. This finishes the proof. □

7. Analysis of $G/Z(G)$ and of G'

The purpose of this section is to prove that, if G is a connected algebraic group over $\overline{\mathbb{Q}}$, then $G/Z(G)$ is definably linear and G' is definably affine, in the ACF-expansion of the pure group G (Theorems 7.13).

Moreover, we obtain some results on $A(G)$ and $W(G)$, particularly when the ground field is of positive caracteristic.

7.1. The subgroups $\mathcal{S}(G)$ and $\mathcal{T}(G)$.

For any group G of finite Morley rank, we denote by $\mathcal{S}(G)$ the connected component of the intersection of the maximal pseudo-tori of G, and by $\mathcal{T}(G)$ the subgroup of G generated by its pseudo-tori.

Remark 7.1. –

- By the Zilber's Indecomposability Theorem, $\mathcal{T}(G)$ is a connected definable subgroup of G, for any group G of finite Morley rank.
- If G is an ACF-group, then $\mathcal{T}(G)$ contains all the tori of G.
- Moreover, $\mathcal{S}(G)$ is central in G°.
  Indeed, it is central in $\mathcal{T}(G)$, and the conjugacy of the maximal pseudo-tori of G (Fact 2.10 (i)) and a Frattini Argument provide $G^\circ = \mathcal{T}(G)N_{G^\circ}(T)$ for each maximal pseudo-tori T of G. So, by Fact 2.10 (ii), we have $\mathcal{S}(G) \leq Z(G^\circ)$.

In any group G of finite Morley rank, we denote by $Q(G)$ the *quasiunipotent radical*, which is the largest normal connected definable nilpotent subgroup of G with no nontrivial radicable torsion subgroup. This subgroup was introduced in [2]. Ideally, in any ACF-group, this subgroup is the unipotent radical. However, this fails for some abelian connected groups, since the pure group $\overline{\mathbb{Q}}_+ \times \overline{\mathbb{Q}}^*$ is abstractly isomorphic to $\overline{\mathbb{Q}}^*$, so its quasiunipotent radical is trivial. In the same way, if K is an algebraically closed field of characteristic $p > 0$, and if we consider the ACF-group $G = K_+ \oplus \mathbb{F}_p$ as a pure group, then G is connected, so its quasiunipotent radical is G, while its unipotent radical is $K_+ \oplus \{0\} < G$.

Lemma 7.2. – *In any connected ACF-group G, we have*

$$G/\mathcal{S}(G) = Q(G/\mathcal{S}(G)) \cdot \mathcal{T}(G)/\mathcal{S}(G).$$

Proof – By Fact 2.10 (iν), we may assume $\mathcal{S}(G) = 1$. Let K be an algebraically closed field such that G is interpretable in K. We may assume that G is an algebraic group over K. In particular, $\mathcal{T}(G)$ is an algebraic subgroup of G containing each torus of G.

By Lemma 4.1, the algebraic group G is connected. We denote by G_L the largest connected linear algebraic subgroup of G. In particular, $G^{\circ\circ}/G_L$ is an abelian variety, so it is a pseudo-torus, and it is covered by $\mathcal{T}(G)$ (Fact 2.10 (iν)). Moreover, we have $G_L = U \rtimes S$ for S a maximal reductive subgroup of G_L and U its unipotent radical. Since S is generated by its tori, we obtain $S \leq \mathcal{T}(G)$.

Now let $Q := d(U)$ denote the smallest definable subgroup of G containing U. Since U is a unipotent group, U is nilpotent and contained in $d(U)^\circ$, so Q is nilpotent and connected. We show that $Q = U$. By Fact 2.10 (iii), Q has a unique maximal pseudo-torus T. In particular T is normal in G and it is contained in each maximal pseudo-torus of G, so we have $T \leq \mathcal{S}(G) = 1$. Thus Q is quasiunipotent and it is contained in $Q(G)$. Hence $G^{\circ\circ}$ is contained in $Q(G)\mathcal{T}(G)$, and we obtain the result by connectedness of G. □

Proposition 7.3. – *Let G be an ACF_p-group where p is either a prime or zero. We assume that if $p = 0$, then G is the ACF-expansion of the pure group associated to an algebraic group over $\overline{\mathbb{Q}}$. Then $\mathcal{T}(G)/\mathcal{S}(G)$ is definably affine.*

Proof – We may assume $G = \mathcal{T}(G)$ and, by Fact 2.10 (iν), we may assume $\mathcal{S}(G) = 1$. Let T be a maximal pseudo-torus of G containing a maximal torus of G, and let B be a Borel subgroup of G, in the algebraic sense, with B containing T. Since $\mathcal{S}(B)$ is central in B (Remark 7.1), it is central in G too, and it is contained in $\mathcal{S}(G) = 1$ by the conjugacy of the maximal pseudo-tori (Fact 2.10 (i)). Moreover, Fact 2.5 says that, for each minimal infinite definable T-normal section $\overline{A}$ of B, the quotient $T/C_T(\overline{A})$ is definably affine. Thus, $W(T)F(B)^\circ$ is nilpotent, and since B' is nilpotent and connected, $W(T)F(B)^\circ$ is contained in $F(B)$.

If $p = 0$, then Theorem 6.3 implies that $W(T)$ is central in B, and Corollary 5.18 says that $W(T)$ is connected. Thus $W(T)$ is contained in $\mathcal{S}(B) = 1$ by the conjugacy of the maximal pseudo-tori in B (Fact 2.10 (i)), and T is definably affine. Now $\mathcal{T}(G)$ is definably affine by Theorem 5.9.

If p is a prime, then each abelian radicable definable subgroup of G has torsion, so it is a decent torus. Therefore Fact 2.10 (iii) says that $W(T)^\circ$ is central in B, and it is contained in $\mathcal{S}(B) = 1$ by the conjugacy of the maximal pseudo-tori in B (Fact 2.10 (i)). Now $W(T)$ is finite, and it is a p-group by Corollary 5.18. But G is an ACF_p-group, so it contains no nontrivial radicable p-subgroup. Hence T has no nontrivial p-subgroup, and $W(T)$ is trivial. Consequently T is definably affine, and $\mathcal{T}(G)$ is definably affine by Theorem 5.9. □

7.2. **More on $W(G)$.**

Proposition 7.4. – *Let A be a G-minimal subgroup of an ACF-group G. If $C_G(A)$ has infinite index in G, then A is definably linear over one interpretable field.*

Proof – Let K be an algebraically closed field interpreting G. We consider G as an algebraic group defined over K. By G-minimality of A, and since $C_G(A)$ has infinite index in G, we have $A = [A, G^{\circ\circ}]$, and A is connected in the pure field K.

First we assume that A is abelian. Let B be a Borel subgroup of G. In particular B contains A. Since $C_G(A)$ has infinite index in G, then A is non-central in $G^{\circ\circ}$, so the intersection $A \cap Z(G^{\circ\circ})$ is finite by G-minimality of A, and $A \cap Z(B)$ is finite too [24, Corollary 22.2.B]. Since B is connected and solvable in the pure field K, then $d(B)$ is connected and solvable in the ACF-group G. Let A_0 be a $d(B)$-minimal subgroup in A. Since $A \cap Z(B)$ is finite, the subgroup A_0 is non-central in $d(B)$. Now Fact 2.5 says that A_0 is definably linear over one interpretable field K_0. Since A is the normal closure of A_0, Theorem 5.9 shows that A is definably linear over K_0.

If A is not abelian, then $Z(A)$ is finite. Let B be a Borel subgroup of A. Since A is connected in the pure field K, then B is a maximal solvable subgroup of A, and B is definable in the ACF-group G. Moreover $Z(B) = Z(A)$ is finite [24, Corollary 22.2.B], so if A_0 is a B-minimal subgroup of B, then A_0 is non-central in B. Now

Fact 2.5 says that A_0 is definably linear over one interpretable field K_0, and since A is the normal closure of A_0, Theorem 5.9 shows that A is definably linear over K_0. □

Lemma 7.5. – *Let G be a group of finite Morley rank. If G is residually definably linear over interpretable fields $K_1, \ldots, K_n$, then G is definably linear over $K_1, \ldots, K_n$.*

Proof – We proceed as in the proof of Lemma 5.17. We find finitely many normal definable subgroups $S_1, \ldots, S_m$ of G such that $\cap_{i=1}^m S_i = 1$ and such that G/S_i is definably linear over $K_1, \ldots, K_n$ for each i. Then G definably embeds in $(G/S_1) \times \cdots \times (G/S_m)$, and the result follows. □

Lemma 7.6. – *Let H be a normal connected definable subgroup of an ACF-group G. If H is definably affine over interpretable fields $K_1, \ldots, K_n$, then the quotient $G/C_G(H)$ is definably linear over $K_1, \ldots, K_n$.*

Proof – We may assume that K_i is not definably isomorphic to K_j for each $i \neq j$. For each i, we denote by H_i the largest connected definable subgroup of H definably affine over K_i (Theorem 5.9). These subgroups are normal in G. By Proposition 5.11, H is generated by $H_1, \ldots, H_n$. Hence, by Lemma 7.5, we may assume that H is definably affine over one interpretable field K. Moreover, we may assume that K is infinite.

Let L be an algebraically closed field such that G is interpretable in the pure field L. By Fact 2.6, the fields K and L are isomorphic, definably in L. We consider $H \rtimes G/C_G(H)$, where $G/C_G(H)$ acts on H by conjugation. It is isomorphic to an algebraic group over L, definably in L. Since K and L are isomorphic, definably in L, we find an isomorphism φ, definable in L, from $H \rtimes G/C_G(H)$ to an algebraic group R over K. Since H is definably affine over K and since K and L are isomorphic, definably in L, the map $\varphi_{|H} : H \to \varphi(H)$ is definable in G. Moreover, since K and L are isomorphic, definably in L, the subgroup $V = \varphi(G/C_G(H))$ is algebraic over K, and it is definable in G. Now the semidirect product $R = \varphi(H) \rtimes V$ is definable in G. Thus, since $\varphi_{|H}$ is definable in G, the graph $\Delta = \{(\overline{g}, v) \in G/C_G(H) \times V \mid \forall h \in H,\ \varphi_{|H}(\overline{g} \cdot h) = v \cdot \varphi_{|H}(h)\}$ of $\varphi_{|G/C_G(H)} : G/C_G(H) \to V$ is definable in G too. From now on, $G/C_G(H)$ is definably isomorphic to an algebraic group V over K.

We verify that V is affine. Since the ACF-group H is connected, and since $\varphi(H)$ and $\varphi_{|H}$ are definable in G, then $\varphi(H)$ is connected in G too, and $\varphi(H)$ is connected in the pure field K. Thus V is an algebraic group acting faithfully over a connected algebraic group, so it is affine by the Rosenlicht's Theorem [31, §5, Théorème 13]. This implies that $G/C_G(H)$ is definably linear over K. □

Fact 7.7. – [19, Proposition 4.3] *Let G be a group of finite Morley rank. If G° is definably linear over interpretable fields $K_1, \ldots, K_n$, then G is definably linear over $K_1, \ldots, K_n$ too.*

Remark 7.8. – If G is a connected algebraic group over an algebraically closed field K, then the elements of $F(G)/F(G)^{\circ\circ}$ are semisimple.

Indeed, if we denote by U the unipotent radical of G, then G/U is a reductive group, and since $F(G)^{\circ\circ}$ contains U [24, §19.5]*, the quotient $G/F(G)^{\circ\circ}$ is reductive too. But G is connected, so $F(G)/F(G)^{\circ\circ}$ is centralized by G, and all the elements of $F(G)/F(G)^{\circ\circ}$ are semisimple.*

Lemma 7.9. – *In any connected ACF-group G, the subgroup $F(G)^\circ$ covers $G/G^{\circ\circ}$.*

Proof – By Lemma 4.1, G has a normal definable connected subgroup I contained in $G^{\circ\circ}$ such that G/I is an abelian p-group. In particular, if U denotes the unipotent radical of $G^{\circ\circ}$, then U covers $G^{\circ\circ}/I$. But U is a normal nilpotent connected closed subgroup of $G^{\circ\circ}$. Hence $d(U)$ is a normal nilpotent connected definable subgroup of G, and it is contained in $F(G)^\circ$. Thus $G^{\circ\circ}$ is contained in the definable subgroup $F(G)^\circ I$, and since G is connected, we obtain $G = F(G)^\circ I$. Since $G^{\circ\circ}$ contains I, we obtain the result. □

Corollary 7.10. – *For each ACF-group G, the quotient $G/F(G)$ is definably linear, and $G/F(G)^\circ$ is definably affine. In particular $W(G)$ is nilpotent.*

Proof – Let S denote the centralizer in G° of all the G°-minimal sections of G°. Let U/V be a G°-minimal section of G°. If G° does not centralize U/V, then U/V is definably linear over one interpretable field by Proposition 7.4, and $G^\circ/C_{G^\circ}(U/V)$ is definably linear by Lemma 7.6. Consequently G°/S is definably linear (Lemma 7.5). Since G° is connected, it centralizes S/S°. But S centralizes all its G°-minimal sections, hence S is nilpotent, that is $F(G^\circ)$ contains S. Conversely, since $F(G^\circ)$ is nilpotent and normal, it centralizes all the G°-minimal sections of G°, so $S = F(G^\circ)$. Thus $G^\circ/F(G^\circ)$ is definably linear, and since $F(G^\circ) = F(G) \cap G^\circ$, the quotient $G^\circ F(G)/F(G)$ is definably linear. We conclude that $G/F(G)$ is definably linear by using Fact 7.7.

Let K be an algebraically closed field interpreting G, and let p be its characteristic (p is a prime or zero). If $p = 0$, then $G/F(G)^\circ$ is definably affine by Corollary 5.18, so we may assume that p is a prime. Viewing G as an algebraic group over K, Remark 7.8 shows that $F(G^{\circ\circ})/F(G^{\circ\circ})^{\circ\circ}$ has no nontrivial unipotent element, so p does not divide its order. Since Lemma 7.9 says that $F(G^\circ)^\circ$ covers $G^\circ/G^{\circ\circ}$, then $F(G^\circ)/F(G^\circ)^\circ$ is isomorphic to $(F(G^\circ) \cap G^{\circ\circ})/(F(G^\circ)^\circ \cap G^{\circ\circ})$. But we have $F(G^{\circ\circ}) = F(G^\circ) \cap G^{\circ\circ}$, and $F(G^\circ)^\circ \cap G^{\circ\circ}$ contains $F(G^{\circ\circ})^{\circ\circ}$, so p does not divides the order of $F(G^\circ)/F(G^\circ)^\circ$. Hence Theorem 5.16 shows that $G^\circ//F(G^\circ)^\circ$ is definably affine, and since $F(G)^\circ = F(G^\circ)^\circ$, the quotient $G/F(G^\circ)^\circ$ is definably affine too by Corollary5.14. □

Corollary 7.11. – *Let G be an ACF_p-group where p is either a prime or zero. We assume that if $p = 0$, then G is the ACF-expansion of the pure group associated to an algebraic group over $\overline{\mathbb{Q}}$. Then $G/C_G(\mathcal{T}(G))$ is definably affine.*

Proof – By Corollary 5.14, we may assume that G is connected. In particular, $\mathcal{S}(G)$ is central in G. We consider $C = C_G(\mathcal{T}(G)/\mathcal{S}(G))$. By Proposition 7.3 and Lemma 7.6, the quotient G/C is definably linear.

Since $[\mathcal{T}(G), C]$ is contained in $\mathcal{S}(G)$, each maximal pseudo-torus of G is normalized by C and, by Fact 2.10 (ii), it is centralized by C°. This implies that C° centralizes $\mathcal{T}(G)$, so $C/C_G(\mathcal{T}(G))$ is a finite group. Then, by Theorem 5.16, we may assume that p is a prime and that p divides the order of $C/C_G(\mathcal{T}(G))$.

Let c be a p-element of $C/C_G(\mathcal{T}(G))$, and let T be a maximal pseudo-torus of G. We consider the semi-direct product $R = T \rtimes \langle c \rangle$, where $\langle c \rangle$ acts on T by conjugation. It is a nilpotent group of class at most two. Thus $[t, c]$ is a p-element for each $t \in T$ and, by the Zil'ber Indecomposable Theorem, $[T, \langle c \rangle]$ is a connected p-subgroup of T, so it is trivial and R is abelian. This implies that c centralizes $\mathcal{T}(G)$, and contradicts that p divides the order of $C/C_G(\mathcal{T}(G))$. □

7.3. More when the ground field is $\overline{\mathbb{Q}}$.

Lemma 7.12. – *Let H be a normal definable subgroup of a connected ACF-group G. If $G/C_G(H)$ is definably affine over the fields $K_1, \ldots, K_n$, then $[G,H]$ is definably affine over $K_1, \ldots, K_n$ too.*

Proof – We consider the semi-direct product $R = H \rtimes G/C_G(H)$, where $G/C_G(H)$ acts on H by conjugation. Then R is an ACF-group, and the subgroup S of R generated by the conjugates of $G/C_G(H)$ is definably affine over $K_1, \ldots, K_n$ (Theorem 5.9). But S contains the definable subgroup $[G,H]$, so $[G,H]$ is definably affine over $K_1, \ldots, K_n$. □

Then we obtain a result on the abstract structure of the algebraic groups over $\overline{\mathbb{Q}}$, and this one is fundamental for us.

Theorem 7.13. – *Let G be a connected algebraic group over $\overline{\mathbb{Q}}$. Then, in the ACF-expansion of the pure group G, the quotient $G/Z(G)$ is definably linear, and G' is definably affine.*

Furthermore, if $U/Z(G)$ (resp. $V/Z(G)$) is the largest connected subgroup of $G/Z(G)$, definably linear over one interpretable field K (resp. L), and if K and L are not definably isomorphic, then $[U,V]$ is trivial and $U' \cap V'$ is finite.

Proof – We show that $G/Z(G)$ is definably affine. Let Q be the preimage of $Q(G/\mathcal{S}(G))$ in G. Since $\mathcal{S}(G)$ is central in G, the group Q is nilpotent, and $Q/Z(Q)$ is definably linear by Theorem 4.3. Moreover, by Corollary 7.11, the quotient $Q/C_Q(\mathcal{T}(G))$ is definably affine, so $QZ(G)/Z(G)$ is definably affine by Lemma 7.2. Since Proposition 7.3 says that $\mathcal{T}(G)/\mathcal{S}(G)$ is definably affine, $\mathcal{T}(G)Z(G)/Z(G)$ is definably affine too. Hence $G/Z(G)$ is definably affine by Theorem 5.9 and Lemma 7.2. Furthermore, Lemma 7.12 says that G' is definably affine.

We show that $G/Z(G)$ is definably linear. By Corollary 5.8, $G/Z(G)$ has a finite normal subgroup $E/Z(G)$ such that G/E is definably linear. In particular, G centralizes $E/Z(G)$, and E is nilpotent. We consider a bounded exponent subgroup B_0 of E covering $E/Z(G)$. Then B_0 is contained in a characteristic bounded exponent subgroup B_1 of E, and B_1 is finite since $\overline{\mathbb{Q}}$ is of characteristic zero. Hence B_1 is central in G, and $E/Z(G)$ is trivial. This proves that $G/Z(G)$ is definably linear.

By Lemma 7.12, the groups $[G,U]$ and $[G,V]$ are definably affine over K and L respectively, so they are definably linear over K and L respectively. By Proposition 5.11, this implies that $U' \cap V' \leq [G,U] \cap [G,V]$ and $[U,V] \leq [G,U] \cap [G,V]$ are finite. Moreover, we have $[U,V] = [U^{\circ}Z(G),V] = [U^{\circ},V]$, so $[U,V]$ is connected, and $[U,V]$ is trivial, as desired. □

7.4. The subgroups $A(G)$ and $W(G)$ in positive characteristic.

Proposition 7.14. – *Let G be a connected ACF_p-group for a prime p. If $Z(G)$ is finite and if p does not divide its order, then G is definably affine.*

Proof – Since $\mathcal{S}(G)$ is connected and central in G, we have $\mathcal{S}(G) = 1$, and $\mathcal{T}(G)$ is definably affine (Proposition 7.3). Then, by Lemma 7.2 and Theorem 5.9, we have just to prove that $W(Q(G))$ is trivial.

If $W(Q(G))$ is finite, then it is central in G, so p does not divides its order. But Corollary 5.18 says that $|W(Q(G))|$ is a power of p, hence we obtain $W(Q(G)) = 1$, as desired.

From now on, we assume toward a contradiction that $W(Q(G))$ is infinite. Let A be a G-minimal subgroup of $W(Q(G))$. Since $Q(G)$ is nilpotent, A is central in $Q(G)$. But $W(Q(G))$ centralizes $\mathcal{T}(G)$ (Corollary 7.11), so A is central in $G = Q(G)\mathcal{T}(G)$ (Lemma 7.2), contradicting that $Z(G)$ is finite. □

We deduce from this a result concerning the structure of ACF_p-groups when p is a prime.

Corollary 7.15. – *If G is a connected ACF_p-group for a prime p, then $W(G)$ is contained in the hypercenter of G.*

Moreover, our study provides a piece of information concerning $A(G)$.

Proposition 7.16. – *If G is a connected ACF_p-group for a prime p, then $G/A(G)$ is nilpotent.*

Proof – Let H be the smallest normal definable subgroup of G such that G/H is nilpotent. Since G is connected, H is connected too. The group $G/\mathcal{T}(G)$ has no nontrivial pseudo-torus (Fact 2.10 $(i\nu)$), so it has no nontrivial radicable torsion subgroup. Since it is constructible over an algebraically closed field, it is nilpotent-by-finite, and by connectedness of the ACF-group G, it is nilpotent. This implies $H \leq \mathcal{T}(G)$ and $H \leq [G, \mathcal{T}(G)]$. Hence Corollary 7.11 and Lemma 7.12 yield the result. □

8. Canonical development of G

In this section, we prove Theorem 1.1 in the special case $K = \overline{\mathbb{Q}}$ (Lemma 8.10 and Theorem 8.14).

For the rest of this section, we fix a nontrivial connected algebraic group G over $\overline{\mathbb{Q}}$, and we consider the ACF-expansion of the pure group G. By Theorem 7.13, we find interpretable fields $K_1, \ldots, K_n$ such that K_i is not definably isomorphic to K_j for each $i \neq j$, and such that $G/Z(G)$ is definably linear over $K_1, \ldots, K_n$. By Lemma 5.6 and Proposition 5.11, if for each i we denote by $G_i/Z(G)$ the largest connected subgroup of $G/Z(G)$ definably linear over K_i, the quotient $G/Z(G)$ is the direct product of the subgroups $G_i/Z(G)$, and $G_i' = (G_i^\circ)'$ is definably linear over K_i for each i (Lemma 7.12). Moreover, Theorem 7.13 says that G is the central product of $G_1, \ldots, G_n$, and we may assume that $G_i/Z(G)$ is nontrivial for each i.

First we consider a purely algebraic lemma.

Lemma 8.1. – *Let $\tilde{G}$ be a connected affine algebraic group over an algebraically closed field K of characteristic zero. Then, for each positive integer n, up to isomorphism of algebraic groups, there is a unique connected affine algebraic extension G^* of $\tilde{G}$, with a normal finite subgroup X of exponent dividing n, and satisfying the following property:*

for each connected affine algebraic group H over K, if H has a normal finite subgroup E of exponent dividing n such that $\tilde{G}$ and H/E are algebraically isomorphic, there is an algebraic surjective homomorphism $\gamma : G^* \to H$ such that $\gamma(X) = E$.

Proof – In this proof, any couple (H, E), where H is a connected affine algebraic group over K and E a normal finite subgroup of H of exponent dividing n, is said to be an *n-extension of $\tilde{G}$* if $\tilde{G}$ and H/E are isomorphic as algebraic groups. We consider the family $\mathscr{E}$ of the n-extensions of $\tilde{G}$ modulo the equivalence relation R defined by the following assertion: *"if (H_1, E_1) and (H_2, E_2)) are two n-extensions* *of $\tilde{G}$, we denote by $(H_1, E_1)R(H_2, E_2)$ the existence of an algebraic isomorphism $\gamma : H_1 \to H_2$ such that $\gamma(E_1) = E_2$."* We identify any n-extension of $\tilde{G}$ with its class modulo R. Moreover, we consider the order relation $\leq$ on $\mathscr{E}$ defined by the following assertion: *"if (H_1, E_1) and (H_2, E_2)) are two elements of $\mathscr{E}$, we denote by $(H_1, E_1) \leq (H_2, E_2)$ the existence of an algebraic surjective homomorphism $\gamma : H_2 \to H_1$ such that $\gamma(E_2) = E_1$."*

First we notice that, for each element (H, E) of $\mathscr{E}$, the order of E is at most n^r, where r denotes the Lie rank of the algebraic group $\tilde{G}$. Indeed, since E is finite and since $\tilde{G}$ and H/E are isomorphic as algebraic groups, the rank of H is r too. But the field K has characteristic zero, and E is finite and normal in H, so E is contained in any maximal torus of H. Consequently, since the exponent of E divides n, the order of E is at most n^r.

Now we have just to prove that, if (M, E) and (N, F) are two maximal elements of $\mathscr{E}$, then $(M, E)R(N, F)$. Since M/E and N/F are isomorphic to $\tilde{G}$ as algebraic groups, we may consider a subgroup Δ of $M \times N$ such that $\Delta/(E \times F)$ is the graph of an isomorphism of algebraic groups from M/E to N/F. We consider $D = \Delta^\circ$ and $A = D \cap (E \times F)$. Then D, M and N have the same dimensions over K, and $D \cap M$ and $D \cap N$ are contained in A. We consider the projection maps $\rho_M : M \times N \to M$ and $\rho_N : M \times N \to N$. Since D, M and N have the same dimensions, and since $D \cap M$ and $D \cap N$ are contained in A, the images of D by ρ_M and ρ_N are M and N respectively. Moreover, we have

$$D \cap \rho_M^{-1}(E) = D \cap (E \times N) = D \cap (\Delta \cap (E \times N)) = D \cap (E \times F) = A$$

and, in the same way, $D \cap \rho_N^{-1}(F) = A$. In particular, (D, A) is an n-extension of $\tilde{G}$ satisfying $(M, E) \leq (D, A)$ and $(N, F) \leq (D, A)$. By the maximalities of (M, E) and (N, F), we obtain $(M, E)R(D, A)$ and $(N, F)R(D, A)$. Finally, we find $(M, E)R(N, F)$, and we can take $(G^*, X) = (M, E)$. □

For each i, we consider an algebraic complement C_i/G'_i of $Z(G)^\circ G'_i/G'_i$ in G_i°/G'_i. Since G is connected, the torsion part of $Z(G) \cap G'$ is finite. Then, for each i, we denote by n_i the exponent of the torsion subgroup X_i of $Z(G)^\circ \cap G'_i$, and by $\overline{C_i}^*$ the unique connected affine algebraic extension of $\overline{C_i} := C_i/X_i$, with a normal finite subgroup of exponent dividing n_i and satisfying the property of Lemma 8.1.

Moreover, we fix an algebraic complement Z to $(Z(G)^\circ \cap G')^\circ$ in $Z(G)^\circ$. We note that, since G is the central product of $G_1, \ldots, G_n$, the group G' is the central product of $G'_1, \ldots, G'_n$, and G is also the central product of Z and $C_1, \ldots, C_n$. Now we obtain the following structural result on G.

Lemma 8.2. – *The group G has a central finite subgroup F_0 such that G/F_0 is the direct product of ZF_0/F_0 and of $\times_{k=1}^n C_k F_0/F_0$.*

Proof – Since G'_j is definably linear over K_j for each j, Lemma 5.6 and Corollary 5.8 imply that the Morley rank of $Z(G) \cap G' = \prod_{j=1}^n (Z(G) \cap G'_j)$ is $\sum_{k=1}^n rk(Z(G) \cap G'_k)$. Then we obtain

$$\begin{array}{rcl} \sum_{k=1}^n rk(C_k) & = & \sum_{k=1}^n (rk(G_k) - rk(Z(G)^\circ) + rk(Z(G)^\circ \cap G'_k)) \\ & = & \sum_{k=1}^n (rk(G_k/Z(G)) + rk(Z(G) \cap G'_k)) \\ & = & rk(G/Z(G)) + rk(Z(G) \cap G'), \end{array}$$

and $rk(Z) + \sum_{k=1}^n rk(C_i) = rk(G/Z(G)) + rk(Z(G)) = rk(G)$. Thus, $Z_0 = Z \cap (\prod_{k=1}^n C_k)$ is finite, and for each j, the group $E_j = C_j \cap (\prod_{k \neq j} C_k)$ is finite too.

Let F_0 be the subgroup generated by Z_0 and the subgroups E_j. Then F_0 is finite and central in G, and G/F_0 is the direct product of ZF_0/F_0 and of $\times_{k=1}^{n} C_k F_0/F_0$. □

It is not clear that C_i is uniquely determined in the pure group G. However, we will see that $\overline{C_i}^*$ is uniquely determined in the pure group G, up to isomorphism of algebraic groups (Corollary 8.9). The proof need the following notion.

Definition 8.3. – *A group A is said to be* centrally indecomposable *if there is no decomposition of A under the form of a central product of two proper subgroups having a finite intersection.*

Lemma 8.4. – *Let G be a connected affine algebraic group over an algebraically closed field K. If G is a central product of two proper closed subgroups U and V with $U \cap V$ finite, then G has a noncontinuous (for the Zariski topology) automorphism α centralizing U.*

In particular, α is nonstandard.

Furthermore, we may choose α such that, for any two infinite closed subgroups U_1 and V_1 of U and V respectively, the isomorphism $\beta : U_1V_1 \to \alpha(U_1V_1)$ induced by α is noncontinuous.

Proof – First we notice that U and V are infinite since $G = UV$ is connected, and since U and V are closed and proper in G. Let μ be a nonconstructible field automorphism of K. Since $U \cap V$ is finite, we may choose μ centralizing $U \cap V$. We consider the automorphism α of G defined by $\alpha(uv) = \mu(u)v$ for each $u \in U$ and each $v \in V$. Then α centralizes U.

Let U_1 and V_1 be two infinite closed subgroups of U and V respectively. We assume toward a contradiction that the isomorphism $\beta : U_1V_1 \to \alpha(U_1V_1)$ induced by α is continuous. We may suppose that U_1 and V_1 are connected and of dimension one. Thus U_1 (resp. V_1) is definably isomorphic either to K_+, or to K^*. Therefore, we may assume that, either U_1 and V_1 are definably isomorphic, or U_1 (resp. V_1) is definably isomorphic to K^* (resp. K_+). Let δ be an injective constructible map from U_1 to V_1 such that $\delta(u) = u$ for each $u \in U_1 \cap V_1$. Then $X = \{u\delta(u) \mid u \in U_1\}$ is a constructible set, and since α is continuous, $\alpha(X) = \{\mu(u)\delta(u) \mid u \in U_1\}$ is constructible too. We notice that If $y \in \alpha(U_1) \cap \alpha(V_1)$,

Then we have $\beta = \gamma \circ \nu$ for ν a field automorphism of K, and γ an isogeny. We obtain $\nu(v) = \gamma^{-1}(v)$ for each $v \in V_1$. Since V_1 is infinite, this implies that ν is definable in the pure field K. Consequently, β is definable in the pure field K too. Now, since U_1 is infinite and since we have $\beta(u) = \mu(u)$ for each $u \in U_1$, the automorphism μ is definable in the pure field K too. In particular, $C_K(\mu)$ is a definable subfield of K, so either μ is trivial, or $C_K(\mu)$ is finite. But the field K has characteristic zero, so μ centralizes an infinite subfield of K. Hence μ is trivial contradicting our choice of μ. □

Lemma 8.5. – *For each i, we have $C_i' = G_i'$ and $Z(C_i)^\circ \leq Z(G)^\circ \cap C_i \leq C_i'$.*

Proof – Since $G_i = G_i^\circ Z(G) = C_i Z(G)$, we have $C_i' = G_i'$. In particular, since G_i°/G_i' is the direct product of C_i/G_i' and of $Z(G)^\circ G_i'/G_i'$, we obtain $Z(G)^\circ \cap C_i \leq C_i'$. Moreover, since G is the central product of $G_1, \ldots, G_n$, we have also $Z(C_i)^\circ \leq Z(G)^\circ \cap C_i$. □

Proposition 8.6. – *For each i, the groups C_i and $\overline{C_i}$ are centrally indecomposable.*

Proof – We assume toward a contradiction that C_i is not centrally indecomposable. Let U and V be two proper subgroups of C_i having a finite intersection and such that C_i is the central product of U and V. Since C_i is connected, U and V are infinite. If U is abelian, then U centralizes G_i, and since G_i centralizes G_j for each $j \neq i$ by Theorem 7.13, we obtain $U \leq Z(G)$. Then $Z(G)^\circ \cap U$ is an infinite subgroup of $Z(G)^\circ \cap C_i \leq C_i' = V'$, contradicting that $U \cap V$ is finite. Hence U is not abelian. In the same way, V is not abelian. In particular, $C_{C_i}(U) = Z(U)V$ and $C_{C_i}(V) = UZ(V)$ are proper in C_i. Moreover, by the connectedness of C_i, we find $C_i = C_U C_V$, for $C_U := C_{C_i}(V)^\circ$ and $C_V := C_{C_i}(U)^\circ$, and this product is central. Thus, C_i is the central product of two proper connected algebraic subgroups C_U and C_V.

We consider an algebraic complement R/C_U' of $(C_U \cap C_V)^\circ C_U'/C_U'$ in C_U/C_U', and an algebraic complement S/C_V' of $(R \cap C_V)^\circ C_V'/C_V'$ in C_V/C_V'. Then C_i is the central product of R and S, and $(R \cap S)^\circ$ is contained in $C_U' \cap C_V' \leq U' \cap V'$, therefore $R \cap S$ is finite. Since $R \leq C_U$ and $S \leq C_V$ are proper in C_i, they are infinite. In the same way that for U and V, the subgroups R and S are not abelian, and since they are connected, R' and S' are infinite.

By Lemma 8.4, the group C_i has a nonstandard automorphism α centralizing $R(C_i \cap F_0)$. Moreover, we may choose α such that the automorphism β of $G_i' = C_i' = R'S'$ induced by α is nonstandard. Since G is the central product of Z and $C_1, \ldots, C_n$, and since $C_i \cap (Z \prod_{k \neq i} C_k)$ is contained in $C_i \cap F_0$, there is an automorphism μ of G centralizing $Z \prod_{k \neq i} C_k$ and whose restriction to C_i is α. In particular, the restriction of μ to C_i' is nonstandard. But $C_i' = G_i'$ is definably linear over K_i, hence this contradicts Fact 3.6, and C_i is centrally indecomposable.

Finally, if $\overline{C_i}$ is the central product of two proper subgroups U_1/X_i and V_1/X_i having a finite intersection, then U_1 and V_1 are two proper subgroups of C_i having a finite intersection, satisfying $C_i = U_1V_1$, and such that $[U_1, V_1] \leq X_i$ is finite and central in C_i. Let V_2 be a subgroup of V_1 such that $C_i = U_1V_2$, and such that $|U_1 \cap V_2|$ is minimal among such subgroups. Then, for each $u \in U_1$, the map $ad_u : V_2 \to X_i$, defined for each $v \in V_2$ by $ad_u(v) = [u, v]$, is an homomorphism. Moreover, its kernel V_3 has finite index in V_2. But $\overline{\mathbb{Q}}$ has characteristic zero and C_i is connected, so C_i has no proper subgroup of finite index. Hence we have $C_i = U_1V_3$, and $V_3 = V_2$ by the choice of V_2. Thus C_i is the central product of U_1 and V_2, contradicting the previous paragraphs. Consequently, $\overline{C_i}$ is centrally indecomposable. □

Proposition 8.7. – *If A is any affine algebraic group over $\overline{\mathbb{Q}}$ abstractly isomorphic to $\overline{C_i}$, then A is isomorphic to $\overline{C_i}$ as algebraic groups.*

Proof – We have to find an isomorphism between A and $\overline{C_i}$, definable in the pure field $\overline{\mathbb{Q}}$. Let α be an abstract isomorphism between A and $\overline{C_i}$. For each isomorphism $\alpha_0 : A \to \overline{C_i}$, we denote by $\overline{\alpha_0}$ the isomorphism between $A/Z(A)^\circ$ and $\overline{C_i}/Z(\overline{C_i})^\circ$, induced by α_0. By Proposition 8.6, the group $\overline{C_i}$ is centrally indecomposable, and since it is connected, then Theorem 7.13 implies that, in the ACF-expansion of the pure group $\overline{C_i}$, the quotient $\overline{C_i}/Z(\overline{C_i})$ is definably linear over one interpretable field. Then $\overline{C_i}/Z(\overline{C_i})^\circ$ is definably linear over one interpretable field too, by Proposition 5.15. Now Fact 3.6 says that the isomorphism $\overline{\alpha} : A/Z(A)^\circ \to \overline{C_i}/Z(\overline{C_i})^\circ$ is standard. Thus, there is an automorphism δ of $\overline{C_i}$, induced by a field automorphism of $\overline{\mathbb{Q}}$, such that $\overline{\alpha} \circ \overline{\delta}^{-1} = \overline{\alpha \circ \delta^{-1}}$ is an isomorphism of algebraic groups. Let $\mu = \alpha \circ \delta^{-1}$, and let Δ be its graph. Then, if $Z_1 = Z(A)^\circ \times Z(\overline{C_i})^\circ$, the group $\Delta Z_1 = \Delta Z(A)^\circ = \Delta Z(\overline{C_i})^\circ$ is definable in the pure field $\overline{\mathbb{Q}}$. This implies that $(\Delta Z_1)' = \Delta'$ is definable in the pure field $\overline{\mathbb{Q}}$ too.

We consider an algebraic complement D/Δ' of $Z_1\Delta'/\Delta'$ in $\Delta Z_1/\Delta'$. Then, since $\Delta Z_1/Z_1$ is the graph of $\overline{\mu}$, we have $D \cap A \leq \Delta Z_1 \cap A = Z_1$. So we obtain

$$D \cap A \leq (D \cap Z_1) \cap A \leq \Delta' \cap A \leq \Delta \cap A.$$

Thus, since Δ is the graph of an isomorphism from A to $\overline{C_i}$, we find $D \cap A = 1$. In the same way, $D \cap \overline{C_i}$ is trivial. If Z_C/X_i denotes the center of $\overline{C_i}$, then $[C_i, Z_C]$ is contained in X_i, and it is connected since C_i is connected, so $Z_C = Z(C_i)$. Hence $Z(\overline{C_i})^\circ$ is contained in $\overline{C_i}'$ by Lemma 8.5. This proves that Δ' contains the graph of the restriction of μ to $Z(A)^\circ$, and it implies that $\Delta' Z_1 = \Delta' Z(A)^\circ = \Delta' Z(\overline{C_i})^\circ$, so $AD \geq A\Delta'$ contains Z_1. Now AD contains $\Delta \leq \Delta Z_1 = DZ_1$, and it contains $\overline{C_i}$ too since Δ is the graph of μ. Consequently, we have $AD = A\overline{C_i}$ and, in the same way, $\overline{C_i}D = A\overline{C_i}$. Since D is an algebraic subgroup satisfying $D \cap A = D \cap \overline{C_i} = 1$ and $AD = \overline{C_i}D = A\overline{C_i}$, it is the graph of an isomorphism of algebraic groups from A to $\overline{C_i}$, and the proof is finished. □

Corollary 8.8. – *Let A be a centrally indecomposable subgroup of G_i containing G_i'. If it is a complement of $Z(G)^\circ G_i'/G_i'$ in G_i°/G_i' then, up to isomorphism of algebraic groups, $\overline{C_i}$ is the only affine algebraic group over $\overline{\mathbb{Q}}$ such that the abstract groups $\overline{A} := A/X_i$ and $\overline{C_i}$ are isomorphic.*

Proof – The group $(Z(G)^\circ \cap G_i')/X_i$ is definable in the pure group G, and it is torsion-free, so it is divisible and it has a complement S/X_i in $Z(G)^\circ/X_i$. Then we have $G_i = AS = C_iS$, and $A/(A \cap S)$ is isomorphic to $C_i/(C_i \cap S)$ as algebraic groups. Moreover, the group $A \cap S$ is contained in $A \cap Z(G)^\circ \leq Z(G)^\circ \cap G_i' = X_i$, so $A \cap S = X_i$. In the same way, we have $C_i \cap S = X_i$, so $\overline{A}$ and $\overline{C_i}$ are isomorphic as abstract groups, and the conclusion follows from Proposition 8.7. □

Corollary 8.9. – *The algebraic groups $\overline{C_i}$ and $\overline{C_i}^*$ depend just on the ACF-group G_i.*

Proof – Since G is the central product of $G_1, \ldots, G_n$, then we have $Z(G_i) = Z(G)$, and the result follows from Corollary 8.8. □

From now on, we consider the direct product $D(G)$ of the groups $\overline{C_1}^*, \ldots, \overline{C_n}^*$, and of an abelian group $T(G)$ which is $\overline{\mathbb{Q}}_+$ if $Z(G)^\circ G'/G'$ is nontrivial and torsion-free, otherwise it is $(\overline{\mathbb{Q}}^*)^r$ where r is the Lie rank of $Z(G)^\circ G'/G'$. We note that $D(G)$ depends just on the pure group G by Corollary 8.9.

By the choice of $T(G)$, there is an integer s such that $Z(G)^\circ G'/G'$ is isomorphic to $T(G) \times \overline{\mathbb{Q}}_+^s$ as algebraic groups, so $T(G) \times \overline{\mathbb{Q}}_+^s$ is isomorphic to $Z \simeq Z(G)^\circ/(Z(G)^\circ \cap G')^\circ$ too. Consequently, since for each i there is a surjective algebraic homomorphism from $\overline{C_i}^*$ to C_i with finite kernel, there is a natural surjective algebraic homomorphism γ_G from $D(G) \times \overline{\mathbb{Q}}_+^s$ to G, with finite kernel F by Lemma 8.2. Moreover, F is central in $D(G) \times \overline{\mathbb{Q}}_+^s$ and, since $\overline{\mathbb{Q}}_+^s$ is torsion-free, F is contained in $D(G)$. Since $T(G)$ and $T(G)/(T(G) \cap F)$ are isomorphic as algebraic groups, we may choose F such that $T(G) \cap F$ is trivial.

Summarize: *G is algebraically isomorphic to $D(G)/F \times \overline{\mathbb{Q}}_+^s$, with F a central finite subgroup such that $T(G) \cap F = 1$.*

We note that we have $[D(G) \times \overline{\mathbb{Q}}_+^s, \gamma_G^{-1}(Z(G))] \leq F$, so $\gamma_G^{-1}(Z(G)) = Z(D(G) \times \overline{\mathbb{Q}}_+^s)$ since $D(G) \times \overline{\mathbb{Q}}_+^s$ is connected. Therefore we find, for each i,

$$\gamma_G^{-1}(G_i) = \overline{C_i}^* Z(D(G) \times \overline{\mathbb{Q}}_+^s).$$

Lemma 8.10. – *There is an abstract isomorphism from G to $D(G)/F$.*

Furthermore, if G' does not contain $Z(G)^\circ$, then there is an abstract isomorphism $\mu : D(G)/F \to D(G)/F \times \overline{\mathbb{Q}}_+^t$ for each integer t.

Proof – We may assume $Z(G)^\circ \not\leq G'$, and $t \geq 1$. Let R be the torsion part of $T(G)$. Then $T(G) = R \times S$, for a direct product S of countably many copies of $\mathbb{Q}_+$. In particular, since $\overline{\mathbb{Q}}_+^t$ is a direct product of countably many copies of $\mathbb{Q}_+$ too, the groups $T(G)$ and $T(G) \times \overline{\mathbb{Q}}_+^t$ are abstractly isomorphic. More precisely, there is an abstract isomorphism $\alpha : T(G) \to T(G) \times \overline{\mathbb{Q}}_+^t$ such that $\alpha(r) = r$ for each $r \in R$. This gives an abstract isomorphism $\alpha^* : D(G) \to D(G) \times \overline{\mathbb{Q}}_+^t$ such that $\alpha^*(x) = x$ for each $x \in (\times_{i=1}^n \overline{C_i}^*) \times R$. But F is finite, so it is contained in $(\times_{i=1}^n \overline{C_i}^*) \times R$, and we obtain the desired isomorphism $\mu : D(G)/F \to D(G)/F \times \overline{\mathbb{Q}}_+^t$. □

However, by the following remark, F depends on the algebraic structure of G, and not just of the abstract structure of G. This is a serious problem for us.

Remark 8.11. – Let $r \geq 5$ be a prime integer, and let $G = H_1 \times H_2$ where H_1 and H_2 are two copies of the group $H = \overline{\mathbb{Q}}_+ \rtimes \overline{\mathbb{Q}}^*$, where the action of $\overline{\mathbb{Q}}^*$ on $\overline{\mathbb{Q}}_+$ is defined by:

$$\text{for each } (a, x) \in \overline{\mathbb{Q}}^* \times \overline{\mathbb{Q}}_+,\ a \cdot x = a^r x.$$

Let Δ be the graph in G of the identity automorphism of H, and let Z be the center of Δ. Now let δ be a field automorphism of $\overline{\mathbb{Q}}$ such that there exists $t \in \overline{\mathbb{Q}}^*$ of order r and satisfying $\delta(t) \neq t$ and $\delta(t) \neq t^{-1}$. This is possible since $r \geq 5$. Let μ be the automorphism of G defined for each $(h_1, h_2) \in H_1 \times H_2$ by $\mu(h_1, h_2) = (\delta(h_1), h_2)$. Then μ induces an abstract isomorphism from G/Z to $G/\mu(Z)$. However, the algebraic groups G/Z and $G/\mu(Z)$ are not isomorphic.

Indeed, suppose toward a contradiction that there is an isomorphism $f : G/Z \to G/\mu(Z)$ of algebraic groups. Then, for $i = 1, 2$, we have either $f(H_iZ/Z) = H_1\mu(Z)/\mu(Z)$, or $f(H_iZ/Z) = H_2\mu(Z)/\mu(Z)$. Let T be a maximal torus of Δ, and let T_0 be a maximal torus of G containing T. Then T contains Z, and T/Z (resp. T_0/Z) is a maximal torus of Δ/Z (resp. G/Z). Moreover, $T_0\mu(Z)/\mu(Z)$ is a maximal torus of $G/\mu(Z)$, and by the conjugacy of the maximal tori in $G/\mu(Z)$, we may assume $f(T_0/Z) = T_0\mu(Z)/\mu(Z)$.

Now let $S/\mu(Z) = f(T/Z)$. Then S is of dimension one over $\overline{\mathbb{Q}}$, and since $S/\mu(Z)$ is a torus, S° is a torus. We have $T \cap H_i \leq \Delta \cap H_i = 1$ for $i = 1, 2$, so $T/Z \cap H_iZ/Z = 1$ for $i = 1, 2$, and $S/\mu(Z) \cap H_i\mu(Z)/\mu(Z) = 1$ for $i = 1, 2$. For $i = 1, 2$, we have $Z \cap H_i \leq \Delta \cap H_i = 1$ so, since μ is an abstract automorphism of G such that $\mu(H_i) = H_i$, we have $\mu(Z) \cap H_i = 1$. Consequently $S \cap H_i \leq \mu(Z) \cap H_i$ is trivial for $i = 1, 2$.

Since T_0 is a maximal torus of G, it has the form $T_0 = T_1 \times T_2$, where T_i is a maximal torus of H_i for $i = 1, 2$. Since S is a torus contained in $T_0\mu(Z)$ and since $\mu(Z)$ is finite, we have $S^\circ \leq T_0$. Now, since T_1, T_2, T and S° are of dimension one over $\overline{\mathbb{Q}}$, the groups T and S° are the graphs of two isomorphisms α and β of algebraic groups from T_1 to T_2. Moreover, since $dim(T_i) = 1$ for $i = 1, 2$, there are precisely *two isomorphisms of algebraic groups from T_1 to T_2, so we have either $\alpha = \beta$ or $\beta(x) = \alpha(x)^{-1}$ for each $x \in T_1$. In the first case, we have $T = S^\circ$, so S contains $Z\mu(Z)$. But the existence of $t \in \overline{\mathbb{Q}}^*$ of order r such that $\delta(t) \neq t$ implies $Z \neq \mu(Z)$, so $Z\mu(Z)$ has order r^2 since r is a prime, and we have $Z\mu(Z) = Z(H_1) \times Z(H_2)$. This contradicts $S \cap H_i = 1$ for $i = 1, 2$. Hence we have $\beta(x) = \alpha(x)^{-1}$ for each $x \in T_1$.*

Let Z_{-1} be the graph in G of the automorphism inversion of $Z(H)$. Since we have $\beta(x) = \alpha(x)^{-1}$ for each $x \in T_1$, we have $Z_{-1} \leq S^\circ$. Moreover, the existence of $t \in \overline{\mathbb{Q}}^*$ of order r such that $\delta(t) \neq t^{-1}$ implies $Z_{-1} \neq \mu(Z)$, so $Z_{-1}\mu(Z)$ has order r^2 since r is a prime, and we have $Z_{-1}\mu(Z) = Z(H_1) \times Z(H_2)$. But we have $S \cap H_i = 1$ for $i = 1, 2$, so the latter implies that S does not contain Z_{-1}. Thus, there is no isomorphism of algebraic groups from G/Z to $G/\mu(Z)$.

Lemma 8.12. – *Let X and Y be two finite subgroups of an algebraic torus T over $\overline{\mathbb{Q}}$. If there is an isomorphism $\delta : X \to Y$, then there is a quasi-standard automorphism φ of T such that $\varphi(x) = \delta(x)$ for each $x \in X$.*

Proof – Let $X = X_1 \times \cdots \times X_r$ be a decomposition of X where, for each i, X_i is cyclic and, if $i > 1$, its order divides the one of X_{i-1}. For each i, we consider $Y_i = \delta(X_i)$. We show, by induction on r, that there exists a direct product $T_1 \times \cdots \times T_r$ of subtori of T with dimension one, such that T_i contains X_i for each i. We may assume that T contains a direct product $S = T_1 \times \cdots \times T_{r-1}$ of subtori with dimension one, such that T_i contains X_i for each $i \leq r-1$. For each prime p dividing $|X_r|$, there are $p^{r-1} - 1$ elements of order p in S, and since $|X_i|$ divides $|X_{i-1}|$ for each $i > 1$, there are $p^{r-1} - 1$ elements of order p in $X_1 \times \cdots \times X_{r-1}$ too. Consequently, $S \cap X_r$ is trivial.

We note that X_r is contained in a subtorus R of T with dimension one. Indeed, otherwise, if we consider a subtorus R_1 of T containg X and with minimal dimension, then R_1 has a subtorus R_2 such that R_1/R_2 has dimension two. Let $\overline{T_1}$ and $\overline{T_2}$ be two subtori of R_1/R_2 such that $R_1/R_2 = \overline{T_1} \times \overline{T_2}$. Let $\overline{x}$ be the image in R_1/R_2 of a generator x of X_r. By the minimality of $dim(R_1)$, we have $\overline{x} \notin \overline{T_i} \setminus \{1\}$ for $i = 1, 2$, so we have $\overline{x} = (\overline{t_1}, \overline{t_2})$ with $\overline{t_i} \in \overline{T_i}$ for $i = 1, 2$. Let μ be an isomorphism of algebraic groups from $\overline{T_1}$ to $\overline{T_2}$. Then we find $\overline{t} \in \overline{T_1}$ and $(a, b) \in \mathbb{N}^2$ such that $\overline{t_1} = \overline{t}^a$ and $\overline{t_2} = \mu(\overline{t})^b$. Now we consider $\overline{R} = \{(\overline{u}^a, \mu(\overline{u})^b) \mid \overline{u} \in \overline{T_1}\}$. Then $\overline{R}$ is the image of a nonzero algebraic homomorphism from $\overline{T_1}$ to R_1/R_2, so $\overline{R}$ is a subtorus with dimension one. But $\overline{R}$ contains $\overline{x}$, hence its preimage R_0 is a subtorus of R_1 containing X_r. Since we have $dim(R_0) = dim(R_1) - 1$, we have a contradiction with the choice of R_1, and X_r is contained in a subtorus R of T with dimension one.

Now SR is a subtorus of T, and we find a subtorus V of SR such that $SR = S \times V$. we have $x = (s, v)$ for $s \in S$ and $v \in V \setminus \{1\}$. Let c_1 and d_1 be the orders of s and v respectively. Then c_1 and d_1 divides $|X_r|$, and we consider $c := |X_r|/c_1$. The previous paragraph applied with $\langle s \rangle$ yields a subtorus S_1 of S with dimension one and containing s. Let $\gamma : S_1 \to V$ be an isomorphism of algebraic groups, and let $y \in S_1$ of order $|X_r|$ such that $y^c = s$. Then there exists an integer e dividing $|X_r|$ such that $\gamma(y)^e = v$. Since $|X_i|$ divides $|X_{i-1}|$ for each $i > 1$, any element of S with order dividing $|X_r|$ belongs to $X_1 \times \cdots \times X_{r-1}$. In particular, we have $y \in X_1 \times \cdots \times X_{r-1}$. Let $Q := \{(u^c, \gamma(u)^e) \mid u \in S_1\}$. Since Q is the image of a nontrivial algebraic homomorphism from S_1 to $S \times V$, it is a subtorus with dimension one. Moreover, Q contains x, and since $S \cap X_r = 1$, no element of $S \cap Q$ has order dividing $|X_r|$. But we have $S \cap Q = S_1 \cap Q = \{u^c \mid u \in S_1,\ u^e = 1\}$, and e divides $|X_r|$, hence each element of $S \cap Q$ has order dividing e and $|X_r|$. This proves that $S \cap Q = 1$, and we may choose T_r to be Q. Thus, as claimed, there exists a direct product $T_1 \times \cdots \times T_r$ of subtori of T with dimension one, such that T_i contains X_i for each i.

In the same way, there exists a direct product $T'_1 \times \cdots \times T'_r$ of subtori of T with dimension one, such that T'_i contains Y_i for each i. Let T_0 (resp. T'_0) be an algebraic complement of $T_1 \times \cdots \times T_r$ (resp. $T'_1 \times \cdots \times T'_r$) in T. Then T_0 and T'_0 are two tori of dimension $dim(T) - r$, so there is an isomorphism of algebraic groups $\varphi_0 : T_0 \to T'_0$. For each $i = 1, \ldots, r$, let $\varphi_i : T_i \to T'_i$ be an isomorphism of algebraic groups. Since $T = T_0 \times T_1 \times \cdots \times T_r = T'_0 \times T'_1 \times \cdots \times T'_r$, the map $\varphi' : T \to T$, defined by $\varphi'(u_i) = \varphi_i(u_i)$ for each i and each $u_i \in T_i$, is an algebraic automorphism of T. Moreover, for each i, the subgroup X_i (resp. Y_i) is the only subgroup of T_i (resp. T'_i) of order $|X_i| = |Y_i|$, so we have $\varphi'(X_i) = Y_i$ for each i.

For each $i = 1, \ldots, r$, let x_i be a generator of X_i. Then, for each i, $\varphi'(x_i)$ and $\delta(x_i)$ are two elements of order $|X_i|$ in T'_i, so there is a field automorphism β_i of $\overline{\mathbb{Q}}$ such that $(\varphi' \circ \beta_i)(x_i) = \delta(x_i)$. Let φ be the automorphism of T, defined by $\varphi(u_0) = \varphi'(u_0)$ for each $u_0 \in T_0$, and by $\varphi(u_i) = (\varphi' \circ \beta_i)(u_i)$ for each $i = 1, \ldots, r$. Thus φ is a quasi-standard automorphism of T, and it satisfies $\varphi(x) = \delta(x)$ for each $x \in X$, as desired. $\square$

Lemma 8.13. – *For each i, the exponent of $Y_i := \gamma_G^{-1}(X_i) \cap \overline{C_i}^* Z(D(G))^\circ$ is n_i.*

Moreover, Y_i is finite and central in $D(G)$, and we have $Y_i = (\gamma_G^{-1}(X_i) \cap \overline{C_i}^*)(\gamma_G^{-1}(X_i) \cap T(G))$.

Proof – Since $X_i \leq Z(G)$ is finite and central in G, and since the kernel F of γ_G is finite, then $Y_i \leq \gamma_G^{-1}(X_i)$ is finite and normal in $D(G)$. Moreover, since $D(G)$ is connected, then Y_i is central in $D(G)$.

By the choice of $\overline{C_i}^*$, the exponent of $Y_i^C := \gamma_G^{-1}(X_i) \cap \overline{C_i}^*$ is n_i. Since we have $X_i \leq C_i = \gamma_G(\overline{C_i}^*)$, we obtain $X_i = \gamma_G(Y_i^C)$. Now we consider $Y_i^T := \gamma_G^{-1}(X_i) \cap T(G)$. Since $T(G) \cap F$ is trivial and since the exponent of X_i is n_i, the exponent of Y_i^T divides n_i, and we have just to prove that $Y_i^C Y_i^T$ contains Y_i.

Let $u \in Y_i$. Then there exist $c \in \overline{C_i}^*$ and $z \in Z(D(G))^\circ$ such that $u = cz$. Since we have $\gamma_G(\overline{C_i}^*) = C_i$ and $\gamma_G(Z(D(G))^\circ) = \gamma_G(Z(D(G)))^\circ = Z(G)^\circ$, we find $\gamma_G(c) \in C_i$ and $\gamma_G(c) = \gamma_G(u)\gamma_G(z^{-1}) \in X_i Z(G)^\circ = Z(G)^\circ$. Thus we obtain $\gamma_G(c) \in C_i \cap Z(G)^\circ = G'_i = (G_i^\circ)' = C'_i$, and there exists $c' \in (\overline{C_i}^*)'$ such that $\gamma_G(c) = \gamma_G(c')$. Now we have $u = (cc'^{-1})(c'z)$ with $cc'^{-1} \in \overline{C_i}^*$ and $c'z \in (\overline{C_i}^*)' Z(D(G))^\circ$. Since we have $\gamma_G(cc'^{-1}) = \gamma_G(c)\gamma_G(c')^{-1} = 1 \in X_i$, we find $cc'^{-1} \in Y_i^C$. Consequently, we have just to prove that $c'z$ belongs to $Y_i^C Y_i^T$.

Since $D(G)$ is the direct product of $\overline{C_1}^*, \ldots, \overline{C_n}^*$ and of $T(G)$, the subgroup $Z(D(G))^\circ$ is the direct product of $Z(\overline{C_1}^*)^\circ, \ldots, Z(\overline{C_n}^*)^\circ$ and of $T(G)$. Moreover Lemma 8.5 implies that $(\overline{C_i}^*)'$ contains $Z(\overline{C_i}^*)^\circ$, therefore $(\overline{C_i}^*)' Z(D(G))^\circ$ is the direct product of $(\overline{C_i}^*)'$, $T(G)$ and of the groups $Z(\overline{C_k}^*)^\circ$ for $k \neq i$. But, for each $k \neq i$, the torsion part of $Z(\overline{C_k}^*)^\circ$ is contained in the maximal torus T_k of $Z(\overline{C_k}^*)^\circ$, and T_k is trivial since $Z(\overline{C_k}^*)^\circ$ is contained in $(\overline{C_k}^*)'$ by Lemma 8.5. Hence the torsion part of $(\overline{C_i}^*)' Z(D(G))^\circ$ is contained in $(\overline{C_i}^*)' \times T(G)$.

Since X_i and F are finite, then $\gamma_G^{-1}(X_i)$ is finite too, and since $\gamma_G(c'z) = \gamma_G(c)\gamma_G(z) = \gamma_G(u)$ belongs to X_i, the element $c'z$ is of finite order. Thus $c'z$ belongs to $(\overline{C_i}^*)' \times T(G)$, and there exist $d \in (\overline{C_i}^*)'$ and $t \in T(G)$ such that $c'z = dt$. Moreover, since $c'z$ has finite order, the elements d and t have finite orders too. Now we find $\gamma_G(t) = \gamma_G(d)^{-1}\gamma_G(c'z) \in C_i'X_i = C_i'$, and $\gamma_G(t) \in C_i' \cap \gamma_G(T(G)) \leq C_i' \cap Z(G)^\circ$, so $\gamma_G(t)$ is a torsion element of $C_i' \cap Z(G)^\circ$ and $\gamma_G(t)$ belongs to X_i. This proves that t belongs to Y_i^T. Furthermore, this proves that $\gamma_G(d) = \gamma_G(c'z)\gamma_G(t)^{-1}$ belongs to X_i, so we obtain $d \in Y_i^C$ and $c'z = dt$ belongs to $Y_i^C Y_i^T$, as desired. □

Theorem 8.14. – *Let H be an affine algebraic group over $\overline{\mathbb{Q}}$. If H and G are abstractly isomorphic, then there is a quasi-standard automorphism α of $D(G)$, and an isomorphism of algebraic groups between H and*

- $D(G)/\alpha(F) \times \overline{\mathbb{Q}}_+^t$ *for an integer t if G' does not contains $Z(G)^\circ$;*
- $D(G)/\alpha(F)$ *if G' contains $Z(G)^\circ$.*

Proof – Let $\delta_0 : G \to H$ be an abstract isomorphism. Since H and G are abstractly isomorphic, and since G is connected, then H is connected too and, in the ACF-expansion of the pure group H, we find interpretable fields $L_1, \ldots, L_n$ such that L_i is not definably isomorphic to L_j for each $i \neq j$, and such that $H/Z(H)$ is definably linear over $L_1, \ldots, L_n$. For each i, we denote by $H_i/Z(H)$ the largest connected subgroup of $H/Z(H)$ definably linear over L_i. We may assume $H_i = \delta_0(G_i)$ for each i. Moreover, since $H \simeq G$, we may assume $D(H) = D(G)$, and even that $T(H) = T(G)$ and that the groups $\overline{C_1}^*, \ldots, \overline{C_n}^*$ for H are the same that for G. Let t be the integer such that $Z(H)^\circ H'/H'$ and $T(H) \times \overline{\mathbb{Q}}_+^t$ are isomorphic as algebraic groups. Then we consider the natural surjective algebraic homomorphism $\gamma_H : D(H) \times \overline{\mathbb{Q}}_+^t \to H$. It satisfies $\gamma_H^{-1}(H_i) = \overline{C_i}^* Z(D(H) \times \overline{\mathbb{Q}}_+^t)$ for each i, and its kernel E is a central finite subgroup of $D(H) = D(G)$, and it verifies $E \cap T(H) = 1$.

By Lemma 8.10, the quotient $D(G)/F$ (resp. $D(G)/E$) is abstractly isomorphic to G (resp. H), so there is an abstract isomorphism $\delta : D(G)/F \to D(G)/E$. We notice that, if G' contains $Z(G)^\circ$, then we have $s = 0$, and H' contains $Z(H)^\circ$ too, so $t = 0$. Hence, in all the cases, we have just to prove that there is a quasi-standard automorphism α of $D(G)$ such that $D(G)/E$ and $D(G)/\alpha(F)$ are isomorphic as algebraic groups. In particular, since G and $D(G)/F$ (resp. H and $D(H)/E$) are abstractly isomorphic by Lemma 8.10, we may assume $G = D(G)/F$ (resp. $H = D(H)/E$), and $\delta = \delta_0$. Thus we have $s = 0$ and $t = 0$, and $H_i = \delta(G_i)$ for each i. From now on, the main difficulty for the proof is to find a quasi-standard automorphism α of $D(G)$ such that $D(G)/Z(D(G))^\circ\alpha(F)$ and $D(G)/Z(D(G))^\circ E$ are isomorphic as algebraic groups.

For each i, we consider $P_i := \overline{C_i}^* Z(D(G)) = \gamma_G^{-1}(G_i) = \gamma_H^{-1}(H_i)$. Then, for each i, we have $G_i = P_i/F$, $H_i = P_i/E$, and $\delta(P_i/F) = P_i/E$. Let $Z_F/F := Z(G) = Z(D(G)/F)$ and $Z_E/E := Z(H) = Z(D(G)/E)$. Then $[D(G), Z_F]$ is a connected subgroup of the finite subgroup F, so Z_F is central in $D(G)$, and since it contains $Z(D(G))$, we obtain $Z_F = Z(D(G))$. In the same way, for each central finite subgroup Y of $D(G)$, we have $Z(D(G)/Y) = Z(D(G))/Y$. In particular, we obtain $Z_E = Z(D(G))$. Now the algebraic groups $P_i/Z(D(G))$, $G_i/Z(G)$ and $H_i/Z(H)$ are isomorphic. Moreover, since $G/Z(G)$ is the direct product of the groups $G_1/Z(G), \ldots, G_n/Z(G)$, we have

$$D(G)/Z(D(G)) \quad = \quad P_1/Z(D(G)) \times \cdots \times P_n/Z(D(G)).$$

But, for each i, in the ACF-expansion of the pure group G, the quotient $G_i/Z(G)$ is definably linear over K_i, so Fact 3.6 says that the isomorphism from $G_i/Z(G)$ to $H_i/Z(H)$ induced by δ is a standard isomorphism. Hence, for each i, since $P_i/Z(D(G))$, $G_i/Z(G)$ and $H_i/Z(H)$ are isomorphic as algebraic groups, the automorphism $\overline{\delta_i^*}$ of $P_i/Z(D(G))$ induced by δ, is a standard automorphism. In other words, for each i, there is a field automorphism φ_i of $\overline{\mathbb{Q}}$, such that the automorphism $\overline{\delta_i^\varphi} := \overline{\delta_i^*} \circ \varphi_i^{-1}$ of $P_i/Z(D(G))$ is algebraic over $\overline{\mathbb{Q}}$.

We fix i, and we consider the preimage Ω of the graph of $\overline{\delta_i^\varphi}$ in $P_i \times Q_i$. It is an algebraic subgroup and, since $P_i/Z(D(G)) \simeq G_i/Z(G)$ is connected, then $\Omega/(Z(D(G)) \times Z(D(G)))$ is connected too, and we have $\Omega = \Omega^\circ(Z(D(G)) \times Z(D(G)))$.

We consider the group $Y_i^H := \gamma_H^{-1}(X_i^H) \cap \overline{C_i}^* Z(D(G))^\circ = \gamma_H^{-1}(X_i^H) \cap P_i^\circ$, where X_i^H is the torsion part of $Z(H)^\circ \cap H_i'$. Then we have $\delta(Y_iF/F) = \delta(\gamma_G^{-1}(X_i)/F \cap \overline{C_i}^* Z(D(G))^\circ F/F) = \delta(\gamma_G^{-1}(X_i)/F) \cap \delta(\overline{C_i}^* Z(D(G))^\circ F/F)$. But we have $G = D(G)/F$, so $\gamma_G^{-1}(X_i)/F$ is precisely X_i, that is the torsion part of $Z(G)^\circ \cap G_i'$. Hence $\delta(\gamma_G^{-1}(X_i)/F)$ is the torsion part of $Z(H)^\circ \cap H_i'$, that is $\delta(\gamma_G^{-1}(X_i)/F) = X_i^H$. Moreover, we have $P_i = \overline{C_i}^* Z(D(G))$, so we find $\delta(\overline{C_i}^* Z(D(G))^\circ F/F) = \delta(P_i^\circ F/F) = P_i^\circ E/E$, and we obtain $\delta(Y_iF/F) = X_i^H \cap P_i^\circ E/E = (\gamma_H^{-1}(X_i^H)/E) \cap (P_i^\circ E/E) = (\gamma_H^{-1}(X_i^H) \cap P_i^\circ)E/E = Y_i^H E/E$. Thus, the isomorphism δ induces an isomorphism $\delta_i^* : P_i^\circ F/Y_iF \to P_i^\circ E/Y_i^H E$ and, since we have $P_i^\circ \cap F \leq Y_i \leq P_i^\circ$ and $P_i^\circ \cap E \leq Y_i^H \leq P_i^\circ$, the isomorphism δ induces an isomorphism $\delta_i^0 : P_i^\circ/Y_i \to P_i^\circ/Y_i^H$. We consider the isomorphism $\delta_i := \delta_i^0 \circ \varphi_i^{-1}$ from P_i°/Y_i to P_i°/Y_i^H. Let $\overline{\delta_i}$ be the isomorphism from $P_i^\circ/Z(D(G))^\circ Y_i$ to $P_i^\circ/Z(D(G))^\circ Y_i^H$ induced by δ_i. Since $Z(D(G))^\circ Y_i$ and $Z(D(G))^\circ Y_i^H$ have finite index in $Z(D(G))$, then the graph of $\overline{\delta_i}$ has finite index in $\Omega/(Z(D(G))^\circ Y_i \times Z(D(G))^\circ Y_i^H)$, because $\Omega/(Z(D(G)) \times Z(D(G)))$ is the graph of $\overline{\delta_i^\varphi}$. Since $\overline{\mathbb{Q}}$ is an algebraically closed field and of characteristic zero, the connected algebraic groups over $\overline{\mathbb{Q}}$ have no proper subgroup of finite index. In particular, $P_i^\circ/Z(D(G))^\circ Y_i$ has no proper subgroup of finite index, so the graph of $\overline{\delta_i}$ has no proper subgroup of finite index. Moreover, by considering $\Omega_Y := \Omega^\circ(Y_i \times Y_i^H)$ and $Z_Y := Z(D(G))^\circ Y_i \times Z(D(G))^\circ Y_i^H$, then the group Ω_Y/Z_Y is connected, so Ω_Y/Z_Y has no proper subgroup of finite index. This implies that the graph of $\overline{\delta_i}$ is Ω_Y/Z_Y. In particular, $\overline{\delta_i}$ is an isomorphism of algebraic groups.

Let Θ denote the preimage in $P_i^\circ \times P_i^\circ$ of the graph of δ_i. Then we have $\Omega_Y = \Theta\, Z_Y$, and $\Omega_Y'(Y_i \times Y_i^H)/(Y_i \times Y_i^H) = \Theta'(Y_i \times Y_i^H)/(Y_i \times Y_i^H)$ is the graph of the restriction of δ_i to $(P_i^\circ)'/Y_i$. Moreover, since $Z_Y\Omega_Y'/\Omega_Y'(Y_i \times Y_i^H)$ is a connected algebraic subgroup of the connected abelian group $\Omega_Y/\Omega_Y'(Y_i \times Y_i^H)$, it has an algebraic complement $W_0/\Omega_Y'(Y_i \times Y_i^H)$.

(1) *The algebraic groups $W := W_0^\circ$ and $\overline{C_i}^*$ are isomorphic, and $W \cap (Z_i \times P_i^\circ)$ and $W \cap (P_i^\circ \times Z_i)$ are trivial. Moreover, the image of the first (resp. the second) projection of $W(Z_i \times Z_i)$ in $P_i^\circ \times P_i^\circ$ is surjective.*

For each k, let $Z_k := T(G) \times (\times_{j \neq k} Z(\overline{C_j}^*)^\circ)$. Then we have $Z(D(G))^\circ = Z(\overline{C_k}^*)^\circ \times Z_k$ for each k. Let $Z_W := Z_iY_i \times Z_iY_i^H \leq P_i^\circ \times P_i^\circ$. Since Lemma 8.5

gives $Z(\overline{C_i}^*)^\circ \leq (\overline{C_i}^*)'$, we obtain $Z_Y \leq ((P_i^\circ)' \times (P_i^\circ)')Z_W$. Then $W_0((P_i^\circ)' \times (P_i^\circ)')Z_W$ contains $\Omega^\circ \leq \Omega_Y$ and, since Ω_Y/Z_Y is the graph of the isomorphism $\overline{\delta_i} : P_i^\circ/Z(D(G))^\circ Y_i \to P_i^\circ/Z(D(G))^\circ Y_i^H$, its first (resp. second) projection in $P_i^\circ \times P_i^\circ$ is surjective. Moreover, since $\Omega_Y'(Y_i \times Y_i^H)/(Y_i \times Y_i^H) = \Theta'(Y_i \times Y_i^H)/(Y_i \times Y_i^H)$ is the graph of the restriction to $(P_i^\circ)'Y_i/Y_i$ of the isomorphism $\delta_i : P_i^\circ/Y_i \to P_i^\circ/Y_i^H$, the first (resp. second) projection of $W_0\Omega_Y' Z_W$ in $P_i^\circ \times P_i^\circ$ is surjective too. But W_0 contains Ω_Y', so we have $(W_0\Omega_Y' Z_W)^\circ = (W_0 Z_W)^\circ \leq W Z_W$, and since P_i° is connected, the first (resp. second) projection of $W Z_W$ in $P_i^\circ \times P_i^\circ$ is surjective.

We verify that $rk(W) = rk(\overline{C_i}^*)$. We have

$$\begin{aligned} rk(W) &= rk(W_0) \\ &= rk(W_0/\Omega_Y'(Y_i \times Y_i^H)) + rk(\Omega_Y'(Y_i \times Y_i^H)) \\ &= (rk(\Omega_Y/\Omega_Y'(Y_i \times Y_i^H)) - rk(Z_Y\Omega_Y'/\Omega_Y'(Y_i \times Y_i^H))) \\ &\quad + rk(\Omega_Y'(Y_i \times Y_i^H)), \end{aligned}$$

and since $Y_i \times Y_i^H$ is finite, we obtain $rk(W) = rk(\Omega_Y) - rk(Z_Y\Omega_Y'/\Omega_Y') = rk(\Omega_Y) - (rk(Z_Y) - rk(Z_Y \cap \Omega_Y'))$. Moreover, by definition of Ω, we have

$$\begin{aligned} rk(\Omega_Y) &= rk(\Omega) \\ &= rk(\Omega/(Z(D(G)) \times Z(D(G)))) + rk(Z(D(G)) \times Z(D(G))) \\ &= rk(P_i/Z(D(G))) + rk(Z_Y) \\ &= rk(P_i^\circ/Z(D(G))^\circ Y_i) + rk(Z_Y) \\ &= rk(\overline{C_i}^* Z(D(G))^\circ/Z(D(G))^\circ) + rk(Z_Y) \\ &= rk(\overline{C_i}^*/(\overline{C_i}^* \cap Z(D(G))^\circ)) + rk(Z_Y) \\ &= rk(\overline{C_i}^*) - rk(\overline{C_i}^* \cap Z(D(G))^\circ) + rk(Z_Y), \end{aligned}$$

and since $D(G)$ is the direct product of $\overline{C_i}^*$ by another connected algebraic group, we obtain $rk(\Omega_Y) = rk(\overline{C_i}^*) - rk(Z(\overline{C_i}^*)^\circ) + rk(Z_Y)$. Since for each central finite subgroup Y of $D(G)$ we have $Z(D(G)/Y) = Z(D(G))/Y$, then we have $Z(D(G)/Y_i) = Z(D(G))/Y_i$. Since $D(G)$ is the central product of the groups $P_1, \ldots, P_n$, we have $Z(P_i^\circ/Y_i) = Z(D(G)/Y_i) \cap P_i^\circ/Y_i$. Therefore, since P_i contains $Z(D(G))$, we obtain

$$Z(P_i^\circ/Y_i)^\circ = (Z(D(G)/Y_i) \cap P_i^\circ/Y_i)^\circ = (Z(D(G))/Y_i \cap P_i^\circ/Y_i)^\circ = Z(D(G))^\circ Y_i/Y_i.$$

In the same way, we have $Z(P_i^\circ/Y_i^H)^\circ = Z(D(G))^\circ Y_i^H/Y_i^H$, so

$$\delta_i(Z(D(G))^\circ Y_i/Y_i) = Z(D(G))^\circ Y_i^H/Y_i^H.$$

Since $\Omega_Y'(Y_i \times Y_i^H)/(Y_i \times Y_i^H) = \Theta'(Y_i \times Y_i^H)/(Y_i \times Y_i^H)$ is the graph of the restriction to $(P_i^\circ)'Y_i/Y_i$ of δ_i, the latter proves that $(Z_Y \cap \Omega_Y')Y_i/Y_i$ is the graph of the restriction to $(Z(D(G))^\circ Y_i \cap (P_i^\circ)'Y_i)/Y_i$ of δ_i. Now, since $Y_i \times Y_i^H$ is finite, we have $rk(Z_Y \cap \Omega_Y') = rk(Z(D(G))^\circ Y_i \cap (P_i^\circ)'Y_i) = rk(Z(D(G))^\circ \cap (P_i^\circ)') = rk(Z(D(G))^\circ \cap (\overline{C_i}^*)') = rk(Z(\overline{C_i}^*)^\circ \cap (\overline{C_i}^*)')$, and since Lemma 8.5 implies that $(\overline{C_i}^*)'$ contains $Z(\overline{C_i}^*)^\circ$, we obtain $rk(Z_Y \cap \Omega_Y') = rk(Z(\overline{C_i}^*)^\circ)$, and

$$\begin{aligned} rk(W) &= rk(\Omega_Y) - rk(Z_Y) + rk(Z_Y \cap \Omega_Y') \\ &= (rk(\overline{C_i}^*) - rk(Z(\overline{C_i}^*)^\circ) + rk(Z_Y)) - rk(Z_Y) + rk(Z(\overline{C_i}^*)^\circ) \\ &= rk(\overline{C_i}^*), \end{aligned}$$

as claimed.

We show that $W/(W \cap (Y_i \times Y_i^H))$ and $\overline{C_i}$ are isomorphic as algebraic groups. Indeed, there is an isomorphism of algebraic groups between $\overline{C_i}$ and

$$\gamma_G^{-1}(C_i)/\gamma_G^{-1}(X_i) = \overline{C_i}^* \gamma_G^{-1}(X_i)/\gamma_G^{-1}(X_i) \simeq \overline{C_i}^*/(\gamma_G^{-1}(X_i) \cap \overline{C_i}^*),$$

and since we have $P_i^\circ = \overline{C_i}^* Z(D(G))^\circ = \overline{C_i}^* \times Z_i$, the groups $P_i^\circ/(\gamma_G^{-1}(X_i) \cap \overline{C_i}^*)Z_i$ and $\overline{C_i}$ are isomorphic as algebraic groups. Thus, since Z_i contains $\gamma_G^{-1}(X_i) \cap T(G) \leq T(G)$, the algebraic groups $\overline{C_i}$ and P_i°/Y_iZ_i are isomorphic by Lemma 8.13, so $\overline{C_i}$ and $(P_i^\circ \times P_i^\circ)/(Y_iZ_i \times P_i^\circ)$ are isomorphic too. Now, since the first projection of WZ_W in $P_i^\circ \times P_i^\circ$ is surjective, that is $WZ_W(\{1\} \times P_i^\circ) = P_i^\circ \times P_i^\circ$, the algebraic groups $\overline{C_i}$ and $WZ_W/(WZ_W \cap (Y_iZ_i \times P_i^\circ))$ are isomorphic. Hence, since Z_W is contained in $Y_iZ_i \times P_i^\circ$, the algebraic groups $\overline{C_i}$ and $W/(W \cap (Y_iZ_i \times P_i^\circ)) = W/(W \cap (WZ_W \cap (Y_iZ_i \times P_i^\circ)))$ are isomorphic. In particular, since we have $rk(W) = rk(\overline{C_i}^*) = rk(\overline{C_i})$ by the previous paragraph, the intersection $W \cap (Y_iZ_i \times P_i^\circ)$ is finite. Since Ω_Y/Z_Y is the graph of $\overline{\delta_i}$, we have $\Omega_Y \cap (Z(D(G))^\circ Y_i \times P_i^\circ) = Z_Y$ and $W \cap (Y_iZ_i \times P_i^\circ) \leq \Omega_Y \cap (Z(D(G))^\circ Y_i \times P_i^\circ)$ is contained in $W \cap Z_Y \leq W_0 \cap Z_Y \leq \Omega'_Y(Y_i \times Y_i^H)$, so we find

$$\begin{array}{rcl} W \cap (Y_iZ_i \times P_i^\circ) & \leq & W \cap Z_Y \\ & \leq & Z_Y \cap \Omega'_Y(Y_i \times Y_i^H)) \\ & \leq & (Z(D(G))^\circ Y_i \cap (P_i^\circ)' Y_i) \times (Z(D(G))^\circ Y_i^H \cap (P_i^\circ)' Y_i^H). \end{array}$$

Let W_P (resp. W_Q) be the image of the finite group $W \cap (Y_iZ_i \times P_i^\circ)$ by the first (resp. the second) projection in $P_i^\circ \times P_i^\circ$. Then W_PF/F is a finite subgroup of

$$\begin{array}{rcl} (Z(D(G))^\circ Y_i \cap (P_i^\circ)' Y_i)F/F & = & (Z(D(G))^\circ Y_iF/F) \cap ((P_i^\circ)' Y_iF/F) \\ & = & (\gamma_G(Z(D(G)))^\circ \gamma_G(Y_i)) \cap ((\gamma_G(P_i)^\circ)' \gamma_G(Y_i)) \\ & \leq & (Z(G)^\circ X_i) \cap ((G_i^\circ)' X_i) \\ & = & Z(G)^\circ \cap G_i', \end{array}$$

so W_PF/F is contained in X_i, and W_P is contained in $\gamma_G^{-1}(X_i) \cap P_i^\circ = Y_i$. In the same way we obtain $W_Q \leq Y_i^H$, so we have $W \cap (Y_iZ_i \times P_i^\circ) \leq Y_i \times Y_i^H$. Thus $W/(W \cap (Y_i \times Y_i^H))$ and $\overline{C_i}$ are isomorphic as algebraic groups.

We prove that W and $\overline{C_i}^*$ are isomorphic as algebraic groups, and that $W \cap (Z_i \times P_i^\circ)$ and $W \cap (P_i^\circ \times Z_i)$ are trivial. Since the exponent of $Y_i \times Y_i^H$ is n_i by Lemma 8.13, and since $W/(W \cap (Y_i \times Y_i^H))$ and $\overline{C_i}$ are isomorphic as algebraic groups, there is an isogeny ι_1 from $\overline{C_i}^*$ to W. Since the first projection of WZ_W in $P_i^\circ \times P_i^\circ$ is surjective, and since P_i° is connected, the first projection of $W(Z_i \times Z_i)$ in $P_i^\circ \times P_i^\circ$ is surjective too. Thus we have $P_i^\circ \times P_i^\circ = (W(Z_i \times Z_i))(\{1\} \times P_i^\circ) = W(Z_i \times P_i^\circ)$. Then, since we have $P_i^\circ = \overline{C_i}^* \times Z_i$, the algebraic groups $W(Z_i \times P_i^\circ)/(Z_i \times P_i^\circ)$ and $\overline{C_i}^*$ are isomorphic, so $W/(W \cap (Z_i \times P_i^\circ))$ and $\overline{C_i}^*$ are isomorphic too. Therefore, since we have $rk(W) = rk(\overline{C_i}^*)$, there is an isogeny ι_2 from W to $\overline{C_i}^*$ with kernel $W \cap (Z_i \times P_i^\circ)$. Now $\iota_2 \circ \iota_1$ is an isogeny from $\overline{C_i}^*$ to $\overline{C_i}^*$. Since $\ker(\iota_2 \circ \iota_1)$ is finite and since $\overline{C_i}^*$ is connected, the preimage of $Z(\overline{C_i}^*)$ by $\iota_2 \circ \iota_1$ is precisely $Z(\overline{C_i}^*)$. In particular, if X_C denotes the torsion part of $Z(\overline{C_i}^*)$, then X_C contains $\ker(\iota_2 \circ \iota_1)$, and $X_C/\ker(\iota_2 \circ \iota_1)$ is isomorphic to X_C. But Lemma 8.5 implies that $(\overline{C_i}^*)'$ contains $Z(\overline{C_i}^*)^\circ$, so $Z(\overline{C_i}^*)^\circ$ has no nontrivial torus, and it is torsion-free. Hence X_C is finite and $\ker(\iota_2 \circ \iota_1)$ is trivial. Thus $\iota_2 \circ \iota_1$ is an automorphism of $\overline{C_i}^*$, and ι_2 is an isomorphism from W to $\overline{C_i}^*$. In particular, W and $\overline{C_i}^*$ are isomorphic as algebraic groups, and since the kernel of ι_2 is $W \cap (Z_i \times P_i^\circ)$, we have $W \cap (Z_i \times P_i^\circ) = 1$. In the same way, we obtain $W \cap (P_i^\circ \times Z_i) = 1$.

(2) *P_i° is the direct product of Z_i by R_i (resp. S_i), the first (resp. the second) projection in $P_i^\circ \times P_i^\circ$ is surjective, and W is the graph of an isomorphism of algebraic groups $\omega_i : R_i \to S_i$.*

Since we have $W \cap (\{1\} \times S_i) \leq W \cap (Z_i \times P_i^\circ) = 1$ and $W \cap (R_i \times \{1\}) \leq W \cap (P_i^\circ \times Z_i) = 1$, the algebraic group W is the graph of an isomorphism $\omega_i : R_i \to S_i$ of algebraic groups. Moreover, since we have $W \cap (Z_i \times P_i^\circ) = 1$ (resp. $W \cap (P_i^\circ \times Z_i) = 1$), then $R_i \cap Z_i$ (resp. $S_i \cap Z_i$) is trivial. Therefore, since the first (resp. the second) projection of $W(Z_i \times Z_i)$ in $P_i^\circ \times P_i^\circ$ is surjective, then P_i° is the direct product of R_i by Z_i (resp. S_i by Z_i).

(3) *$D(G)$ is the direct product of $T(G)$ and of the groups R_k (resp. S_k) for $k \in \{1, \ldots, n\}$.*

We note that, for each k, since we have $P_k^\circ = \overline{C_k}^* \times Z_k = R_k \times Z_k$, we find $(\overline{C_k}^*)' = R_k'$, $Z(\overline{C_k}^*) = Z(P_k^\circ) \cap \overline{C_k}^*$ and $Z(R_k) = Z(P_k^\circ) \cap R_k$. Therefore, since $(\overline{C_k}^*)'$ contains $Z(\overline{C_k}^*)^\circ$ (Lemma 8.5), then $R_k \geq (\overline{C_k}^*)'$ contains $Z(\overline{C_k}^*)^\circ$, and $Z(R_k)$ contains $Z(\overline{C_k}^*)^\circ$. But we have $Z(P_k^\circ) = Z(\overline{C_k}^*) \times Z_k = Z(R_k) \times Z_k$, hence we obtain $Z(R_k)^\circ = Z(\overline{C_k}^*)^\circ$ for each k. In particular, the product $T(G)(\prod_{j=1}^n R_j)$ contains $Z_k = T(G) \times (\times_{j \neq k} Z(R_j)^\circ)$ for each k, and since P_k° is the direct product of R_k by Z_k for each k, this product contains $\overline{C_k}^*$ for each k. Thus we have $D(G) = T(G)(\prod_{j=1}^n R_j)$. Since $D(G)$ is the central product of $T(G)$ and of the groups $P_j = \overline{C_j}^* Z(D(G))$ for $j \in \{1, \ldots, n\}$, and since R_j is contained in P_j for each j, then $D(G)$ the central product of $T(G)$ and of the groups R_j for $j \in \{1, \ldots, n\}$. This implies that that $Z(D(G))$ is the product of the groups $T(G)$ and $Z(R_j)$ for $j \in \{1, \ldots, n\}$. Moreover, since the algebraic groups R_k and $\overline{C_k}^*$ are isomorphic for each k, and since $D(G)$ is the direct product of $T(G)$ and of $\overline{C_j}^*$ for $j \in \{1, \ldots, n\}$, we have $rk(D(G)) = rk(T(G)) + \sum_{j=1}^n rk(\overline{C_j}^*) = rk(T(G)) + \sum_{j=1}^n rk(R_j)$. Therefore, for each k, the intersection $I_k := R_k \cap (T(G) \prod_{j \neq k} R_j)$ is finite. Since $D(G)$ is the central product of $T(G)$ and of the groups R_j for $j \in \{1, \ldots, n\}$, then I_k is central in R_k.

For each k, since $(\overline{C_k}^*)'$ contains $Z(\overline{C_k}^*)^\circ$ by Lemma 8.5, the maximal torus of $Z(\overline{C_k}^*)^\circ$ is trivial, and $Z(\overline{C_k}^*)^\circ$ is torsion-free. Consequently, for each k, the torsion part J_k^C of $Z(\overline{C_k}^*)$ is finite, and $Z(\overline{C_k}^*)$ is the direct product of J_k^C by $Z(\overline{C_k}^*)^\circ$. But, for each k, the algebraic groups R_k and $\overline{C_k}^*$ are isomorphic, hence $Z(R_k)$ is the direct product of its torsion part $J_k \simeq J_k^C$ by $Z(R_k)^\circ$. Since $D(G)$ is the direct product of $T(G) \leq Z(D(G))^\circ$ and of the groups $\overline{C_j}^*$ for $j \in \{1, \ldots, n\}$, then $Z(D(G))$ is the direct product of $T(G)$ and of the groups $Z(\overline{C_j}^*)$ for $j \in \{1, \ldots, n\}$. Thus $Z(D(G))$ is the direct product of $Z(D(G))^\circ$ and of the groups J_j^C for $j \in \{1, \ldots, n\}$. In particular, its Morley degree is $\sum_{j=1}^n |J_j^C| = \sum_{j=1}^n |J_j|$. Since $Z(D(G))$ is the product of the groups $T(G)$ and $Z(R_j)$ for $j \in \{1, \ldots, n\}$, and since $Z(R_j)$ is the direct product of J_j by $Z(R_j)^\circ$ for each j, then $Z(D(G))$ is the product of $Z(D(G))^\circ$ and of the groups J_j for $j \in \{1, \ldots, n\}$. But its Morley degree is $\sum_{k=1}^n |J_j|$, hence $Z(D(G))$ is the direct product of $Z(D(G))^\circ$ and of the groups J_j for $j \in \{1, \ldots, n\}$.

Now we consider $k \in \{1, \ldots, n\}$ and $x \in I_k$. Then there are $u \in T(G)$ and $x_j \in R_j$ for each $j \neq k$ such that $x = u \prod_{j \neq k} x_j$. Let $x_k := x^{-1}$. Then we have $u \prod_{j=1}^{n} x_j = 1$ and, for each j, we find $x_j = u^{-1} \prod_{l \neq j} x_l \in I_j$. For each j, since I_j is a finite central subgroup of R_j, we have $I_j \leq J_j$. But we have $u \in T(G) \leq Z(D(G))^\circ$, and $Z(D(G))$ is the direct product of $Z(D(G))^\circ$ and of the groups J_j for $j \in \{1, \ldots, n\}$. Hence we find $u = 1$ and $x_j = 1$ for each j. In particular, x and I_k are trivial. Thus $D(G)$ is the direct product of $T(G)$ and of the groups R_k for $k \in \{1, \ldots, n\}$, as desired. In the same way, $D(G)$ is the direct product of $T(G)$ and of the groups S_k for $k \in \{1, \ldots, n\}$.

(4) *Determination of the quasi-standard automorphism α of $D(G)$.*

Since, for each j, the group $Z(\overline{C_j}^*)^\circ$ is torsion-free, and since $Z(D(G))^\circ$ is the direct product of $T(G)$ and of the groups $Z(\overline{C_j}^*)^\circ$ for $j \in \{1, \ldots, n\}$, then the torsion part S of $T(G)$ is the one of $Z(D(G))^\circ$. Since we have $Z_F = Z_E = Z(D(G))$, then SF/F (resp. SE/E) is the torsion part of $Z(D(G)/F)^\circ$ (resp. $Z(D(G)/E)^\circ$), and we obtain $\delta(SF/F) = SE/E$. Moreover, since $F \cap T(G) = E \cap T(G) = 1$, the automorphism δ induces an automorphism δ_S of S such that $\delta(xF) = \delta_S(x)E$ for each $x \in S$.

Let $X := T(G) \cap D(G)'F$. Since we have $T(G) \cap D(G)' = T(G) \cap (\overline{C_1}^* \times \cdots \times \overline{C_n}^*)' = 1$, then X is a finite subgroup of $T(G)$, and it is contained in $S \leq T(G)$. If $S = 1$, we denote by φ_X the identity map of $T(G)$. Otherwise $T(G)$ is a torus, and Lemma 8.12 provides a quasi-standard automorphism φ_X of $T(G)$ such that $\varphi_X(x) = \delta_S(x)$ for each $x \in X$. We consider the automorphism α of $D(G)$ defined as follows: *for each j, for each $x \in R_j$, $\alpha(x) = (\omega_j \circ \varphi_j)(x)$; for each $x \in T(G)$, $\alpha(x) = \varphi_X(x)$.* Thus α is a quasi-standard automorphism of $D(G)$.

(5) *$D(G)/Z(D(G))^\circ\alpha(F)$ and $D(G)/Z(D(G))^\circ E$ are isomorphic as algebraic groups.*

Let $\overline{\alpha} : D(G)/F \to D(G)/\alpha(F)$ be the abstract isomorphism induced by α, and let $\nu = \delta \circ \overline{\alpha}^{-1}$. For each i, we have

$$\overline{\alpha}(P_i/F) = \alpha(P_i)/\alpha(F) = \alpha(R_i Z(D(G)))/\alpha(F) = S_i Z(D(G))/\alpha(F) = P_i/\alpha(F),$$

and $\nu(P_i/\alpha(F)) = \delta(P_i/F) = P_i/E$. We show that the map

$$\overline{\nu} : D(G)/Z(D(G))^\circ\alpha(F) \to D(G)/Z(D(G))^\circ E$$

induced by ν is an isomorphism of algebraic groups. We consider the abstract isomorphism $\overline{\alpha}^Z : D(G)/Z(D(G))^\circ F \to D(G)/Z(D(G))^\circ\alpha(F)$ induced by α, and let $\overline{\delta} : D(G)/Z(D(G))^\circ F \to D(G)/Z(D(G))^\circ E$ be the abstract isomorphism induced by δ. For each i, we denote by $\overline{\nu}_i : P_i^\circ\alpha(F)/Z(D(G))^\circ\alpha(F) \to P_i^\circ E/Z(D(G))^\circ E$ the restriction of $\overline{\nu}$ to $P_i^\circ\alpha(F)/Z(D(G))^\circ\alpha(F)$. For each i, since the subgroup R_i (resp. S_i) covers $P_i^\circ F/Z(D(G))^\circ F$ (resp. $P_i^\circ\alpha(F)/Z(D(G))^\circ\alpha(F)$), and since $\omega_i : R_i \to S_i$ satisfies

$$\omega_i(R_i \cap Z(D(G))^\circ F) = \omega_i(R_i) \cap \alpha(Z(D(G))^\circ F) = S_i \cap Z(D(G))^\circ\alpha(F),$$

then ω_i induces an isomorphism of algebraic groups

$$\overline{\omega}_i : P_i^\circ F/Z(D(G))^\circ F \to P_i^\circ\alpha(F)/Z(D(G))^\circ\alpha(F).$$

Thus, for each i and each $\overline{x} \in P_i^\circ\alpha(F)/Z(D(G))^\circ\alpha(F)$, we have $\overline{\nu}_i(\overline{x}) = (\overline{\delta} \circ (\overline{\alpha}^Z)^{-1})(\overline{x}) = (\overline{\delta} \circ \varphi_i^{-1} \circ \overline{\omega}_i^{-1})(\overline{x})$, so we obtain

$$\overline{\nu}_i = (\overline{\delta} \circ \varphi_i^{-1}) \circ \overline{\omega}_i^{-1}.$$

But, for each i, we have

$$Y_i = \gamma_G^{-1}(X_i) \cap \overline{C_i}^* Z(D(G))^\circ \leq \gamma_G^{-1}(Z(G)^\circ) \cap P_i^\circ = Z(D(G))^\circ F \cap P_i^\circ,$$

and $F \cap P_i^\circ \leq \gamma_G^{-1}(X_i) \cap \overline{C_i}^* Z(D(G))^\circ = Y_i$, so we have $Z(D(G))^\circ Y_i = Z(D(G))^\circ F \cap P_i^\circ$, and in the same way, $Z(D(G))^\circ Y_i^H = Z(D(G))^\circ E \cap P_i^\circ$. Hence the isomorphism $\overline{\delta} \circ \varphi_i^{-1} : P_i^\circ F/Z(D(G))^\circ F \to P_i^\circ E/Z(D(G))^\circ E$ is induced by $\overline{\delta_i}$, and it is algebraic. Since $\overline{\omega}_i$ is an isomorphism of algebraic groups too, this proves that $\overline{\nu}_i$ is an isomorphism of algebraic groups. Since $D(G)$ is generated by the subgroups P_i° for $i = 1, \ldots, n$, the map $\overline{\nu}$ is an isomorphism of algebraic groups, as desired.

(6) *Final argument.*

Let Δ be the preimage of the graph of ν in $D(G) \times D(G)$, and let $Z_D = Z(D(G))^\circ\alpha(F) \times Z(D(G))^\circ E$. Then $\Delta Z_D/Z_D$ is the graph of $\overline{\nu}$, so it is an algebraic group and, since $D(G)/Z(D(G))^\circ\alpha(F)$ is connected, $\Delta Z_D/Z_D$ is connected. Moreover, since $Z_D/(\alpha(F) \times E)$ is connected too, $\Delta Z_D/(\alpha(F) \times E)$ is connected. We have $(\Delta Z_D/(\alpha(F) \times E))' = \Delta'(\alpha(F) \times E)/(\alpha(F) \times E)$, so $\Delta'(\alpha(F) \times E)/(\alpha(F) \times E)$ is a connected algebraic subgroup of $\Delta Z_D/(\alpha(F) \times E)$.

We consider the diagonal Δ_0 of $T(G) \times T(G)$, and let ρ be the isomorphism from $T(G)\alpha(F)/\alpha(F)$ to $T(G)E/E$ whose graph is $\Delta_0(\alpha(F) \times E)/(\alpha(F) \times E)$. Let $\overline{u} \in D(G)'\alpha(F)/\alpha(F) \cap T(G)\alpha(F)/\alpha(F)$. Then there exists $x \in X$ such that $\overline{\alpha}^{-1}(\overline{u}) = xF$. Hence we have $\nu(\overline{u}) = \delta(xF) = \delta_S(x)E = \varphi_X(x)E = \alpha(x)E$, and

$$(\overline{u}, \nu(\overline{u})) = (\alpha(x)\alpha(F), \alpha(x)E) \in \Delta_0(\alpha(F) \times E)/(\alpha(F) \times E).$$

Consequently, for each $\overline{u} \in D(G)'\alpha(F)/\alpha(F) \cap T(G)\alpha(F)/\alpha(F)$, we have $\nu(\overline{u}) = \rho(\overline{u})$, and we can define an isomorphism

$$\nu_\rho : D(G)'T(G)\alpha(F)/\alpha(F) \to D(G)'T(G)E/E$$

by $\nu_\rho(\overline{x}) = \nu(\overline{x})$ for each $\overline{x} \in D(G)'\alpha(F)/\alpha(F)$, and $\nu_\rho(\overline{x}) = \rho(\overline{x})$ for each $\overline{x} \in T(G)\alpha(F)/\alpha(F)$. Then the graph of ν_ρ is

$$\Delta_1/(\alpha(F) \times E) := \Delta'\Delta_0(\alpha(F) \times E)/(\alpha(F) \times E).$$

Moreover, since $\Delta'(\alpha(F) \times E)/(\alpha(F) \times E)$ and $\Delta_0(\alpha(F) \times E)/(\alpha(F) \times E)$ are two algebraic groups, Δ_1 is algebraic, and ν_ρ is an isomorphism of algebraic groups.

Since $\Delta_1/(\alpha(F) \times E)$ is the graph of an isomorphism of algebraic groups between two connected algebraic groups, it is connected. Since $\Delta Z_D/(\alpha(F) \times E)$ is connected, then $\Delta Z_D/\Delta_1$ is connected too. But Δ_1 contains Δ', so $\Delta Z_D/\Delta_1$ is abelian. Hence, since $\Delta Z_D/\Delta_1$ and $Z_D\Delta_1/\Delta_1$ are connected, there is an algebraic complement Δ_2/Δ_1 of $Z_D\Delta_1/\Delta_1$ in $\Delta Z_D/\Delta_1$. Now, since $\Delta Z_D/Z_D$ is the graph of $\overline{\nu}$, we have $\Delta Z_D/Z_D \cap (D(G) \times Z(D(G))^\circ E)/Z_D = 1$ and

$$\begin{aligned} \Delta_2 \cap (D(G) \times E) &\leq \Delta Z_D \cap (D(G) \times E) \\ &= Z_D \cap (D(G) \times E) \\ &= Z(D(G))^\circ\alpha(F) \times E. \end{aligned}$$

Consequently we obtain

$$\begin{array}{rcl}\Delta_2 \cap (D(G)\times E) & = & \Delta_2 \cap (Z(D(G))^\circ \alpha(F)\times E) \\ & \leq & \Delta_2 \cap Z_D \\ & \leq & \Delta_1.\end{array}$$

Finally, , since $\Delta_1/(\alpha(F)\times E)$ is the graph of an isomorphism of algebraic groups from $D(G)'T(G)\alpha(F)/\alpha(F)$ to $D(G)'T(G)E/E$, we find

$$\Delta_2 \cap (D(G)\times E) = \Delta_1 \cap (D(G)\times E) = \alpha(F)\times E.$$

In the same way, we obtain $\Delta_2 \cap (\alpha(F)\times D(G)) = \alpha(F)\times E$.

Since Lemma 8.5 implies that $(\overline{C_i}^*)'$ contains $Z(\overline{C_i}^*)^\circ$ for each i, then the subgroup $D(G)'T(G)$ contains $Z(D(G))^\circ$. Therefore, for each $\overline{x} \in Z(D(G))^\circ F/F = Z(D(G)/F)^\circ$, there exist $\overline{d} \in (D(G)'F/F)\cap(Z(D(G))^\circ F/F)$ and $\overline{u} \in T(G)F/F$ such that $\overline{x} = \overline{d}\,\overline{u}$, and we have

$$\nu_\rho(\overline{x}) = \nu_\rho(\overline{d})\nu_\rho(\overline{u}) = \nu(\overline{d})\rho(\overline{u}) \in Z(D(G)/E)^\circ \cdot T(G)E/E = Z(D(G))^\circ E/E.$$

The latter proves that $\nu_\rho(Z(D(G))^\circ F/F) = Z(D(G))^\circ E/E$. Thus, since the graph of ν_ρ is $\Delta_1/(\alpha(F)\times E)$, the group $(\Delta_1 \cap Z_D)/(\alpha(F)\times E)$ is the graph of the restriction of ν_ρ to $Z(D(G))^\circ F/F$. Consequently, we obtain

$$rk(\Delta_1 \cap Z_D) = rk((\Delta_1 \cap Z_D)/(\alpha(F)\times E)) = rk(Z(D(G))^\circ F/F) = rk(Z(D(G))).$$

This yields

$$\begin{array}{rcl} rk(Z_D\Delta_1) & = & rk(Z_D) + rk(\Delta_1) - rk(\Delta_1 \cap Z_D) \\ & = & 2rk(Z(D(G))) + rk(\Delta_1) - rk(Z(D(G))) \\ & = & rk(Z(D(G))) + rk(\Delta_1).\end{array}$$

Moreover, since $\Delta Z_D/Z_D$ is the graph of $\overline{\nu}$, we have

$$rk(\Delta Z_D/Z_D) = rk(D(G)/Z(D(G))^\circ \alpha(F)) = rk(D(G)) - rk(Z(D(G)))$$

and

$$rk(\Delta Z_D) = (rk(D(G)) - rk(Z(D(G)))) + rk(Z_D) = rk(D(G)) + rk(Z(D(G))).$$

Now, since Δ_2/Δ_1 is a complement of $Z_D\Delta_1/\Delta_1$ in $\Delta Z_D/\Delta_1$, we find

$$\begin{array}{rcl} rk(\Delta_2) & = & rk(\Delta Z_D/\Delta_1) - rk(Z_D\Delta_1/\Delta_1) + rk(\Delta_1) \\ & = & rk(\Delta Z_D) - rk(Z_D\Delta_1) + rk(\Delta_1) \\ & = & (rk(D(G)) + rk(Z(D(G)))) - (rk(Z(D(G))) + rk(\Delta_1)) + rk(\Delta_1) \\ & = & rk(D(G)).\end{array}$$

Since we have $\Delta_2 \cap (D(G)\times E) = \alpha(F)\times E$ and $\Delta_2 \cap (\alpha(F)\times D(G)) = \alpha(F)\times E$ by the previous paragraph, the latter implies that $\Delta_2/(\alpha(F)\times E)$ is the graph of an isomorphism β from $D(G)/\alpha(F)$ to $D(G)/E$. Moreover, β is algebraic since Δ_2 is an algebraic group.

Finally, $D(G)/\alpha(F)$ and $D(G)/E$ are isomorphic as algebraic groups, as desired. □

We note that Theorem 8.14 together with Lemma 8.10 provide Theorem 1.1 in the case $K = \overline{\mathbb{Q}}$.

9. The final argument

In this section, we fix a connected affine ACF_0-group $\mathcal{M} = (G, \cdot, ^{-1}, 1, \cdots)$, interpretable in the pure field $\overline{\mathbb{Q}}$, and an elementary substructure $\mathcal{M}^* = (G^*, \cdots)$ of $\mathcal{M}$. We note that the structures $\mathcal{M}$ and $\mathcal{M}^*$ have finite Morley rank, so any infinite field, interpretable in $\mathcal{M}$ or $\mathcal{M}^*$, is algebraically closed by a theorem of A. Macintyre [5, Theorem 8.1].

Lemma 9.1. – *Let U^*/V^* be a definable section of G^*, and let U/V be its canonical extension to G. If U/V is definably isomorphic to K_+ (resp. K^*) for an infinite interpretable field K, then $U = U^*V$.*

Proof – In this case, there is an infinite field L, interpretable in $\mathcal{M}^*$, and a definable morphism α from U^* to L_+ (resp. L^*), with kernel V^*. Then we have $L = M_0/R_0$ for M_0 a definable subset of $(G^*)^n$, where n is an integer, and R_0 a definable equivalence relation over M_0. We consider the extensions L_1, M_1, R_1 and α_1 to G of L, M_0, R_0 and α respectively. Since L_1 is an infinite field, interpretable in $\overline{\mathbb{Q}}$, then Fact 2.6 says that L_1 is isomorphic to $\overline{\mathbb{Q}}$. Moreover, L is infinite and interpretable in $\mathcal{M}^*$, so it is algebraically closed. Since $\overline{\mathbb{Q}}$ has no proper algebraically closed subfield, this implies that the canonical embedding of L in L_1 is an isomorphism: the subset M_0 of M_1 covers $L_1 = M_1/R_1$. Thus, for each $u \in U$, there is $u^* \in U^*$ such that $\alpha_1(u) = \alpha_1(u^*)$. Since α_1 is a morphism with kernel V, this yields the result. □

We note that, since G is connected and since its ground field is algebraically closed and of characteristic zero, there is no proper subgroup of finite index in G.

Proposition 9.2. – $G = G^*Z(G)^\circ$.

Proof – Since there is no proper subgroup of finite index in G, we have just to prove that $G = G^*Z(G)$. Let B^* be a Borel subgroup of G^*, that is a maximal solvable connected definable subgroup, and let B be its extension to G. Then B is a maximal solvable connected definable subgroup of G, so it is an algebraic Borel subgroup of G by [5, Corollary 5.38] and Lemma 4.1. In particular, we have $Z(B) = Z(G)$, so $Z(B^*) = Z(G^*)$. Moreover, since B is the extension of B^*, we have $B = d(B^*)$. Let $\overline{D} = D/Z(G)$ be the largest connected definable subgroup of $\overline{B} = B/Z(G)$ contained in $\overline{B^*} = B^*Z(G)/Z(G)$. Then $\overline{D}$ is normal in $\overline{B^*}$, so $\overline{D}$ is normal in $\overline{B}$.

If $\overline{D} = \overline{B^*}$, then we have $\overline{B^*} = \overline{B}$. Since the conjugates of B cover G, the ones of B^* in G^* cover G^*. In particular, $\overline{G^*} = G^*Z(G)/Z(G)$ is generated by the conjugates of $\overline{D}$ in $\overline{G^*}$. Now, since $\overline{D}$ is definable in $\overline{G}$ and connected, $\overline{G^*}$ is a connected definable subgroup of $\overline{G}$. Since G^* is an elementary substructure of G, this proves that $\overline{G^*} = \overline{G}$, and $G = G^*Z(G)$. Hence we may assume $\overline{D} \neq \overline{B^*}$.

Let $D_0^*/Z(G^*)$ be the largest connected definable subgroup of $B^*/Z(G^*)$, with D_0^* contained in D, and let D_0 be its extension to G. Then D_0^* is normal in B^*, and $D_0/Z(G)$ is a normal connected definable subgroup of $B/Z(G)$, and D_0 is contained in D. Let U^*/D_0^* be a B^*-minimal section of B^*, and let U be the extension of U^* to G. Then U/D_0 is a B-minimal section of B. By choice of D_0^*, the subgroups U^* and U are not contained in D, and $U^*/Z(G^*)$ and $U/Z(G)$ are connected.

If U/D_0 is definably isomorphic to K_+ or K^* for an infinite interpretable field K, then Lemma 9.1 gives $U = U^*D_0$. But this implies that U is contained in B^*D, hence $U/Z(G)$ is contained in $\overline{B^*}$, contradicting the maximality of $\overline{D}$.

If the Fitting subgroup $F(B)$ covers U/D_0, then U/D_0 is definably isomorphic to a section of $F(B)/Z(B)$. But $F(B)/Z(B)$ is a $\widetilde{U}$-group by Fact 2.21, hence U/D_0 is a $\widetilde{U}$-group by Fact 2.19. Since $\overline{\mathbb{Q}}$ is algebraically closed of characteristic zero, there is no infinite definable group of bounded exponent in $F(B)/Z(B)$, and Fact 2.17 shows that $F(B)/Z(B)$ is torsion-free. In particular, U/D_0 is torsion-free. Consequently, by B-minimality of U/D_0, Fact 2.17 provides an interpretable algebraically closed field L of characteristic zero such that U/D_0 is definably isomorphic to L_+. This contradicts the previous paragraph, so $F(B)$ does not cover U/D_0, and $(F(B) \cap U)D_0/D_0$ is finite. But D_0 contains $Z(B)$ and $F(B)/Z(B)$ is torsion-free, so $F(B)$ avoids U/D_0.

Since B is a connected solvable algebraic group, $B/F(B)^\circ$ is abelian, and $F(B)$ is a maximal nilpotent subgroup of B, so $F(B)$ is the intersection of the centralizers of the B-minimal sections of B. Then we find finitely many B-minimal sections $R_1/S_1, \ldots, R_n/S_n$ of B such that $F(B)$ is the intersection of their centralizers in B. Now, for each $i = 1, \ldots, n$, either B centralizes R_i/S_i, or $B/C_B(R_i/S_i)$ is definably isomorphic to K_i^* for an interpretable algebraically closed field K_i (Fact 2.5). Consequently, for each $i = 1, \ldots, n$ such that $C_B(R_i/S_i)$ does not cover U/D_0, there is a finite subgroup F_i/D_0 of U/D_0 such that U/F_i is definably isomorphic to an infinite definable subgroup of K_i^*. Morover, in $\overline{\mathbb{Q}}$, the fields K_i and $\overline{\mathbb{Q}}$ are definably isomorphic (Fact 2.6), so K_i^* has Morley rank one (in $\overline{\mathbb{Q}}$). This implies that U/F_i is definably isomorphic to K_i^*.

If i and j are two elements of $\{1, \ldots, n\}$ such that $C_B(R_i/S_i)$ and $C_B(R_j/S_j)$ do not cover U/D_0, then U/F_iF_j is definably isomorphic to K_i^* and K_j^*, so K_i and K_j are definably isomorphic (Lemma 5.4). Moreover, since $F(B)$ avoids U/D_0, the intersection of the subgroups F_i of this form is D_0. Thus, there is $i \in \{1, \ldots, n\}$ such that U/D_0 is definably isomorphic to a subgroup of $(K_i^*)^n$. By minimality of U/D_0, this implies that U/D_0 is definably isomorphic to K_i^*, contradicting that U/D_0 is definably isomorphic to no group of the form K^* for an infinite interpretable field K. This finishes the proof. □

Corollary 9.3. – *If G' contains $Z(G)^\circ$, then $G = G^*$, otherwise G is abstractly isomorphic to $G^* \times \overline{\mathbb{Q}}_+$.*

Proof – In the first case, Proposition 9.2 gives $G' = (G^*)'$, so $G^* \geq G'$ contains $Z(G)^\circ$, and again Proposition 9.2 provides the result.

In the second case, we note that, since the ground field of G is of characteristic zero, there are finitely many elements of order n in $Z(G)$ for each $n \in \mathbb{N}$. Hence $Z(G^*)$ contains the torsion of $Z(G)$. Moreover, since the ground field of G is algebraically closed and of characteristic zero, $Z(G)^\circ$ is divisible, so $Z(G^*)^\circ$ is divisible too. Therefore $Z(G^*)^\circ = G^* \cap Z(G)^\circ$ has a complement D in $Z(G)^\circ$, and D is divisible and torsion-free since $Z(G)^\circ$ is divisible and since $Z(G^*)$ contains the torsion of $Z(G)$. Now Proposition 9.2 gives $G = G^* \times D$. Since D is divisible and torsion-free, it is a direct product of countably many copies of $\mathbb{Q}$, and $\overline{\mathbb{Q}}_+$ is isomorphic to $\overline{\mathbb{Q}}_+ \times D$. But $Z(G)^\circ$ is not contained in G', hence Lemma 8.10 shows that G is abstracly isomorphic to

$$G \times \overline{\mathbb{Q}}_+ \simeq G^* \times D \times \overline{\mathbb{Q}}_+ \simeq G^* \times \overline{\mathbb{Q}}_+,$$

and the result follows. □

Corollary 9.4. – *Let H be another connected affine algebraic group over $\overline{\mathbb{Q}}$. If the pure groups G and H are elementarily equivalent, then they are abstractly isomorphic.*

PROOF – We assume that $\mathcal{M} = (G, \cdot, ^{-1}, 1)$, and we consider the structure $\mathcal{N} = (H, \cdot, ^{-1}, 1)$. Let T be the theory of $\mathcal{M}$ and $\mathcal{N}$. By [27, Theorem 4.2.20], there is an elementary substructure $\mathcal{M}_0 = (G_0, \cdot, ^{-1}, 1)$ of $\mathcal{M}$ (resp. $\mathcal{N}_0 = (H_0, \cdot, ^{-1}, 1)$ of $\mathcal{N}$), where $\mathcal{M}_0$ (resp. $\mathcal{N}_0$) is a prime model of T. Then $\mathcal{M}_0$ and $\mathcal{N}_0$ are isomorphic by [27, Corollary 4.2.16]. Now Corollary 9.3 says that,

- either G' contains $Z(G)^\circ$, therefore H' contains $Z(H)^\circ$, and we have $G = G_0 \simeq H_0 = H$;
- or G' does not contain $Z(G)^\circ$, therefore H' does not contain $Z(H)^\circ$, and we have
$$G \simeq G^* \times \overline{\mathbb{Q}}_+ \simeq H^* \times \overline{\mathbb{Q}}_+ \simeq H.$$

□

From now on, we are ready for the proof of Theorem 1.1 and, simultaneously, for the one of Theorem 1.6 in the affine case.

PROOF OF THEOREM 1.1 IN THE GENERAL CASE, AND OF THEOREM 1.6 FOR AFFINE GROUPS – Let K be an algebraically closed field of characteristic zero, and let G be a connected affine algebraic $\overline{\mathbb{Q}}$-group. We consider an elementary substructure K_1 of the pure field K, with K_1 isomorphic to $\overline{\mathbb{Q}}$. Let G_1 be the elementary substructure of $G(K)$ in K_1. Then Lemma 8.10 and Theorem 8.14 provide the existence of a connected affine algebraic group D_1 over K_1 and of a finite central subgroup F of D_1, such that the affine algebraic groups over K_1 abstractly isomorphic to G_1 are the following ones, up to isomorphism of algebraic groups:

- $D_1/\alpha_1(F)$ if G_1' contains $Z(G_1)^\circ$;
- $D_1/\alpha_1(F) \times (K_1)_+^s$ if $Z(G_1)^\circ$ is not contained in G_1',

where α_1 is a quasi-standard automorphism of D_1, and s is an integer. We may assume that there is an integer s such that G_1 and $D_1/F \times (K_1)_+^s$ are isomorphic as algebraic groups, and we may assume that D_1 is constructed as $D(G_1)$ in §8. We consider the elementary extension $D(K)$ of D_1 to K. We note that, since the extension of F to K is F, the groups $G(K)$ and $D(K)/F \times K_+^s$ are isomorphic as algebraic groups.

Firstly, if $G(K)'$ contains $Z(G(K))^\circ$, then for each quasi-standard automorphism α of $D(K)$, the groups $D(K)/F$ and $D(K)/\alpha(F)$ are abstractly isomorphic. Secondly, if $Z(G(K))^\circ$ is not contained in $G(K)'$, we consider an integer r and a quasi-standard automorphism α of $D(K)$, and we show that $D(K)/\alpha(F) \times K_+^r$ is abstractly isomorphic to $G(K)$. Since $Z(G(K))^\circ$ is not contained in $G(K)'$, then $Z(G_1)^\circ$ is not contained in G_1', and the subgroup T_1 of D_1, corresponding to $T(G_1)$ in §8, is either $(K_1)_+$, or $(K_1^*)^k$ for a positive integer k. Hence $D(K)$ has an algebraic subgroup A such that we have either $D(K) = A \times K_+$, or $D = A \times (K^*)^k$ for a positive integer k. Since K has characteristic zero, then for each integer l, the groups K_+ (resp. $(K^*)^k$) and $K_+ \times K_+^l$ (resp. $(K^*)^k \times K_+^l$) are abstractly isomorphic. This implies that $D(K)/F$ is abstractly isomorphic to $D(K)/F \times K_+^l$ for each integer l. In particular, the groups $G(K)$, $D(K)/F$ and $D(K)/F \times K_+^r$ are abstractly isomorphic. Since $D(K)/F$ and $D(K)/\alpha(F)$ are abstractly isomorphic too, we obtain the abstract isomorphy between $G(K)$ and $D(K)/\alpha(F) \times K_+^r$.

Now we consider a connected affine algebraic $\overline{\mathbb{Q}}$-group H, and we assume that the pure groups $G(K)$ and $H(K)$ are elementarily equivalent. Let H_1 be the elementary substructure of $H(K)$ in K_1. Then the pure groups G_1 and H_1 are elementarily equivalent, and Corollary 9.4 shows that G_1 and H_1 are abstractly isomorphic. Consequently, there exists a quasi-standard automorphism α_1 of D_1 such that:

- either G_1' contains $Z(G_1)^\circ$, and H_1 and $D_1/\alpha_1(F)$ are isomorphic as algebraic groups;
- or G_1' does not contain $Z(G_1)^\circ$, and there is an integer r such that H_1 and $D_1/\alpha_1(F) \times (K_1)_+^r$ are isomorphic as algebraic groups.

Since $\alpha_1(F)$ is finite, its extension to K is $\alpha_1(F)$, and one of the following two conditions is satisfied:

- either $Z(G(K))^\circ \leq G(K)'$, and $H(K)$ and $D(K)/\alpha_1(F)$ are isomorphic as algebraic groups;
- or $Z(G(K))^\circ \not\leq G(K)'$, and $H(K)$ and $D(K)/\alpha_1(F) \times K_+^r$ are isomorphic as algebraic groups.

Moreover, since α_1 is a quasi-standard automorphism of D_1, there is two decompositions $D_1 = R_1^* \times \cdots \times R_n^*$ and $D_1 = S_1^* \times \cdots \times S_n^*$ of D_1, where $R_1^*, \ldots, R_n^*, S_1^*, \ldots, S_n^*$ are some algebraic subgroups of D_1 such that $\alpha_1(R_i^*) = S_i^*$ for each i, and such that the isomorphism $\alpha_1^i : R_i^* \to S_i^*$ induced by α_1 is standard for each i. Thus, for each $i = 1, \ldots, n$, there is a field automorphism δ_i^* of K_1 and an isomorphism $\mu_i^* : R_i^* \to S_i^*$ of algebraic groups such that $\alpha_1^i = \mu_i^* \circ \delta_i^*$. Now we consider the extensions $R_1, \ldots, R_n, S_1, \ldots, S_n$ to K of $R_1^*, \ldots, R_n^*, S_1^*, \ldots, S_n^*$ respectively, and the ones $\mu_1, \ldots, \mu_n$ of $\mu_1^*, \ldots, \mu_n^*$ respectively. Moreover, for each i, we consider an extension δ_i to K of the automorphism δ_i^* of K_1. Then, for each i, the isomorphism $\alpha^i = \mu_i \circ \delta_i$ from R_i to S_i is standard, and it satisfies $\alpha^i(x) = \alpha_1^i(x)$ for each $x \in R_i^*$. Now the automorphism α of $D(K)$ defined by $\alpha(x) = \alpha^i(x)$ for each $x \in R_i$ is quasi-standard, and it satisfies $\alpha(F) = \alpha_1(F)$. Finally, $H(K)$ has the desired form. □

10. Bi-interpretability and standard isomorphisms

From Steinberg [34, Theorem 30 p.158], it is well-known that all the automorphisms of a simple algebraic group over a perfect field are standard. From our work, we obtain an algebraic characterization of the algebraic groups over an algebraically closed field of positive characteristic all of whose abstract automorphisms are standard (Theorem 10.1 below). This is the main result of this section.

However, this characterisation does not work when the ground field is of characteristic zero but not isomorphic to $\overline{\mathbb{Q}}$, since Examples 3.1 (1) and (3) yield counterexamples. Then we provide a sufficient algebraic condition under which all the abstract automorphisms of a connected algebraic group over an algebraically closed field of characteristic zero are standard (Theorem 10.9).

Furthermore, we find model-theoretical characterizations of the algebraic groups all of whose abstract automorphisms are standard (Theorems 10.6 and 10.7).

First we give our main result, where the meaning of the condition (1) is detailed in Remark 10.2 below.

Theorem 10.1. – *Let G be a nontrivial connected algebraic group over an algebraically closed field K, such that K has no nonzero derivations (*i.e. *either its characteristic is positive, or $K \simeq \overline{\mathbb{Q}}$). Then the following conditions are equivalent:*

- *any isomorphism α from G to another algebraic group over an algebraically closed field is standard;*
- *any automorphism of G is standard;*
- (1) *(Remark 10.2) there is no nonzero homomorphism from G to $Z(G)$;*
  (2) *the group G is not central product of two proper closed subgroups U and V with $U \cap V$ finite.*

Proof – If G satisfies (1) and (2), we consider the ACF-expansion of the pure group G. Then G is definably affine by (1) and Lemma 10.5, and G is definably linear over one interpretable field by (2) and Proposition 5.11. In this case, Fact 3.6 shows that any isomorphism α from G to another algebraic group over an algebraically closed field is standard. Moreover, this implies that any automorphism of G is standard. Hence we may assume that any automorphism of G is standard, and we have just to prove that G satisfies the conditions (1) and (2). Since this follows from Lemmas 10.4 and 8.4, we obtain the result. □

Remark 10.2. – For the main results of this section, we are concerned by connected algebraic groups G over an algebraically closed field K for which there is no nonzero homomorphism from G to $Z(G)$. If p denotes the characteristic of K, by [24, §19.5], this means that,

- either $p > 0$, the center $Z(G)$ has no nontrivial torus, and either G/G' is a torus (*i.e.* G is generated by its tori), or $Z(G)$ has no nontrivial unipotent element;
- or $p = 0$ and either G is perfect (*i.e.* $G = G'$), or $Z(G)$ is finite.

10.1. **Positive characteristic and bi-interpretability.** In this part, we demonstrate strong links between bi-interpretability and standard automorphisms, since we characterize the groups all of whose abstract isomorphisms are standard by using bi-interpretability (Theorems 10.6 and 10.7).

At first, we show that we may build a nonstandard automorphism for some algebraic groups, in the same vein than Lemma 8.4 (Lemmas 10.3 and 10.4). Although these results seem natural, it is not clear that there exists direct proofs.

Lemma 10.3. – *Let G be a connected affine algebraic group over an algebraically closed field K. If G' does not contain $Z(G)^{\circ\circ}$, then G has a nonstandard automorphism α such that $G = T \times C_G(\alpha)$ for a subgroup T of $Z(G)^{\circ\circ}$.*

Proof – Let p be the characteristic of K. Let M be a maximal proper connected closed subgroup of G containing G' and not $Z(G)^{\circ\circ}$. Then there is a connected closed subgroup $A \leq Z(G)^{\circ\circ}$ of dimension one over K such that $G = AM$, and $A \cap M$ is finite. If A is radicable with torsion, let q be a prime integer with $q > p$ such that A has an element of order q, and such that q does not divide $|A \cap M|$. Let T be a subgroup of A satisfying $T \simeq \mathbb{Z}(q^\infty)$, where $\mathbb{Z}(q^\infty)$ denotes $\{x \in \mathbb{C} \mid \exists n \in \mathbb{N},\ x^{q^n} = 1\}$. If A is radicable and torsion-free, let T be a subgroup of A satisfying $T \simeq \mathbb{Q}_+$. If A is not radicable, then p is a prime, $A \simeq K_+$ has exponent p, and we consider a subgroup T of order p^2 in A satisfying $T \cap M = 1$. Then we have $T \cap M = 1$, and by using [5, Fact 1.1] when A is radicable, we find a subgroup H of G containing M such that $H \cap T = 1$. Moreover, there is an automorphism τ of T such that $C_T(\tau) = 1$.

Now we consider the automorphism α of G defined by $\alpha(ht) = h\tau(t)$ for each $h \in H$ and each $t \in T$. Then we have $G = T \times C_G(\alpha)$. We assume toward a contradiction that $\alpha = \beta \circ \mu$ for μ a field automorphism of K, and β an isogeny. Then we have $\mu(x) = \beta^{-1}(x)$ for each $x \in M$. Thus, if M is nontrivial, then μ is constructible, so either it is trivial, or p is a prime and μ is a power of the Frobenius automorphism. In particular, the condition $M \neq 1$ implies that α is constructible, so $H = C_G(\alpha)$ is definable, and $G/H \simeq T$ is definable too. But this is impossible since either T is radicable and isomorphic to $\mathbb{Q}_+$ or $\mathbb{Z}(q^\infty)$, or $|T| = p^2$ contradicting that G is connected. Hence we have $M = 1$ and $G = A$.

Thus we may assume that either $A = K_+$ or $A = K^*$. Moreover, we may assume that, if K_0 denotes the prime subfield of K, then $P := (K_0)_+$ is contained in H in the first case, and $P := K_0^*$ is contained in H in the second case. In particular, for each $x \in P$, we have $\beta(x) = \alpha(x) = x$. On the other hand, if p is a prime, we denotes by K_1 the subfield of K of order p^3, and we may assume that $Q := K_1^*$ is contained in H when $A = K^*$. We show that α is a field automorphism. If $A = K_+$, there exists $a \in K^*$ and $n \in \mathbb{N}$ such that $\beta(x) = ax^{p^n}$ for each $x \in A$, so we have $ax^{p^n} = x$ for each $x \in P$, and we obtain $a = 1$. Thus β and α are field automorphisms. If $A = K^*$ and if $p = 0$, then either $\beta(x) = x$ for each $x \in A$, or $\beta(x) = x^{-1}$ for each $x \in A$. Since $\beta(x) = x$ for each $x \in P$, we obtain $\beta(x) = x$ for each $x \in A$ and $\alpha = \mu$ is a field automorphism. If $A = K^*$ and if p is a prime, there exists $s \in \mathbb{N}$ such that, for $r = p^s$ or $r = -p^s$, we have $\beta(x) = x^r$ for each $x \in A$. Since the map $\beta^* : K \to K$ defined by $\beta^*(x) = x^{p^s}$ for each $x \in K$ is a power of the Frobenius automorphism, by considering $\beta^* \circ \mu$ instead of μ, we may assume that, either $\beta(x) = x$ for each $x \in A$, or $\beta(x) = x^{-1}$ for each $x \in A$. In the first case, $\alpha = \mu$ is a field automorphism, so we may assume $\beta(x) = x^{-1}$ for each $x \in A$. Then, for each $x \in Q$, we have $x^{-1} = \beta^{-1}(x) = \beta^{-1} \circ \alpha(x) = \mu(x)$. But either $\mu(x) = x$ for each $x \in Q$ or $\mu(x) = x^p$ for each $x \in Q$ or $\mu(x) = x^{p^2}$ for each $x \in Q$, hence we have either $x^{-1} = x$ for each $x \in Q$, or $x^p = x^{-1}$ for each $x \in Q$, or $x^{p^2} = x^{-1}$ for each $x \in Q$. This contradicts that Q has an element of order $p^3 - 1$, so α is a field automorphism.

Now $C := C_K(\alpha)$ is a subfield of K, and K is C-vector space of dimension ≥ 2. If $A = K_+$, this implies that H imbeds in $A/H \simeq T$, so K_+ imbeds in $T \times T$. This is impossible if p is a prime since T is finite, and this is impossible if $p = 0$ too because, in this case, H is a direct product of infinitely many copies of $\mathbb{Q}_+$ and T is isomorphic to $\mathbb{Q}_+$. Thus we have $A = K^*$ and $T \simeq \mathbb{Z}(q^\infty)$ for a prime $q > p$. If p is a prime, we consider the smallest finite subfield F of K with an element of order q. Let p^n be its order, let $m \in \mathbb{N}$ such that $p^m = |F \cap C|$, and let F_1 be the subfield of K of order p^{2n}. In particular, q divides $p^n - 1$, so q does not divide $p^n + 1$ since $q > p \geq 2$. Moreover, q does not divide $p^m - 1$ by minimality of F. Let $r \in \mathbb{N}$ be the largest integer such that q^r divides $p^n - 1$. Since $A = T \times H$ with $H = C_A(\alpha)$, we have $\frac{p^n-1}{p^m-1} = q^r$, and F has q^r q-elements. Since q does not divide $\frac{p^{2n}-1}{p^n-1} = p^n + 1$, all the q-elements of F_1 lies in F, and F_1 has q^r-elements. Thus, if s is the integer such that $p^s = |F_1 \cap C|$, then $q^r = \frac{p^{2n}-1}{p^s-1}$, and since $F \cap C$ is a subfield of $F_1 \cap C$, there is $k \in \mathbb{N}$ such that $s = km$. This implies that $\frac{p^{2n}-1}{p^{km}-1} = \frac{p^n-1}{p^m-1}$, hence we obtain $p^n + 1 = p^{(k-1)m} + p^{(k-2)m} + \cdots + 1$, so we find $k = 1$ and $n = m$, contradicting the choices of m and n. Finally, we have $p = 0$. In particular, $K_0^* \simeq \mathbb{Q}^*$ has an element of order 2, so C has a nontrivial 2-torus and an element i of order 4. On the other hand, we may assume $q = 3$, so H has no element of order 3. This contradicts that $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is an element of order 3 in $H = C^*$. $\square$

Lemma 10.4. – *Let G be a connected affine algebraic group over an algebraically closed field K. If there is a nonzero homomorphism from G to $Z(G)$, then G has a nonstandard automorphism.*

Proof – We assume toward a contradiction that any automorphism of G is standard. Then Lemma 10.3 shows that G' contains $Z(G)^{\circ\circ}$, so $Z(G)$ has no nontrivial torus. First we suppose that K is of positive characteristic. Then G/G' is not a torus and $Z(G)$ has a nontrivial unipotent element. Since G' contains $Z(G)^{\circ\circ}$, we find a subgroup H of index p in G and containing $G'Z(G)$. Since $Z(G)$ has a nontrivial unipotent element, there is an homomorphism $\gamma : G \to Z(G)$ with kernel H. Then the map $\alpha : G \to G$, defined by $\alpha(x) = x\gamma(x)$ for each $x \in G$, is an automorphism of G. Therefore α is standard and we have $\alpha = \beta \circ \mu$ for μ a field automorphism of K and β an isogeny. Since G' contains $Z(G)^{\circ\circ}$, the group G is not abelian and G' is infinite. But we have $\alpha(x) = x$ for each $x \in G'$, so μ is constructible. Hence α is constructible too, and $H = C_G(\alpha)$ is definable, contradicting that G is connected and that H is of index p in G. Thus we may assume that K is of characteristic zero.

From now on, G/G' is nontrivial and $Z(G)$ is infinite. Since G' contains $Z(G)^{\circ\circ}$, the groups $G/Z(G)G'$ and $Z(G)$ have subgroups $P/Z(G)G' \simeq \mathbb{Q}_+$ and $Q \simeq \mathbb{Q}_+$ respectively. Now $P/Z(G)G'$ has a complement $X/Z(G)G'$ in $G/Z(G)G'$ [5, Fact 1.1], and there is an epimorphism γ from G to Q with kernel X. Let α be the automorphism of G defined, for each $x \in G$, by $\alpha(x) = x\gamma(x)$. Therefore α is standard and we have $\alpha = \beta \circ \mu$ for μ a field automorphism of K and β an isogeny. Since G' contains $Z(G)^{\circ\circ}$, the group G is not abelian and G' is infinite. But we have $\alpha(x) = x$ for each $x \in G'$, so μ is constructible. Hence α is constructible too, and $X = C_G(\alpha)$ is definable. Now $G/X \simeq \mathbb{Q}_+$ is an abelian connected algebraic group over K. This contradicts that, since K is algebraically closed, because any infinite abelian algebraic group over K contains a copy of $\mathbb{Q}_+ \oplus \mathbb{Q}_+$. This finishes the proof. □

By Theorem 10.6, there is a strong link between biinterpretability and standard automorphisms. Its proof uses the following lemma.

Lemma 10.5. – *Let G be a nontrivial connected algebraic group over an algebraically closed field K. We assume that K has no nonzero derivations (*i.e.* either its characteristic is positive, or $K \simeq \overline{\mathbb{Q}}$). If there is no nonzero homomorphism from G to $Z(G)$, then the ACF-expansion of the pure group G is definably affine.*

Furthermore, if the characteristic of K is positive, then the pure group G is definably affine.

Proof – If $K \simeq \overline{\mathbb{Q}}$, then either G is perfect or $Z(G)$ is finite, so Theorems 5.16 and 7.13 imply that G is definably affine. Hence we may assume that the characteristic of K is positive, and we have just to prove that the pure group G is definably affine. If $Z(G)$ has a nontrivial torus, then G/G' has a nontrivial torus [24, §19.5], contradicting our hypothesis over G. But $\mathcal{S}(G)$ is a connected subgroup of a pseudo-tori of the pure group G, so $\mathcal{S}(G)$ is a radicable closed subgroup of G and it is a tori. Hence, since $\mathcal{S}(G)$ is central in G, it is trivial. In particular, this shows that, if G is generated by its tori, then G is definably affine (Proposition 7.3). Consequently, we may assume that $Z(G)$ has no nontrivial unipotent element. That is, $Z(G)$ is finite and p does not divide its order. Then Proposition 7.14 says that G is definably affine, finishing the proof. □

Theorem 10.6. – *Let G be a nontrivial connected affine algebraic group over an algebraically closed field K. Then the following conditions are equivalent:*

- *the field K and the ACF-expansion of the pure group G are biinterpretable;*
- *the algebraic group G and the ACF-expansion of the pure group G are biinterpretable;*
- *the ACF-expansion of the pure group G is definably linear over one interpretable field;*
- *any automorphism of G is standard.*

Proof – By Fact 2.6, to say that K and the ACF-expansion of the pure group G are biinterpretable is equivalent to say that, in the ACF-expansion of the pure group G, there an interpretable field L and a definable isomorphism α from G to an algebraic group over L. That is, G is definably linear over one interpretable field or, equivalently, the algebraic group G and the ACF-expansion of the pure group G are biinterpretable. In particular, by Fact 3.6, this implies that any automorphism of G is standard.

Now we assume that any automorphism of G is standard, and we consider the ACF-expansion of the pure group G. We show that G is definably linear over one interpretable field. By Proposition 5.11 and Lemma 8.4, it is sufficient to prove that G is definably affine. By Lemma 10.5, we may assume that K is of characteristic zero. Since any automorphism of G is standard, the image of any closed subgroup of G by any automorphism of G is a closed subgroup. Hence each closed subgroup of G is definable. On the other hand, Lemma 10.4 show that either G is perfect or $Z(G)$ is finite. Let T be a maximal torus of G. It is definable, and we show that it is definably affine. Indeed, let B be a Borel subgroup of G containing T, and let U be its unipotent radical. These subgroups are definable, and we have $B = U \rtimes T$ and $Z(B) = Z(G)$ [24, §19.3 and §22.2]. If $T \cap Z(B)$ is infinite, then G' is perfect, contradicting [24, §19.5]. Since each torus of $F(B)$ is central in B, this implies that $T \cap F(B)$ is finite. From now on, the intersection of the centralizers of the B-minimal sections $\overline{A}$ of U is finite. But, for each B-minimal section $\overline{A}$, either it is centralized by T, or $T/C_T(\overline{A})$ is definably linear (Fact 2.5). Hence T is definably affine (Theorem 5.16 and Lemma 5.17). Consequently, the (definable) subgroup R generated by the tori of G is definably affine (Theorem 5.9). Thus, we may assume that G is not perfect, that is $Z(G)$ is finite. Let V be the unipotent part of G. It is definable, generated by its closed subgroup of dimension one, and satisfies $G = VR$. Moreover, each closed subgroup A of dimension one in V is definable, radicable and with no nontrivial proper definable subgroup. By Lemma 7.6, either $A \simeq AC_G(R)/C_G(R)$ is definably affine, or it centralizes R. In the second case, since $Z(G)$ is finite and since $G = VR$, we have $A \not\leq Z(V)$. In particular, A is not a pseudo-torus (Fact 2.10 (iii)), so it is definably isomorphic to L_+ for an interpretable field L. Thus A is definably affine, and Theorem 5.9 says that V and $G = VR$ are definably affine too, as desired. □

When the ground field is of positive characteristic, we have a similar result by considering the pure group rather than the ACF-expansion of the pure group.

Theorem 10.7. – *Let G be a nontrivial connected affine algebraic group over an algebraically closed field K of positive characteristic. Then the following conditions are equivalent:*

- *the field K and the pure group G are biinterpretable;*

- *the algebraic group G and the pure group G are biinterpretable;*
- *the pure group G is definably linear over one interpretable field;*
- *any automorphism of G is standard.*

Proof – We prove the equivalence between the three first assertions as in the proof of Theorem 10.6. Now, we consider the pure group G, and we show that, if any automorphism of G is standard, then G is definably linear over one interpretable field. We notice that Lemmas 10.4 and 8.4 say that,

(1) there is no nonzero homomorphism from G to $Z(G)$
(2) G is not a central product $G = U \cdot V$ of two proper closed subgroups U and V with $U \cap V$ finite.

In particular, by the second condition and Proposition 5.11, it is sufficient to prove that G is definably affine. Hence the first condition and Lemma 10.5 yields the result. □

Remark 10.8. – We notice that the previous result fails when the ground field is of characteristic zero.

Indeed, if we consider the field $K = \overline{\mathbb{Q}}$ and

$$G = \left\{ \begin{pmatrix} t & a & u \\ 0 & t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in K^*,\ (a,u,v) \in K^3 \right\},$$

then all the automorphisms of G are standard by Lemma 3.10. However, the field K and the pure group G are not bi-interpretable (see Example 3.1 (2)).

10.2. **Characteristic zero.** Unlike the positive characteristic, we fail to find in zero characteristic an algebraic characterization of the algebraic groups all of whose all the abstract automorphisms are standard. However, the main result of this section provides algebraic conditions under which any automorphism of a connected algebraic group, defined over an algebraically closed field of characteristic zero, is standard.

Theorem 10.9. – *Let G be a connected algebraic group over an algebraically closed field K of characteristic zero. Then all of its automorphisms are standard if it satisfies the following three conditions:*

(1) *either G is perfect or $Z(G)$ is finite;*
(2) *G is not a central product of two proper closed subgroups U and V with $U \cap V$ finite;*
(3) *for each characteristic abelian connected closed subgroup A of G, and each maximal torus T of G, the centralizer $C_A(T)$ is central in G,*

Furthermore, under these conditions, the algebraic group G and the pure group G are biinterpretable.

Its proof requires the following results.

Lemma 10.10. – *Let G be a connected ACF_0-group. If G is perfect or if $Z(G)$ is finite, then $F(G)^\circ$ is a torsion-free $\widetilde{U}$-group.*

Proof – We fix an algebraically closed field K of characteristic zero interpreting G, and we consider G as an algebraic group over K. Since either G is perfect or $Z(G)$ is finite, the algebraic group G is affine. Moreover, we note that $F(G)^\circ$ is unipotent, otherwise the maximal torus T of $F(G)^\circ$ would be nontrivial, and since

$G' \cap T$ is finite [24, §19.5], the center $Z(G)$ would be finite by our hypothesis, contradicting [24, Corollary 16.3]. This implies that $F(G)^\circ$ is torsion-free, and that $F(G)^\circ$ is the unipotent radical of G.

Let B be a Borel subgroup of G. In particular B contains $F(G)^\circ$. Then $F(B)/Z(B)$ is a $\widetilde{U}$-group by Fact 2.21, and since $Z(B) = Z(G)$ by [24, Corollary 22.2.B], the quotient $F(G)^\circ/(F(G)^\circ \cap Z(G))$ is a $\widetilde{U}$-group (Fact 2.19). In particular, since $F(G)^\circ$ is torsion-free, then if $Z(G)$ is finite, the subgroup $F(G)^\circ$ is a $\widetilde{U}$-group. Hence we may assume that G is perfect.

Let R be a maximal reductive subgroup of G. Then we have $G = F(G)^\circ \rtimes R$, so $G' = [F(G)^\circ, G] \rtimes R'$, and since G is perfect, we find $F(G)^\circ = [F(G)^\circ, G]$. Now Fact 2.22 says that $F(G)^\circ$ is a $\widetilde{U}$-group. □

Proposition 10.11. – *Let G be a connected ACF_0-group, and let K be an algebraically closed field of characteristic zero interpreting G. We assume that either G is perfect or $Z(G)$ is finite. If the maximal tori of G, viewed as an algebraic group over K, are definable in the ACF-group G, then G is definably affine.*

Proof – Let T be a maximal torus of the algebraic group G over K, and let $U = F(G)^\circ$. Then U is torsion-free by Lemma 10.10. In particular, it is a unipotent subgroup of the algebraic group G, so $T \cap U$ is trivial. By Corollary 7.10, the quotient G/U is definably affine. Therefore T is definably affine, and the subgroup $\mathscr{T}$ of G generated by the tori of G is definably affine too (Theorem 5.9). Hence we may assume that $\mathscr{T}$ is proper in G. In particular, G is not perfect since $G/\mathscr{T}$ is a unipotent group, so $Z(G)$ is finite.

Moreover $G/C_G(\mathscr{T})$ is definably linear by Lemma 7.6, so $G/C_U(\mathscr{T})$ is definably affine by Lemma 5.17, and we may assume that $C_U(\mathscr{T})$ is nontrivial. In particular, since $C_U(\mathscr{T})$ is a normal subgroup of the nilpotent group U, the group $C_{Z(U)}(\mathscr{T})$ is nontrivial. But $G/\mathscr{T}$ is a unipotent group, so it is covered by the unipotent radical of G, and since this one is contained in $U = F(G)^\circ$, we find $G = U\mathscr{T}$. Hence $C_{Z(U)}(\mathscr{T})$ is central in G, contradicting that U is torsion-free and $Z(G)$ is finite. □

Proof of Theorem 10.9 – We assume toward a contradiction that G is a countexample to Theorem 10.9. By Fact 3.6, the pure group G is not definably linear over an interpretable field. By (2) and Proposition 5.11, the pure group G is not definably affine. Now $G/Z(G)$ is not definably affine by (1), and by using Theorem 5.16 if $Z(G)$ and Lemma 7.12 if G is perfect.

Let T_0 be the torsion subgroup of a maximal torus T of G, and let $D = d(T_0)$ be the definable hull of T_0 in the pure group G. We note that T is the Zariski closure of T_0, so D contains T, and since $T_0 \leq T$ is abelian, D is abelian too. Moreover, since $G/Z(G)$ is not definably affine, we have $TZ(G) < DZ(G)$ by Proposition 10.11.

Let $U = F(G)^\circ$. It is a torsion-free $\widetilde{U}$-group by Lemma 10.10, and G/U is definably affine by Corollary 7.10. In particular, U is a unipotent group, and U contains $W(G)$ (Corollary 5.18). Since $TW(G)/W(G)$ is definable by Corollary 5.12, then $TW(G) = DW(G)$, and since $TZ(G) < DZ(G)$, the intersection $D \cap W(G)$ is noncentral in G.

Let $E/(D \cap Z(G) \cap W(G))$ be a D-minimal subgroup of $(D \cap W(G))/(D \cap Z(G) \cap W(G))$. Since U is a $\widetilde{U}$-group, $E/(D \cap Z(G) \cap W(G))$ is definably isomorphic to L_+ for an algebraically closed field L. If T_1 is a maximal radicable abelian torsion subgroup of G, then T_1 is formed by semisimple elements, and its Zariski closure

$\overline{T_1}$ is a torus. Actually, by maximality of T_1, the torus $\overline{T_1}$ is maximal, and T_1 is a the torsion subgroup of $\overline{T_1}$. Hence T_1 is conjugate with T_0, so $d(T_1)$ is conjugate with D as well, and since $Z(G) \cap W(G)$ is a characteristic in G, we conclude that $D \cap Z(G) \cap W(G)$ is a characteristic subgroup of G. Let A be the subgroup of $W(G)$ generated by all the images of E by the automorphisms of the pure group G. Since E is definable and connected in the pure group G, then A is definable and connected in the pure group G too by the Zil'ber Indecomposability Theorem ([28, Theorem 2.9] or [5, Theorem 5.26]). Moreover, A is a characteristic subgroup of G, and $A/(D \cap Z(G) \cap W(G))$ is definably affine over L by Theorem 5.9. Consequently $W(G)$ centralizes $A/(D \cap Z(G) \cap W(G))$ (Lemma 7.6), and $A/(D \cap Z(G) \cap W(G))$ is abelian. At this stage, if $Z(G)$ is finite, then $D \cap Z(G) \cap W(G)$ is a finite subgroup of the torsion-free subgroup U, so A is abelian and $C_A(T)$ is central in G by (3), contradicting that $E \leq D \leq C_A(T)$ is noncentral in G. Thus by (1), the group G is perfect.

Since $\overline{A} = A/(D \cap Z(G) \cap W(G))$ is definably affine over L, it is definably linear over L (Theorem 5.16), and since it is torsion free and abelian, it is definably isomorphic isomorphic to an L-vector space. Then we find vector subspaces $\overline{A_1}, \ldots, \overline{A_k}$ of $\overline{A}$ of dimension one normalized by T and such that $\overline{A}$ is the direct sum of $\overline{A_1}, \ldots, \overline{A_k}$. Since $\overline{A}$ is definably linear over L, these vector subspaces are definable. Moreover, since $W(G)$ centralizes $\overline{A}$ and since $TW(G) = DW(G)$, they are normalized by D as well. If T centralizes each subspace $\overline{A_i}$, then T centralizes $\overline{A}$, and since T is a torus and $A \leq U$ is unipotent, the torus T centralizes A. But G is perfect, so it is generated by its maximal tori, which are conjugate in G, so G centralizes A, contradicting $E \not\leq Z(G)$. Hence T does not centralizes all the subpaces $\overline{A_i}$. For each i, we denote by A_i the preimage of $\overline{A_i}$ in A. We fix $j \in \{1, \ldots, k\}$ such that T does not centralizes $\overline{A_j}$. Since $A_j \leq U$ is torsion-free, and since the dimension of $A_j/Z(A_j)$ is at most one, then A_j is abelian. Now, since D is abelian, the group A_jD is solvable of class two. But the codimension of D in A_jD is one since the dimension of $\overline{A_j} = A_j/(D \cap Z(G) \cap W(G))$ is one, so since D does not centralizes $\overline{A_j}$, it is a Carter subgroup of A_jD. By Fact 6.1 (iν), we find a definable connected characteristic abelian subgroup B_j of A_jD such that $A_jD = B_j \rtimes D$. Since D covers $A_jD/(A_jD)'$ (Fact 6.1 (iii)), the subgroup B_j is contained in $(A_jD)' \leq A_j$ and it is torsion-free. Moreover, since the codimension of D in A_jD is one, the Zariski dimension of B_j is one, and since D does not cover $\overline{A_j}$, the subgroup B_j covers $\overline{A_j}$. This implies that B_j is definably isomorphic to $L_+ \leq \overline{A_j}$. Now, if B denotes the largest connected subgroup of A definably linear over L (Theorem 5.9), then B covers $\overline{A_j}$, and B is central in $A \leq W(G)$ (Lemma 7.6). In particular, B is a characteristic abelian connected closed subgroup of G, and by (3), the centralizer $C_B(T)$ is central in G, so B does not contain E.

By the choice of B, the torus T centralizes all the section $\overline{A_i}$ not covered by B, so T centralizes A/B. Hence, since G is generated by its maximal tori, the section A/B is centralized by G, and G normalizes $(D \cap A)B \geq EB > B$. Since the maximal radicable abelian torsion subgroups of G are conjugates, and since T_0 is one of them with $D = d(T_0)$, then the subgroup $(D \cap A)B$ is characteristic in G, and since it contains E, we find $A = (D \cap A)B$. Thus, since B is central in A and since D is abelian, the group A is abelian, and we have our final contradiction with (3) since $C_A(T)$ contains E. □

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Laboratoire de Mathématiques et Applications,
UMR 7348 du CNRS, Université de Poitiers, Téléport 2 - BP 30179, Bd Marie et Pierre Curie, 86962 Futuroscope Chasseneuil Cedex, France
E-mail address: olivier.frecon@math.univ-poitiers.fr