HOMOGENEITY IN RELATIVELY FREE GROUPS

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ABSTRACT. We prove that any torsion-free, residually finite relatively free group of infinite rank is not $\aleph_1$-homogeneous. This generalizes R. Sklinos' result that a free group of infinite rank is not $\aleph_1$-homogeneous, and, in particular, gives a new simple proof of that result.

1. Introduction

A. Ould Houcine [6] and independently C. Perin and R. Sklinos [9] proved that any free group of finite rank is strongly homogeneous. Also, it is shown in [9] that any free group is strongly $\aleph_0$-homogeneous. R. Sklinos [12] showed that any free group of uncountable rank is not $\aleph_1$-homogeneous. His proof was based on the deep Z. Sela's result [11] on stability of the theory of free groups and used some sophisticated technique of model-theoretic stability theory.

The aim of this note is to give a simple direct proof of a more general result: any torsion-free, residually finite relatively free group F of infinite rank is not $\aleph_1$-homogeneous.

As a by-product of the proof we show that F has an isomorphic elementary substructure G such that no basis of G can be extended to a basis of F. This generalizes R. Sklinos' result [12] who showed that a free group of countable rank has an elementary substructure which is not a free factor. Note that it is well-known and easy to see that in a relatively free group any infinite subset of a basis always generates an elementary substructure. A motivation for his result was C. Perin's theorem [8] that in a non-cyclic free group of finite rank any elementary substructure is a non-cyclic free factor. Note that a non-cyclic free factor of any free group is always an elementary substructure — this is a hard fundamental result due to Z. Sela [10] and O. Kharlampovich–A. Myasnikov [4].

2. Preliminaries

We recall some well-known definitions and facts.

A partial map f from $\mathcal{M}$ to $\mathcal{M}$ is called *elementary* in a structure $\mathcal{M}$ if $\phi(\bar{a})$ holds in $\mathcal{M}$ iff $\phi(f(\bar{a}))$ holds in $\mathcal{M}$, for any first order formula $\phi(\bar{v})$ and tuple $\bar{a}$ in dom(f). Obviously, any restriction of any automorphism is an elementary map. For a cardinal κ, a structure is called *strongly κ-homogeneous*

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if any elementary map f with $|\mathrm{dom}(f)| < \kappa$ extends to an automorphism. A structure is called *κ-homogeneous* if, for any of its elements a, any elementary map f with $|\mathrm{dom}(f)| < \kappa$ extends to an elementary map h with $a \in \mathrm{dom}(h)$. Clearly, strong κ-homogeneity implies κ-homogeneity; in general, the converse fails. However, it is known that if $\kappa = |M|$ then $\mathcal{M}$ is κ-homogeneous iff $\mathcal{M}$ is strongly κ-homogeneous. A structure $\mathcal{M}$ is called homogeneous if it is κ-homogeneous for $\kappa = |M|$.

Let $\mathcal{V}$ be a group variety. A subset X of a group G is called *$\mathcal{V}$-independent* if, for any group word $w(u_1, \ldots, u_n)$ and different $x_1, \ldots, x_n \in X$, the equality $w(x_1, \ldots, x_n) = 1$ holds in G only if $w(u_1, \ldots, u_n) \equiv 1$ is an identity in $\mathcal{V}$. For a $\mathcal{V}$-group F, a generating $\mathcal{V}$-independent subset of F is called a *$\mathcal{V}$-basis* of F. A $\mathcal{V}$-group having a $\mathcal{V}$-basis is called $\mathcal{V}$-free. It is easy to see that a subset X of a $\mathcal{V}$-free group F is a $\mathcal{V}$-basis iff X generates F, and for any $\mathcal{V}$-group H any map from X to H extends to a homomorphism from G to H. It is known that any two $\mathcal{V}$-bases of a $\mathcal{V}$-free group are of the same cardinality; that cardinality is called the *rank* of the $\mathcal{V}$-free group. For any cardinal κ there is a unique, up to isomorphism, $\mathcal{V}$-free group of rank κ. The groups that are $\mathcal{V}$-free for some variety $\mathcal{V}$ are called *relatively free.*

For a subset Y of a group, we denote by $\langle Y \rangle$ and $\langle\langle Y \rangle\rangle$ the subgroup and normal subgroup of the group generated by Y, respectively.

Here are some known simple properties of $\mathcal{V}$-bases we will need.

(1) If $Y \sqcup Z$ is $\mathcal{V}$-independent then $\langle Y \rangle \cap \langle Z \rangle = \{1\}$.
(2) If $Y \sqcup Z$ is a $\mathcal{V}$-basis of F, and $\alpha \in \mathrm{Aut}\langle Y \rangle$, $\beta \in \mathrm{Aut}\langle Z \rangle$, then $\alpha \cup \beta$ extends to an automorphism of F.
(3) If $Y \sqcup Z$ is a $\mathcal{V}$-basis of F, and $K = \langle\langle Z \rangle\rangle$ then $\bar{F} = F/K$ is $\mathcal{V}$-free with $\mathcal{V}$-basis $\{\bar{a} : a \in Y\}$, where $\bar{a} = aK$.
(4) Let X be a $\mathcal{V}$-basis of F, and $N = \langle\langle R \rangle\rangle$, where R is a set of group words over X. Then, for any map γ from X to a $\mathcal{V}$-group G such that $w(\gamma(x_1), \ldots, \gamma(x_n)) = 1$ in G for all $w(x_1, \ldots, x_n) \in R$, there is a homomorphism $\rho : F/N \to G$ such that $\rho(xN) = \gamma(x)$ for $x \in X$.

For basics of model theory, see [2]. The facts on group varieties we need can be found in [5]. For the results on abelian groups we will use, see [1].

3. Non-$\aleph_1$-homogeneity of relatively free groups

Our goal is to prove the following

Theorem 1. *Let $\mathcal{V}$ be a group variety of exponent 0 such that all $\mathcal{V}$-free groups are residually finite. Then any $\mathcal{V}$-free group of infinite rank is not $\aleph_1$-homogeneous.*

Remark 1. Examples of such $\mathcal{V}$ are

- the variety of all groups,
- the variety of all abelian groups,
- the variety of all solvable groups of a given class,
- the variety of all poly-nilpotent groups of a given class,

- any nilpotent group variety of exponent 0;

see [5]. I do not know whether relatively free groups of infinite rank are not $\aleph_1$-homogeneous for an arbitrary variety of exponent 0. Note that there exist varieties of exponent 0 not satisfying the assumption of the theorem; in fact, there exists a variety of exponent 0 in which all relatively free groups of rank more than one are non-hopfian [3].

Remark 2. The assumption that $\mathcal{V}$ is of exponent 0 is essential: *In the variety $\mathcal{V}$ of abelian groups of exponent $n > 1$ all $\mathcal{V}$-free groups are saturated* and, in particular, homogeneous. Indeed, if $n = q_1 \dots q_s$, where $q_1, \dots, q_s$ are powers of different primes, then any $\mathcal{V}$-free group F of infinite rank κ is of the form $A_1 \oplus \dots \oplus A_s$, where A_i is a direct sum of κ cyclic groups of order q_i. Such groups F are known to be saturated.

Remark 3. The assumption of infinity of rank in the theorem is essential: *any free abelian group F of finite rank is strongly κ-homogeneous for any κ.* Indeed, let f be an elementary map in F. Let A and B be the pure subgroups of F generated by the domain and range of f, respectively. Since F is a free abelian group of finite rank, A and B are free abelian groups of finite rank. The map f extends to an elementary isomorphism h between A and B. It is known that a pure subgroup H of an abelian group G is a direct summand in G if G/H is finitely generated. Therefore A, B have direct complements A', B' in F; clearly,

$$\mathrm{rk}(A') = \mathrm{rk}(F) - \mathrm{rk}(A) = \mathrm{rk}(F) - \mathrm{rk}(B) = \mathrm{rk}(B').$$

Hence $A' \simeq B'$. Therefore h extends to an automorphism of F.

An absolutely free group F of finite rank is strongly κ-homogeneous for any κ as well. Indeed, let f be an elementary map in F. Then f extends to an elementary isomorphism h between the subgroups acl(dom(f)) and acl(range(f)). A. Ould Houcine and D. Vallino [7] proved that the subgroup $\mathrm{acl}(C)$ is finitely generated for any subset C of F. Since F is strongly homogeneous [6],[9], the map h extends to an automorphism of F.

Remark 4. *Free abelian groups of infinite rank are strongly $\aleph_0$-homogeneous*, even though they are not $\aleph_1$-homogeneous by the theorem. Indeed, let F be a free abelian group of infinite rank, and f an elementary map in F with a finite domain. Let A and B be the pure subgroups of F generated by the domain and range of f, respectively; they are free abelian groups of finite rank. Clearly, A and B are contained in a finitely generated direct summand C of F. The map f extends to an elementary isomorphism h between A and B. As above, A and B are direct summands of C, and so of F; therefore h extends to an automorphism of F.

Remark 5. *For an arbitrary variety $\mathcal{V}$, any $\mathcal{V}$-free group F of infinite rank κ is not κ^+-homogeneous.* Indeed, let X be a $\mathcal{V}$-basis of F, and $e \in X$, and $f : X \setminus \{e\} \to X$ a bijection. The map f is elementary in F because the restriction of f on any finite set extends to a permutation of X and so to an automorphism of F. However, f cannot be extended to an elementary map h defined on e. Suppose not. Since X generates F, there are a group word $w(u_1, \dots, u_n)$ and $e_1, \dots, e_n \in X$ such that $h(e) = w(e_1, \dots, e_n)$. Let $e_i = f(c_i)$, where $c_i \in X \setminus \{e\}$. Then $e_i = h(c_i)$. So $h(e) = w(h(c_1), \dots, h(c_n))$, and hence $e = w(c_1, \dots, c_n)$, contrary to $e \notin \langle X \setminus \{e\} \rangle$ which holds by property (1) in Preliminaries.

Now we pass to a proof of Theorem 1.

Proof. Let F be a $\mathcal{V}$-free group of infinite rank. Let X be a $\mathcal{V}$-free basis of F, and $e_1, e_2, \dots$ be a sequence of distinct elements of X. Consider the map

$$f : \{e_1, e_2, \dots\} \to F, \quad f(e_i) = e_i e_{i+1}^{-(i+1)}.$$

To prove that F is not $\aleph_1$-homogeneous, it suffices to show that the map f is elementary but cannot be extended to an elementary map h such that $e_1 \in \operatorname{range}(h)$.

First we show that f is elementary. For $n \geq 1$ denote

$$Y_n = \{e_1, \dots, e_n, e_{n+1}\}, \quad G_n = \langle Y_n \rangle, \quad H_n = \langle f(e_1), \dots, f(e_n), e_{n+1} \rangle.$$

Clearly, H_n is contained in G_n and contains $e_{n+1}, e_n, \dots, e_1$; hence $H_n = G_n$. Then there exists an epimorphism $\alpha_n : G_n \to G_n$ such that $\alpha_n(e_{n+1}) = e_{n+1}$ and $\alpha_n(e_i) = f(e_i)$ for $1 \leq i \leq n$. Since G_n is $\mathcal{V}$-free, it is residually finite. By classical Maltsev's theorem, any finitely generated residually finite group is hopfian, that is, every surjective endomorphism of it is an automorphism. Thus G_n is hopfian, and so α_n is an automorphism of G_n. Let β_n be the identity automorphism of $\langle X \setminus Y_n \rangle$. By property (2) in Preliminaries, $\alpha_n \cup \beta_n$ extends to an automorphism of F, and so is elementary. Let $q : X \to F$ be the map that extends f and is the identity on $X \setminus \operatorname{dom}(f)$. We show that q is elementary; in particular, f is elementary. Any finite subset S of X is contained in the union of $\{e_1, \dots, e_n\}$ and $X \setminus \operatorname{dom}(f)$, for some n. Since q and $\alpha_n \cup \beta_n$ coincide on S, the restriction of q on S is elementary. Therefore q is elementary.

Towards a contradiction, suppose that f extends to an elementary map h such that $e_1 \in \operatorname{range}(h)$, and $e_1 = h(a)$, where $a \in F$.

Let $N = \langle\langle R \rangle\rangle$, where

$$R = \operatorname{range}(q) = \{e_i e_{i+1}^{-(i+1)} : i = 1, 2, \dots\} \cup (X \setminus \operatorname{dom}(f)).$$

We show that $e_1 \notin N$. For $g \in F$ denote $\bar{g} = gN$. Then $\bar{e}_i = \bar{e}_{i+1}^{i+1}$ in F/N, for all i. Since $\mathcal{V}$ is of zero exponent, it contains a nontrivial divisible group D because the infinite cyclic groups are in $\mathcal{V}$ and, being the union of an increasing chain of cyclic subgroups, the additive group of rationals is in $\mathcal{V}$. Choose in D elements $a_1, a_2, \dots$ such that $a_1 \neq 1$ and $a_i = a_{i+1}^{i+1}$ for all i. By the property (4) in Preliminaries, there exists a homomorphism $\rho : F/N \to D$ such that $\rho(\bar{e}_i) = a_i$ for all i. Since $\rho(\bar{e}_1) = a_1 \neq 1$, we have $\bar{e}_1 \neq \bar{1}$ and so $e_1 \notin N$. In particular, $e_1 \notin \langle\langle f(e_i) : 1 \leq i < \omega \rangle\rangle$.

Let Γ be the set of formulas of the form

$$(\exists v_1 \dots v_k) u = (v_1^{-1} u_1^{\epsilon_1} v_1) \dots (v_k^{-1} u_k^{\epsilon_k} v_k),$$

where all ϵ_i are ± 1. For any subset C of F and $g \in F$, we have $g \in \langle\langle C \rangle\rangle$ iff $F \models \phi(g, c_1, \dots, c_k)$ for some $\phi(u, u_1, \dots, u_k) \in \Gamma$ and $c_1, \dots, c_k \in C$.

Applying this to $C = \{f(e_i) : 1 \leq i < \omega\}$, we have that for all $i_1, \dots, i_k$ and all $\phi(u, u_1, \dots, u_k) \in \Gamma$

$$F \models \neg\phi(e_1, f(e_{i_1}), \dots, f(e_{i_k})),$$

that is,

$$F \models \neg\phi(h(a), h(e_{i_1}), \dots, h(e_{i_k})),$$

that is,

$$F \models \neg\phi(a, e_{i_1}, \dots, e_{i_k}).$$

Hence $a \notin K = \langle\langle e_i : 1 \leq i < \omega \rangle\rangle$.

For any n the element e_1 satisfies in F the formula

$$\exists v_2 \dots v_{n+1} \bigwedge_{1 \leq i \leq n} v_i = f(e_i) v_{i+1}^{i+1}$$

with the free variable v_1; the elements $e_2, \dots, e_{n+1}$ witness that. Since h is an elementary map, $e_1 = h(a)$, and $f(e_i) = h(e_i)$ for all i, the element a satisfies in F the formula

$$\exists v_2 \dots v_{n+1} \bigwedge_{1 \leq i \leq n} v_i = e_i v_{i+1}^{i+1}.$$

Then in $\bar{F} = F/K$ the element $\bar{a} = aK$ is nontrivial and satisfies the formula

$$\theta_n(v_1) := \exists v_2 \dots v_{n+1} \bigwedge_{1 \leq i \leq n} v_i = v_{i+1}^{i+1}$$

for any $n \geq 1$. By the property (3) in Preliminaries, the group $\bar{F}$ is $\mathcal{V}$-free, and hence residually finite. Then there is a homomorphism τ from $\bar{F}$ to a finite group B such that $\tau(\bar{a}) \neq 1$. Clearly, the element $\tau(\bar{a})$ satisfies in B the formula $\theta_n(v_1)$ for every $n \geq 1$. Let $n = |B| - 1$; then $b^{n+1} = 1$ for every $b \in B$. Since

$$\tau(\bar{a}) = b_2^2, \qquad b_2 = b_3^3, \quad \dots \quad b_n = b_{n+1}^{n+1},$$

for some $b_2, \dots, b_{n+1} \in B$, we have $\tau(\bar{a}) = 1$ in B. Contradiction. □

Theorem 2. *Let $\mathcal{V}$ be a group variety of exponent 0 such that all $\mathcal{V}$-free groups are residually finite. Then any $\mathcal{V}$-free group F of infinite rank has an isomorphic elementary substructure G such that no $\mathcal{V}$-basis of G can be extended to a $\mathcal{V}$-basis of F.*

Proof. The elementary map $q : X \to F$ from the proof of Theorem 1 uniquely extends to an elementary isomorphism between the groups F and $G = \langle \text{range}(q) \rangle$. Then $G \prec F$. Towards a contradiction, suppose some $\mathcal{V}$-basis Z of G extends to a $\mathcal{V}$-basis $Y \sqcup Z$ of F. We have

$$N = \langle\langle \text{range}(q) \rangle\rangle = \langle\langle G \rangle\rangle = \langle\langle Z \rangle\rangle.$$

Therefore, by (3) in Preliminaries,

$$F/N \simeq F/\langle\langle Z\rangle\rangle \simeq \langle\langle Y\rangle\rangle,$$

and so the group F/N is $\mathcal{V}$-free, and therefore residually finite. As $\bar{e}_1 \neq \bar{1}$ and $\bar{e}_i = \bar{e}_{i+1}^{i+1}$ in F/N for all i, the subgroup generated by $\bar{e}_1, \bar{e}_2, \ldots$ is nontrivial, abelian, and divisible. But a residually finite group cannot have a nontrivial divisible subgroup because a nontrivial finite group cannot be divisible. Contradiction. □

Remark 6. As we mentioned in the introduction, for an absolutely free group F of finite rank the assertion of the theorem fails [8]. For free abelian groups of finite rank the theorem fails as well but for a different reason: *a free abelian group F of finite rank has no proper elementary substructures.* Indeed, if $G \prec F$ then G is a proper pure subgroup with finitely generated F/G, and hence G is a direct factor of F. Then G is a free abelian subgroup of smaller rank, and hence $G/2G$ and $F/2F$ are of different finite orders, which is impossible because $G \equiv F$.

Remark 7. The assumption that $\mathcal{V}$ is of exponent 0 is essential. For example, if $\mathcal{V}$ is the variety of abelian groups of prime exponent p then Theorem 2 fails because any basis of a subspace of a vector space over $\mathbb{F}_p$ extends to a basis of the space.

In fact, Theorem 2 fails for any variety of abelian groups of prime power exponent because the following holds. *Let $\mathcal{V}$ be the variety of abelian groups of prime power exponent q, and F, G be $\mathcal{V}$-free groups. If G is a pure subgroup of F then every $\mathcal{V}$-basis of G extends to a $\mathcal{V}$-basis of F.* Indeed, the $\mathcal{V}$-groups are exactly the direct sums of cyclic groups of exponent q, and the number of cyclic factors of each type is an isomorphism invariant of the group. A group is $\mathcal{V}$-free iff it is a direct sum of cyclic groups of order q. Since G is a pure subgroup of F, and F/G is a direct sum of cyclic groups, G has a direct complement H in F. In any decomposition of H into a direct sum of cyclic groups of exponent q all summands are of order q; otherwise F admits a decomposition into a direct sum of cyclic subgroups in which not all summands are of order q. Thus H is $\mathcal{V}$-free. If Y, Z are $\mathcal{V}$-bases of G, H then $Y \sqcup Z$ is a $\mathcal{V}$-basis of F.

Note that *Theorem 2 still holds for the variety $\mathcal{V}$ of abelian groups of exponent $n > 1$ if n is not a prime power.* Let $n = q_1 \ldots q_s$, where q_i are powers of different primes, $s > 1$. Any $\mathcal{V}$-free group F of infinite rank κ is of the form $A_1 \oplus A_2 \oplus \cdots \oplus A_s$, where

$$A_i = \bigoplus_{\gamma<\kappa} \langle a_{i,\gamma}\rangle,$$

and each $a_{i,\gamma}$ is an element of order q_i. Let $G = A_1^- \oplus A_2 \oplus \ldots A_s$, where

$$A_1^- = \bigoplus_{0<\gamma<\kappa} \langle a_{1,\gamma}\rangle.$$

It is easy to see that G is an isomorphic elementary substructure of F. If some $\mathcal{V}$-basis of G extended to a $\mathcal{V}$-basis of F then, by (3), F/G would be $\mathcal{V}$-free. But F/G is a cyclic group of order q_1, which is not $\mathcal{V}$-free.

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