Model theoretic aspects of the Ellis semigroup

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Abstract

Let G be a group definable in a theory T. For every model M of T, the space $S_G(M)$ of the complete G-types over M is a G^M-flow. We compare the Ellis semigroups related to the flows $S_G(M)$ and $S_G(N)$ when $M \prec^* N$, focusing particularly on the groups into which the minimal left ideals in these semigroups split. In the case where T is an o-minimal expansion of the theory of reals and G is definably compact we show that these groups are isomorphic to the quotient group G/G^{00}.

Introduction

Let T be a complete theory in a countable language L and we work within a monster model $\mathfrak{C}$ of T. Assume the language L contains a predicate symbol and function symbols defining in $\mathfrak{C}$ a group G and the operations of group multiplication and group inverse in G. For $M \prec \mathfrak{C}$ let G^M be $G \cap M$, to simplify notation we assume that $G^M = M$. Given a (partial) type (or a formula) $p(x)$ over $\mathfrak{C}$, $[p(x)]$ denotes the class of all types over $\mathfrak{C}$, in variable x, containing $p(x)$.

Stable group theory is the central part of geometric model theory. Unfortunately, outside the stable context the main notions of stable model theory, like generic types and forking, do not work well. In [N1, N2] it was pointed out how some notions of topological dynamics may successfully serve as a counterpart and generalization of the notion of generic type in the general unstable case.

In topological dynamics, given an abstract group G and a point-transitive G-flow X, the topological dynamics of X is explained by the properties of the enveloping Ellis semigroup $E(X)$. Of particular importance are minimal left ideals I of $E(X)$. Each such ideal splits into a disjoint union of groups. All these groups are isomorphic, also for distinct minimal ideals in $E(X)$.

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Returning to the model-theoretic set-up, we consider the space $S(M)$ as a G^M-flow, the action of G^M on $S(M)$ is induced by left translation. In [N1] we considered also a larger G^M-flow $S_M(\mathfrak{C})$, consisting of the types in $S(\mathfrak{C})$ finitely satisfiable in M. We proved that $S_M(\mathfrak{C})$ is isomorphic (as a G^M-flow) to its Ellis semigroup, which induces a semi-group operation on $S_M(\mathfrak{C})$. As mentioned in the previous paragraph, we find a family of isomorphic disjoint subgroups H^M of $S_M(\mathfrak{C})$ such that every minimal left ideal in $S_M(\mathfrak{C})$ splits into a disjoint union of these groups . We proved [N1] that the group G/G^{00}_M is a homomorphic image of each of the groups H^M. This provides a new outlook on the group G/G^{00}_M. We conjectured that the groups H^M are closely related to the group G/G^{00}_M, in nice situations (like NIP) even isomorphic to it.

The model-theoretic set-up of topological dynamics raised also some specific questions. Namely, assume N is an elementary extension of M. So besides the G^M-flow $S(M)$ and the related objects $S_M(\mathfrak{C})$ and H^M we have also the G^N-flow $S(N)$ and the related objects $S_N(\mathfrak{C})$ and H^N. Model theoretically, N is in many ways similar to M. In fact, the properties common to all models of T are considered inherent to T. It is natural to compare the topological dynamics of $S(M)$ and $S(N)$. This comparison may point to those aspects of the flow $S(M)$ that are inherent to T and not just to M.

G^{00}_N is a subgroup of G^{00}_M, hence G/G^{00}_M is a homomorphic image of G/G^{00}_N. Because of our conjecture relating H^M and G/G^{00}_M we would expect that the groups H^M and H^N are related, too. In fact, in [N1] we proved that $S_M(\mathfrak{C})$ embeds as a sub-semigroup into $S_N(\mathfrak{C})$, however unfortunately the images of the groups H^M under this embedding need not be the groups H^N. Instead, we would rather expect the groups H^M to be homomorphic images of the groups H^N (since G/G^{00}_M is a homomorphic image of G/G^{00}_N).

The main result of this paper states that for $M \prec^* N$, H^M is a homomorphic image of a subgroup of H^N, assuming additionally that generic points exist in $S_N(\mathfrak{C})$. This assumption holds e.g. in the case where T is an o-minimal expansion of the theory of the field of reals and G is definably compact. Admittedly, in this case we can prove that the groups H^M are downright isomorphic to the groups H^N and to the group G/G^{00}, partially confirming our conjecture from [N1].

In the proof of the main result of this paper we use a Boolean interpretation of the semigroup $S_M(\mathfrak{C})$. This enables us to understand better the groups H^M and find a well-behaved subgroup of $S_N(\mathfrak{C})$, mapping homomorphically onto H^M. Then, under the additional assumption of existence of generic points in $S_N(\mathfrak{C})$, we present H^M as a homomorphic image of a subgroup of H^N.

The paper is organized as follows . Below we recall the basic notions of topological dynamics relevant to our results. In Section 1 we set up the model-theoretic context and interpret the Ellis semigroup of the flow $S_M(\mathfrak{C})$ as the semigroup od automorphisms of certain algebra of subsets of M. In Sections 2 and 3 we construct a subgroup of $S_N(\mathfrak{C})$ with homomorphic image G^M. Section 4 contains the main result of the paper, also we discuss there the case of a definably compact G.

Given an abstract group G, in topological dynamics by a G-flow we mean a compact topological space X acted upon by G by homeomorphisms. By a subflow of X we mean any closed subspace Y of X, closed under the action of G. In particular, for any $p \in X$ the set $cl(Gp)$ is a subflow of X. We say that a G-flow is minimal if it has no proper subflows. We call a point $p \in X$ almost periodic if the flow $cl(Gp)$ is minimal. We say that X is point-transitive if it contains a dense G-orbit. There is a largest point-transitive G-flow: the space βG of ultrafilters on G, acted upon by G by left translation. Namely, for $\mathcal{U} \in \beta G$ and $g \in G$ we put $g\mathcal{U} = \{gU : U \in \mathcal{U}\}$.

The properties of a point-transitive G-flow X are expressed by means of its Ellis semigroup $E(X)$. Namely, every $g \in G$ determines the homeomorphism $\pi_g : X \to X$. Let $E(X) = cl\{\pi_g : g \in G\}$ be the topological closure of the set $\{\pi_g : g \in G\}$ in the space X^X considered with the Tychonov product topology (i.e. the topology of pointwise convergence). $E(X)$ is a semigroup with respect to composition of functions and the semigroup operation is continuous in the first coordinate. It is also a point-transitive G-flow: for $g \in G$ and $f \in E(X)$ let $g * f = \pi_g \circ f$; the functions $\pi_g, g \in G$, form a dense G-orbit. Also, $E(E(X))$ is isomorphic to $E(X)$, as a semigroup and a G-flow.

We say that $I \subseteq E(X)$ is a (left) ideal (symbolically: $I \lhd E(X)$) if $fI \subseteq I$ for every $f \in E(X)$. Of particular importance are minimal ideals $I \lhd E(X)$ (denoted by $I \lhd_m E(X)$), since they are exactly the minimal subflows of $E(X)$. Moreover, they determine the minimal subflows of X as follows. Let $I \lhd_m E(X)$. Then the minimal subflows of X are of the form $Ip, p \in X$. Also, $p \in X$ is almost periodic iff $p \in Ip$.

We may consider a more general situation of a compact topological space S carrying a semigroup operation that is continuous in the first coordinate (just like $E(X)$). In this case we use the analogous notation $I \lhd S, I \lhd_m S$ to denote (left) ideals and minimal (left) ideals I in S. We say that $j \in S$ is an idempotent if $j^2 = j$. We will use the following fundamental observation of Ellis.

Theorem 0.1 ([E]) *Assume S is a compact topological space carrying a semigroup operation that is continuous in the first coordinate.*

(1) The set J of idempotents of S is non-empty.

(2) Given $I \lhd_m S$, the set $J(I) = J \cap I$ is non-empty.

(3) For every $I \lhd S$ and $j \in J(I)$, the set jI is a subgroup of I and I is the disjoint union of the groups $jI, j \in J(I)$.

(4) The groups $jI, I \lhd_M S, j \in J(I)$, are all isomorphic.

In particular, Theorem 0.1 applies to $E(X)$.

βG is the largest point-transitive G-flow, meaning that any point-transitive G-flow is an image of βG via a G-mapping (that is, a mapping respecting the action of G). Also, βG is naturally isomorphic to its Ellis semigroup $E(\beta G)$. Namely, for $\mathcal{U}, \mathcal{V} \in \beta G$ let

$$\mathcal{U} * \mathcal{V} = \{U \subseteq G : \{g \in G : g^{-1}U \in \mathcal{V}\} \in \mathcal{U}\}$$

and let $l_{\mathcal{U}} : \beta G \to \beta G$ be given by $l_{\mathcal{U}}(\mathcal{V}) = \mathcal{U} * \mathcal{V}$. Then $*$ is a semigroup operation on βG, $E(\beta G) = \{l_{\mathcal{U}} : \mathcal{U} \in \beta G\}$ and the mapping $\mathcal{U} \mapsto l_{\mathcal{U}}$ is an isomorphism of βG and $E(\beta G)$, both as G-flows and semigroups. For more background on topological dynamics and the Ellis semigroup the reader may consult [E, A, G].

1 The model-theoretic set-up

In the model-theoretic setting in this paper we consider a G^M-flow $S(M)$. In [N1] we considered the G^M-flow $S_M(\mathfrak{C})$ consisting of the types in $S(\mathfrak{C})$ finitely satisfiable in M. This flow turned out to be isomorphic to its Ellis semigroup and served as a model-theoretic counterpart of βG^M. Here we will re-define $S_M(\mathfrak{C})$ by means of externally definable subsets of M.

We say that $U \subseteq M$ is externally definable if $U = \varphi(\mathfrak{C}, \bar{a}) \cap M$ for some formula $\varphi(x, \bar{y})$ of L and some $\bar{a} \subseteq \mathfrak{C}$. In this case we write $U \subseteq_{ext} M$. Let $Def_{ext}(M) = \{U : U \subseteq_{ext} M\}$. This is a Boolean algebra of sets. For every $U \subseteq_{ext} M$ and $g \in M$ we have that gU is also externally definable in M. Hence $Def_{ext}(M)$ is a G^M-algebra of sets (meaning it is closed under the left translation by elements of G^M).

Let $S_{ext}(M) = S(Def_{ext}(M))$ be the Stone space of ultrafilters in $Def_{ext}(M)$. The sets $[U] = \{p \in S_{ext}(M) : U \in p\}, U \subseteq_{ext} M$, form a basis of the topology in $S_{ext}(M)$. Also $S_{ext}(M)$ is a point-transitive G^M-flow.

Lemma 1.1 *$S_{ext}(M)$ and $S_M(\mathfrak{C})$ are isomorphic as G^M-flows.*

Proof. For $\mathcal{U} \in \beta G^M$ and $A \subseteq \mathfrak{C}$ containing M let

$$p^A_{\mathcal{U}} = \{\varphi(x, \bar{a}) : \varphi(x, \bar{y}) \in L, \bar{a} \subseteq A \text{ and } \varphi(\mathfrak{C}, \bar{a}) \cap M \in \mathcal{U}\}.$$

So $p^A_{\mathcal{U}} \in S(A)$. Also, $p^A_{\mathcal{U}}$ depends only on $\mathcal{U}' = \mathcal{U} \cap Def_{ext}(M)$ and $\mathcal{U}' \in S_{ext}(M)$, so we may write $p^A_{\mathcal{U}'}$ in place of $p^A_{\mathcal{U}}$. Clearly, the mapping $\mathcal{U}' \mapsto p^{\mathfrak{C}}_{\mathcal{U}'}$ is an isomorphism of G^M-flows $S_{ext}(M)$ and $S_M(\mathfrak{C})$. $\square$

In [N1] we defined a semi-group operation $*$ on $S_M(\mathfrak{C})$ by: $p^{\mathfrak{C}}_{\mathcal{U}} * p^{\mathfrak{C}}_{\mathcal{V}} = p^{\mathfrak{C}}_{\mathcal{U}*\mathcal{V}}$. Then $E(S_M(\mathfrak{C})) = \{f_{\mathcal{U}} : \mathcal{U} \in \beta G^M\}$, where $f_{\mathcal{U}}(p^{\mathfrak{C}}_{\mathcal{V}}) = p^{\mathfrak{C}}_{\mathcal{U}} * p^{\mathfrak{C}}_{\mathcal{V}}$. The mapping $p^{\mathfrak{C}}_{\mathcal{U}} \mapsto f_{\mathcal{U}}$ is an isomorphism of G^M-flows and semi-groups $S_M(\mathfrak{C})$ and $E(S_M(\mathfrak{C}))$ (it is a counterpart of the isomorphism between βG and $E(\beta G)$ mentioned above).

The semi-group operation on $S_M(\mathfrak{C})$ induces a semi-group operation on $S_{ext}(M)$, via the isomorphism from Lemma 1.1. We will give an explicit definition of it. We will do it via a Boolean interpretation of $S_{ext}(M)$ as the semi-group of G^M-endomorphisms of $Def_{ext}(M)$. In the case of βG this was already done by Ellis [E, page 74], here we adapt his construction to the model-theoretic set-up. We use this Boolean interpretation to describe better the ideals and groups in $S_{ext}(M)$.

For $p \in S_{ext}(M)$ and $U \subseteq_{ext} M$ let

$$d_p U = \{g \in M : U \in gp\} = \{g \in M : g^{-1}U \in p\}.$$

Lemma 1.2 *For $U \subseteq_{ext} M$ we have that $d_p U \subseteq_{ext} M$.*

Proof. Assume $U = \varphi(\mathfrak{C}, \bar{a}) \cap M$ for some suitable φ and $\bar{a}$. Let $p' \in S_M(\mathfrak{C})$ be the type corresponding to p via the isomorphism from Lemma 1.1. Let b realize p' (in some elementary extension $\mathfrak{C}'$ of $\mathfrak{C}$). For $g \in M$ we have:

$$g \in d_p U \iff U \in gp \iff \varphi(x, \bar{a}) \in gp'$$

$$\iff \mathfrak{C}' \models \varphi(gb, \bar{a}) \iff g \in \varphi'(\mathfrak{C}', \bar{a}, b) \cap M,$$

where $\varphi'(x, \bar{y}, z) = \varphi(xz, \bar{y})$. So $d_p U = \varphi'(\mathfrak{C}', \bar{a}, b) \cap M$ is externally definable in M. $\square$

Lemma 1.3 *$d_p : Def_{ext}(M) \to Def_{ext}(M)$ is a homomorphism of G^M-algebras of sets.*

Proof. It is obvious that d_p preserves the Boolean operations. We check that it preserves the left translation by elements of G^M. So let $g, h \in G^M$ and $U \subseteq_{ext} M$. we have:

$$g \in d_p(hU) \iff hU \in gp \iff U \in h^{-1}gp \iff h^{-1}g \in d_p U \iff g \in h d_p U.$$

$\square$

Now we define the operation $*$ on $S_{ext}(M)$ by: $U \in p * q \iff d_q U \in p$. Also, for $p, q \in S_{ext}(M)$ we define the functions $l_p, r_q : S_{ext}(M) \to S_{ext}(M)$ by $l_p(q) = r_q(p) = p * q$.

Lemma 1.4 *Let $p, q, r \in S_{ext}(M)$.*
(1) $d_{q*r} = d_q \circ d_r$.
(2) $*$ is associative, i.e. $l_{p*q} = l_p \circ l_q$.

Proof. Let $g \in M$ and $U \subseteq_{ext} M$. We have:

$$g \in d_{q*r}(U) \iff g^{-1}U \in q * r \iff d_r(g^{-1}U) \in q$$

$$\iff g^{-1} d_r U \in q \iff g \in d_q(d_r U),$$

so (1) follows. Also,

$$U \in (p * q) * r \iff d_r U \in p * q \iff d_q(d_r U) \in p$$

$$\iff d_{q*r}(U) \in p \iff U \in p * (q * r),$$

so (2) follows. $\square$

We see that $*$ is a semigroup operation on $S_{ext}(M)$. We can consider $g \in G^M$ as a principal ultrafilter in $S_{ext}(M)$. Then for $p \in S_{ext}(M)$ we have $gp = g * p$, i.e. $l_g : S_{ext}(M) \to S_{ext}(M)$ is the homeomorphism given by the usual action of G^M on $S_{ext}(M)$. We know that $E(S_{ext}(M)) = cl(\{l_g : g \in G^M\})$.

Lemma 1.5 *(1) For $p \in S_{ext}(M)$ we have that $l_p = \lim_p l_g$, the limit of the functions l_g over the ultrafilter p in the pointwise convergence topology in the space of functions $S_{ext}(M) \to S_{ext}(M)$.*
(2) $E(S_{ext}(M)) = \{l_p : p \in S_{ext}(M)\}$. Also, the mapping $p \mapsto l_p$ is an isomorphism of the semigroups $S_{ext}(M)$ and $E(S_{ext}(M))$.
(3) The operation $*$ in $S_{ext}(M)$ is continuous in the first coordinate. In fact, for $q \in S_{ext}(M)$ and $U \subseteq_{ext} M$ we have that $r_q^{-1}([U]) = [d_q U]$.

Proof. (1) Let $q \in S_{ext}(M)$ and $U \subseteq_{ext} M$. We have:

$$U \in l_p(q) \iff d_q U \in p \iff \{g \in G^M : U \in l_g(q)\} \in p.$$

So $U \in l_p(q)$ iff for some $V \in p$ we have that $U \in l_g(q)$ for every $g \in V$. This means that $l_p = \lim_p l_g$.

(2) follows from (1) and Lemma 1.4. Also, the mapping $p \mapsto l_p$ is $1-1$, since $p = p * e$, where e is the identity element of G^M.

(3) follows from the definitions. $\square$

The isomorphism between $S_{ext}(M)$ and $E(S_{ext}(M))$ described in Lemma 1.5 agrees with the isomorphism between $S_M(\mathfrak{C})$ and $E(S_M(\mathfrak{C}))$ described in [N1], via the isomorphism from Lemma 1.1.

The next proposition was known to Ellis in the case of βG^M [E, page 74]. Let $\mathcal{E}_G(M) = \mathrm{End}_G(Def_{ext}(M))$ be the semi-group of endomorphisms of the G^M-algebra of sets $Def_{ext}(M)$ (the semi-group operation being the composition of functions). We define the function $d : S_{ext}(M) \to \mathcal{E}_G(M)$ by $d(p) = d_p$.

Proposition 1.6 *d is an isomorphism of semi-groups.*

Proof. By Lemma 1.4(1), d is a homomorphism. To see that d is $1-1$, consider $p \neq q \in S_{ext}(M)$. Choose $U \in p$ with $U^c \in q$. Then $e \in d_p U$ and $e \notin d_q U$, so $d_p U \neq d_q U$.

To see that d is onto, consider any $f \in \mathcal{E}_G(M)$. Let $p = \{U \subseteq_{ext} (M) : e \in f(U)\}$. Clearly, $p \in S_{ext}(M)$. Also, for every $U \subseteq M$ and $g \in M$ we have:

$$g \in d_p U \iff g^{-1}U \in p \iff e \in f(g^{-1}U) \iff e \in g^{-1}f(U) \iff g \in f(U),$$

hence $d_p U = f(U)$ and $f = d_p$. $\square$

The identification of $S_{ext}(M)$ and $E(S_{ext}(M))$ with $\mathcal{E}_G(M)$ via the function d enables us to understand better the decomposition of the minimal ideals in $S_{ext}(M)$ into disjoint unions of isomorphic groups. Given a $p \in S_{ext}(M)$ we consider

$$\mathrm{Ker}(d_p) = \{U \subseteq_{ext} (M) : d_p U = \emptyset\} \text{ and } \mathrm{Im}(d_p) = \{d_p U : U \subseteq_{ext} (M)\}.$$

$\mathrm{Ker}(d_p)$ is a G^M-ideal in $Def_{ext}(M)$, meaning that it is an ideal in the Boolean algebra $Def_{ext}(M)$, closed under multiplication by elements of G^M: if $U \in \mathrm{Ker}(d_p)$ and $h \in G^M$, then $hU \in \mathrm{Ker}(d_p)$. $\mathrm{Im}(d_p)$ is a G^M-subalgebra of $Def_{ext}(M)$. These objects are crucial to our understanding of the groups in $\mathcal{E}_G(M)$. The identity elements of these groups are idempotents. The next lemma explains their nature.

Lemma 1.7 *Assume $u \in \mathcal{E}_G(M)$. The following are equivalent.*

(1) u is an idempotent.

(2) For every $U \subseteq_{ext} (M)$ we have that $U \triangle u(U) \in Ker(u)$ and $Ker(u) \cap Im(u) = \{\emptyset\}$.

Moreover, if u is an idempotent then $u \restriction_{Im(u)} = id_{Im(u)}$ and $Im(u)$ is a section of the family of cosets of $Ker(u)$ in $Def_{ext}(M)$, whence $Def_{ext}(M)/Ker(u) \cong Im(u)$.

The next lemma explains the nature of groups contained in $\mathcal{E}_G(M)$.

Lemma 1.8 *Assume $H \subseteq \mathcal{E}_G(M)$ is a group.*

(1) There is a G^M-ideal $K \subseteq Def_{ext}(M)$, a common kernel of all $f \in H$.

(2) There is a G^M-subalgebra $R \subseteq Def_{ext}(M)$, a common image of all $f \in H$.

(3) $K \cap R = \{\emptyset\}$, R is a section of the family of K-cosets in $Def_{ext}(M)$, $Def_{ext}(M)/K \cong R$ and for every $f \in H$ we have that $f \restriction_R$ is a G^M-automorphism of R.

(4) The mapping $f \mapsto f \restriction_R$ is an embedding of H into the group $Aut(R) \cong Aut(Def_{ext}(M)/K)$.

Proof. The starting point is an easy observation that for $f, g \in \mathcal{E}_G(M)$, $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(fg)$ and $\mathrm{Im}(f) \subseteq \mathrm{Im}(fg)$.

Let $f, g \in H$ be arbitrary. Choose $h_0, h_1 \in H$ with $f = h_0 g = g h_1$. So $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$ and $\mathrm{Im}(g) \subseteq \mathrm{Im}(f)$. Since f, g are arbitrary, we get that $\mathrm{Ker}(f) = \mathrm{Ker}(g)$ and $\mathrm{Im}(f) = \mathrm{Im}(g)$. This proves (1) and (2).

Let $K = \mathrm{Ker}(f), R = \mathrm{Im}(f)$. (3) follows from (1),(2) and Lemma 1.7, since $K = \mathrm{Ker}(e)$ and $R = \mathrm{Im}(e)$ for the identity element e of H. The rest is easy. $\Box$

The next lemma shows that (left) ideals in $S_{ext}(M)$ correspond (via d) to some G^M-ideals in $Def_{ext}(M)$.

Lemma 1.9 *Let $p, q \in S_{ext}(M)$.*

(1) $cl(Gp) \subseteq cl(Gq) \iff Ker(d_p) \supseteq Ker(d_q)$.

(2) $cl(Gp) = cl(Gq) \iff Ker(d_p) = Ker(d_q)$.

(3) p is almost periodic iff $Ker(d_p)$ is maximal among the ideals in $Def_{ext}(M)$ of the form $Ker(f), f \in \mathcal{E}_G(M)$.

Proof. Let $U \subseteq_{ext} M$. The following are equivalent:

(a) $U \in \mathrm{Ker}(d_p)$.

(b) $gp \notin [U]$ for every $g \in G^M$.

(c) $gp \in [U^c]$ for every $g \in G^M$.

(d) $Gp \subseteq [U^c]$.

So the lemma follows. $\Box$

Corollary 1.10 *Let $I \lhd_m S_{ext}(M)$. There is a common kernel K od all $d_p, p \in I$. Also, $I = \{p \in S_{ext}(M) : Ker(d_p) = K\}$.*

For $I \lhd_m S_{ext}(M)$ let K_I be the common kernel of $d_p, p \in I$ and $\mathcal{R}_I = \{\mathrm{Im}(d_p) : p \in I\}$. Let $J(I) = \{p \in I : p^2 = p\}$. By Theorem 0.1 we know that I is the disjoint union of the groups $uI, u \in J(I)$. Each such group uI is determined by K_I (the common kernel) and $R = \mathrm{Im}(d_u)$ (the common image of $d_p, p \in uI$). u is the identity element of uI.

Remark 1.11 *For every $R_1, R_2 \in \mathcal{R}_I$, if $R_1 \subseteq R_2$, then $R_1 = R_2$.*

Proof. By Lemma 1.8(3), both R_1 and R_2 are sections of the cosets of K_I in $Def_{ext}(M)$. $\square$

The next lemma shows that the family $\mathcal{R}_I$ od subalgebras of $Def_{ext}(M)$ does not depend on the choice of $I \lhd_m S_{ext}(M)$. Henceafter we denote $\mathcal{R}_I$ by $\mathcal{R}$.

Lemma 1.12 *Assume $I_1, I_2 \lhd_m S_{ext}(M)$. Then $\mathcal{R}_{I_1} = \mathcal{R}_{I_2}$.*

Proof. Let $p_1 \in I_1, p_2 \in I_2$. So $I_2 p_1 = I_1$, whence for every $q \in I_2$ we have that $r := qp_1 \in I_1$, i.e. $\mathrm{Im}(d_q) \supseteq \mathrm{Im}(d_r)$ for some $r \in I_2$. It follows that

($*$) for every $R_2 \in \mathcal{R}_{I_2}$ there is an $R_1 \in \mathcal{R}_{I_1}$ with $R_1 \subseteq R_2$.

Symmetrically,

($**$) for every $R_1 \in \mathcal{R}_{I_1}$ there is an $R_2 \in \mathcal{R}_{I_2}$ with $R_2 \subseteq R_1$.

Choose any $R_2 \in \mathcal{R}_{I_2}$ and let R_1 be any element of $\mathcal{R}_{I_1}$ contained in R_2 (as in ($*$)). By ($**$), choose $R_2' \in \mathcal{R}_{I_2}$ contained in R_1. So $R_2' \subseteq R_1 \subseteq R_2$ hence $R_2' \subseteq R_2$. By Remark 1.11 we get $R_2 = R_2' = R_1$ belongs to $\mathcal{R}_{I_1}$. It follows that $\mathcal{R}_{I_2} \subseteq \mathcal{R}_{I_1}$ and (symmetrically) $\mathcal{R}_{I_1} \subseteq \mathcal{R}_{I_2}$, hence $\mathcal{R}_{I_1} = \mathcal{R}_{I_2}$. $\square$

By Theorem 0.1, the groups $uI, I \lhd_m S_{ext}(M), u \in J(I)$, are isomorphic. The identification of $S_{ext}(M)$ with $\mathcal{E}_G(M)$ clarifies this fact. Below we give an alternative proof of it.

First consider a fixed ideal $I \lhd_m S_{ext}(M)$. Let $K = K_I$. By Lemma 1.8, every $d_p, p \in I$, induces an automorphism $\widetilde{d}_p$ of $Def_{ext}(M)/K$. Hence we get a function $\widetilde{d} : I \mapsto \mathrm{Aut}(Def_{ext}(M)/K)$, mapping p to $\widetilde{d}_p$. Let $\mathcal{H}_I = \widetilde{d}[I]$.

Proposition 1.13 *(1) $\widetilde{d}$ is a $*$-homomorphism, i.e. $\widetilde{d}_{p*q} = \widetilde{d}_p \circ \widetilde{d}_q$ for all $p, q \in I$.*

(2) $\mathcal{H}_I$ is a subgroup of $Aut(Def_{ext}(M)/K)$ and for every $u \in J(I)$, $\widetilde{d}$ is a group isomorphism of uI and $\mathcal{H}_I$. In particular, the groups $uI, u \in J(I)$, are all isomorphic.

Proof. (1) follows from Lemma 1.4. In particular, $\mathcal{H}_I$ is closed under composition.

(2) Let $p \in I$. Since $\widetilde{d}$ is a $*$-homomorphism, the function $l_p : I \to I$ induces (via $\widetilde{d}$) the function $\mathcal{H}_I \to \mathcal{H}_I$ of left translation by $\widetilde{d}_p$, so that the following diagram commutes.

$$\begin{tikzcd} I \arrow[r, "l_p"] \arrow[d, "\widetilde{d}"'] & I \arrow[d, "\widetilde{d}"] \\ \mathcal{H}_I \arrow[r, "\widetilde{d}_p \cdot"] & \mathcal{H}_I \end{tikzcd}$$

When $p = u \in J(I)$ is an idempotent, $\widetilde{d}_u$ is the identity, hence $\widetilde{d}$ commutes with $l_u : I \to I$.

Let $u_1, u_2 \in J(I)$. We have that the function $l_{u_2} : u_1 I \to u_2 I$ commutes with $\widetilde{d}$. It follows that $\widetilde{d}[u_1 I] \subseteq \widetilde{d}[u_2 I]$, and by symmetry we get $\widetilde{d}[u_1 I] = \widetilde{d}[u_2 I]$.

Since I is the union of the groups $uI, u \in J(I)$, we get that $\mathcal{H}_I = \widetilde{d}[uI]$ for every $u \in J(I)$ and it is a group. By Lemma 1.9, $\widetilde{d}$, restricted to each $uI, u \in J(I)$, is a monomorphism, hence it is an isomorphism of the groups uI and $\mathcal{H}_I$. We see also that $l_{u_2} : u_1 I \to u_2 I$ is an isomorphism of groups. $\square$

Now fix an $R \in \mathcal{R}$. For every $I \lhd_m S_{ext}(M)$ choose $u_I \in J(I)$ with $R = \mathrm{Im}(d_{u_I})$. By Lemma 1.8, every $d_p, p \in u_I I$, induces an isomorphism d^*_p of R. Hence we get a function $d^* : \bigcup\{u_I I : I \lhd_m S_{ext}(M)\} \to \mathrm{Aut}(R)$, mapping p to d^*_p. Let $\mathcal{H}_R$ be the range of d^*.

Proposition 1.14 *(1) d^* is a $*$-homomorphism, i.e. $d^*_{p*q} = d^*_p \circ d^*_q$.*

(2) $\mathcal{H}_R$ is a subgroup of $Aut(R)$ and for every $I \lhd_m S_{ext}(M)$, d^* is a group isomorphism of $u_I I$ and $\mathcal{H}_R$. In particular, the groups $u_I I, I \lhd_m S_{ext}(M)$, are all isomorphic. Actually, given $I_1, I_2 \lhd_m S_{ext}(M)$, the function $r_{u_{I_1}}$ is an isomorphism of the groups $u_{I_2} I_2$ and $u_{I_1} I_1$.

Proof. Similar as in Proposition 1.14, only we consider right translation instead of left translation. $\square$

2 Transfer between models: weak heirs

In the next section we will compare the Ellis semigroups $S_{ext}(M)$ for various models $M \prec \mathfrak{C}$, focusing on the groups $uI, I \lhd_m S_{ext}(M), u \in J(I)$. Since the definition of $S_{ext}(M)$ involves externally definable subsets of M, we will be considering pairs of structures $M \prec N$, where the algebras $Def_{ext}(M)$ and $Def_{ext}(N)$ are related in a prescribed way. We explain this now.

Consider a model $M \prec \mathfrak{C}$. For every $U \subseteq_{ext} M$ let P_U be a new relation symbol. Let $L_{ext,M} = L \cup \{P_U : U \subseteq_{ext} M\}$. Let M_{ext} be the expansion of M to an L_{ext}-structure, where P_U is interpreted as U. For simplicity we sometimes write U in place of P^M_U and P_U. Let N^0 be an elementary extension of M_{ext}. So $N := N \restriction_L$ is an elementary extension of M and we may assume $N \prec \mathfrak{C}$. For every $U \subseteq_{ext} M$ let $U^N = P_U(N^0)$.

Lemma 2.1 *For every $U \subseteq_{ext} M$, the set U^N is externally definable in N.*

Proof. Say, $U = \varphi(\mathfrak{C}, \bar{a}) \cap M$ for some L-formula $\varphi(x, \bar{y})$ and $\bar{a} \subseteq \mathfrak{C}$. This means that for every n, in M_{ext} the following sentence holds:

$$(\forall x_1, \ldots, x_n)\,(\forall x'_1, \ldots, x'_n)\left(\bigwedge_{1\le i\le n} (P_U(x_i) \wedge \neg P_U(x'_i)) \to \exists \bar{y} \bigwedge_{1\le i\le n} (\varphi(x_i, \bar{y}) \wedge \neg\varphi(x'_i, \bar{y}))\right).$$

Since $M_{ext} \prec N^0$, this sentence is true also in N^0. This means that the following set of formulas:

$$\{\varphi(b, \bar{y}) : b \in U^N\} \cup \{\neg\varphi(b, \bar{y}) : b \in N \setminus U^N\}$$

is finitely satisfiable in N. Hence this set has a realization $\bar{a}'$ in $\mathfrak{C}$. Clearly, $U^N = \varphi(\mathfrak{C}, \bar{a}') \cap N$. $\square$

Let N_{ext} be the expansion of N to an $L_{ext,N}$-structure, where for $U \subseteq_{ext} M$ we identify P_{U^N} with P_U. We express the relationship between M_{ext} and N_{ext} described above, writing $M \prec^* N$. So, for arbitrary $M \prec N \prec \mathfrak{C}$, $M \prec^* N$ means the new relational symbols of $L_{ext,M}$ are identified with some new relational symbols of $L_{ext,N}$ so that $M_{ext} \prec N_{ext} \restriction_{L_{ext,M}}$. In this situation, for $U \subseteq_{ext} M$, U^N denotes $P_U(N_{ext})$. For the rest of this section we fix two models $M \prec^* N \prec \mathfrak{C}$. We will compare the semigroups $S_{ext}(M)$ and $S_{ext}(N)$.

The mapping $U \mapsto U^N$ for $U \subseteq_{ext} M$ defines an embedding $Def_{ext}(M) \to Def_{ext}(N)$. Henceforth we consider $Def_{ext}(M)$ as a subalgebra of $Def_{ext}(N)$. We have that $Def_{ext}(M)$ is a G^M-algebra of sets and $S_{ext}(M)$ is a G^M-flow, while $Def_{ext}(N)$ is a G^N-algebra and $S_{ext}(N)$ is a G^N-flow.

We can consider any $p \in S_{ext}(M)$ as a complete quantifier-free $L_{ext,M}$-type over M_{ext}. The same applies also to any $q \in S_{ext}(N)$. We have a natural restriction function $r : S_{ext}(N) \to S_{ext}(M)$, mapping a qf-type $q \in S_{ext}(N)$ to is $L_{ext,M}$-restriction to M. Formally, r maps any ultrafilter $q \in S_{ext}(N) = S(Def_{ext}(N))$ to $q \cap Def_{ext}(M)$, an element of $S_{ext}(M)$.

For $p \in S_{ext}(M)$ and $q \in S_{ext}(N)$ we write $p \subseteq q$ if $r(q) = p$. In this situation we say that q extends p and p is a restriction of q. Sometimes we denote $r(q)$ by $q \restriction_M$.

Let $p \in S_{ext}(M)$. p is *definable*, meaning that for every $U \subseteq_{ext} M$, the set $d_pU = \{g \in M : g^{-1}U \in p\}$ is still externally definable in M (see Lemma 1.2). Let $Def^0_{ext}(N)$ be the G^N-subalgebra of $Def_{ext}(N)$ generated by $Def_{ext}(M)$ (regarded as a subalgebra of $Def_{ext}(N)$) and let $S^0_{ext}(N)$ be $S(Def^0_{ext}(N))$.

Lemma 2.2 *Let $p \in S_{ext}(M)$. The family of sets*

$$\Phi = \{g^{-1}U^N : g \in G^N, U \subseteq_{ext} M \text{ and } g \in (d_pU)^N\}$$

generates an ultrafilter in $Def^0_{ext}(N)$.

Proof. First we prove that Φ has the finite intersection property. So let $U_1, \ldots, U_n \subseteq_{ext} M$, $g_i \in (d_pU_i)^N$ for $i = 1, \ldots, n$ and we want to show that

$$(*) \quad \bigcap_{i=1}^{n} g_i^{-1} U_i^N \neq \emptyset.$$

However, for any $g_i' \in d_pU_i$ $(i = 1, \ldots, n)$ we have that $\bigcup_{i=1}^{n} (g_i')^{-1} U_i$ belongs to p, hence is non-empty. So the following sentence is true in M_{ext}:

$$(\forall x_1, \ldots, x_n) \left(\bigwedge_{1 \leq i \leq n} x_i \in d_pU_i \to \exists x \bigwedge_{1 \leq i \leq n} x \in x_i^{-1} U_i \right).$$

Since $M \prec^* N$, this sentence holds also in N_{ext}, hence we have $(*)$.

Notice that for every $g \in N$ and $U \subseteq_{ext} M$, either $g \in (d_pU)^N$ or $g \in (d_pU^c)^N$, where $U^c = M \setminus U$ (since $M \prec^* N$). Also $(U^c)^N = N \setminus U^N$. If $g \in (d_pU)^N$, then $g^{-1}U^N \in \Phi$. If $g \notin (d_pU)^N$, then $g \in (d_pU^c)^N$, hence $N \setminus g^{-1}U^N = g^{-1}(U^c)^N \in \Phi$.

We see that for every $g \in N$ and $U \subseteq_{ext} M$, either $g^{-1}U^N \in \Phi$ or $N \setminus g^{-1}U^N \in \Phi$. Since the sets $g^{-1}U^N$, $U \subseteq_{ext} M, g \in N$, generate $Def^0_{ext}(N)$, we see that Φ generates an ultrafilter in $Def^0_{ext}(N)$. $\square$

Given $p \in S_{ext}(M)$ let p_N denote the ultrafilter in $Def^0_{ext}(N)$ generated by Φ from Lemma 2.2. Even though M_{ext}, N_{ext} are structures in different languages $L_{ext,M} \subseteq L_{ext,N}$, the notion of heir extension from $S_{ext}(M)$ to $S_{ext}(N)$ still makes sense. Namely, assume $p \in S_{ext}(M), q \in S_{ext}(N)$ and $p \subseteq q$. We regard p, q as complete qf-types over M, N in languages $L_{ext,M}, L_{ext,N}$ respectively. We say that q is an *heir* of p (or just: an heir over M) if for every quantifier-free $L_{ext,M}$-formula $\varphi(x, \bar{y}, \bar{z})$ and $\bar{m} \subseteq M, \bar{n} \subseteq N$ we have that if $\varphi(x, \bar{m}, \bar{n}) \in q$, then $\varphi(x, \bar{m}, \bar{n}') \in p$ for some $\bar{n}' \subseteq M$. The usual argument shows that every $p \in S_{ext}(M)$ has an heir in $S_{ext}(N)$. Also we say that q is a *weak heir* of p (or just: a weak heir over M) if for every $U \subseteq_{ext} M$ we have that $d_qU^N = (d_pU)^N$.

Remark 2.3 *(1) q is a weak heir of p iff $p_N \subseteq q$.*

(2) If q is an heir of p, then q is a weak heir of p.

Proof. (1) is obvious. (2) Suppose q is an heir and not a weak heir of p. Hence for some $U \subseteq_{ext} M$ and $g \in N$ we have that $g^{-1}U^N \in q$ and $g \notin (d_pU)^N$. Let $V = d_pU$ and $\varphi(x, y) = P_U(y^{-1} \cdot x) \wedge \neg P_V(x)$. So $\varphi(x, y)$ is a quantifier-free $L_{ext,M}$-formula and $\varphi(x, g) \in q$. Since q is an heir of p, for some $g' \in M$ we have $\varphi(x, g') \in p$, meaning that $(g')^{-1}U \in p$ and $g' \notin d_pU$, a contradiction. $\square$

Heirs were used in [N1, Proposition 2.3] to prove that almost periodic types in $S(M)$ have almost periodic extensions in $S(N)$. In the proof in [N1] we start from an heir $q_0 \in S(N)$ of an almost periodic type $p \in S(M)$ and then shift q_0 a bit to get an almost periodic type $q \in S(N)$ extending p. We can essentially repeat this argument for types in $S_{ext}(M)$ and $S_{ext}(N)$, even though M_{ext}, N_{ext} are structures in different languages. In [N1] we asked if the shift from q_0 to q is needed, i.e. if p has an heir $q_0 \in S(N)$ that is already almost periodic (without a need for a further shift). This may be false, but may be more plausible if we require that an almost periodic type $q_0 \in S(N)$ is just a weak heir of p rather than an heir. Later in this paper we shall see that for example in the case of the circle group S^1 the shift is needed even then, answering negatively our question from [N1]. Anyway, the next lemma shows that weak heirs are still strong enough to yield almost periodic extensions.

Lemma 2.4 *(1) Assume $q \in S_{ext}(N)$ is a weak heir of $p \in S_{ext}(M)$. Then $r[cl(G^Nq)] \subseteq cl(G^Mp)$. If moreover p is almost periodic, then $r[cl(G^Nq)] = cl(G^Mp)$.*

(2) Assume $p \in S_{ext}(M)$ is almost periodic. Then p extends to an almost periodic $q \in S_{ext}(N)$.

Proof. (1) Suppose $r[cl(G^Nq)] \not\subseteq cl(G^Mp)$. Then for some non-empty $U \subseteq_{ext} M$ we have that $S_{ext}(M) \cap [U] \cap cl(G^Mp) = \emptyset$ and for some $h \in G^N$, $hq \in [U]$, i.e. $h \in d_qU$.

But $d_qU = (d_pU)^N$. Since $(d_pU)^N \neq \emptyset$ and $M \prec^* N$, also $d_pU \neq \emptyset$, i.e. for some $g \in G^M$ we have that $gp \in [U]$. So $[U]$ and $cl(G^Mp)$ are not disjoint, a contradiction.

In the case where p is almost periodic, $cl(G^Mp)$ is a minimal G^M-flow. Since $r[cl(G^Nq)]$ is a G^M-invariant, closed subset of $cl(G^Mp)$, we get that $r[cl(G^Nq)] = cl(G^Mp)$.

(2) The proof is similar as in [N1], so we will be brief. Let $q_0 \in S_{ext}(N)$ be a weak heir of p. By (1), $r[cl(G^Nq_0)] = cl(G^Mp)$ and similarly also $r[Y] = cl(G^Mp)$ for any minimal flow $Y \subseteq cl(G^Nq)$. So for any such Y there is a $q \in Y$ extending p. $\Box$

Let $S_{ext,M}(N)$ be the set of types in $S_{ext}(N)$ that are weak heirs over M. For $p \in S_{ext}(M)$ let

$$S_{ext,p}(N) = S_{ext}(N) \cap [p_N] = S_{ext,M}(N) \cap [p],$$

this is the set of weak heirs of p in $S_{ext}(N)$. So

$$S_{ext,M}(N) = \bigcup_{p \in S_{ext}(M)} S_{ext,p}(N).$$

Lemma 2.5 *(1) Let $q \in S_{ext}(N)$ and $s \in S_{ext,M}(N)$. Then $r(q * s) = r(q) * r(s)$.*

(2) $S_{ext,M}(N)$ is closed under $*$, i.e. it is a sub-semigroup of $S_{ext}(N)$.

(3) $r : S_{ext,M}(N) \to S_{ext}(M)$ is a $*$-epimorphism.

Proof. (1) Let $U \subseteq_{ext} M$. Since s is a weak heir over M, we have:

$$U \in r(q * s) \iff U^N \in q * s \iff d_sU^N \in q \iff$$

$$(d_{r(s)}U)^N \in q \iff d_{r(s)}U \in r(q) \iff U \in r(q) * r(s).$$

(2) Let $p, q \in S_{ext,M}(N)$ and let $U \subseteq_{ext} M$. By (1), $r(p * q) = r(p) * r(q)$. Since both p and q are weak heirs,

$$d_{p*q}(U^N) = d_p(d_q(U^N)) = d_p(d_{r(q)}U)^N = (d_{r(p)}(d_{r(q)}U))^N = (d_{r(p)*r(q)}U)^N = (d_{r(p*q)}U)^N,$$

hence $p * q$ is a weak heir over M.

(3) follows from (1),(2). $\Box$

Unfortunately, $S_{ext,M}(N)$ neet not be a G^N-subflow of $S_{ext}(N)$. There is even no reason why it should be closed. However, for a fixed $p \in S_{ext}(M)$, the set $S_{ext,p}(N) = S_{ext}(N) \cap [p_N]$ is closed in $S_{ext}(N)$. In Lemma 2.4 we proved that every almost periodic $p \in S_{ext}(M)$ extends to an almost periodic $p' \in S_{ext}(N)$. Every such type p is an element of the group uI, where $I = cl(G^Mp)$ and $u \in I$ is an idempotent with $\mathrm{Im}(d_u) = \mathrm{Im}(d_p)$, the identity element of uI. Assume moreover that $p = u$ is an idempotent. Does u extend to an almost periodic $u' \in S_{ext}(N)$ that is an idempotent, too ?

3 Transfer between models: Ellis semigroups

In this section we assume that $M \prec^* N \prec \mathfrak{C}$. We have two Ellis semigroups: $S_{ext}(M)$ and $S_{ext}(N)$ and the restriction function $r : S_{ext}(N) \to S_{ext}(M)$, a surjective mapping. Unfortunately, r need not be a $*$-homomorphism. At least, by Lemma 2.5, r restricted to $S_{ext,M}(N)$ is a $*$-epimorphism.

We are interested in comparing minimal ideals and their partitions into groups in $S_{ext}(M)$ and $S_{ext}(N)$. Let $I \lhd_m S_{ext}(M)$ and let $J(I)$ be the set of idempotents in I. So $uI, u \in J(I)$, are disjoint groups and I is a union of them. By Lemma 2.4 every type in I extends to an almost periodic type in $S_{ext}(N)$. Moreover, there is an $I' \lhd_m S_{ext}(N)$ with $I = r[I']$.

One would expect that the groups $uI', u \in J(I')$ are related to the groups $uI, u \in J(I)$. Unfortunately we were not able to find any relationship between them in general. Instead, given the group uI (for some $u \in J(I)$), in this section we shall find inside $S_{ext,M}(N)$ a group mapped homomorphically by r onto uI. Later in this paper we will show however that assuming additionally existence of generic types in $S_{ext}(N)$ we have that every $uI, u \in J(I)$, is a homomorphic image of a subgroup of some $u'I', u' \in J(I')$.

As in Section 2, we consider $Def_{ext}(M)$ as a G^M-subalgebra of $Def_{ext}(N)$ (via the identification of $U \in Def_{ext}(M)$ with $U^N \in Def_{ext}(N)$). $Def_{ext}(M)$ generates an intermediate G^N-subalgebra $Def^0_{ext}(N) \subseteq Def_{ext}(N)$. By Lemma 2.2, every $q \in S_{ext}(M)$ determines an ultrafilter $q_N \in S^0_{ext}(N)$ such that for $U \subseteq_{ext} M$ and $g \in G^N$, $g^{-1}U^N \in q_N$ iff $g \in (d_qU)^N$.

Lemma 3.1 *Let $q, s \in S_{ext}(M)$ and $q' \in S_{ext,q}(N)$.*

(1) $d_{q'} : Def_{ext}(N) \to Def_{ext}(N)$ preserves $Def^0_{ext}(N)$ and $d_{q'} \restriction_{Def^0_{ext}(N)}$ does not depend on the choice of $q' \in S_{ext,q}(N)$. We denote $d_{q'} \restriction_{Def^0_{ext}(N)}$ by d_{q_N}.

(2) For $U \subseteq_{ext} M$, $d_{q_N}U^N = (d_qU)^N$. Also, for $V \in Def^0_{ext}(N)$, $d_{q_N}V = \{g \in N : g^{-1}V \in q_N\}$.

(3) $d_{q_N} \restriction_{Def_{ext}(M)} = d_q$.

(4) $Ker(d_{q_N}) \cap Def_{ext}(M) = Ker(d_q)$.

(5) $Im(d_{q_N}) \cap Def_{ext}(M) = Im(d_q)$.

(6) $Im(d_{q_N})$ is the G^N-subalgebra of $Def^0_{ext}(N)$ generated by $Im(d_q)$.

(7) $d_{(q*s)_N} = d_{q_N} \circ d_{s_N}$.

Proof. Let $U \subseteq_{ext} (M)$. Then $d_{q'}U^N = (d_qU)^N$ belongs to $Def_{ext}(M)$ (regarded as a subalgebra of $Def_{ext}(N)$) and

$$d_{q'}U^N = \{g \in N : g^{-1}U^N \in q'\} = \{g \in N : g^{-1}U^N \in q_N\}.$$

Similarly, for $h \in N$,

$$d_{q'}(hU^N) = h(d_qU)^N = \{g \in N : g^{-1}hU^N \in q_N\}.$$

The sets $hU^N, h \in N, U \subseteq_{ext} N$, generate $Def^0_{ext}(N)$. So $d_{q'} \restriction_{Def^0_{ext}(N)}$ is determined by q_N, which justifies denoting it by d_{q_N}.

Let $V \in Def^0_{ext}(N)$ be arbitrary. Then V is a Boolean combination of some $h_1U_1^N, \dots, h_kU_k^N$, where $h_i \in N, U_i \subseteq_{ext} M$. Write V as $f(h_1U_1^N, \dots, h_kU_k^N)$, where $f(x_1, \dots, x_k)$ is a Boolean term. Since $d_{q'}$ is a G^N-algebra homomorphism, we get that $d_{q'}V =$

$$d_{q'}f(h_1U_1^N, \dots, h_kU_k^N) = f(d_{q'}h_1U_1^N, \dots, dq'h_kU_k^N) = f(h_1(d_qU_1)^N, \dots, h_k(d_qU_k)^N),$$

hence d_qV belongs to $Def^0_{ext}(N)$ and equals the set $\{g \in N : g^{-1}V \in q_N\}$. So we have proved (1) and (2).

(3) and (4) are obvious.

(5) $\supseteq$ follows from (3).

$\subseteq$: Assume $V \subseteq_{ext} M$ and $V^N \in \mathrm{Im}(d_{q_N})$, i.e. $V^N = d_{q_N}W$ for some $W \in Def^0_{ext}(N)$. Write W as $f(h_1U_1^N, \dots, h_kU_k^N)$ for some Boolean term $f(x_1, \dots, x_k)$, $h_i \in N$ and $U_i \subseteq_{ext} M$. We have $d_{q_N}W = f(h_1(d_qU_1)^N, \dots, h_k(d_qU_k)^N)$. Let $V_i = d_qU_i$, $i = 1, \dots, k$. So $d_{q_N}W = V^N$ means

$$(*) \quad V^N = f(h_1V_1^N, \dots, h_kV_k^N).$$

$(*)$ is expressible by an $L_{ext,M}$-sentence $\varphi(h_1, \dots, h_k)$ with parameters $h_1, \dots, h_k$. So

$$N_{ext} \models (\exists h_1, \dots, h_k)\varphi(h_1, \dots, h_k).$$

Since $M \prec^* N$, this sentence holds also in M_{ext}, meaning that for some $g_1, \dots, g_k \in M$ we have that $V = f(g_1V_1, \dots, g_kV_k)$. Let $W' = f(g_1U_1, \dots, g_kU_k)$. We have that $W' \subseteq_{ext} M$ and

$$d_qW' = f(g_1d_qU_1, \dots, g_kd_qU_k) = f(g_1V_1, \dots, g_kV_k) = V.$$

Hence $V \in \mathrm{Im}(d_q)$.

(6),(7) Easy. $\square$

Now we list some properties of types preserved in the transition from q to q_N. We need the following definition.

Assume $q, s \in S_{ext}(M)$. We say that s is an inverse of q if $\mathrm{Im}(d_s) = \mathrm{Im}(d_q)$, $\mathrm{Ker}(d_s) = \mathrm{Ker}(d_q)$ and the functions d_s, d_q restricted to $\mathrm{Im}(d_s)$ are inverse to each other. If an inverse of q exists, it is unique. Also, in this case $d_q : \mathrm{Im}(d_q) \to \mathrm{Im}(d_q)$ is a bijection, hence $\mathrm{Im}(d_q)$ is a section of the cosets of $\mathrm{Ker}(d_q)$ in $Def_{ext}(M)$.

Lemma 3.2 *Assume $q \in S_{ext}(M)$.*

(1) If d_q is $1-1$ on $Im(d_q)$, then d_{q_N} is $1-1$ on $Im(d_{q_N})$.

(2) If q is almost periodic, then q has an inverse in $S_{ext}(M)$.

(3) If $u \in S_{ext}(M)$ is an idempotent, then $d_{u_N} \circ d_{u_N} = d_{u_N}$.

(4) If s is the inverse of q, then s_N is the inverse of q_N. In particular, in this case $d_{q_N} : Im(d_{q_N}) \to Im(d_{q_N})$ is a bijection and $Im(d_{q_N})$ is a section of the cosets of $Ker(d_{q_N})$ in $Def^0_{ext}(N)$.

Proof. (1) It is enough to prove that $\mathrm{Ker}(d_{q_N}) \cap \mathrm{Im}(d_{q_N}) = \{\emptyset\}$, that is: for $V \in Def^0_{ext}(N)$, $d_{q_N} \circ d_{q_N}(V) = \emptyset$ implies $d_{q_N}(V) = \emptyset$.

Write V as $f(h_1U_1^N, \ldots, h_kU_k^N)$ for some Boolean term $f(x_1, \ldots, x_k)$, $h_i \in N$ and $U_i \subseteq_{ext} M$. Let $V_i = d_qU_i$ and $W_i = d_qV_i$. Then

$$d_{q_N}V = f(h_1V_1^N, \ldots, h_kV_k^N) \text{ and } d_{q_N} \circ d_{q_N}(V) = f(h_1W_1^N, \ldots, h_kW_k^N).$$

Since $d_q \circ d_q(f(g_1U_1, \ldots, g_kU_k)) = \emptyset$ implies $d_q(f(g_1U_1, \ldots, g_kU_k)) = \emptyset$, we have that the sentence:

$$(\forall g_1, \ldots, g_k)(f(g_1W_1, \ldots, g_kW_k) = \emptyset \rightarrow f(g_1V_1, \ldots, g_kV_k) = \emptyset)$$

holds in M_{ext}. Since $M \prec^* N$, this sentence holds also in N_{ext}, and we are done.

(2) Since q is almost periodic, it belongs to a group $H \subseteq S_{ext}(M)$. Let $s \in H$ be the group-inverse of q. By Lemma 1.8, s is the inverse of q in the sense of $S_{ext}(M)$.

(3) By assumptions, d_u is identity on $\mathrm{Im}(d_u)$. It is enough to show that d_{u_N} is identity on $\mathrm{Im}(d_{u_N})$. This is proved similarly as (1), using the fact that $M \prec^* N$.

(4) Assume s is the inverse of q. We have that d_{q*s} and d_{s*q} are identity on $\mathrm{Im}(d_s) = \mathrm{Im}(d_q)$. As in (3) it implies that $d_{(q*s)_N}$ and $d_{(s*q)_N}$ are identity on $\mathrm{Im}(d_{s_N})$ and on $\mathrm{Im}(d_{q_N})$. By Lemma 3.1(7), $d_{(q*s)_N} = d_{q_N} \circ d_{s_N}$ and $d_{(s*q)_N} = d_{s_N} \circ d_{q_N}$. This implies easily that $\mathrm{Im}(d_{q_N}) = \mathrm{Im}(d_{s_N})$ and d_{q_N}, d_{s_N} restricted to $\mathrm{Im}(d_{q_N})$ are inverse to each other. $\square$

Now let $I \lhd_m S_{ext}(M)$ and fix $u \in J(I)$. So uI is a group. By Lemma 1.8 there are $K, R \subseteq Def_{ext}(M)$, a common kernel and image of $d_q, q \in uI$. By Lemma 2.5, the set $S_{ext,u}(N)$ is a closed sub-semigroup of $S_{ext}(N)$. By Theorem 0.1, any minimal ideal $I' \lhd_m S_{ext,u}(N)$ is the disjoint union of isomorphic groups $u'I', u' \in J(I')$ (where $J(I')$ is the set of idempotents in I'). Any group of the form $u'I', u' \in J(I')$, is determined by the common kernel and image of $d_{q'}, q' \in u'I'$. The next lemma explains what the minimal ideals $I' \lhd_m S_{ext,u}(N)$ are.

Lemma 3.3 *(1) For any ideal $I^+ \lhd S_{ext}(N)$, the set $I^+ \cap S_{ext,u}(N)$ is an ideal in $S_{ext,u}(N)$, provided it is non-empty.*

(2) Let $I' \lhd_m S_{ext,u}(N)$, $q' \in I'$ and $I^+ = cl(G^Nq')$. Then $I^+ \lhd S_{ext}(N)$, I^+ does not depend on the choice of $q' \in I'$ and $I^+ \cap S_{ext,u}(N) = I'$.

Proof. (1) is obvious.

(2) Since the semigroup operation is continuous in the first coordinate, it is easy to see that $I^+ = S_{ext}(N) * q'$. Hence obviously $I' \subseteq I^+ \cap S_{ext,u}(N)$. We will prove the reverse inclusion.

First consider the case where q' is an idempotent. Assume $s' \in S_{ext}(N)$ and $s' * q' \in I^+ \cap S_{ext,u}(N)$. Thus

$$s' * q' = s' * (q' * q') = (s' * q') * q' \in I',$$

so we are done.

Now let $q'' \in I'$ be arbitrary and let $I^{++} = cl(G^N q'') = S_{ext}(N)q''$. It is enough to prove that $I^{++} = I^+$. But $I^{++} \subseteq I^+$ (since $q'' \in I^+$) and $I' \subseteq I^{++} \cap S_{ext,u}(N)$ (since $I' \lhd_m S_{ext,u}(N)$), hence $q' \in I^{++}$. It follows that $I^+ \subseteq I^{++}$ and $I^+ = I^{++}$. $\square$

Fix $I' \lhd_m S_{ext,u}(N)$. By Lemma 3.3(2), there is a common kernel $K' \subseteq Def_{ext}(N)$ of all $d_{q'}, q' \in I'$. Also, let $\mathcal{R}' = \{\mathrm{Im}(d_{q'}) : q' \in I'\}$. By Theorem 0.1 and a variant of Lemma 1.8, I' splits into a disjoint union of groups corresponding to elements of $\mathcal{R}'$. Hence every $R' \in \mathcal{R}'$ is a section of the family of cosets of K' in $Def_{ext}(N)$.

Lemma 3.4 *(1) $K' \cap Def_{ext}(M) = K$ and $K' \cap Def^0_{ext}(N) = Ker(d_{u_N})$.*
(2) Let $R' \in \mathcal{R}'$. Then $R' \cap Def_{ext}(M) = R$ and $R' \cap Def^0_{ext}(N) = Im(d_{u_N})$.

Proof. (1) is straightforward.
(2) We know that I' is the disjoint union of the groups $u'I', u' \in J(I')$. Given $u' \in J(I')$, the images $\mathrm{Im}(d_{q'}), q' \in u'I'$, are all equal. Hence in our situation there is an idempotent $u' \in J(I')$ with $R' = \mathrm{Im}(d_{u'})$. Since u' is an idempotent, we have that $R' = \{U \subseteq_{ext} N : d_{u'}U = U\}$. Since $d_u = d_{u'} \restriction_{Def_{ext}(M)}$ and $d_{u_N} = d_{u'} \restriction_{Def^0_{ext}(N)}$, the conclusion follows. $\square$

By Lemma 3.3 we choose an ideal $I^+ \lhd Def_{ext}(N)$ with $I^+ \cap S_{ext,u}(N) = I'$ and $I^+ = cl(G^N q')$ for every $q' \in I'$. For every $q \in I$ let $I'_q = S_{ext,q}(N) \cap I^+$. We shall use the following variant of Remark 1.10.

Remark 3.5 *For every $R_1, R_2 \in \mathcal{R}'$, $R_1 \subseteq R_2$ implies $R_1 = R_2$.*

Lemma 3.6 *Let $q \in I$.*
(1) $I'_q \neq \emptyset$. If q is an idempotent, then $I'_q \lhd_m S_{ext,q}(N)$.
(2) For every $q' \in I'_q$ we have that $I^+ = cl(G^N q')$, $Ker(d_{q'}) = K'$ and $K' \cap Im(d_{q'}) = \{\emptyset\}$.
(3) Assume $q \in uI$. Then the set $\mathcal{R}'_q := \{Im(d_{q'}) : q' \in I'_q\}$ equals $\mathcal{R}'$.

Proof. (1) First consider the case where q is an idempotent. Then $q * u = q$. Let $q' \in S_{ext,q}(N)$ and $u' \in I'$. Then $q' * u' \in S_{ext,M}(N)$. Also, by Lemma 3.2(5), $r(q' * u') = r(q') * r(u') = q * u = q$, so $q' * u' \in S_{ext,M}(N) \cap [q] = S_{ext,q}(N)$. Moreover, $q' * u' \in I^+$, so $q' * u' \in I'_q$, hence I'_q is non-empty.

Since q is an idempotent, by Lemma 3.3(1), I'_q is an ideal in $S_{ext,q}(N)$. Suppose for a contradiction that I'_q is not minimal. Choose $q'' \in I'_q$ that generates a minimal ideal $I''_q \lhd S_{ext,q}(N)$, properly contained in I'_q. Let I^{++} be the ideal in $S_{ext}(N)$ generated by I''_q. Be Lemma 3.3, $I^{++} \cap S_{ext,q}(N) = I''_q$ and of course $I^{++} \subseteq I^+$. Also, by what we have already proved for q, switching the roles of q and u we get that $I^{++} \cap S_{ext,u}(N)$ is a non-empty ideal in $S_{ext,u}(N)$ contained in I'. Since I' is minimal we get that $I^{++} \cap S_{ext,u}(N) = I'$. Since I' generates I^+, we have that $I^+ = I^{++}$ and $I''_q = I'_q$, a contradiction.

Now consider the case, where $q \in I$ is arbitrary. Choose an idempotent $q_0 \in J(I)$ with $q \in q_0 I$. We have that $I'_{q_0} \lhd_m S_{ext,q_0}(N)$ and I^+ is the ideal in $S_{ext}(N)$ generated by I'_{q_0}. Since $q * q_0 = q$, we get that $I'_q \neq \emptyset$ similarly as in the case where q is an idempotent.

(2) First we prove that

(∗) for every $q' \in I'_q$ we have that $I^+ = cl(G^N q')$.

By (1) and Lemma 3.4 this is true when q is an idempotent. Now assume $q \in I$ is arbitrary. Let $q' \in I'_q$ and $I^{++} = cl(G^N q')$. Clearly $I^{++} \subseteq I^+$. Choose an idempotent $q_0 \in J(I)$ with $q = q_0 I$. Choose $s \in q_0 I$ such that $s * q = q_0$ (remember that $q_0 I$ is a group with the identity element q_0). Let $s' \in S_{ext,s}(N)$. We have that

$$s' * q' \in I^{++} \cap S_{ext,q_0}(N) \subseteq I^+ \cap S_{ext,q_0}(N) = I'_{q_0}.$$

Since $I'_{q_0} \lhd_m S_{ext,q_0}(N)$ and I'_{q_0} generates I^+, we have that $I^{++} \cap S_{ext,q_0}(N) = I'_{q_0}$ and $I^{++} = I^+$ and (∗) is proved.

(∗) implies immediately that $\mathrm{Ker}(d_{q'}) = K'$ for every $q' \in I'_q$. To prove the last clause of (2), suppose for a contradiction that for some $q' \in I'_q$ we have $K' \cap \mathrm{Im}(d_{q'}) \neq \{\emptyset\}$. Then $K' = \mathrm{Ker}(d_{q'}) \subsetneq \mathrm{Ker}(d_{q'*q'})$. But $q * q \in I$ and $q' * q' \in I'_{q*q}$, hence $\mathrm{Ker}(d_{q'*q'}) = K'$, a contradiction.

(3) Choose $s \in uI$ with $q * s = u$. Let $q' \in S_{ext,q}(N), s' \in S_{ext,s}(N)$. Then $q' * s' \in I'$ and $\mathrm{Im}(d_{q'*s'}) \in \mathcal{R}'$. Also, $\mathrm{Im}(d_{q'*s'}) \subseteq \mathrm{Im}(d_{q'})$. However, by Lemma 1.8 $\mathrm{Im}(d_{q'*s'})$ meets every coset of K' in $Def_{ext}(N)$ just once and by (2), $\mathrm{Im}(d_{q'})$ meets every such coset at most once. So $\mathrm{Im}(d_{q'*s'}) = \mathrm{Im}(d_{q'})$, hence $\mathrm{Im}(d_{q'}) \in \mathcal{R}'$ and $\mathcal{R}'_q \subseteq \mathcal{R}'$.

For the reverse inclusion consider any $u' \in I'$. Since $u * q = q$, by Lemma 2.5 we have $u' * q' \in I'_q$ and $\mathrm{Im}(d_{u'*q'}) \subseteq \mathrm{Im}(d_{u'})$. Since $\mathcal{R}'_q \subseteq \mathcal{R}'$ we have that $\mathrm{Im}(d_{u'*q'}) \in \mathcal{R}'$ and also $\mathrm{Im}(d_{u'}) \in \mathcal{R}'$. By Remark 3.5 we have that $\mathrm{Im}(d_{u'*q'}) = \mathrm{Im}(d_{u'})$, hence $\mathcal{R}' \subseteq \mathcal{R}'_q$, and we are done. □

Fix an idempotent $u' \in I'$ and let $R' = \mathrm{Im}(d_{u'})$. Let

$$\mathcal{H} = \{q' \in \bigcup_{q \in uI} I'_q : \mathrm{Im}(d_{q'}) = R'\}.$$

The next proposition is the main result of this section.

Proposition 3.7 *$\mathcal{H}$ is a group and $r : \mathcal{H} \to uI$ is an epimorphism of groups, with the kernel $u'I'$.*

Proof. By Lemmas 2.5 and 3.6, $\mathcal{H}$ is closed under $*$ and $r : \mathcal{H} \to uI$ is a $*$-epimorphism. We need to check that $\mathcal{H}$ is a group.

Clearly $u' \in \mathcal{H}$ is the identity element in $\mathcal{H}$. Let $q' \in \mathcal{H}$. We want to find a group inverse of q' in $\mathcal{H}$. $q' \in I'_q$, where $q = r(q')$. We have that $q \in uI$ and uI is a group, so there is an $s \in uI$ inverse to q. Let $s' \in I'_s$. Then $s' * q'$ and $q' * s'$ belong to $u'I'$, that is a group. So there is an $u'' \in u'I'$ such that $q' * s' * u'' = u'$.

Let $s'' = s' * u''$. So $s'' \in I'_s$ and $q' * s'' = u'$. Similarly we find an $s''' \in I'_s$ with $s''' * q' = u'$. It follows that $d_{q'}$, restricted to R', is an automorphism of R' and both $d_{s''}$ and $d_{s'''}$ are inverse to $d_{q'}$ on R'. Hence $s'' = s'''$ is a two-sided group inverse to q' in $\mathcal{H}$. □

One can show that up to isomorphism the group $\mathcal{H}$ we arrived at does not depend on the choice of $I \lhd_m S_{ext}(M), u \in J(I)$ and $u' \in J(I')$. The proof is similar to the one we gave for isomorphism of the groups $uI, I \lhd_m S_{ext}(M), u \in J(I)$, in Section 1.

Besides the group $\mathcal{H}$, whose homomorphic image under r is uI, inside $S_{ext}(N)$ there is also a subgroup $\mathcal{H}'$ downright isomorphic to uI. Namely, by [N1, Remark 4.6], there is a $*$-monomorphism $j : S_{ext}(M) \to S_{ext}(N)$, mapping any $p \in S_{ext}(M)$ to the only $q \in S_{ext}(N)$ such that $U \cap M \in p$ for every $U \in q$. Then $\mathcal{H}' = j[uI]$ is isomorphic to uI. However, we are interested in comparing the group uI to the groups $u'I', I' \lhd_m S_{ext}(N), u' \in J(I')$. We will do it in the next section, under an additional assumption on $S_{ext}(N)$, using the group $\mathcal{H}$ as an intermediate tool. $\mathcal{H}'$ would not serve here well, since, unlike in the case of $\mathcal{H}$, we do not have a good description of the common kernel and image of $d_p, p \in \mathcal{H}'$.

4 Groups with external generics

In this section we continue analyzing the situation considered in Section 3. So let $M \prec^* N$. Let H^M be any of the groups $uI, I \lhd_m S_{ext}(M), u \in J(I)$, and H^N any of the groups $u'I', I' \lhd_m S_{ext}(N), u' \in J(I')$. We think these groups should be strongly related algebraically. We were able to prove this only under some additional assumptions, the weakest one being the existence of generic types in $S_{ext}(N)$. The main result of this section is the following theorem.

Theorem 4.1 *Assume there are generic types in $S_{ext}(N)$. Then H^M is a homomorphic image of a subgroup of H^N.*

Before the proof we need some preparatory analysis.

Lemma 4.2 *The following conditions are equivalent.*

(1) There is a generic type in $S_{ext}(N)$.

(2) There is a single minimal ideal $I \lhd_m S_{ext}(N)$.

(3) If $I \lhd_m S_{ext}(N)$, $U \subseteq_{ext} N$ and $[U] \cap I = \emptyset$, then $[d_q U] \cap I = \emptyset$ for every $q \in S_{ext}(N)$.

(4) Ker(d_p) is closed under d_q for every almost periodic $p \in S_{ext}(N)$ and every $q \in S_{ext}(N)$.

Proof. (1) $\Leftrightarrow$ (2) By [N1], for every point-transitive G-flow X, in X there is a generic point iff all weak generic points in X are generic iff in X there is a single minimal subflow. But minimal subflows in $S_{ext}(N)$ are just the minimal ideals $I \lhd_m S_{ext}(N)$.

(3) $\Leftrightarrow$ (4) As in the proof of Lemma 1.9, for every almost periodic $p \in S_{ext}(N)$, the set $cl(G^N p)$ equals $\bigcap\{[U] \cap S_{ext}(N) : U^c \in \mathrm{Ker}(d_p)\}$, it is a minimal ideal in $S_{ext}(N)$, and every $I \lhd_m S_{ext}(N)$ is of this form. The rest is just revealing of definitions.

(2) $\Rightarrow$ (4) Let $I \lhd_m S_{ext}(N)$. We have that $I = cl(G^N p)$ for any $p \in I$. Let $U \in \mathrm{Ker}(d_p)$, meaning that $[U] \cap I = \emptyset$ and $I \subseteq [U^c]$. We want to show that $I \subseteq [d_q U^c]$.

We have that $I * q$ is a minimal ideal, so by our assumptions we get $I * q = I$, i.e. $r_q[I] = I$, where r_q is the right multiplication by q. This implies $r_q[I] \subseteq [U^c]$, i.e. $I \subseteq r_q^{-1}[U^c]$. But $r_q^{-1}[U^c] = [d_q U^c]$ (see Lemma 1.5(3)), so we are done.

(4) $\Rightarrow$ (2) Suppose I_0, I_1 are two distinct minimal ideals in $S_{ext}(N)$. Let $q_1 \in I_1$. So $I_1 = I_0 * q_1$. Choose $U \subseteq_{ext} N$ with $I_1 \subseteq [U]$ and $[U] \cap I_0 = \emptyset$.

Let $p \in I_0$. So $U \in \mathrm{Ker}(d_p)$. By our assumptions also $d_q U \in \mathrm{Ker}(d_p)$, so $U \in \mathrm{Ker}(d_{p*q})$. But $p * q \in I_1$, so $[U] \cap I_1 = \emptyset$, a contradiction. $\square$

Next we show that the existence of a generic type in $S_{ext}(N)$ yields some homomorphisms between the groups in $S_{ext}(N)$ and H^N. From now on, until the end of the proof of Theorem 4.1, we assume there is a generic type in $S_{ext}(N)$.

Let $I_0 \lhd_m S_{ext}(N)$ be the only minimal ideal in $S_{ext}(N)$ and let K_0 be the common kernel of $d_q, q \in I_0$. As explained in Section 1, for every $q \in I_0$, d_q induces an automorphism $\widetilde{d}_q : Def_{ext}(N)/K_0 \to Def_{ext}(N)/K_0$, and the function $\widetilde{d} : I_0 \to \mathrm{Aut}(Def_{ext}(N))$ mapping q to $\widetilde{d}_q$ is a $*$-homomorphism. Let $\mathcal{H}^N$ be $\widetilde{d}[I_0]$. By Proposition 1.13, $\mathcal{H}^N$ is a group and $\widetilde{d}$ is an isomorphism between any of the groups $uI_0, u \in J(I_0)$ and $\mathcal{H}^N$.

Since there is a generic type in $S_{ext}(N)$, by Lemma 4.2(4) we have that d_q preserves K_0 for every $q \in S_{ext}(N)$. In particular, d_q induces a function $\widetilde{d}_q$: $Def_{ext}(N)/K_0 \to Def_{ext}(N)/K_0$ and we can extend the function $\widetilde{d}$ defined above to all of $S_{ext}(N)$. Now $\widetilde{d} : S_{ext}(N) \to \mathrm{End}(Def_{ext}(N)/K_0)$ mapping q to $\widetilde{d}_q$ is still a $*$-homomorphism. We have the following lemma.

Lemma 4.3 *$\widetilde{d}$ maps $S_{ext}(N)$ onto $\mathcal{H}^N$.*

Proof. Assume $q \in S_{ext}(N)$ and let $K = \mathrm{Ker}(d_q)$. Let $I^* = cl(G^N q) = S_{ext}(N) * q$. I^* is the ideal and G^N-flow generated by q. By Lemma 4.2, $I_0 \subseteq I^*$, hence by Lemma 1.9, $K \subseteq K_0$. It follows that $\widetilde{d}_q$ is an endomorphism of $Def_{ext}(N)/K_0$. We must show yet that $\widetilde{d}_q$ belongs to $\mathcal{H}^N$.

Let $s \in J(I_0)$. Then $q * s \in I_0$ hence $\widetilde{d}_{q*s} \in \mathcal{H}^N$. But $\widetilde{d}_{q*s} = \widetilde{d}_q \circ \widetilde{d}_s$ and $\widetilde{d}_s$ is the identity, so $\widetilde{d}_q = \widetilde{d}_{q*s} \in \mathcal{H}^N$. $\square$

In particular, for every group $\mathcal{H}$ contained in $S_{ext}(N)$ we have that $\widetilde{d}$ maps $\mathcal{H}$ onto a subgroup of $\mathcal{H}^N$. Let us return to the notation from Section 3. That is, we fix $I \lhd_m S_{ext}(M)$ and $u \in J(I)$ and then an $I' \lhd_m S_{ext,u}(N)$. Let I^+ be the ideal in $S_{ext}(N)$ generated by I' and let K' be the common kernel of $d_{q'}, q' \in I'$. Let $K \subseteq Def_{ext}(M)$ be the common kernel of $d_q, q \in I$. Recall that we have $Def_{ext}(M) \subseteq Def_{ext}(N)$ via the function $U \mapsto U^N$.

Lemma 4.4 *$K_0 \cap Def_{ext}(M) = K$.*

Proof. Since there is a generic type in $S_{ext}(N)$, we have that $K' \subseteq K_0$, so by Lemma 3.4(1), $K = K' \cap Def_{ext}(M) \subseteq K_0 \cap Def_{ext}(M)$. Since $I \lhd_m S_{ext}(M)$, we have that K is a maximal G^M-ideal $\mathcal{K}$ in $Def_{ext}(M)$ such that

$$(*) \quad \bigcap_{U \in \mathcal{K}} [U^c] \cap S_{ext}(M) \text{ is nonempty.}$$

Since $\mathcal{K} = K_0 \cap Def_{ext}(M)$ also satisfies $(*)$ (as $M \prec^* N$), we have that $K = K_0 \cap Def_{ext}(M)$. $\square$

Proof of Theorem 4.1. We have that $H^M \cong uI$. Let $u', R', \mathcal{H}$ be as in Proposition 3.6. The restriction function r is a group epimorphism $\mathcal{H} \to uI$. By Lemma 4.3 we have a homomorphism $\widetilde{d} : \mathcal{H} \to \mathcal{H}^N$. Let $\mathcal{H}' = \widetilde{d}[\mathcal{H}]$, this is a subgroup of $\mathcal{H}^N$. We will show that there is a group epimorphism $f : \mathcal{H}' \to uI$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{H} \arrow[rr, "\widetilde{d}"] \arrow[dr, "r"'] & & \mathcal{H}' \arrow[dl, "f"] \\ & uI & \end{tikzcd}$$

We have that $\mathrm{Ker}(r) = \mathcal{H} \cap S_{ext,u}(N)$, so the existence of f is equivalent to

$$(**) \quad \mathrm{Ker}(\widetilde{d}) \subseteq S_{ext,u}(N).$$

Now we prove $(**)$. Let $q' \in \mathcal{H}$. So $q' \in S_{ext,q}(N)$ for some $q \in uI$. Assume $q' \in \mathrm{Ker}(\widetilde{d})$, that is $\widetilde{d}_{q'}$ is the identity. We want to show that $q = u$, i.e. d_q induces the identity on $Def_{ext}(M)/K$. By Lemma 3.1(3) we have that $d_q = d_{q'} \restriction_{Def_{ext}(M)}$.

Let $U \subseteq_{ext} M$. Since $\widetilde{d}_{q'}(U/K_0) = U/K_0$, we have that $U = d_{q'}U (\mathrm{mod}\ K_0)$. Since $d_{q'} \restriction_{Def_{ext}(M)} = d_q$, we get $d_{q'}U \in Def_{ext}(M)$. So we have

$$U \triangle d_{q'}U \in K_0 \cap Def_{ext}(M).$$

By Lemma 4.4, $U = d_qU (\mathrm{mod}\ K)$, hence d_q induces the identity on $Def_{ext}(M)/K$ and $q = u$. $\square$

Theorem 4.1 has the assumption that there are generic types in $S_{ext}(N)$. Now we shall analyze this assumption. We want to consider this assumption as a property of $Th(N)$ rather than N itself. So we say that G has *external generic types* if for every $M \models T$ here are generic types in the G^M-flow $S_{ext}(M)$. The next lemma clarifies this notion and shows it has a *local character*, i.e. in order to verify it we need to check only boundedly many models of T.

Lemma 4.5 *Let κ be an infinite cardinal. The following conditions are equivalent.*

(1) G has external generic types.

$(2)_\kappa$ *For every model $M \models T$ of power κ there are generic types in the G^M-flow $S_{ext}(M)$.*

Proof. $(1) \Rightarrow (2)_\kappa$ is obvious.

$(2)_\kappa \Rightarrow (1)$ Suppose for some $M \models T$ there are no generic types in $S_{ext}(M)$. So choose $U \subseteq_{ext} M$ such that both U and $M \setminus U$ are not generic.

First consider the case where M has power $\geq \kappa$. Choose an elementary submodel $(M', U^{M'}) \prec (M, U)$ of power κ. Clearly, $U^{M'} = U \cap M$ and $M' \setminus U^{M'}$ are both externally definable in M' and not generic. So there are no generic types in $S_{ext}(M')$.

Next consider the case where M has power $< \kappa$. Choose an M' of power κ with $M \prec^* M'$. Then again $U^{M'}$ and $M' \setminus U^{M'}$ are externally definable in M' and not generic. $\square$

In the last part of the paper we shall see that every definably compact group G definable in an o-minimal expansion of the field of reals has external generic types. This will follow rather directly from compact domination of G.

So from now on assume that M is any model of $T = Th(\mathbb{R}^*)$, where $\mathbb{R}^*$ is an o-minimal expansion of the ordered field of reals, and G is a group definable in T, that is moreover definably compact [HPP]. Hrushovski and Pillay proved in [HP] that G is compactly dominated, meaning that for every set $U \subseteq G$ definable in $\mathfrak{C}$, the set of G^{00}-cosets meeting both U and $G \setminus U$ has Haar measure zero. Also, in [HPP] it is proved that G has finitely satisfiable generic types (fsg), meaning that there are global generic types in G and every such type is finitely satisfiable in every small model M. fsg implies also that the left generic sets coincide in G with the right generic sets. In fact, fsg follows from compact domination.

Since now the universe of G is not necessarily equal to the universe of the model of T, we modify our notation. Given $M \models T$ and a group G^M 0-definable in M, we consider the G^M-flow $S_G(M)$ of all complete G-types in $S(M)$. Then we consider $Def_{ext,G}(M)$, an algebra of externally definable subsets of G^M and its Stone space of ultrafilters $S_{ext,G}(M)$, another G^M-flow, corresponding to $S_{ext}(M)$ in the old set-up.

In fact, we will show that compact domination implies also that the groups of the form $uI, I \lhd_m S_{ext,G}(M), u \in J(I)$, are isomorphic to G/G^{00}. Still, just for the record we have the following proposition.

Proposition 4.6 *Assume G is a definably compact group, definable in the theory of an o-minimal expansion of the reals. Then G has external generic types.*

Proof. Assume $U \subseteq \mathfrak{C}$ is a definable generic subset of G. It is enough to show that $U \cap M$ is a generic subset of G^M. Let

$$X_U = \{x/G^{00} : x/G^{00} \text{ meets } U\} \text{ and } X_{U^c} = \{x/G^{00} : x/G^{00} \text{ meets } U^c\}.$$

Both sets X_U and X_{U^c} are closed in G/G^{00} (in the logic topology) and by compact domination, the set $X_U \cap X_{U^c}$ has Haar measure zero. Also, the set X_U is generic in G/G^{00}. So the set $X_U \setminus X_{U^c}$ has non-empty interior in G/G^{00}.

In particular there is an M-definable set $U' \subseteq U$ such that $X_{U'} \setminus X_{U'^c}$ has non-empty interior, too. This implies that U' is a generic subset of G, hence also $U \cap M$ is a generic subset of G^M. $\square$

In order to prove that in our case the groups uI are isomorphic to G/G^{00}, we must recall some of the results and set-up from [N1].

Given $p \in S_{ext,G}(M)$ and $A \supseteq M$ we define $p^A \in S_G(A)$ as the set of formulas $\varphi(x)$ over A such that $\varphi(\mathfrak{C}) \cap M \in p$. Let $i_{M,A} : S_{ext,G}(M) \to S_G(A)$ be the function mapping p to p^A. Clearly, $i_{M,A}$ is continuous and if every n-type over M is realized in A, then $i_{M,A}$ is 1-1, hence in this case $i_{M,A}$ is a continuous embedding.

Let $S_{M,G}(A)$ be the range of $i_{M,A}$. It consists of those complete G-types over A that are finitely satisfiable in M.

Now assume $A = M'$ is an $\|M\|^+$-saturated model containing M. So $i_{M,M'} : S_{ext,G}(M) \to S_{M,G}(M')$ is a homeomorphism. In $S_{M,G}(M')$ we can recover the semigroup operation $*$ from $S_{ext,G}(M)$ as follows. Let $p, q \in S_{ext,G}(M)$. Let b realize $q^{M'}$

and a realize $p^{M'b}$. Then $a \cdot b$ realizes $(p * q)^{M'}$. So if we put $p^{M'} * q^{M'} = tp(a \cdot b/M')$ then we get a semigroup operation $*$ in $S_{G,M}(M')$ such that $i_{M,M'}$ is a $*$-isomorphism.

In the next lemma we collect some facts linking the generic types and G/G^{00} in our situation.

Lemma 4.7 *Assume $M \prec M'$, M' is $\|M\|^+$-saturated and $p \in S_{ext,G}(M)$ is generic.*
(1) $p^{M'}(\mathfrak{C})$ is contained in a single G^{00}-coset.
(2) p is the only generic extension of p^M in $S_{ext,G}(M)$.
(3) $Stab_L(p) = Stab_L(p^M) = G^{00} \cap M$.
(4) $Stab_R(p) = Stab_R(p^M) = G^{00} \cap M$.

Proof. (1) is implicit in the proof of [N1, Proposition 4.4]. (2) follows from compact domination. (3) appears in [HP] and also in [N1] (since fsg implies that the number of generic types is bounded). (4) follows since fsg implies that left generic definable subsets of G coincide with right generic definable subsets of G [HPP]. $\square$

Proposition 4.8 *Assume G is a definably compact group definable in the theory of an o-minimal expansion of the reals. Assume M' is an $\|M\|^+$-saturated elementary extension of M, $I \lhd_m S_{ext,G}(M)$ and $u \in J(I)$. The function $\pi : uI \to G/G^{00}$ mapping p to the G^{00}-coset containing $p^{M'}(\mathfrak{C})$ is a group isomorphism.*

Proof. By the proof of [N1, Proposition 4.4], the function π is a group epimorphism. To see that π is 1-1, consider any $p \in uI$ with $p^{M'}(\mathfrak{C}) \subseteq G^{00}$. It is enough to prove that $p = u$.

We have $u * p = p$ (since u is the identity element of the group uI). On the other hand choose an $\|M'\|^+$-saturated model $M'' \succ M'$. Let $b \in M''$ realize $p^{M'}$ and let a realize $u^{M''}$. Then $a \cdot b$ realizes $(u * p)^{M'}$.

By compact domination, $u^{M''}$ is generic, so by Lemma 4.7(4), $b \in G^{00} \cap M''$ implies $b \in Stab_R(u^{M''})$ and $ab \models u^{M''}b = u^{M''}$. It follows that $(u * p)^{M'} = u^{M'}$, hence $u * p = u$ and $p = u$. $\square$

Lemma 4.7 and Proposition 4.8 show that in the case of a definably compact group G definable in an o-minimal expansion of the reals we can naturally interpret the groups $uI, I \lhd_m S_{ext,G}(M), u \in J(I)$, inside $S_G(M)$. Let $Gen(M)$ be the set of generic types in $S_G(M)$. By Lemma 4.7(2), any $p', q' \in Gen(M)$ determine unique generic $p, q \in S_{ext,G}(M)$ with $p' = p^M, q' = q^M$. So we can define $p' * q'$ as $(p * q)^M$. Actually, we can define $p * q$ more explicitly as the type $tp(a \cdot b/M)$, where b realizes q' and a realizes p^{Mb}.

The function $p \mapsto p^M$ is a $*$-isomorphism between the semigroup of generic points in $S_{ext,G}(M)$ and $Gen(M)$. So we have that $Gen(M)$ splits into a disjoint union of groups isomorphic to G/G^{00}. The neutral elements of these groups are precisely the generic types from $Gen(M)$ in G^{00}. So the number of these groups equals the number of generic types in $S_G(M)$ inside G^{00}. Unlike in the stable case, there may be more than one of them. For example, when our group G is the circle S^1, $Gen(M)$ is the union of two such groups and when G is the torus $S^1 \times S^1$, there are infinitely many of them.

In fact, Propositions 4.6 and 4.8 and Lemma 4.7 remain true for a definable group G satisfying the following property of topological domination, weaker than compact domination. We say that a definable group G is *topologically dominated* if G^{00} exists and for every definable set $U \subseteq G$, the set of G^{00}-cosets meeting both U and U^c is nowhere dense in G/G^{00}.

Now assume $M \prec^* N$ are models of our o-minimal theory T and G is a definably compact group definable in T. In Theorem 4.1 we have shown a subgroup $\mathcal{H}'$ of $\mathcal{H}^N$ and an epimorphism $f : \mathcal{H}' \to uI$. One can see that in our case $\mathcal{H}' = \mathcal{H}^N$ and the function f is an isomorphism commuting with the functions π from Proposition 4.8.

Finally we shall consider in greater detail the case where G is the circle group S^1, written multiplicatively. Assume M is an o-minimal expansion of the field of reals $\mathbb{R}$. Since $\mathbb{R}$ is Dedekind complete, we have that $Def_{ext,G}(M) = Def_G(M)$ and consequently $S_{ext,G}(M) = S_G(M)$.

Let us fix the anti-clockwise orientation of $G = S^1$. For $b \neq c \in S^1$ let (b, c) denote the open arc in S^1, from b to c, according to our orientation. Similarly we define the arcs $[b, c), [b, c]$ and $(b, c]$.

For $a \in G^M$ let p_a^+ be the non-algebraic type in $S_G(M)$ generated by the arcs (a, b), while p_a^- — the non-algebraic type generated by $(b, a), b \in G^M \setminus \{a\}$. So the types $p_a^+, p_a^-, a \in G^M$, are the generic types in $S_G(M)$, forming the only minimal ideal in $I \lhd_m S_G(M)$.

Let $a \in G^M$. If $b \neq c \in G^M$ and U is any arc in G^M from b to c, then

$$d_{p_a^+}(U) = (ac^{-1}, ab^{-1}] \text{ and } d_{p_a^-}(U) = [ac^{-1}, ab^{-1}).$$

We see that $\mathrm{Im}(d_{p_a^+})$ is the subalgebra of $Def_G(M)$ generated by the arcs $(b, c]$, while $\mathrm{Im}(d_{p_a^-})$ is generated by the arcs $[b, c), b \neq c \in G^M$. All $d_{p_a^+}, d_{p_a^-}$ have a common kernel consisting of non-generic sets in $Def_G(M)$.

The idempotents in I are the types p_1^+, p_1^- and I splits into two groups $p_1^+ I = \{p_a^+ : a \in G^M\}$ and $p_1^- I = \{p_a^- : a \in G^M\}$.

Now let us assume that $M \prec^* N$ for some sufficiently saturated model N. Let $q \in S_{ext,G}(N)$ be a weak heir of p_a^+. Let $U = (b, c)$ for some $b \neq c \in G^M$. So

$$d_q U^N = (d_{p_a^+} U)^N = (ac^{-1}, ab^{-1}]^N.$$

Let $d \in (ac^{-1}, ab^{-1}]^N$ be infinitesimally close to ac^{-1} and $e = ab^{-1}$. So $d^{-1}(b, c)^N \cap e^{-1}(b, c)^N$ belongs to q and is not generic.

We see that in our case no weak heir of a generic type $p_a^\pm \in S_G(M)$ in $S_{ext,G}(N)$ is generic, so the shift in the proof of Lemma 2.4 is needed.

One can see that in the case of a definably compact group G definable in the theory of an o-minimal expansion of the reals, when $M \prec N$ and $p \in S_{ext,G}(M)$ is generic, then its co-heir extension $p^N \in S_G(N)$ is generic, too. However, it is not true for arbitrary definable group, even in an o-minimal theory.

For example, let G be the additive group of reals, considered as a definable group in the ordered field of reals. Let M be the field of reals. In $S_G(M)$ there are just two almost periodic types: $p_{-\infty}$ and $p_{+\infty}$. Their co-heirs are not almost periodic in $S_G(N)$ for any proper extension $N \succ M$.

References

[A] J.Auslander, **Minimal flows and their extensions**, North Holland, Amsterdam 1988.

[E] R.Ellis, **Lectures on topological dynamics**, Benjamin, New York 1969.

[G] E.Glasner, *Enveloping semigroups in topological dynamics*, Topology Appl. 154(2007), 2344-2363.

[HPP] E.Hrushovski, Y.Peterzil, A.Pillay, *Groups, Measures and the NIP*, J.AMS 21(2008), 563-596.

[HP] E.Hrushovski, A.Pillay, *On NIP and invariant measures*, preprint 2008.

[N1] L.Newelski, *Topological dynamics of definable group actions*, J. Symb.Logic 74(2009), 50-72.

[N2] L.Newelski, *Bounded orbits and measures on a group*, Israel J.Math., to appear.

[PP] Y.Peterzil, A.Pillay, *Generic sets in definably compact groups*, Fund.Math. 193(2007), 153-170.

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