A logical construction of a model category

notes by misha gavrilovich

Abstract: In 1967 Quillen introduced *model categories* "to cover in a uniform way" "a large number of arguments [in the different homotopy theories encountered] that were formally similar to well-known ones in algebraic topology". We show the same formalism "covers in a uniform way" a number of arguments in (naive) set theory. We argue that the formalism is curious as it suggests to look at a *homotopy-invariant* variant of Generalised Continuum Hypothesis which has less independence of ZFC, and first appeared in PCF theory independently but with a similar motivation.

1. This is a dense announcement of results partly reporting on joint work with Assaf Hasson, and shall eventually appear in the form of a joint paper. Proofs, speculations and motivations may be found in a more verbose report [Gavrilovich] which also contains some questions.

1.1. We write $A \xrightarrow{g} B \,\rightthreetimes\, X \xrightarrow{h} Y$ iff for any $A \longrightarrow X$ and $B \longrightarrow Y$ there exists $B \longrightarrow X$ and $A \longrightarrow Y$ such that

($\boxtimes$) if the square of solid arrows commutes, then the whole diagram commutes.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, dashed] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

We read $A \longrightarrow B \,\rightthreetimes\, X \longrightarrow Y$ as: the morphism (or arrow) $A \longrightarrow B$ lifts wrt $X \longrightarrow Y$. We use dotted arrows to indicate an existential quantifier. In the categories we shall consider there is at most one morphism between any two objects, and therefore any diagram that can drawn is necessarily commutative, e.g. we would not need to check the condition ($\boxtimes$) above.

1.2. Let $StNaamen$ be the following labelled category. Its objects are arbitrary sets, and there is at most one arrow between any two objects. An arrow carries none or some of the three labels c, w, f, and we write e.g. $A \xrightarrow{(wc)} B$ to indicate that the arrow $A \longrightarrow B$ carries labels w, c and possibly f. Auxilary items $(\longrightarrow)_0$, $(wc)_0$, $(c)_0$ define notation used to bootstrap the definition, and labels $(wc)_0$, $(c)_0$ are not part of structure of $StNaamen$. We define:

notes by misha gavrilovich. Parts of these notes, especially those connecting Quillen's model categories with Shelah's approach to cardinal arithmetic, arose in the course of a joint work with Assaf Hasson, and will eventually appear in the form of a joint paper. Any help in proofreading is appreciated.

$(\longrightarrow)_0$ $\{A\} \longrightarrow \{B\}$ iff $A \subseteq B$

$(wc)_0$ $\{A\} \xrightarrow{(wc)_0} \{B\}$ iff the difference $B \setminus A$ is finite (and $A \subseteq B$)

$(c)_0$ $\{A\} \xrightarrow{(c)_0} \{B\}$ iff $\operatorname{card} A = \operatorname{card} B$ or $\operatorname{card} B \leqslant \aleph_0$ (and $A \subseteq B$)

$(\longrightarrow)$ $X \longrightarrow Y$ iff $\forall x \in X\ \exists y \in Y\ (x \subseteq y)$

(f) $X \xrightarrow{(f)} Y$ iff $\{a\} \longrightarrow \{b\} \rightthreetimes X \longrightarrow Y$ whenever $\{a\} \xrightarrow{(wc)_0} \{b\}$

(wf) $X \xrightarrow{(wf)} Y$ iff $\{a\} \longrightarrow \{b\} \rightthreetimes X \longrightarrow Y$ whenever $\{a\} \xrightarrow{(c)_0} \{b\}$

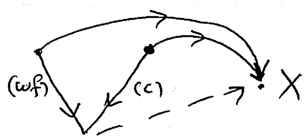
(c) $A \xrightarrow{(c)} B$ iff $A \longrightarrow B \rightthreetimes X \longrightarrow Y$ whenever $X \xrightarrow{(wf)} Y$

(wc) $A \xrightarrow{(wc)} B$ iff $A \longrightarrow B \rightthreetimes X \longrightarrow Y$ whenever $X \xrightarrow{(f)} Y$

(w) $A \xrightarrow{(w)} Y$ iff it decomposes as $A \overset{(wc)}{\dashrightarrow} \cdot \overset{(wf)}{\dashrightarrow} Y$

1.3. Let $QtNaamen \subseteq StNaamen$ be the full subcategory of $StNaamen$ consisting of sets X such that any of the following equivalent conditions hold for any M, a, b and arrows $A \xrightarrow{(c)} B$, $B' \xrightarrow{(wf)} B$, $X \longrightarrow Y$:

(Qt_1) $A \xrightarrow{(c)} B \rightthreetimes X \longrightarrow Y$ or $B' \xrightarrow{(wf)} B \rightthreetimes X \longrightarrow Y$



(Qt'_1) $\cup\{X'' : X \dashleftarrow X_0 \overset{(c)}{\dashleftarrow} X'' \overset{(wf)}{\dashrightarrow} X' \dashrightarrow X\} \dashrightarrow X$

(Qt_2) if $\{a\} \rightarrow X$, $B' \longrightarrow X$ and $\{a\} \xrightarrow{(c)} \{b\}$ and $B' \xrightarrow{(wf)} \{b\}$, then $\{b\} \longrightarrow X$

(Qt_3) $M^{\leqslant\aleph_0} \longrightarrow X$ implies $M^{\leqslant \max_{x \in X} \operatorname{card}(x \cap M)} \longrightarrow X$ (where $M^{\leqslant\lambda} := \{L \subseteq M : \operatorname{card} L \leqslant \lambda\}$)

2. The above c-w-f-arrow notation allows one to use in set theory the language of commutative diagrams of model categories, e.g. to draw an analogy between a fibre bundle $V \longrightarrow B$ and an inductive construction $\{M_i\} \xrightarrow{(f)} \{\cup_i M_i\}$. Here f stands for fibration, c stands for cofibration, and w stands for weak (homotopy) equivalence.

Below we develop an example to show that the language of model categories retains some of its powers. Diagram chasing along with basic set theory arguments gives:

2.1. The category $QtNaamen$ with these labels is a model category (see the appendix for the definition).

2.2. Let $\mathcal{A}, \mathcal{B}$ be quasi-partially ordered sets considered as categories where $x \longrightarrow y$ iff $x \leqslant y$. Then a (covariant) *functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a non-decreasing function

$F : \mathcal{A} \longrightarrow \mathcal{B}$. If both $\mathcal{A}$ and $\mathcal{B}$ are also equipped with a c-w-f labelling, we say that a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is *homotopy-invariant* iff for any arrow $X \xrightarrow{(w)} Y$ (weak homotopy equivalence), it holds $F(X) \xrightarrow{(w)} F(Y)$. An initial object $\perp$ of $\mathcal{A}$ is a minimal element of $\mathcal{A}$ (whenever such exists). (As any diagram is commutative in these categories, we need not state the conditions that the functors have to respect commutative diagrams.)

2.3. Let On be the category of ordinals where each arrow is labelled (cf) and each isomorphism is labelled (cwf). For a *function* $F : \mathcal{A} \longrightarrow On$, define (minimum is taken over all finite sequences labelled as shown)

$$\mathbb{L}_c F(X) = \min \left\{ F(Y) : \begin{tikzcd} & X_1 & & X_3 & & X_n \arrow[r, dashed] & Y \\ X \arrow[ur, dashed] & & X_2 \arrow[ul, dashed, "(w)"'] \arrow[ur, dashed] & & \dots \arrow[ul, dashed, "(w)"'] \arrow[ur, dashed] & & \perp \arrow[u, dashed, "(c)"'] \end{tikzcd} \right\}$$

2.4. $\mathbb{L}_c F(X)$ is a homotopy invariant functor "closest from the left"(Quillen, I:4.1) to the function $F : StNaamen \longrightarrow On$, by which is meant: for any homotopy-invariant functor $G : StNaamen \longrightarrow On$ such that $G(X) \longrightarrow F(X)$ for any object X such that $\perp \xrightarrow{(c)} X$, it holds that $G(Y) \longrightarrow \mathbb{L}_c F(Y)$ for any $\perp \xrightarrow{(c)} Y$ (note then there is a natural transformation from functor G to functor $\mathbb{L}_c F$).

In particular, the function $\mathbb{L}_c F : StNaamen \longrightarrow On$ is the left derived functor of $F : StNaamen \longrightarrow On$ provided that F is a functor.

2.5. Take $F = \mathrm{card}$ to be the cardinality function. Arguably, the model category formalism suggests we view $\mathbb{L}_c\mathrm{card} : StNaamen \longrightarrow On$ as an analogue of a cofibrantly replaced left derived functor of the “forgetful functor” $\mathrm{card} : StNaamen \longrightarrow On$. Then homotopy yoga suggests we view values of $\mathbb{L}_c\mathrm{card}$, e.g. $\mathbb{L}_c\mathrm{card}(\{\aleph_\alpha\}) = \mathbb{L}_c\mathrm{card}(\{X : X \subseteq \aleph_\alpha\})$, as (homotopy-invariant and therefore) more robust and interesting invariants, as compared to the non-homotopy-invariant values $\mathrm{card}(\{X : X \subseteq \aleph_\alpha\})$.

2.6. And indeed, it is for the reasons of being more robust and less prone to change by forcing that the values of $\mathbb{L}_c\mathrm{card}(\{\aleph_\alpha\})$ (for limit $\aleph_\alpha$) have been introduced in set theory (Shelah, Cardinal Arithmetic). Set-theoretically, $\mathbb{L}_c\mathrm{card}(\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2)$ is the least size of a family X of countable subsets of $\aleph_\alpha$, such that every countable subset of $\aleph_\alpha$ is a subset of a set in the family X. This may used, for example, to study the cardinality $(\aleph_\alpha)^{\aleph_0}$ of the set of countable subsets of $\aleph_\alpha$, via the bound $(\aleph_\alpha)^{\aleph_0} \leqslant \mathrm{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2) + 2^{\aleph_0}$, by decomposing it into a "noise" "non-homotopy-invariant" part $2^{\aleph_0}$ whose value is known to be highly independent of ZFC (and easy to force to change), and a homotopy-invariant part $\mathrm{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2)$ which admit bounds in ZFC (and is harder to force to change).

2.7. A short calculation gives $\mathbb{L}_c\mathrm{card}(\{X : X \subseteq \aleph_0\}) = 1$ (in ZFC) whereas it is known that there are models of ZFC where e.g. $\mathrm{card}(\{X : X \subseteq \aleph_0\}) = 2^{\aleph_0} > \aleph_{\omega_\omega}$. Meanwhile, non-trivially, Shelah (Cardinal Arithmetic, IX:4) proves $\mathbb{L}_c\mathrm{card}(\{\aleph_\omega\}) <$

$\aleph_{\omega_4}$. Similar upper bounds exist on $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\})$ for (most) $\aleph_\alpha$ limit (excepting $\aleph_\alpha = \alpha$), and are provided by PCF theory.

2.8. Arguably, the above justifies saying that the homotopy-invariant version of Generalised Continuum Hypothesis has less independence of ZFC, as suggested by homotopy theory.

2.9. **Remarks.** These remarks are explained in more details in [Gavrilovich].

2.9.1. Gromov [Ergosystems] writes that «The category/functor modulated structures can not be directly used by ergosystems, e.g. because the morphisms sets between even moderate objects are usually unlistable. But the ideas of the category theory show that there are certain (often non-obviuos) rules for generating proper concepts.» Curiously, in our categories where this obstruction does not arise, all definitions we make seem to be a result of a rather direct and automatic, straightforward repeated application of the lifting property to basic concepts of naive set theory, and the axioms of a model category admit a functional semantics whereby they are interpreted as rules to draw arrows and add labels on labelled graphs.

2.9.2. Shelah explicitly states his ideology of PCF theory in Shelah (Logical Dreams), e.g. Thesis 5.10, and we find it remarkably similar to the model category ideology as applied to StNaamen. It is unclear whether a deeper connection with PCF theory exists, e.g. whether the sequence of PCF generators is a (non-pointed) analogue of a (co)fibration sequence, or whether $X \longmapsto \{X\}$ and $X \longmapsto \cup_{x\in X} x$ can be usefully viewed as analogues of suspension $X \longmapsto \Sigma X$ and loop $X \longmapsto \Omega X$ spaces.

2.9.3. Manin (A course in logic, 2010, p.174) discusses the Continuum Hypothesis and the possibility for a need to "try to find alternative languages and semantics". It would seem that the connection between homotopy theory (in the model category formalism) and set theory (in ZFC or NF, or similar formalisms) we suggest, may provide for such an alternative language and semantics.

2.9.4. Our original motivation was to associate a model category (via the class of families of models) to an uncountably categorical theory and, more generally, to an excellent abstract elementary class (Shelah, Classification theory of non-elementary classes). In particular, we wanted to use the language of homotopy theory to perform the model-theoretic analysis of complex exponentiation $(\mathbb{C}, +, *, \exp)$ (Zilber, Pseudo-exponentiation on algebraically closed fields of characteristic zero) and covers of semi-Abelian varieties ([Bays] and references therein). These results claim there exist a unique, up to an appropriate notion of isomorphism (*not* respecting topology), function $\mathrm{e}x\ : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying $\mathrm{e}x\,(x+y) = \mathrm{e}x\,(x)\mathrm{e}x\,(y)$, the Schanuel conjecture and a dual thereto; Bays replaces $\mathbb{C}$ and $\mathrm{e}x$ by an elliptic curve and its cover $\mathrm{e}x_E : \mathbb{C} \longrightarrow \mathbb{C}/\Lambda$. Their analysis leads to a number- and geometric-theoretic conditions on semi-Abelian varieties (Mumford-Tate, Kummer theory, Mordell-Weil, Schanuel Conjecture); we wanted an analysis covering more general algebraic varieties which would to lead to geometric conditions in place of those above.

2.10. Thanks. I thank my Mother and Father for support, patience and more. I also thank Artem Harmaty for attention to this work, and encouraging conversations, and Martin Bays for reading and discussing. Detailed thanks are in the report [Gavrilovich].

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AXIOMS OF A MODEL CATEGORY IN LABELLED COMMUTATIVE DIAGRAMMES NOTATION.

We state the axioms of Quillen of a model category in their original form. In particular, we follow the axiom numeration of Quillen(Homotopical Algebra).

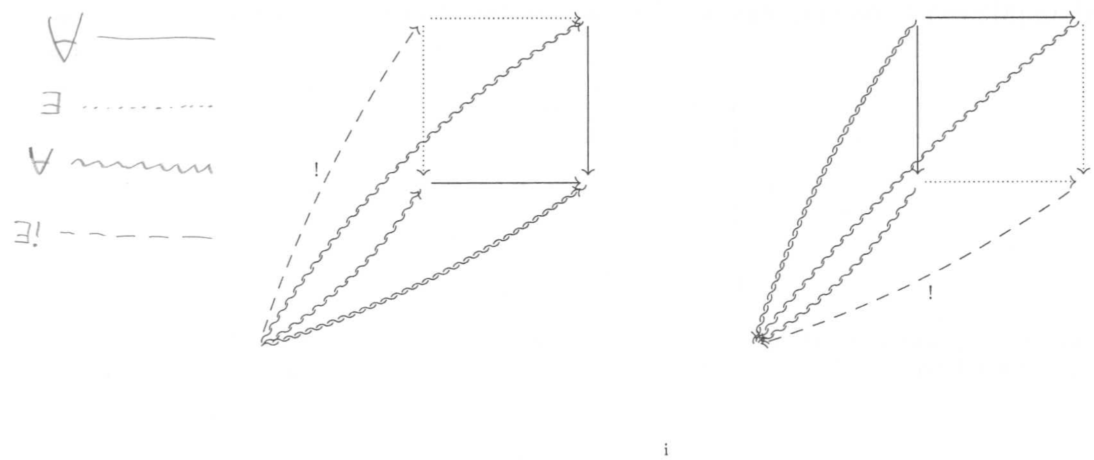
Notation (Commutative diagrammes). *Commutative diagrammes will be used systematically throughout this note. Most importantly, diagrammes will be used to introduce new definitions. We introduce our notation for commutative diagrams. The properties defined are always properties of arrows. To distingish the arrows in the diagrammes which are the object of the definition we will denote them by ◄ or ►. We will mostly use commutative diagrammes to introduce* $\forall\exists$*-definitions. In such cases solid arrows will be universally quantified and dashed arrows will be existentially quantified. Whenever definitions involving higher quantifier depth (such as in Figure) a legend will be provided. As in Figure 1, we will use the notation* $X \xrightarrow{\therefore(\cdot)} Y$ *to mean "if the commutative diagram is true, then* $X \longrightarrow Y$ *is labeled* $(\cdot)$*". Notation* $X \xrightarrow{!} Y$ *indicates uniqueness. A legend on the right might be used to indicate the quantifiers and their order (from top to bottom). Unless stated otherwise, solid arrows are quantified universally, and dotted arrows are quantified existentially.*

Axiom ($M0$). *The category* $\mathcal{C}$ *is closed under finite projective and injective limits. It is known that it is enough to require existence of initial objects, terminal objects and pullbacks and pushouts.*

$\forall$ ——

$\exists$ - - - -

$$\perp \dashrightarrow \bullet \xrightarrow{(id)} \bullet \qquad\qquad \bullet \xrightarrow{(id)} \bullet \dashrightarrow \top$$



Axiom (M1). *The two following lifting properties for labeled arrows hold:*

$\forall$ ——
$\exists$ ·········

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "(wc)"'] & \cdot \arrow[d, "(f)"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d, "(c)"'] & \cdot \arrow[d, "(wf)"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd}$$

Axiom (M2). *The following two ∀∃-diagrams hold:*

$\forall$ ——
$\exists$ ·········

$$\begin{tikzcd} & \cdot \arrow[dr, dotted, "(f)"] & \\ \cdot \arrow[ur, dotted, "(wc)"] \arrow[rr] & & \cdot \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[dr, dotted, "(wf)"] & \\ \cdot \arrow[ur, dotted, "(c)"] \arrow[rr] & & \cdot \end{tikzcd}$$

Axiom (M3(ccc,fff)). *Fibrations and cofibrations are stable under compositions. Namely, the following two ∀∃-diagrams hold:*

$\forall$ ——
$\exists$ ·········

$$\begin{tikzcd} & \cdot \arrow[dr, "(f)"] & \\ \cdot \arrow[ur, "(f)"] \arrow[rr, dotted, "(f)"'] & & \cdot \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[dr, "(c)"] & \\ \cdot \arrow[ur, "(c)"] \arrow[rr, dotted, "(c)"'] & & \cdot \end{tikzcd}$$

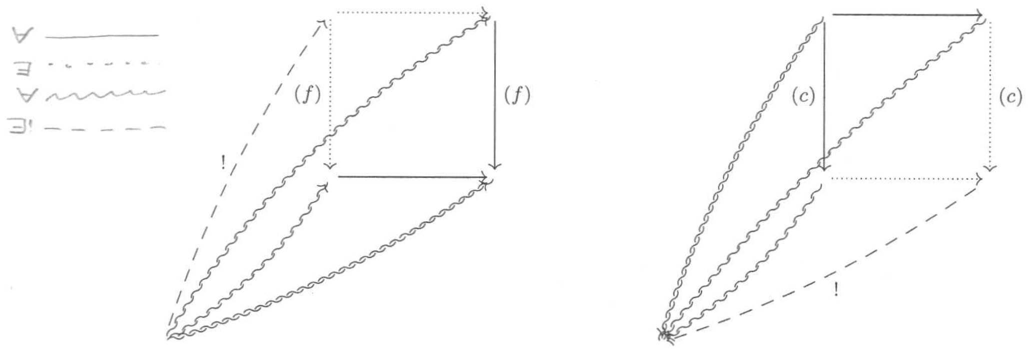
Axiom (M2(cwf)). *Isomorphisms are fibrations, co-fibrations and weak equivalences:*

$\forall$ ——
$\exists$ ·········

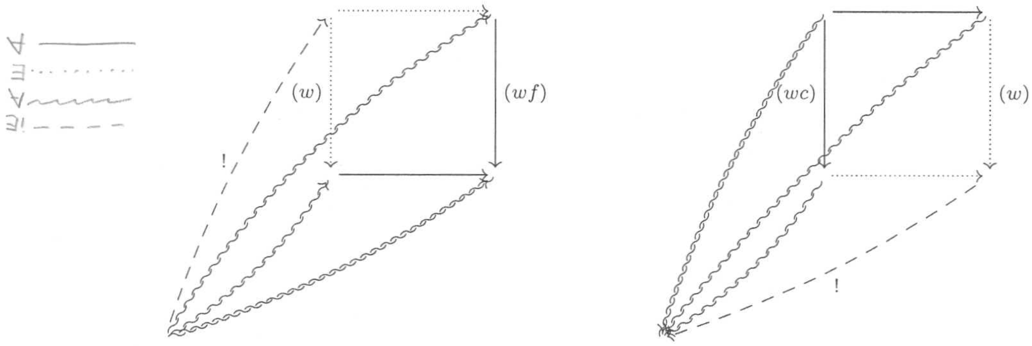
$$\begin{tikzcd} X \arrow[r] \arrow[d, "(wcf)\therefore"'] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Figure 1: The figure reads: if the commutative ∀∃-diagramme is true then *the left arrow is labeled (wcf).*

Axiom (M3($f \leftarrow f, c \rightarrow c$)). *Fibrations and cofibrations are stable under base change and co-base change respectively. I.e. the following diagrammes are true:*



Axiom (M4($wf \leftarrow w, wc \rightarrow w$)). *The base extension of an arrow labeled (wc) and the co-base extension of an arrow labeled (wf) are both labeled (w):*



The last axiom assures that weak equivalence is close enough to being transitive:

Axiom (M5, Two out of three). *In a triangluar diagram, if any two of the arrows are labeled (w) so is the third*