CHARACTERIZATIONS OF MODULES DEFINABLE IN O-MINIMAL STRUCTURES

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Abstract. Let $\mathfrak{M}$ be an o-minimal expansion of a densely linearly ordered set and $(S, +, \cdot, 0_S, 1_S)$ be a ring definable in $\mathfrak{M}$. In this article, we develop two techniques for the study of characterizations of S-modules definable in $\mathfrak{M}$. The first technique is an algebraic technique. More precisely, we show that every S-module definable in $\mathfrak{M}$ is finitely generated. For the other technique, we prove that if S is an infinite ring without zero divisors, every S-module definable in $\mathfrak{M}$ admits a unique definable S-module manifold topology. As consequences, we obtain the following: (1) if S is finite, then a module A is isomorphic to an S-module definable in $\mathfrak{M}$ if and only if A is finite; (2) if S is an infinite ring without zero divisors, then a module A is isomorphic to an S-module definable in $\mathfrak{M}$ if and only if A is a finite dimensional free module over S; and (3) if S is an infinite ring without zero divisors, then every S-module definable in $\mathfrak{M}$ is connected with respect to the unique definable S-module manifold topology.

Throughout this paper, let $\mathfrak{M}$ be a fixed (but arbitrary) *o-minimal* expansion of a densely linearly ordered set $(M, <)$ (that is, every unary definable set is a finite union of open intervals and points). We assume the reader's familiarity with basic model theory and o-minimality. (We refer to [1] and [7] for more on model theory and [2], [13], [5], and [14] for more on o-minimality). Here, the word "definable" means "definable in $\mathfrak{M}$ possibly with parameters" and the word "0-definable" means "definable in $\mathfrak{M}$ without parameters". Recall that we may equip M with the ordered topology induced by $<$; therefore, every subset of M^n can be equipped with the subspace topology induced by the product topology on M^n. Unless indicated otherwise, topological properties on a subset of M^n are considered with respect to this topology. For natural numbers $m \leq n$, let $\Pi(n, m)$ denote the set of all coordinate projections from M^n to M^m. For any set $X \subseteq M^n$, let $\dim X$ denote the largest natural number m where there exists $\pi \in \Pi(n, m)$ such that the image $\pi(X)$ has nonempty interior.

Let $(G, *, e)$ be a group with the group operation $*$ and the identity e. We say that the group $(G, *, e)$ is a *definable group* if the set G and the group operation $*$ are definable. We will simply write G if the group operation and the identity are clear from the context. Note that every finite group is isomorphic to a definable group. In [12], A. Pillay introduced definable group manifolds and used them to study characterizations of infinite definable groups.

Let X be a definable set and τ be a topology on X. We say that τ is a *definable topology* if there is a definable collection of subsets of X that generates τ. We call every element of τ

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a τ-open set. A map from X^n to X^m is *τ-continuous* if the map is continuous with respect to the product topologies on X^n and X^m generated by τ. Next, let G be a definable group. Obviously, we may equip G with the subspace topology induced by the ordered topology on $(M, <)$ or the discrete topology. These topologies are definable topologies on G. In addition, for each $k \in \mathbb{N}$, we say that a definable topology τ on G is a *definable group k-manifold topology* if both the group operation and the inversion map are τ-continuous, and there exist definable τ-open subsets $D_1, \ldots, D_n$ of G and definable maps $\phi_1, \ldots, \phi_n$ such that $\bigcup\{D_i : i = 1, \ldots, n\} = G$ and each $\phi_i \colon D_i \to M^k$ is a homeomorphism from D_i onto its image. Interestingly, by [12], we obtained that every definable group admits a unique definable group $\dim G$-manifold topology, τ_G. In [15], V. Razenj proved that if $\dim G = 1$ and G is definably τ_G-connected, then G is isomorphic to either $\bigoplus_{i \in I} \mathbb{Q}$ or $\bigoplus_{p \in \mathbb{P}} \mathbb{Z}_{p^\infty} \oplus \bigoplus_{i \in I} \mathbb{Q}$ where $\mathbb{P}$ is the set of all primes. Characterizations of 2-dimensional and 3-dimensional definable groups are studied in [8]. We know that if $\dim G = 2$ and G is a definably τ_G-connected, non-abelian definable group, then there is a real closed field T such that G is isomorphic to a semidirect product of the additive group of T and the multiplicative group of the positive elements of T; and if $\dim G = 3$ and G is a non-solvable, centerless, definably τ_G-connected definable group, then there is a real closed field T such that G is isomorphic to either $PSL_2(T)$ or $SO_3(T)$. In [3], M. Edmundo introduce a notion of definable G-modules and used them to study definable solvable groups.

Analogously, definable rings are also studied in [9]. Let $(S, +, \cdot, 0_S, 1_S)$ (or simply write S if it is clear from the context) be a ring. We say that S is a *definable ring* if the set S, the addition $+$ and the multiplication $\cdot$ are definable. For each $k \in \mathbb{N}$, a topology τ on S is a *definable ring k-manifold topology* if the addition, the additive inversion and the multiplication are τ-continuous and there exist definable τ-open subsets $D_1, \ldots, D_n$ of G and definable maps $\phi_1, \ldots, \phi_n$ such that $\bigcup\{D_i : i = 1, \ldots, n\} = S$ and each $\phi_i \colon D_i \to M^k$ is a homeomorphism from D_i onto its image. We also know that S admits a unique definable ring $\dim S$-manifold topology, τ_S. In [10], Y. Peterzil and C. Steinhorn proved that if S is an infinite definable ring without zero divisors, then there is a real closed field T such that S is definably isomorphic to either T, $T(\sqrt{-1})$, or $\mathbb{H}(T)$ where $\mathbb{H}(T)$ denote the ring of quaternions over T; therefore, S is a division ring.

Inspired by these results, we are interested in an intermediate step. To be more precise, the main question of this article is to find characterizations of definable modules. Let $(S, +, \cdot, 0_S, 1_S)$ be a definable ring and $(A, \oplus, 0_A, \lambda_S)$ be a left (right) S-module where $\lambda_S \colon S \times A \to A$ is the left (right) scalar multiplication. We say that A is a *definable left (right) S-module* if $(A, \oplus, 0_A)$ is a definable group and λ_S is definable. For the sake of readability, we will write λ instead of λ_S if the ring S is clear from the context. To study characterizations of definable S-modules, we develop two techniques. For the first approach, we consider the generators of A as S-module. The key step is to show that every definable S-module is finitely generated (see Section 1). As a result, we obtain:

Theorem A. *(1) If S is a finite ring and A is an S-module, then A is isomorphic to a definable S-module if and only if A is finite.*

(2) Suppose S is an infinite definable ring without zero divisors and A is an S-module. Then A is isomorphic to a definable S-module if and only if A is a finite dimensional free module over S.

In addition, by the Fundamental Theorem of Finite Abelian Groups, the characterization of infinite definable rings without zero divisors, and Theorem A, we have:

Corollary A. *(1) Suppose S is a finite ring and A is a definable S-module. Then A is isomorphic to a direct product of cyclic groups of prime-power order.*

(2) Suppose S is an infinite definable ring without zero divisors and A is a definable S-module. Then there exist a definable real closed field T and a natural number k such that T is a subring of S and A is definably isomorphic (as S-modules) to either T^k, $T(\sqrt{-1})^k$ or $\mathbb{H}(T)^k$.

Next, since manifold topologies on algebraic structures are important tools to study characterizations, we also develop a result on the existence of definable module manifold topologies, which will be introduced in Section 2, and use it to give an alternative proof of (2) in Theorem A. Interestingly, this proof implies that every definable module over infinite definable ring without zero divisors is connected with respect to the unique definable group manifold topology.

Conventions and Notations

Throughout this paper, d, k, m, n and p will range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers. Let $\bar{a} = (a_1, \dots, a_n) \in M^n$. For notational simplicity, we also use $\bar{a}$ to denote the set $\{a_1, \dots, a_n\}$.

1. Generators of Modules

Let $a_1, \dots, a_n \in A$. The *span* of $\{a_1, \dots, a_n\}$ is the set

$$\mathrm{Span}_S\{a_1, \dots, a_n\} = \{\lambda(s_1, a_1) \oplus \cdots \oplus \lambda(s_n, a_n) : s_1, \dots, s_n \in S\}.$$

We will say that A is *finitely generated* if there exist $a_1, \dots, a_n \in A$ such that $\mathrm{Span}_S\{a_1, \dots, a_n\} = A$. It is easy to see that if A is a definable S-module, then $\mathrm{Span}_S\{a_1, \dots, a_n\}$ is a definable subgroup of A.

In [11], Y. Peterzil and S. Starchenko proved:

1.1. **Lemma.** [11, Lemma 2.16] *Suppose $\mathfrak{M}$ is $\aleph_0$-saturated and G is a definable group. Then there exist $a_1, \dots, a_k \in G$ such that the only definable subgroup of G containing $a_1, \dots, a_n$ is G.*

Note that such $a_1, \dots, a_k$ in the above lemma are not generators of the group G in the sense of classical group theory since every finitely generated group must be countable. However, when we consider in the context of definable S-modules, the above result gives us more descriptive information.

Theorem B. *Suppose A is a definable S-module. Then A is finitely generated.*

Proof. Let $\bar{b} \in M^k$ and $\varphi(\bar{x}, \bar{z}), \psi(\bar{y}, \bar{z})$ be formulas such that $\varphi(\bar{x}, \bar{b})$ defines the ring S and $\psi(\bar{y}, \bar{b})$ defines the S-module A. Let $\mathfrak{N}$ be an elementary extension of $\mathfrak{M}$ that is $\aleph_0$-saturated. Then $\varphi(\bar{x}, \bar{b})$ defines a ring S' in $\mathfrak{N}$ and $\psi(\bar{y}, \bar{b})$ defines an S'-module A' in $\mathfrak{N}$. By Lemma 1.1, there exist $d_1, \dots, d_k \in A'$ such that the only definable subgroup of A' containing $d_1, \dots, d_k$ is A'. Since $\mathrm{Span}_{S'}\{d_1, \dots, d_k\}$ is a definable subgroup of A', we have

$$\mathrm{Span}_{S'}\{d_1, \dots, d_k\} = A'.$$

Then $\bar{y} \in A'$ if and only if there exist $\bar{x_1}, \dots, \bar{x_k} \in S'$ such that $\bar{y} = \lambda(\bar{x_1}, d_1) \oplus \cdots \oplus \lambda(\bar{x_k}, d_k)$. Let $\chi(\bar{y}, \bar{y_1}, \dots, \bar{y_k})$ be the formula representing

$$\psi(\bar{y}, \bar{b}) \leftrightarrow \exists \bar{x_1} \dots \exists \bar{x_k}, \bigwedge_{i=1}^{k} \varphi(\bar{x_i}, \bar{b}) \wedge \bar{y} = \lambda(\bar{x_1}, \bar{y_1}) \oplus \cdots \oplus \lambda(\bar{x_k}, \bar{y_k})$$

Therefore,

$$\mathfrak{N} \models \exists \bar{y_1} \dots \exists \bar{y_k} \forall \bar{y}, \chi(\bar{y}, \bar{y_1}, \dots, \bar{y_k}).$$

Since $\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$ and $\bar{b}$ is in M,

$$\mathfrak{M} \models \exists \bar{y_1} \dots \exists \bar{y_k} \forall \bar{y}, \chi(\bar{y}, \bar{y_1}, \dots, \bar{y_k}).$$

Therefore, A is finitely generated. □

We now give the first proof of Theorem A.

Proof of Theorem A. Obviously, every finite S-module is isomorphic to a definable S-module. If S is finite and A is a definable S-module, by Theorem B, we have that A is also finite. Therefore, we obtain (1) in Theorem A.

To prove (2), suppose S is an infinite definable ring without zero divisors. Obviously, each S^k is a definable S-module and every finite dimensional free module over S is isomorphic to S^k (for some k) as S-modules. Suppose A is isomorphic to a definable S-module. Without loss of generality, we assume that A is a definable S-module. Recall that every infinite definable ring without zero divisors is a division ring and every module over a division ring is free. By Theorem B, we have A is a finitely generated module over S; hence, A is a finite dimension free module over S. □

In addition, Theorem B also provides information about definable ideals of S. Observe that every definable ideal of S is a definable S-module with respect to the induced operators from S. The following is an immediate consequence of Theorem B and this observation.

Corollary B. *Every definable ideal of S is a finitely generated ideal.*

2. Definable S-Module Manifold Topologies

From now, we assume A is a definable S-module. For each topology τ, we say a map $f \colon S \times A \to A$ is *τ-continuous* if f is continuous with respect to the product topology $\tau_S \times \tau$ on $S \times A$ and the topology τ on A. Let $k \in \mathbb{N}$. A definable topology τ on A is a *definable S-module k-manifold topology* if the addition, the additive inversion, and the scalar multiplication are τ-continuous and there exist definable τ-open subsets $D_1, \dots, D_n$ of A and definable maps $\phi_1, \dots \phi_n$ such that $\bigcup\{D_i : i = 1, \dots, n\} = A$ and each $\phi_i \colon D_i \to M^k$ is a homeomorphism from D_i onto its image.

For a definable topology τ, we say that a set is *definably τ-connected* if it is not a disjoint union of two definable τ-open sets. Observe that for definable topologies τ_1 and τ_2, the product of a definably τ_1-conneced set and a definably τ_2-connected set is definably $(\tau_1 \times \tau_2)$-connected. We know that, by [12, Corollary 2.10] and Cell Decomposition Theorem, if τ is a definable group $\dim A$-manifold topology on A, then the definably τ-connected component containing the identity 0_A, denoted by A^0, exists.

2.1. **Lemma.** *If A admits a definable S-module $\dim A$-manifold topology, then A^0 is a definable S-submodule of A.*

Proof. Let τ be a definable S-module $\dim A$-manifold topology on A. By [12, Proposition 2.12], we have A^0 is the smallest definable subgroup of finite index in A. Therefore, $\dim A^0 = \dim A$. Recall that S has only finitely many definably τ_S-connected components. Let $S_1, \ldots, S_k$ enumerate all definably τ_S-connected components of S. Therefore, each $S_i \times A^0$ is definably $(\tau_S \times \tau)$-connected. Since λ is τ-continuous and $0_A \in A^0$, each image $\lambda(S_i \times A^0)$ is a definably τ-connected set containing 0_A. Therefore, $\lambda(S \times A^0) = \bigcup\{\lambda(S_i \times A^0) : i = 1, \ldots, k\} \subseteq A^0$. It follows immediately that A^0 is an S-submodule of A. □

Recall that definable groups admit the descending chain condition on definable subgroups, i.e., every descending family $(G_i)_{i\in\mathbb{N}}$ of definable groups is eventually constant (see e.g. [12, Remark 2.13]). As a consequence of this result, we obtain:

2.2. **Lemma.** *Let G be a definable subgroup of A^0. Assume that there is $b \in A^0$ such that $kb \notin G$ for every positive integer k. Then there exists the smallest definable subgroup G' of A^0 containing $G \cup \{b\}$. In addition, we have $\dim G < \dim G' \leq \dim A^0$.*

Proof. Suppose to the contrary that there is no smallest definable subgroup of A^0 containing $G \cup \{b\}$. We recursively define a sequence $(A_i)_{i\in\mathbb{N}}$ of definable subgroups of A^0 as follows:

Set $A_0 = A^0$. Suppose $A_0, \ldots, A_i$ have been constructed. Then there exists a definable subgroup A'_i of A^0 containing $G \cup \{b\}$ such that A_i is not a subgroup of A'_i. Set $A_{i+1} = A_i \cap A'_i$. Then A_{i+1} is a proper definable subgroup of A_i containing $G \cup \{b\}$.

Therefore $(A_i)_{i\in\mathbb{N}}$ is an infinite proper descending chain of definable subgroups of A^0. This contradicts the descending chain condition of definable groups.

Let G' be the smallest definable subgroup of A^0 containing $G \cup \{b\}$. Since there is no positive interger k such that $kb \in G$, we have G is of infinite index in G'. By [12, Lemma 2.11], we have $\dim G < \dim G' \leq \dim A^0$. □

By the above lemmas, we can prove a key step towards an alternative prove of (2) in Theorem A.

2.3. **Lemma.** *If A admits a definable S-module $\dim A$-manifold topology, then A is a finitely generated module over S. Moreover, if A is a free module over S, then A is a finite dimensional free module over S.*

Proof. Without loss of generality, we assume that $\mathfrak{M}$ is $\aleph_1$-saturated. Note that A^0 is infinite and abelian.

Claim. Let G be a definable subgroup of A^0. Suppose for any $a \in A^0$, there is $k \in \mathbb{N} \setminus \{0\}$ such that $ka \in G$. Then $G = A^0$.

Proof of Claim. By saturation and Compactness Theorem, there is no positive integer k such that $ka \in G$ for all $a \in A^0$. Since $k(a \oplus G) = ka \oplus G = G$ for all $a \in G$, the quotient group A^0/G is of bounded exponent. By [16, Lemma 5.7], we have A^0/G is finite. Since A^0 is a subgroup of A of finite index, G also has finite index in A. Since $G \subseteq A^0$ and A^0 is the smallest definable subgroup of A of finite index, we have $G = A^0$. □

We recursively construct a sequence $(a_i)_{i\in\mathbb{N}}$ as follows:

Set $a_0 = 0_A$. Suppose $a_0, \dots, a_i$ have been constructed. If the smallest definable subgroup of A^0 containing $a_0, \dots, a_i$ is A^0, then let $a_{i+1} = 0_A$. Otherwise, by the above claim, let $a_{i+1} \in A^0$ such that ka_{i+1} does not contain in the smallest definable subgroup containing $a_0, \dots, a_i$ for any positive integer k.

For each $i \in \mathbb{N}$, let A_i be the smallest definable subgroup of A^0 containing $a_0, \dots, a_i$. By minimality and Lemma 2.2, we have $A_i \subseteq \mathrm{Span}_S\{a_0, \dots, a_i\}$ and $\dim A_i < \dim A_{i+1} \leq \dim A^0$ for every $i \in \mathbb{N}$. Since $\dim A^0 = n'$, we have $\dim A_j = n'$ for every $j \geq n'$ and it follows that $A_{n'} = A^0$. Since $A_{n'} \subseteq \mathrm{Span}_S\{a_0, \dots, a_{n'}\}$ and $a_0, \dots, a_{n'} \in A^0$, by Lemma 2.1, we get $A^0 = \mathrm{Span}_S\{a_0, \dots, a_{n'}\}$. Since A^0 is of finite index in A, there exist $b_0, \dots, b_p \in A$ such that $A = \bigcup_{j=0}^{p} b_j \oplus A^0$. Hence $A = \mathrm{Span}_S\{a_0, \dots, a_{n'}, b_0, \dots, b_p\}$ and therefore A is finitely generated. □

Remark. Since every finite dimensional free module over S is isomorphic to S^k for some $k \in \mathbb{N}$, if S is definably τ_S-connected, then A is definably τ_A-connected.

To complete this alternative proof of (2) of Theorem A, it suffices to prove the following:

Theorem C. *If S is an infinite definable ring without zero divisors, then A admits a unique definable S-module* $\dim A$*-manifold topology.*

Note that, by the uniqueness of definable group $\dim A$-manifold topology on A, the topologies obtained in Theorem C and [12, Proposition 2.5] coincide. Due to more sophisticated conditions on the scalar multiplication, we refine the construction to guarantee the continuity of the scalar multiplication. We will explicitly construct the topology of the module A in Section 3.

We end this section by an immediate consequence of Theorem C and the remark after Lemma 2.3.

2.4. **Corollary.** *If S is an infinite definable ring without zero divisors, then A is definably τ_A-connected.*

3. Proof of Theorem C

In [4], E. Hrushovski showed that an algebraic group can be recovered from birational data. Inspired by this result, A. Pillay gave a construction of definable group manifold topologies on definable groups (see [12]). In addition, M. Otero et al. showed an analog of the statement for definable rings in [9]. Here, we adopt these ideas.

Throughout the rest of this section, we assume that $\mathfrak{M}$ is very saturated. Let $B \subseteq M$ and $\bar{a} = (a_1, \dots, a_n) \in M^n$. The *definable closure* of B (denoted by $\mathrm{dcl}\, B$) is the set

$$\mathrm{dcl}\, B := \{x \in M : \{x\} \text{ is } B\text{-definable}\}.$$

We say that $\bar{a}$ is *independent* over B if $a_i \notin \operatorname{dcl}(B \cup (\bar{a} \setminus \{a_i\}))$ for every $i \in \{1, \ldots, n\}$. The *dimension* of $\bar{a}$ over B, denoted by $\dim(\bar{a}/B)$, is the least cardinality of a subset I of $\bar{a}$ such that $\bar{a} \subseteq \operatorname{dcl}(B \cup I)$; equivalently, the cardinality of maximal independent subtuples of $\bar{a}$ over B (see [12, Lemma 1.2]). Let $X \subseteq M^n$ be B-definable. By saturation, we have that

$$\dim X = \max\{\dim(\bar{a}/B) : \bar{a} \in X\}.$$

An element $\bar{a} \in X$ is a *generic* of X over B, if $\dim X = \dim(\bar{a}/B)$. We know that if $Q \subseteq M^{m+n}$ is B-definable, then the set $\{\bar{b} \in M^m : (\bar{b}, \bar{a}) \in Q$ for every generic $\bar{a}$ of X over $\bar{b}\}$ is B-definable. Let $Y \subseteq X \subseteq M^n$ be definable. We say that Y is *large* in X if $\dim(X \setminus Y) < \dim X$.

3.1. **Lemma.** [12, Lemma 1.12] *Let $Y \subseteq X$ be definable. Then Y is large in X if and only if for every $B \subseteq M$ over which X and Y are defined, every generic point of X over B is in Y.*

Recall that $(A, \oplus, 0_A)$ is a definable abelian group.

3.2. **Lemma.** [12, Lemma 2.1] *Let $b \in A$ and let a be a generic of A over b. Then $a \oplus b$ is a generic of A over b.*

3.3. **Lemma.** [12, Lemma 3.2] *Let $f \colon A \to A$ be a B-definable endomorphism of A with finite kernel (that is, the pre-image $f^{-1}(0_A)$ is finite). Then the image $f(A)$ has finite index in A. In particular, if a is a generic of A over B, then $f(a)$ is also a generic of A over B.*

3.4. **Lemma.** [12, Lemma 2.4] *Let V be a large definable subset of A. Then finitely many translates of V cover A.*

In addition, assume $A \subseteq M^n$ is a definable S-module, $S \subseteq M^m$ has no zero divisors, $\dim A = n' \leq n$ and $\dim S = m' \leq m$. Hence S is a division ring. To construct a definable left S-module manifold topology on A, we first introduce a special 5-tuple (V, W, X, Y, P), which is a main ingredient in our construction. We say that a 5-tuple (V, W, X, Y, P) of 0-definable sets has the property $(*)$ if

- *(i)* V is open and large in A, and V is a finite disjoint union of sets that are homeomorphic to an open subset of $M^{n'}$ under some coordinate projection from M^n to $M^{n'}$;
- *(ii)* $\ominus : V \to V$ is a 0-definable continuous bijection;
- *(iii)* W is open and large in $A \times A$, and $\oplus \colon W \to V$ is 0-definable and continuous;
- *(iv)* for any $v_1 \in V$, if v_2 is a generic of A over v_1, then $(v_2, v_1) \in W$ and $(\ominus v_2, v_1 \oplus v_2) \in W$;
- *(v)* X is open and large in S, and both $-, {}^{-1} : X \to X$ are 0-definable continuous bijections;
- *(vi)* Y is open and large in $S \times S$, and both $+, \cdot \colon Y \to X$ are 0-definable and continuous;
- *(vii)* for any $x_1 \in X$, if x_2 is a generic of S over x_1, then $(x_2, x_1) \in Y$, $(-x_2, x_1 + x_2) \in Y$ and $(x_2^{-1}, x_2 x_1) \in Y$;
- *(viii)* P is open and large in $S \times A$, and $\lambda \colon P \to V$ is 0-definable and continuous; and
- *(ix)* for any $x \in X$, if v is a generic of A over x, then $(x, v) \in P$ and $(x^{-1}, \lambda(x, v)) \in P$.

Recall that the the topological properties in $(*)$ are considered with respect to the topology induced from the ambient space.

3.5. **Proposition.** *There exists a special 5-tuple* (V,W,X,Y,P) *with the property* $(*)$*.*

3.1. **Definable manifold topologies on A.** We postpone the proof of the above proposition and suppose for now that we have a special 5-tuple (V,W,X,Y,P) with the property $(*)$. We define the topology τ_A on A and τ_S on S by

$$U \subseteq A \text{ is } \tau_A\text{-open if and only if for any } a \in A, (a \oplus U) \cap V \text{ is open in } V; \text{ and}$$
$$U \subseteq S \text{ is } \tau_S\text{-open if and only if for any } s \in S, (a + U) \cap X \text{ is open in } X.$$

By the same arguments as in [12] and [9], we obtain:

3.6. **Lemma.** *Let* $U \subseteq V$ *and* $a \in A$*. Then* $a \oplus U$ *is* τ_A*-open if and only if* U *is open in* V*.*

3.7. **Lemma.** *Let* $U \subseteq X$ *and* $s \in S$*. Then* $s + U$ *is* τ_S*-open if and only if* U *is open in* X*.*

3.8. **Lemma.** τ_A *is a definable group* n'*-manifold topology on* A*.*

3.9. **Lemma.** τ_S *is a definable ring* m'*-manifold topology on* S*.*

Next, we show that the scalar multiplication $\lambda : S \times A \to A$ is τ_A-continuous.

3.10. **Lemma.** *Let* $s \in S$ *be nonzero and* a *be a generic of* A *over* s*. Then* $\lambda(s,a)$ *is a generic of* A *over* s*.*

Proof. Let $f : A \to A$ be a function defined by $f(x) = \lambda(s,x)$. Then f is a $\{s\}$-definable endomorphism. Since A is a free S-module and $s \neq 0$, f is injective. So f has a finite kernel. Since a is a generic of A over s, by Lemma 3.3, $\lambda(s,a)$ is a generic of A over s. □

For $\mathcal{O} \subseteq X \times V$ and $(t,b) \in S \times A$, let

$$\Gamma^{t,b}_{\mathcal{O}} := \{(t+s, b \oplus a) \in S \times A : (s,a) \in \mathcal{O}\}.$$

3.11. **Lemma.** *Let* $\mathcal{O} \subseteq X \times V$ *and* $(t,b) \in S \times A$*. Then* $\Gamma^{t,b}_{\mathcal{O}}$ *is* $(\tau_S \times \tau_A)$*-open if and only if* $\mathcal{O}$ *is open in* $X \times V$*.*

Proof. Assume $\Gamma^{t,b}_{\mathcal{O}}$ is $(\tau_S \times \tau_A)$-open. We may write $\Gamma^{t,b}_{\mathcal{O}} = \bigcup\{S_i \times A_i : i \in I\}$ where $S_i \in \tau_S$ and $A_i \in \tau_A$, for some index set I. Observe that

$$\mathcal{O} = \bigcup\{\big((-t) + S_i\big) \times \big((\ominus b) \oplus A_i\big) : i \in I\}.$$

Since each S_i is τ_S-open and $(-t) + S_i \subseteq X$, by Lemma 3.7, each $(-t) + S_i$ is open in X. Similarly, by Lemma 3.6, each $(\ominus b) \oplus A_i$ is open in V. Therefore, $\mathcal{O}$ is open in $X \times V$. The converse can be proved by a similar argument. □

Remark. For $\mathcal{O} \subseteq X \times V$, since $\mathcal{O} = \Gamma^{0_S,0_A}_{\mathcal{O}}$, we have $\mathcal{O}$ is $(\tau_S \times \tau_A)$-open if and only if $\mathcal{O}$ is open in $X \times V$.

For $Q \subseteq M^{m+n}$ and $c \in M^m$, we define the *fiber of a set Q over c* by

$$Q_c := \{y \in M^n : (c,y) \in Q\}$$

3.12. **Lemma.** *Let $a, b \in A$ and $s \in S$. Then the set*

$$D = \{(x, v) \in X \times V : b \oplus \lambda(x + s, v \oplus a) \in V\}$$

is open in $X \times V$.

Proof. Fix $(x_0, v_0) \in D$. Since S is infinite, there exists $n_0 \in S$ such that $x_0 + s + n_0 \neq 0_S$ and $x_0 + s + n_0 + 1_S \neq 0_S$. To show that D is open in $X \times V$, we will find an open neighborhood of (x_0, v_0) contained in D. Let t be a generic of S over $\{s, n_0, x_0\}$ and c be a generic of A over $\{a, b, s, t, n_0, x_0, v_0\}$. Let

$$\begin{aligned}
U_0 &= \{(x, v) \in X \times V : (t, x) \in Y, tx + ts + tn_0 \in X, (c \oplus a, v) \in W\},\\
U_1 &= \{(x, v) \in U_0 : (tx + ts + tn_0, c \oplus a \oplus v) \in P\},\\
U_2 &= \{(x, v) \in U_1 : \lambda(t, c \oplus b) \oplus \lambda(tx + ts + tn_0, c \oplus a \oplus v) \in V\}, \text{ and}\\
U_3 &= \{(x, v) \in U_2 : (t^{-1}, \lambda(t, c \oplus b) \oplus \lambda(tx + ts + tn_0, c \oplus a \oplus v) \in P\}.
\end{aligned}$$

We define a subset U_4 of U_3 by $(x, v) \in U_4$ if and only if

$$(\ominus\lambda(x + s + n_0 + 1_S, c) \ominus \lambda(n_0, a \oplus v), c \oplus b \oplus \lambda(x + s + n_0, c \oplus a \oplus v)) \in W.$$

Since $\oplus(W) \subseteq V$, $U_4 \subseteq D$. Next, we will show that $(x_0, v_0) \in U_4$. Since $x_0 \in X$, $x_0 + s + n_0 \neq 0_S$ and t is a generic of S over $\{s, n_0, x_0\}$, by (vii), we have $(t, x_0) \in Y$ and $tx_0 + ts + tn_0 = t(x_0 + s + n_0) \in X$. Note that $c \oplus a \in V$. Since $v_0 \in V$ and $c \oplus a$ is a generic of A over v_0, by (iv), $(c \oplus a, v_0) \in W$, i.e. $(x_0, v_0) \in U_0$. From $tx_0 + ts + tn_0 \in X$ and $c \oplus a \oplus v_0$ is a generic of A over $\{s, t, n_0, x_0\}$, by (ix), $(tx_0 + ts + tn_0, c \oplus a \oplus v_0) \in P$, i.e. $(x_0, v_0) \in U_1$. By the genericity of c, we have (x_0, v_0) lies in both U_2 and U_3 (we also use (ix) for the latter result). By Lemma 3.10, $\lambda(x_0 + s + n_0 + 1_S, c)$ is a generic of A. Note that $c \oplus b \oplus \lambda(x_0 + s + n_0, c \oplus a \oplus v_0) = (b \oplus \lambda(x_0 + s, a \oplus v_0)) \oplus \lambda(x_0 + s + n_0 + 1_S, c) \oplus \lambda(n_0, a \oplus v_0)$. It follows that $(x_0, v_0) \in U_4$.

It remains to prove that U_4 is open in $X \times V$. Consider $g_1, g_3 : X \times V \to S \times A$, $g_2 : X \times V \to A$ and $g_4 : X \times V \to A \times A$ defined by

$$\begin{aligned}
g_1(x, v) &= (tx + ts + tn_0, c \oplus a \oplus v),\\
g_2(x, v) &= \lambda(t, c \oplus b) \oplus \lambda(tx + ts + tn_0, c \oplus a \oplus v),\\
g_3(x, v) &= (t^{-1}, \lambda(t, c \oplus b) \oplus \lambda(tx + ts + tn_0, c \oplus a \oplus v), \text{ and}\\
g_4(x, v) &= (\ominus\lambda(x + s + n_0 + 1_S, c) \ominus \lambda(n_0, a \oplus v), c \oplus b \oplus \lambda(x + s + n_0, c \oplus a \oplus v)).
\end{aligned}$$

By Lemmas 3.8, 3.9 and 3.11, we have that for each $i \in \{0, 1, 2, 3\}$, g_i is continuous on U_{i+1}. Observe that $U_0 = (X \cap Y_t \cap \left(\cdot^{-1}\left((+^{-1}(X))_{ts+tn_0}\right)\right)_t) \times (V \cap W_{c\oplus a})$, $U_1 = U_0 \cap g_1^{-1}(P)$, $U_2 = U_1 \cap g_2^{-1}(V)$, $U_3 = U_2 \cap g_3^{-1}(P)$ and $U_4 = U_3 \cap g_4^{-1}(W)$. Hence, U_4 is open in $X \times V$. □

Remark. Immediately from Lemma 3.12, the definable map $(x, v) \mapsto b \oplus \lambda(x + s, v \oplus a)$ is continuous from $D \to V$.

3.13. **Lemma.** *The scalar multiplication λ is τ_A-continuous.*

Proof. Let $U \subseteq A$ be τ_A-open. We shall show that $\lambda^{-1}(U) = \{(s, a) \in S \times A : \lambda(s, a) \in U\}$ is $(\tau_S \times \tau_A)$-open. By Lemma 3.4, we may assume that $U \subseteq c \oplus V$ for some $c \in A$. By Lemma 3.6, $(\ominus c) \oplus U$ is open in V. To show $\lambda^{-1}(U)$ is $(\tau_S \times \tau_A)$-open, it suffices to prove that for any $t \in S$ and $b \in A$, $K := \{(s, a) \in (t + X) \times (b \oplus V) : \lambda(s, a) \in U\}$ is $(\tau_S \times \tau_A)$-open. Let $\mathcal{O} = \{(s, a) \in X \times V : \lambda(t + s, b \oplus a) \in U\}$. Observe that $\Gamma^{t,b}_{\mathcal{O}} = K$. By Lemma 3.12, $\mathcal{O} = \{(s, a) \in X \times V : \ominus c \oplus \lambda(t + s, b \oplus a) \in \ominus c \oplus U\}$ is open in $X \times V$. By Lemma 3.11, $\Gamma^{t,b}_{\mathcal{O}}$ is $(\tau_S \times \tau_A)$-open. □

Therefore, to complete the proof of Theorem C, we only need to give a construction of this special 5-tuple with the property $(*)$.

3.2. **Construction of special 5-tuples.** We now start our construction. Cell Decomposition Theorem is a powerful tool in the study of o-minimal structures. The following result follows immediately from this theorem.

3.14. **Lemma.** *Suppose $E \subseteq M^n$ is 0-definable and $\dim E = n'$.*

(1) There exist pairwise disjoint 0-definable subsets $E_1, \dots, E_p$ of E such that (i) $E_1 \cup \dots \cup E_p$ is large in E; (ii) for each $i \in \{1, \dots, p\}$, E_i is open in E and is homeomorphic to an open subset of $M^{n'}$ under some coordinate projection from M^n to $M^{n'}$; and (iii) for each $i \neq j$, $\operatorname{cl} E_i \cap E_j = \emptyset$.

(2) For every definable map $f : E \to M^k$, there exists a large open dense 0-definable subset E' of E such that $f \restriction E'$ is continuous.

Recall that $A \subseteq M^n$ with $\dim A = n' \leq n$ and $S \subseteq M^m$ with $\dim S = m' \leq m$. For a set $X \subseteq M^n$, we denote by $\operatorname{cl} X$ the closure of X with respect to the induced topology from the ambient space.

Throughout the rest of this section, we fix pairwise disjoint 0-definable $E_1, \dots, E_p \subseteq A$ and $T_1, \dots, T_q \subseteq S$ such that

(1) each E_i is open in A and is homeomorphic to an open subset of $M^{n'}$ under some coordinate projection from M^n to $M^{n'}$;
(2) for each $i \neq j$, $\operatorname{cl} E_i \cap E_j = \emptyset$;
(3) $V_0 := E_1 \cup \dots \cup E_p$ is large in A;
(4) each T_j is open in S and is homeomorphic to an open subset of $M^{m'}$ under some coordinate projection from M^m to $M^{m'}$;
(5) for each $i \neq j$, $\operatorname{cl} T_i \cap T_j = \emptyset$; and
(6) $X_0 := T_1 \cup \dots \cup T_q$ is large in S.

3.15. **Lemma.** *There exist 0-definable sets V_1, W_1, X_1, Y_1 and P_1 such that*

(1) $V_1 \subseteq V_0$ is a large open subset of A and $\ominus \restriction V_1$ is a 0-definable continuous bijection onto V_1;

(2) $W_1 \subseteq V_0 \times V_0$ is a large open subset of $A \times A$ and $\oplus \restriction W_1$ is a 0-definable continuous map from W_1 into V_0;

(3) $X_1 \subseteq X_0$ is a large open subset of S and $- \restriction X_1$ and ${}^{-1} \restriction X_1$ are 0-definable continuous bijections onto X_1;

(4) $Y_1 \subseteq X_0 \times X_0$ is a large open subset of $S \times S$ and $+ \restriction Y_1$ and $\cdot \restriction$ are 0-definable continuous maps from Y_1 into V_0; and

(5) $P_1 \subseteq X_0 \times V_0$ is a large open subset of $X_0 \times V_0$ such $\lambda \restriction P_1$ is a 0-definable continuous map from P_1 into V_0.

Proof. We only focus on the constructions of X_1 and Y_1. We can follow the following argument to obtain V_1 and W_1. By Lemma 3.14, let $\tilde{X_0}$ be a 0-definable large open dense subset of X_0 such that $- \upharpoonright \tilde{X_0}$ and $^{-1} \upharpoonright \tilde{X_0}$ are continuous. Set $X_1 = \tilde{X_0} \cap (-\tilde{X_0}) \cap \tilde{X_0}^{-1} \cap (-\tilde{X_0})^{-1}$. It is clear that X_1 is open in X_0. To show that X_1 is large in S, let s be a generic of S over $\emptyset$. Since $\tilde{X_0}$ is large in S, $s \in \tilde{X_0}$. Note that $-s, s^{-1}$ and $(-s)^{-1}$ are also a generic of S over $\emptyset$. We can show that $s \in (-\tilde{X_0}) \cap \tilde{X_0}^{-1} \cap (-\tilde{X_0})^{-1}$. Therefore, $s \in X_1$ and so we have that X_1 is large in S. This completes the construction of X_1.

By Lemma 3.14 again, we obtain a 0-definable large open dense subset Y_1 of $X_0 \times X_0$ such that $+ \upharpoonright Y_1$ and $\cdot \upharpoonright Y_1$ is continuous. To obtain P_1, just apply Lemma 3.14 to P_0 and λ. □

Fix sets V_1, W_1, X_1, Y_1, P_1 as in Lemma 3.15. We now construct a 5-tuple (V, W, X, Y, P) that satisfies the property $(*)$.

3.16. **Lemma.** *Let $\bar{s}, \bar{t} \in S$. If $\bar{s}$ is a generic of S over $\bar{t}$ and $\bar{t}$ is a generic of S over $\emptyset$, then $\bar{t}$ is also a generic of S over $\bar{s}$.*

Proof. Assume $\bar{s}$ is a generic of S over $\bar{t}$ and $\bar{t}$ is a generic of S over $\emptyset$. Suppose to the contrary that $\bar{t}$ is not a generic of S over $\bar{s}$. Without loss of generality, we may assume that $\bar{s} = (s_0, \bar{s'})$ and $\bar{t} = (t_0, \bar{t'})$ where $t_0 \in \operatorname{dcl}(\bar{s} \cup \bar{t'})$ but $t_0 \notin \operatorname{dcl}(\bar{s'} \cup \bar{t'})$. Since the Exchange Lemma holds in $\mathfrak{M}$ (see e.g. [6, Theorem 2.2.2]), we have $s_0 \in \operatorname{dcl}(\bar{s'} \cup \bar{t})$, which is absurd. □

3.17. **Lemma.** *There exist a tuple (V, W, X, Y) of 0-definable sets that satisfies $(i) - (vii)$ in the property $(*)$.*

Proof. We only focus on the constructions of X and Y. We can follow the following argument to obtain V and W as desired. Let X_2 be the subset of S such that $t \in X_2$ if and only if:

(i) $t \in X_1$;
(ii) for every generic s of S over t, $(s, t) \in Y_1$, $(-s, s+t) \in Y_1$ and $(s^{-1}, st) \in Y_1$; and
(iii) for every generic a of A over t, $(t, a) \in P_1$ and $(t^{-1}, \lambda(t, a)) \in P_1$.

Note that X_2 is 0-definable. To show X_2 is large in S, let t be a generic of S over $\emptyset$. Since X_1 is 0-definable and large in S, $t \in X_1$. Let s be a generic of S over t. Then (s, t) is a generic point of $S \times S$ over $\emptyset$. Since Y_1 is large in $S \times S$, $(s, t) \in Y_1$. By Lemmas 3.16 and 3.2, s is a generic of S over $s + t$, and so $(-s, s+t), (s^{-1}, st) \in Y_1$. By the same argument, if a is a generic of A over t, by Lemma 3.10, $(t, a) \in P_1$ and $(t^{-1}, \lambda(t, a)) \in P_1$. So we have $t \in X_2$ and hence X_2 is large in S. Apply Lemma 3.14, we obtain a 0-definable subset $X_3 \subseteq X_2$ such that X_3 is large in S and open in X_0. Set $X = X_3 \cap (-X_3) \cap X_3^{-1} \cap (-X_3)^{-1}$. This completes the construction of X.

Next, define $Y = (X \times X) \cap \{(s, t) \in Y_1 : s + t \in X \text{ and } st \in X\}$. Since X is open in X_0, by Lemma 3.15, the $+ \upharpoonright Y$ and $\cdot \upharpoonright Y$ are 0-definable continuous maps from Y into X and Y is open in $X_0 \times X_0$. Lastly, we will verify that Y is large in $S \times S$, let (s_1, s_2) be a generic of $S \times S$ over $\emptyset$. Therefore $s_1 + s_2$ and $s_1 s_2$ are generics of S over $\emptyset$, i.e. $(s_1, s_2) \in Y$. It follows that Y is large in $S \times S$. □

We now complete:

Proof of Proposition 3.5. Let (V, W, X, Y) be a tuple obtained by Lemma 3.17. Define $P = (X \times V) \cap \{(s, a) \in P_1 : \lambda(s, a) \in V\}$. By Lemma 3.15, since $X \times V$ is open in $X_0 \times V_0$, $\lambda \restriction P$ is a 0-definable continuous map from P into V and P is open in $X_0 \times V_0$. To verify that P is large in $S \times A$, let (s, a) be a generic of $S \times A$ over $\emptyset$. By Lemma 3.10, $\lambda(s, a)$ is a generic of A over s. We have $\lambda(s, a) \in V$ and so $(s, a) \in P$, it follows that P is large in $S \times A$. Next, let $t \in X$ and a be a generic of A over t. Since $t \in X$, by Lemma 3.17, $(t, a) \in P_1$ and $(t^{-1}, \lambda(t, a)) \in P_1$. By Lemma 3.10, $\lambda(t, a) \in V$, i.e. $(t, a) \in P$. Since $\lambda(t^{-1}, \lambda(t, a) = a \in V)$, $(t^{-1}, \lambda(t, a)) \in P$. □

4. Open questions

4.1. *Suppose S is an infinite ring. Here, we obtain a complete characterization of definable S-modules when S has no zero divisors. However, the question is still open when S (possibly) has zero divisors.*

4.2. *Suppose A is a definable abelian group. Obviously, if $|A| = n$ for some positive integer n, then A is an $\mathbb{Z}/n\mathbb{Z}$-module. This gives rise to the question:*

If A is infinite, how to determine whether A is a definable S-module for some definable ring S?

Throughout the rest of this paper, we fix a definable ring $(S, +, \cdot, 0_S, 1_S)$ and a left S-module $(A, \oplus, 0_A, \lambda_S)$. Note that the arguments given next will also work for right S-modules.

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