POLYNOMIAL INVARIANTS OF GRAPHS AND TOTALLY CATEGORICAL THEORIES

J.A. MAKOWSKY* AND B. ZILBER**

Abstract. In the analysis of the structure of totally categorical first order theories, the second author showed that certain combinatorial counting functions play an important role. Those functions are *invariants of the structures* and are always polynomials in one or many variables, depending on the number of independent dimensions of the theory in question.

The first author introduced the notion of graph polynomials definable in Monadic Second Order Logic, and showed that the Tutte polynomial and its generalization, the matching polynomial, the cover polynomial and the various interlace polynomials fall into this category. This definition can be extended to allow definability in full second order, or even higher order Logic.

The purpose of this paper is to show that many graph polynomials and combinatorial counting functions of graph theory do occur as combinatorial counting functions of totally categorical theory. We also give a characterization of polynomials definable in Second Order Logic.

1. Introduction

1.1. **Graph invariants and graph polynomials.** A *graph invariant* is a function from the class of (finite) graphs $\mathcal{G}$ into some domain $\mathcal{D}$ such that ismorphic graphs gave the same picture. Usually such invariants are meant to be uniformly defined in some formalism. If $\mathcal{D}$ is the two-element boolean algebra we speak of *graph properties*. Examples are the properties of being connected, planar, Eulerian, Hamiltonian, etc. If $\mathcal{D}$ consists of the natural numbers, we speak of *numeric graph invariants*. Examples are the number of connected components, the size of the largest clique or independent set, the diameter, the chromatic number, etc. But $\mathcal{D}$ could also be a polynomial ring $\mathbb{Z}[\bar{X}]$ over $\mathbb{Z}$ with a

Date: Last revised, December, 8, 2006.

* Partially supported by a Grant of the Fund for Promotion of Research of the Technion–Israel Institute of Technology.

** Partially supported by MODNET: Marie Curie Research Training Network MRTN-CT-2004-512234 .

set of indeterminates $\bar{X}$. Here examples are the charatcteristic polynomial, the chromatic polynomial, the Tutte polynomial. There are many more graph invariants discussed in the literature, which are polynomials in $\mathbb{Z}[\bar{X}]$, but there are hardly any papers discussing classes of graph polynomials as an object of study in its generality. An outline of such a study was presented in [Mak06]. In [Mak04] the first author has introduced the **MSOL**-definable and the **SOL**-definable graph polynomials, the class of graph polynomials where the range of summation is definable in (monadic) second order logic. He has verified that all the examples of graph polynomials discussed in the literature are actually **SOL**-polynomials over some expansions (by adding order relations) of the graph, cf. also [Mak06]. In some cases this is straight forward, but in some cases it follows from intricate theorems. This definition can be extended in two ways: by allowing some additional combinatorial functions as monomials, and by allowing higher order logic formulas. For n-th order logic formulas $\mathbf{HOL}^n$ we call the corresponding polynomials *extended* $\mathbf{HOL}^n$*-polynomials*. For higher order logic formulas of $\mathbf{HOL} = \bigcup_n \mathbf{HOL}^n$ we call the corresponding polynomials *extended* **HOL***-polynomials*. It is easy to define (artificial) **HOL**-polynomials which are provably not **SOL**-polynomials.

The purpose of this paper is to present two related unified frameworks for defining graph invariants, and more generally, invariants of finite first order τ-structures for arbitrary vocabularies (similarity types), which are all polynomials. Both frameworks are model-theoretic. The first framework, which we call *counting functions of generalised colorings*, uses finite model theory. k-vertex-colorings of a graph $G = (V, E)$ with colors from a set $\{0, \ldots, k-1\} = [k]$ are functions from $f : V \rightarrow [k]$ such that no two vertices connected by an edge have the same value. A simple case of generalised colorings are the ϕ-colorings, where $\phi(F)$ is a first order formula over graphs with an additional r-ary function symbol F, and we allow all functions f which are interpretations F satisfying $\phi(F)$. To define a ϕ-coloring, the formula has to be subject to certain semantic restrictions such as invariance under permutation of the colors, the existence of a bound on the colors used, and independence of the colors not used. More complicated cases arise by expanding the graph, allowing several color sets, and replacing functions by relations. The associated counting function $\chi_\phi(k)$ counts the number ϕ-colorings as a function of k.

The second framework which we call *model theoretic invariants*, or short *MT-invariants*, uses ω-categorical ω-stable structures. The totally categorical structures are a special example. In the analysis of the structure of models of totally categorical theories, the second author showed that certain combinatorial counting functions play an important role, cf. [Zil93]. This was later extended to ω-categorical ω-stable theories by G. Cherlin, L. Harrington and A. Lachlan [CHL80]. Those functions are invariants of the structure, which we call *model theoretic invariants*, or short *MT-invariants*. From the structure theory of ω-categorical, ω-stable structures we have, cf. [Zil93, Proposition 5.2] and [CH03, Theorem 6]:

Theorem A (Zilber). *Every MT-invariant is a polynomial in one or many variables, depending on the number of independent dimensions of the structure in question.*

For the sake of brevity, we call polynomials, which do occur as combinatorial counting functions in such structures *MT-polynomials*.

We note that, for the purposes of this paper, we need a simplified version of the general theorem. This version has a direct combinatorial proof. In fact this proof is equivalent to the direct proof of Corollary C below, which is given in section 2. and was suggested to us by A. Blass after having been shown the general theorem[1].

1.2. **Main results.** Our main results here are:

Theorem B. *Every counting function of a generalised coloring is a MT-invariant.*

Using Theorem A we get:

Corollary C. *Every counting function of a generalised coloring is a polynomial graph invariant, hence a MT-polynomial.*

To see that both our frameworks are very general we show next:

Theorem D. *Every extended* $\mathbf{HOL}^n$*-polynomial over some* τ*-structure* $\mathcal{A}$ *is a counting function of a generalised coloring over some expansion of* $\mathcal{A}$*. In particular every extended* **SOL***-polynomial of graphs is a counting function of a generalised coloring definable in* **SOL**.

Actually, a converse is also true:

Theorem E. *Every counting function of a generalised coloring of graphs definable in* **SOL** *is an extended* **SOL***-polynomial of graphs.*

It seems at first sight that there are more MT-invariants than counting functions of generalised colorings. However, it is conceavable that every MT-invariant is an **HOL**-definable graph polynomial.

[1]We wish to thank A. Blass for allowing us to use and further elaborate his suggestion.

1.3. **Outline of the paper.** We assume the reader is familiar with the basics of graph theory as, say, presented in [Die96, Bol99]. We also assume the reader is familiar the basics of finite and infinite model theory as, say, presented in [EFT94, EF95, Hod93, Rot95].

Section 2 is a prelude to our general discussion. In it we discuss the chromatic polynomial and explain how it fits into the various frameworks.

In Section 3 we introduce our notion of counting functions of generalised colorings definable in **HOL**. We state and give a direct proof of a generalization of Corollary C.

In Section 4 we give precise definition of **SOL**-definable and **HOL**-definable polynomials prove both Theorem D and Theorem E.

In Section 5 we finally we show how all this fits into the framework of totally categorical structures (for polynomials in one variable) and of ω-categorical ω-stable structures (in the case of several variables), and we prove Theorem B.

In Section 6 we draw conclusions and discuss some open problems.

Acknowledgements. The first author would like to thank I. Averbouch, A. Blass, B. Courcelle, B. Godlin, E. Hrushovski, S. Shelah and M. Ziegler for valuable discussions and suggestions. We would like to thank A. Blass for allowing us to incorporate the simple proof of Theorem 3.7, which he suggested.

2. Prelude: The chromatic polynomial

2.1. **Four themes.** Before we introduce our general definitions, we discuss the oldest graph polynomial studied in the literature, the classical *chromatic polynomial* $\chi_G(k)$. It has a very rich literature. For an excellent and exhaustive monograph, cf. [DKT05].

We denote by $\mathcal{G}$ the set of graph of the form $G = ([n], E)$. A k-vertex-coloring of G is a function $f : [n] \to [k]$ such that whenever $u, v \in E$ then $f(u) \neq f(v)$. $\chi_G(k)$ denotes the number of k-vertex-colorings of G. $\chi_G(k)$ defines, for each graph, a function

$$\chi_G(\lambda) : \mathcal{G} \to \mathbb{N}$$

which turns out to be a polynomial in λ.

We isolate four themes:

(i) A recursive definition of $\chi_G(k)$ (using an order on the vertices or edges).

(ii) A uniform static definition of $\chi_G(k)$ over the graph using a second order logic formalism.

(iii) We associate with each $k \in \mathbb{N}$ a two-sorted structure $\mathcal{G}_k = \langle G, [k] \rangle$ and interpret $\chi_G(k)$ as counting the number of expansions $\mathcal{G}_k, F$ satisfying some first order formula $\phi(F)$.
(iv) We can also replace the family of structures $\mathcal{G}_k = \langle G, [k] \rangle$ by a single infinite structure $\mathcal{M}(G)$ which is totally categorical, with a strongly minimal set X, and interpret $\chi_G(k)$ as the number of elements f satisfying some first order formula $\hat{\phi}(f)$ in the algebraic closure in $\mathcal{M}(G)$ of $Y \subset X$ where $| Y |= k$.

In [CGM0x] the relationship between recursive and static definitions is studied. There, a framework is provided which allows to show that every recursive definition of a graph polynomial also allows a static definition. The converse is open but seems not to be true. Here we are interested in the relationship between static definition, counting expansions, and the size of definable sets in $\mathcal{M}(G)$.

2.2. **Uniform family.** We note that $\chi_G(\lambda)$ really denotes a family of polynomials indexed by graphs from $\mathcal{G}$. This family is furthermore uniformly defined based on some of the properties of the graph G. Below, we are interested in various formalisms in which such uniform definitions can be given.

2.3. **Recursive definition.** The first proof that $\chi_G(\lambda)$ is a polynomial used the observation that $\chi_G(\lambda)$ has a recursive definition using the order of the edges, which can be taken as the order induced by the lexical ordering on $[n]^2$. However, the object defined does not depend on the particular order of the edges. For details, cf. [Big93, Bol99]. The essence of the proof is as follows:

For $e = (v_1, v_2)$, we put $G - e = (V, E')$ with $E' = E - \{e\}$, and $G/e = (V', E')$ $V' = V - \{v_2\}$ and $E' = (E \cap (V')^2) \cup \{(v_1, v); (v_2, v) \in E\}$. The operation passing from G to $G - e$ is called *edge removal*, and the operation passing from G to G/e is called *edge contraction*.

Lemma 2.1. *Let e, f be two edges of G. Then we have*
(i) $(G - e) - f = (G - f) - e$.
(ii) $(G/e) - f = (G - f)/e$.
(iii) $(G - e)/f = (G/f) - e$.
(iv) $(G/e)/f = (G/f)/e$.

Let $E_n = ([n], \emptyset)$. We have
(i) $\chi_{E_n}(\lambda) = \lambda^n$.
(ii) For any edge $e \in E$ we have

$$\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda).$$

Let $E = (e_0, e_1, \ldots, e_m)$ be the enumeration of the edges in this lexicographic order. Using the order on the edges, this allows us to compute $\chi_G(\lambda)$. It also turns out, using Lemma 2.1, that the result is *independent* of the ordering of the edges.

2.4. **Static descriptions.** There are other proofs that $\chi_G(\lambda)$ is a polynomial[2].

Proof. We first observe that any coloring uses at most n of the λ colors. For any $m \leq n$, let $c(m)$ be the number of colorings, with a fixed set of m colors, which are vertex colorings and use all m of the colors. Then, given λ colors, the number of vertex colorings that use exactly m of the λ colors is the product of $c(m)$ and the binomial coefficient $\binom{\lambda}{m}$. So

$$\chi_G(\lambda) = \sum_{m \leq n} \binom{\lambda}{m} c(m)$$

The right side here is a polynomial in λ, because each of the binomial coefficients is. We also use that for $\lambda \leq m$ we have $\binom{\lambda}{m} = 0$.

If both the set of colors and the set of vertices are initial segments of the natural numbers with their order, we can also rewrite this in the following way:

$$\text{(chrom-1)} \qquad \chi_G(\lambda) = \sum_{A: init(A,V)} \sum_{f: ontocol(f,A)} \binom{\lambda}{card(A)}$$

where $init(A, V)$ says that A is an initial segment of V, and $ontocol(f, A)$ says that f is a vertex coloring using all the colors of A. □

Equation chrom-1 is an example of a static definition of the chromatic polynomial.

In [DKT05, Theorem 1.4.1] another static description of $\chi_G(\lambda)$ is given: Let $a(G, m)$ be the number of partitions of V into m independent sets, and let

$$(\lambda)_m = \lambda \cdot (\lambda - 1) \cdot \ldots \cdot (\lambda - m + 1)$$

Then

$$\chi_G(\lambda) = \sum_m a(G, m) \cdot (\lambda)_m$$

This again be written as

$$\text{(chrom-2)} \qquad \chi_G(\lambda) = \sum_{P: indpart(P, A_P, V)} (\lambda_{card(A_P)})$$

[2]This one was pointed out to us by A. Blass, who kindly allowed us to reproduce it here.

where $indpart(P, A_P, V)$ says that P is an equivalence relation on V and A_P consists of the first elements (with respect to the order on $V = [n]$) of each equivalence class.

A third static description for $\chi_G(\lambda)$ is given in [DKT05, Theorem 2.2.1]. It can be obtained from a two-variable polynomial $Z_G(\lambda, V)$ defined by

$$Z_G(\lambda, V) = \sum_{S:S\subseteq E} \left(\prod_{v:fcomp(v,S)} \lambda \cdot \prod_{e:e\in S} V \right) = \sum_{S:S\subseteq E} \left(\lambda^{k(S)} \cdot \prod_{e:e\in S} V \right)$$

where $fcomp(v, S)$ is the property "v is the first vertex in the order of V of some connected component of the spanning subgraph $< S : V >$ on V induced by S", and $k(S)$ is the number of connected components of $< S : V >$. Now we have

(chrom-3) $$\chi_G(\lambda) = Z_G(\lambda, -1)$$

The three static descriptions of the chromatic polynomial chrom-1, chrom-2, chrom-3 have several properties in common:

(i) They satisfy the same recursive definition.
(ii) They are of the form $\sum_k A_k(G)P_k(\lambda)$ where $P_k(\lambda)$ is a polynomial in λ with integer coefficients of degree k.
(iii) The coefficients $A_k(G)$ are positive and have a combinatorial interpretation.
(iv) The coefficients can be alternatively obtained by collecting the terms $P_k(\lambda)$ of a summation over certain relations definable in second order logic over the graph with an order on the vertices and interpreting k as the cardinality of such a relation.
(v) Although the order on the vertices is used in the static description of the polynomial, the polynomial is *invariant under permutations of the ordering.*

There are also signifcant differences.

(i) In chrom-1 it is important that the set of colors and the set of vertices are initial segments of the natural numbers with their natural order. The summation involves *one unary relation* and *one unary function.*
(ii) In chrom-2 The summation involves a *binary relation* on the vertices which is not a subset of the edge relation, but of its complement. The order relation is only needed to identify equivalence classes.
(iii) In chrom-3 we actually use a two-variable polynomial and then substitute for one variable -1. The summation involves a *binary relation* on vertices which is a subset of the edge relation.

It can be also viewed as a *unary relation* on the set of edges. The order relation is only needed to identify connected components.

2.5. **Counting expansions.** A k-vertex-coloring of G is a function $f : [n] \to [k]$ such that whenever $u, v \in E$ then $f(u) \neq f(v)$. Let F be a unary function symbol and let $\phi(F, E)$ be the formula which says that F is a vertex-coloring for the edge relation E. Then $\chi_G(k) = \chi_\phi(G, k)$ is the number of functions in $\langle [n], [k], E, F \rangle$ which satisfy $\phi(F, E)$. We note that

(i) a coloring is invariant under permutations of the colors,
(ii) the number of colors is bounded by the size of V, and
(iii) the property of being a coloring is independent of the colors not used.

This is readily generalised to other formulas $\psi(F, E)$ satisfying similar properties, and will be the starting point for our notion of generalised coloring.

2.6. **Totally categorical structure.** Here we assume the reader is familiar with basic model theory as described in [Hod93, Rot95]. We now define an infinite structure $\mathcal{M}(G)$ in the following way. For $G = (V, E)$ the universe M of $\mathcal{M}(G)$ consists of three disjoint sets V, X and X^V with $V = [n]$ and $X = \mathbb{N}$. We assume we have unary predicates P_V, P_X identifying these sets. The additional relations are $E \subseteq V^2$ and $R \subseteq V \times X^V \times X$. E is the edge relation of G and $(v, f, x) \in R$ iff $f(v) = x$.

We note:

(i) Let T_G be the complete first order theory of $\mathcal{M}(G)$. Two models M_1, M_2 of T_G are isomorphic iff $P_X(M_1)$ and $P_X(M_2)$ have the same cardinality, hence T_G is totally categorical. P_X defines a strongly minimal set. The algebraic closure of P_X is the whole model.
(ii) Let $\hat{\phi}(f)$ be the first order formula

$$\forall u, v \, ((E(u, v) \wedge S(u, f, x) \wedge S(v, f, y)) \to x \neq y)$$

In models of T_G the formula $\hat{\phi}(f)$ says that f is a vertex-coloring of G with colors in P_X.
(iii) For $Y \subset X$ in $\mathcal{M}(G)$, denote by $acl(Y)$ the algebraic closure of Y in $\mathcal{M}(G)$. Let

$$count(\hat{\phi}(f), Y) = \{f \in cl(Y) : \mathcal{M}(G) \models \hat{\phi}(f)\}$$

It is easy to check that for any $Y \subseteq X$ with $\mid Y \mid= k$, we have $\chi_G(k) = count(\hat{\phi}(f), Y)$.

(iv) Using Theorem A, this gives another proof that $\chi_G(k)$ is a polynomial.

3. Generalised chromatic polynomial

3.1. **Generalised colorings.** Let $\mathcal{M}$ be a τ-structure with universe M, and denote by $\mathcal{M}_k$ the two-sorted structure $\langle \mathcal{M}, [k] \rangle$ for the vocabulary τ_1. We denote relation symbols by bold-face letters, and their interpretation by the corresponding roman-face letter.

Definition 3.1 (Coloring property). *Let $\tau_R = \tau_1 \cup \{\mathbf{R}\}$, where is $\mathbf{R}$ is a two-sorted relation symbol of arity $r = s+t$. A class of τ_R- structures $\mathcal{P}$ is a* coloring property *if*

(i) *$\mathcal{P}$ is closed under τ_R-isomorphisms,*

(ii) *Let $\mathcal{M}$ be fixed. Then $\mathcal{M}_k$ is a substructure of $\mathcal{M}_n$ for each $n \geq k$. Let R_0 be a fixed relation on $\mathcal{M}_k$. If $\langle \mathcal{M}_k, R_0 \rangle \in \mathcal{P}$ and $n \geq k$ then also $\langle \mathcal{M}_n, R_0 \rangle \in \mathcal{P}$.*

(iii) *Let $R \subseteq M^s \times [k]^t$ be a fixed relation on $\mathcal{M}_k$. For π is a permutation of $[k]$, We define*

$$R_\pi = \{(\bar{m}, \pi(\bar{a})) \in M^\times [k]^t : (\bar{m}, \bar{a}) \in R\}.$$

Then $\langle \mathcal{M}_k, R \rangle \in \mathcal{P}$ iff $\langle \mathcal{M}_k, R_\pi \rangle \in \mathcal{P}$.

We refer to $\mathbf{R}$ and its interpretations R as coloring predicates*.*

Remark 3.2. *It does not matter whether we define coloring properties for fixed $\mathcal{M}$ or not, as the union of a family of coloring properties over the same vocabulary is again a coloring property.*

Definition 3.3 (Bounded coloring properties)**.** *.*

(i) *A coloring property is* bounded*, if for every $\mathcal{M}$ there is a number N_M such that for all $k \in \mathbb{N}$ the set of colors*

$$\{x \in [k] : \exists \bar{y} \in M^m R(\bar{y}, x)\}$$

has size at most N_M.

(ii) *A coloring property is* range bounded*, if its range is bounded in the following sense: There is a number $d \in \mathbb{N}$ such that for every mathcalM and $\bar{y} \in M^m$ the set $\{x \in [k] : R(\bar{y}, x)\}$ has at most d elements.*

Clearly, if a coloring property is range bounded, it is also bounded.

Definition 3.4 (Coloring formula)**.** *A first order (or second order) formula $\phi(\mathbf{R})$ is a* coloring formula*, if the class of its models, which are of the form of the form $\langle \mathcal{M}, [k], R \rangle$, is a coloring property.*

We will discuss coloring formulas definable in second order logic **SOL** in more detail in Subsection 3.7.

Definition 3.5 (Generalised colorings).
Let $\mathcal{P}$ be a bounded coloring property. A relation $R_M \subset M^m \times [k]$ is a generalised $k - \mathcal{P}$-coloring *if $\langle \mathcal{M}_k, R \rangle \in \mathcal{P}$.*
We denote by $\chi_{\mathcal{P}}(\mathcal{M}, k)$ the number of generalised $k - \mathcal{P}$-coloring R on $\mathcal{M}$. If $\mathcal{P}$ is definable by some formula $\phi(\mathbf{R})$ we also write $\chi_{\phi(R)}(\mathcal{M}, k)$.

Example 3.6. *Let $\mathbf{F}$ be a unary function symbol which serves as the coloring predicate. A vertex coloring of a graph $G = (V, E)$ is a map $F : V \rightarrow [k]$ for some k,*

(i) *A vertex coloring F is* proper*, if it satisfies the coloring formula*

$$\forall u, v(\mathbf{E}(u, v) \rightarrow \mathbf{F}(u) \neq \mathbf{F}(v)).$$

Clearly, this does define a coloring property.

(ii) *If we require that a vertex coloring F uses all the colors, then this is not a coloring property. It violates (ii) of Definition 3.1.*

(iii) *A vertex coloring is* pseudo-complete*, if it satisfies the formula*

$$\forall x, y \exists u, v(\mathbf{F}(u) = x \wedge \mathbf{F}(v) = y.$$

For the same reason as above this is not a coloring formula.

(iv) *A vertex coloring is* complete*, if it is both proper and pseudo-complete. Complete colorings are studied in the context of the* achromatic number of a graph *G which is the largest number k such that G has a complete coloring with k colors. The achromatic number of G and the number of complete colorings is a function of G but not of k. In other words, complete colorings are not colorings in our sense. The achromatic number was introduced in* [HHR67]. *For a survey of recent work, cf.* [HM97].

Theorem 3.7. *Let $\mathcal{P}$ be a bounded coloring property. For every $\mathcal{M}$ the number $\chi_{\mathcal{P}}(\mathcal{M}, k)$ is a polynomial in k of the form*

$$\sum_{j=0}^{d \cdot |M|^m} c_{\phi(R)}(\mathcal{M}, j) \binom{k}{j}$$

where $c_{\phi(R)}(\mathcal{M}, j)$ is the number of generalised $k - \phi$-colorings R with a fixed set of j colors.

Proof. We first observe that any generalised coloring R uses at most N_M of the k colors, if it is bounded. Furthermore, $N_M = d \cdot \mid M \mid^m$, if it is range bounded. For any $j \leq N$, let $c_{\mathcal{P}}(\mathcal{M}, j)$ be the number of colorings, with a fixed set of j colors, which are generalised vertex colorings and use all j of the colors. We use the properties of the coloring property. So any permutation of the set of colors used is also a coloring. Therefore, given k colors, the number of vertex colorings that use exactly j of the k colors is the product of $c_{\mathcal{P}}(\mathcal{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_{\mathcal{P}}(\mathcal{M}, k) = \sum_{j \leq N} c_{\mathcal{P}}(\mathcal{M}, j) \binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k \leq j$ we have $\binom{k}{j} = 0$. $\square$

In the light of this theorem we call $\chi_{\mathcal{P}}(\mathcal{M}, k)$ a *generalised chromatic polynomial.*

Remark 3.8. *The restriction to coloring properties in Theorem 3.7 is essential. Let* $\chi_{complete}(G, k)$ *be the number of complete colorings of a graph* $G - (V, E)$ *with* k *colors. Clearly, this is not a polynomial in* k *is for* $k \geq \binom{|E|}{2}$ *it always vanishes, so it should be constant* 0*.*

3.2. **Properties of counting polynomials.**

Definition 3.9 (Counting polynomials)**.**

(i) Newton polynomials *are of the form*

$$p(\lambda) = \sum_{j \leq N} b_j \binom{\lambda}{j}$$

(ii) *We call a polynomial* $p(\lambda) \in \mathbb{Z}[\lambda]$ *a* counting polynomial *if* $p(k)$ *is a non-negative integer for non-negative integers* k*.*

Clearly, if $p(\lambda)$ is a counting polynomial, $p(\lambda)$ tends to infinity with λ, unless it is constant. Polynomials of the form $p(\lambda) = \sum_{j \leq N} a_j \lambda^j$, or, as they are called in [GKP94], *Newton polynoials*, with all the coefficients a_j, respectively b_j non-negative integers, are counting polynomials. The polynomials $p(\lambda)$ obtained in Theorem 3.7 are all counting polynomials. In fact they are Newton polynomials with non-negative coefficients.

There are counting polynomials which have negative coefficents:

$$p(\lambda) = (\lambda - 2)^2 = \lambda^2 - 4\lambda + 4 = \binom{\lambda}{2} - 3\binom{\lambda}{1} + 2\binom{\lambda}{0}$$

The characteristic polynomial of a graph G with adjacency matrix A_G is defined by $P_G(\lambda) = \det(\lambda \cdot \mathbf{1} - A_G)$ and is not a counting polynomial. To see this we note that, for a graph of n vertices and $P_G(\lambda) = \sum_{k=0}^{n} c_k \lambda^k$, we have $c_{n-1} = 0$, $-c_{n-2}$ is the number of edges, and $-c_{n-3}$ is twice the number of triangles of G, hence $c_{n-2} \leq 0$ and $c_{n-3} \leq 0$, cf. [Big93, Proposition 2.3.]. Hence, for $G = K_3$ we get $P_{K_3}(\lambda) = \lambda^2 - 3\lambda + 1$ which gives $P(1) = -1$. However, $P_G(\lambda)$ can be written as the difference of two counting polynomials $P_G^+(\lambda) - P_G^-(\lambda)$, and we put $\bar{P}_G(\lambda, \mu) = P_G^+(\lambda) + \mu P_G^-(\lambda)$. We shall see that in the sequel that $\bar{P}_G(\lambda, \mu)$ is a two variable generalized chromatic polynomial, and, hence, $P_G(\lambda) = \bar{P}_G(\lambda, -1)$ is a substitution instance of $\bar{P}_G(\lambda, \mu)$.

3.3. **Generalised multi-colorings.** To construct also graph polynomials in several variables, we extend the definition as follows.

Let $\mathcal{M}$ be a τ-structure with universe M, and denote by $\mathcal{M}_{k_1,\ldots,k_\alpha}$ the $(1+\alpha)$-sorted structure $\langle \mathcal{M}, [k_1], \ldots [k_\alpha] \rangle$ for the vocabulary τ_α. We put $\bar{k}^\alpha = (k_1, \ldots, k_\alpha)$. The notions of a *multi-coloring property* $\mathcal{P}$, *bounded* and *range bounded multi-coloring properties* are defined exactly like for the coloring properties.

Definition 3.10 (Generalised multi-colorings).
Let $\mathcal{P}$ be a bounded multi-coloring property for structures $\mathcal{M}_{k_1,\ldots,k_\alpha}$. A relation

$$R_0 \subset M^m \times ([k_1]^{m_1} \sqcup \ldots \sqcup [k_\alpha]^{m_\alpha})$$

is a generalised $\bar{k^\alpha} - \mathcal{P}$-multi-coloring *if* $\langle \mathcal{M}_{k_1,\ldots,k_\alpha}, R_0 \rangle \in \mathcal{P}$.
We denote by $\chi_{\mathcal{P}}(\mathcal{M}, k_1, \ldots, k_\alpha)$ the number of generalised $k - \mathcal{P}$-multi-coloring R on $\mathcal{M}$. If $\mathcal{P}$ is definable by some formula $\phi(\mathbf{R})$ we also write $\chi_{\phi(R)}(\mathcal{M}, k_1, \ldots, k_\alpha)$.

Theorem 3.11. *Let $\mathcal{P}$ be a bounded multi-coloring property with bound N. In the case of range bounded multi-colorings $N = d \cdot \mid M \mid^m$. For every $\mathcal{M}$ the number $\chi_{\mathcal{P}}(\mathcal{M}, k_1, \ldots, k_\alpha)$ is a polynomial in $k_1, \ldots, k_\alpha$ of the form*

$$\sum_{j=0}^{N} c_{\phi(R)}(\mathcal{M}, \bar{j}^\alpha) \prod_{1 \leq \beta \leq \alpha} \binom{k_\beta}{j_\beta}$$

where $c_{\mathcal{P}}(\mathcal{M}, \bar{j}^\alpha)$ is the number of generalised $\bar{k}^\alpha - \phi$-colorings R with fixed sets of j_β colors respectively.

Proof. Similar to the one variable case. □

We shall call multi-coloring properties and multi-coloring simply also coloring properties and colorings, if the situation is clear from the context.

3.4. **Several simultaneous colorings.** Let $\mathcal{M}$ be a τ-structure and $\mathcal{M}_k$ as before. Assume we have a formula $\phi(\mathbf{F}_1, \ldots, \mathbf{F}_s)$ with s function variables for generalised colorings which specifies the functions simultaeously. If we fix the interpreation of the first $s-1$ function variables and denote these by $F_1, \ldots, F_{M-1}$ we have a new structure

$\mathcal{N} = \langle \mathcal{M}_k, F_1, \ldots, F_{s-1} \rangle$ in which we count just one generalised coloring for each interpretation $F_1, \ldots, F_{s-1}$. The general counting is obtained by summing over all interpretations. Hence, as the sum of polynomials is a polynomial, this again gives us a polynomial. The same argument works, if we allow relations on the structure $\mathcal{M}$, which do not involve the sort $[k]$ in $\mathcal{M}_k$, and provided the range of these relations is bounded in the sense of Definition 3.3. We call polynomials so obtained also *generalised chromatic polynomials*.

3.5. **Closure properties.** The following will be useful.

Proposition 3.12 (Sums and products)**.** *The sum and product of two generalised chromatic polynomials $\chi_{\phi(\mathbf{F})}(G, \lambda)$ and $\chi_{\psi(\mathbf{F})}(G, \lambda)$ is again a generalised chromatic polynomial.*

Proof. For the sum we take $\chi_{\theta_1}(G, \lambda)$ with

$$\theta_1(\mathbf{F}) = ((\phi(\mathbf{F}) \wedge \neg\psi(\mathbf{F})) \vee (\psi(\mathbf{F}) \wedge \neg\phi(\mathbf{F})) \vee (\phi(\mathbf{F}) \wedge \psi(\mathbf{F}))) \,.$$

For the product we take $\chi_{\theta_2}(G, \lambda)$ where we use two distinct function symbols $\mathbf{F}$ and $\mathbf{F}'$ and $\theta_2(\mathbf{F}, \mathbf{F}') = (\phi(\mathbf{F}) \wedge \psi(\mathbf{F}'))$. □

3.6. **Examples.** We now show how many graph polynomials can be viewed as generalised chromatic polynomials.

Combinatorial polynomials. The following combinatorial polynomials can be thought of as generalised chromatic polynomials:

(i) For the polynomial λ^n we take all maps $[n] \to [k]$ for $\lambda = k$. So $\lambda^n = \chi_{true(f)}$ where $true(f)$ is $\forall v(f(v) = f(v)$. It is a first order definable bounded coloring property.

(ii) Similarly, for $\lambda_{(n)} = \lambda{\cdot}(\lambda{-}1){\cdot}\ldots{\cdot}(\lambda{-}n{+}1)$ we take all injective maps, which is easily expressed by a first order formula. which defines a bounded coloring property.

(iii) Finally, for $\binom{\lambda}{n}$ we take the ranges of injective maps. This is a bounded coloring property of a second order formula $\phi(\mathbf{P})$ which says that $P \subseteq [k]$ is the range of an injective map $f : [n] \to [k]$.

Connected components. We denote by $k(G)$ the number of connected components of G. The polynomial $\lambda^{k(G)}$ can be written as $\chi_{\phi_{connected}}(G, \lambda)$ with $\phi_{connected}(f)$ the formula

$$((u, v) \in E \to f(u) = f(v)).$$

Hypergraph colorings and mixed hypergraph colorings. A hypergraph G consists of a set of vertices $V(G)$ and a family $E(V)$ of subsets of V, called the hyperedges. To make this into a first order structure we have two sorts of elements, the elements of V and of E, together with the membership relation, which satisfies extensionality. A mixed hypergraph G has two kinds of hyperedges, $D(G)$ and $E(G)$. Mixed hypergraph colorings come in several flavours. For a recent exhaustive survey, cf. [Vol02]. We discuss here two cases:

(i) A *weak mixed hypergraph coloring with k colors* is a mapping $f : V \to [k]$ such that
$$\forall u, v \in d \in D(G) \to f(u) = f(v)$$
and
$$\forall e \in E(G) \exists u, v \in e \in E(G) \to f(u) \neq f(v).$$

(ii) A *strong mixed hypergraph coloring with k colors* is a mapping $f : V \to [k]$ such that
$$\forall u, v \in d \in D(G) \to f(u) = f(v)$$
and
$$\forall e \in E(G) \forall u, v \in e \in E(G) \to f(u) \neq f(v).$$

We denote by $\chi_{weak}(G, k)$ and $\chi_{strong}(G, k)$ respectively the number of weak (strong) mixed hypergraph colorings with at most k colors.

Proposition 3.13 (V.L. Voloshin). *$\chi_{weak}(G, k)$ and $\chi_{strong}(G, k)$ are polynomials in k.*

Clearly, this is a corollary to our Theorem 3.7.

Matching polynomial. Let $G = (V, E)$ be a graph. A subset $M \subseteq E$ is a matching of no two edges in E have a common vertex. The matching polynomial of G is given by
$$g(G, \lambda) = \sum_j \mu(G, j) \lambda^j$$
where $\mu(G, j)$ is the number of of matchings of size j.

We look at the structure G_k and at pairs (M, F) with $M \subset E$ and $F : E \to [k]$ such that M is a matching and the domain of F is M, which can be expressed by a formula $match(M, F)$. We have
$$\chi_{match(M,F)}(G, k) = \sum_j \mu(G, j) k^j = g(G, k)$$
There are two close relatives to the matching polynomial, cf. [God93].

(i) The *acyclic polynomial*

$$m(G,k) = \sum_j (-1)^j \mu(G,j) k^{n-2j} = k^n g(G, -k^{-2})$$

and

(ii) the rook polynomial $r(G,k)$, which defined for bipartite graphs only. We have

$$r(G,k) = \sum_j \mu(G,j) k^{n-j} = k^n g(G, -x^{-1}).$$

The rook polynomial is a substitution instance of $g(G,k)$, and the acyclic polynomial is a product of k^n with a substitution instance of $g(G,k)$.

Tutte polynomial. We use the Tutte polynomial in the following form:

$$Z(G,q,v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|}$$

where $conn(A)$ is the number of connected components of the spanning subgraph (V,A). This form of the Tutte polynomial is discussed in [Sok05]. For this purpose we look at the 4-sorted structure

$$G_{k,l} = \langle V, [k], [l], \wp(E), E \rangle$$

and at the triples (A, F_1, F_2) with $A \in \wp(E)$, $F_1 : V \times \wp(E) \to [k]$ and $F_2 : A \to [l]$ such that for $(u,v) \in A \to F_1(A,u) = F_1(A,v)$. This is expressed in the formula $tutte(A, F_1, F_2)$. Now we have

$$\chi_{tutte(A,F_1,F_2)}(G,k,l) = \sum_{A \subseteq E} k^{conn(A)} l^{|A|}$$

which is the evaluation of $Z(G,q,v)$ for $q = k, v = l$.

3.7. **Definability in Higher Order Logic.** In our definition of generalised chromatic polynomials we have often requested that the generalised coloring be specified by a formula of first order logic $\mathbf{FOL}(\tau)$. This is not necessary. One example we have seen was the combinatorial function $\binom{k}{n}$. We now introduce more formally higher order logics. The formulas of $\mathbf{SOL}(\tau)$ are defined like the ones of $\mathbf{FOL}$, with the addition that we allow countably many variables for n-ary relation symbols $U_{n,\alpha}$ for $\alpha \in \mathbb{N}$, for each $n \in \mathbb{N}$, and quantification over these. Monadic second order logic $\mathbf{MSOL}(\tau)$ is the restriction of $SOL(\tau)$ to unary relation variables and quantification over these. We can also introduce higher order logic $\mathbf{HOL}^n(\tau)$. $\mathbf{FOL}(\tau) = \mathbf{HOL}^1(\tau)$. $\mathbf{SOL}(\tau) = \mathbf{HOL}^2(\tau)$.

In $\mathbf{HOL}^3(\tau)$ we have additionally variables for relations over relations. We do not need the details here.

Definition 3.14. *A generalised chromatic polynomial is definable in* $\mathbf{SOL}(\tau)$*, respectively in* $\mathbf{MSOL}(\tau)$ *or* $\mathbf{HOL}^n(\tau)$*, if it is of the form* $\chi_\phi(\mathcal{M}, \bar{\lambda})$*, where* $\phi \in \mathbf{SOL}$*, respectively in* $\mathbf{MSOL}(\tau)$ *or* $\mathbf{HOL}^n(\tau)$*, and defines a bounded coloring property.*

Here is a generalization of Theorem 3.11.

Theorem 3.15. *Every counting function of a generalised* $\mathbf{HOL}^n$*-definable coloring is a polynomial, which we call also generalised chromatic polynomial.*

Also Proposition 3.12 remains true with the same proof.

Proposition 3.16 (Sums and products)**.** *The sum and product of two generalised* $\mathbf{SOL}(\tau)$*-definable chromatic polynomials* $\chi_{\phi(f)}(G, \lambda)$ *and* $\chi_{\psi(f)}(G, \lambda)$ *is again a generalised* $\mathbf{SOL}(\tau)$*-definable chromatic polynomial.*

Remark 3.17. *If we allow second order quantification over the sorts* $[k_1], \ldots, [k_\alpha]$ *the Theorem 3.15 is false. We could define a coloring which is a proper vertex-colorin with k colors, if k is even, and a coloring of the connected components with k colors, if k is odd. Clearly this is not a polynomial.*

As our structures $\mathcal{M}$ are finite, the issue of higher order logic can be circumvented, by adding the appropriate power sets to corresponding expansions of $\mathcal{M}$. However, the class of generalised **SOL**-definable chromatic polynomials will be of special interest.

3.8. **Is there a syntactic equivalent for coloring properties?** We have introduced coloring properties via a semantic definition. It is natural to ask whether there are syntactic conditions which are equivalent to it. More precisely,

Question 1. *Is there a syntactically defined class of* **FOL***-formulas* $COLOR$ *such that for any* $\phi(\mathbf{R}) \in \mathbf{FOL}(\tau_1 \cup \{\mathbf{R}\})$ *the formula* $\phi(\mathbf{R})$ *defines a coloring property if and only if* $\phi(\mathbf{R})$ *is logically equivalent, or logically equivalent over finite structures, to a formula* $\psi(\mathbf{R}) \in COLOR$*? The same question can be asked for formulas of* **SOL** *or even higher order logic* **HOL***.*

There are obvious syntactic restrictions which imply that $\phi(\mathbf{R})$ defines a coloring property. One of them is, for **FOL**, that $\phi(\mathbf{R})$ does not contain any variables of the sort $[k]$. However, this seems to be too restrictive. We leave the question of finding an appropriate characterization as an open problem.

4. SOL-definable graph polynomials

4.1. **SOL(τ)-polynomials.** We are now ready to introduce the **HOL**-definable polynomials. For a a more detailed discussion, cf. [CGM0x]. Let $\mathcal{R}$ be a commutative semi-ring, which contains the semi-ring of the integers $\mathbb{N}$. For our discussion $\mathcal{R} = \mathbb{N}$ or $\mathcal{R} = \mathbb{Z}$ suffices, but the definitions generalize. Our polynomials have a fixed finite set of variables (indeterminates, if we distinguish them from the variables of **HOL**), $\mathbf{X}$.

Definition 4.1 (**SOL**-monomials)**.** *Let $\mathcal{M}$ be a τ-structure. We first define the* **SOL**-*definable* $\mathcal{M}$-monomials. *inductively.*

(i) *Elements of $\mathbb{N}$ are* **SOL**-*definable $\mathcal{M}$-monomials.*
(ii) *Elements of $\mathbf{X}$ are* **SOL**-*definable $\mathcal{M}$-monomials.*
(iii) *Finite products of monomials are* **SOL**-*definable $\mathcal{M}$-monomials.*
(iv) *Let $\phi(\bar{a})$ be a $\tau \cup \{\bar{a}\}$-formula in* **SOL**, *where $\bar{a} = (a_1, \ldots, a_m)$ is a finite sequence constant symbols not in τ. Let t be a $\mathcal{M}$-monomial. Then*

$$\prod_{\bar{a}:\langle \mathcal{M}, \bar{a}\rangle \models \phi(\bar{a})} t$$

is a **SOL**-*definable $\mathcal{M}$-monomial.*

The polynomial t may depend on relation or function symbols occuring in ϕ.

We note that the degree of a $\mathcal{M}$-monomial is polynomially bounded by the cardinality of $\mathcal{M}$.

Definition 4.2 (**SOL**-polynomials)**.** *The $\mathcal{M}$-polynomials definable in* **SOL** *are defined inductively:*

(i) *$\mathcal{M}$-monomials are* **SOL**-*definable $\mathcal{M}$-polynomials.*
(ii) *Let $\phi(\bar{a})$ be a $\tau \cup \{\bar{a}\}$-formula in* **SOL** *where $\bar{a} = (a_1, \ldots, a_m)$ is a finite sequence of constant symbols not in τ. Let t be a $\mathcal{M}$-polynomial. Then*

$$\sum_{\bar{a}:\langle \mathcal{M}, \bar{a}\rangle \models \phi(\bar{a})} t$$

is a **SOL**-*definable $\mathcal{M}$-polynomial.*

(iii) *Let* $\phi(\bar{R})$ *be a* $\tau \cup \{\bar{R}\}$*-formula in* **SOL** *where* $\bar{R} = (R_1, \dots, R_m)$ *is a finite sequence of relation symbols not in* τ*. Let* t *be a* $\mathcal{M}$*-polynomial definable in* **SOL***. Then*

$$\sum_{\bar{R}:\langle \mathcal{M}, \bar{R}\rangle \models \phi(\bar{R})} t$$

is a **SOL***-definable* $\mathcal{M}$*-polynomial.*

The polynomial t *may depend on relation or function symbols occuring in* ϕ*.*

An $\mathcal{M}$-polynomial $p_{\mathcal{M}}(\mathbf{X})$ is an expression with parameter $\mathcal{M}$. The family of polynomials, which we obtain from this expression by letting $\mathcal{M}$ vary over all τ-structures, is called, by abuse of terminology, a $SOL(\tau)$-polynomial.

In [Mak], cf. also [CGM0x], the following is shown:

Proposition 4.3. *The sum and product of two* $SOL(\tau)$*-polynomials is again a* $SOL(\tau)$*-polynomial.*

4.2. **Proof of Theorem D.** We first prove the theorem for **SOL**.

Theorem 4.4. *Every* $\mathbf{SOL}(\tau)$*-polynomial over some* τ*-structure* $\mathcal{A}$ *is a counting function of a generalised coloring definable in* $\mathbf{SOL}(\tau)$ *in a suitable expansion of* $\mathcal{A}$*.*

Proof. We proceed by induction and use extensively Propositions 3.12, 3.16 and 4.3.

(i) For $n \in \mathbb{N}$ we use the structure with n elements $\langle [n] \rangle$ and count the number of ways we can interpret a constant (0-ary function) on $[n]$.

(ii) For a specific indeterminate $X \in \mathbf{X}$ we again use the structure with n elements $\langle [n] \rangle$ and count the number of functions $f : [n] \to [k]$ which map all elements of $[n]$ onto a single element of $[k]$.

(iii) Let $\phi(\bar{a})$ be a $\tau \cup \{\bar{a}\}$-formula in **SOL**, where $\bar{a} = (a_1, \dots, a_m)$ is a finite sequence constant symbols not in τ. Let $t(\bar{A})$ be a $\langle \mathcal{M}, \bar{a}\rangle$-monomial which, by induction hypothesis, is of the form $\chi_{\theta(\bar{a}, F_{\bar{a}})}(\langle \mathcal{M}, \bar{a}\rangle, \mathbf{X})$, with $\theta(\bar{a}, F_{\bar{a}}) \in \mathbf{SOL}(\tau \sqcup \{\bar{a}, F_{\bar{a}}\})$. F is a function reperesenting a generalised coloring. For several functions or bounded relations, the argument is the same.

We have to show that

$$\prod_{\bar{a}:\langle \mathcal{M}, \bar{a}\rangle \models \phi(\bar{a})} t$$

is a **SOL**-definable $\mathcal{M}$-monomial. But this follows from Proposition 3.16 and the fact, that we can assure that all the symbols $\bar{a}$ and $F_{\bar{a}}$ are distinct.

(iv) Finally, we have to deal with the sums. With the same notation as before, we keep the function symbols $F_{\bar{a}} = F$ the same. For

$$\sum_{\bar{a}:\langle \mathcal{M},\bar{a}\rangle \models \phi(\bar{a})} t$$

we count the functions for different $\bar{a}$'s.

In contrast to this, for $\sum_{\bar{R}:\langle \mathcal{M},\bar{R}\rangle \models \phi(\bar{R})} t$ we count the functions for different interpretations of R.

□

The proof of Theorem D is similar.

4.3. **Extended SOL(τ)-polynomials.** We have seen in Section 3 that the combinatorial functions λ^n, $\lambda_{(n)}$ and $\binom{\lambda}{n}$ can be viewed as generalised chromatic polynomials of the empty graph with n verties. Are they also **SOL**(τ)-polynomials for τ the empty vocabulary?

For λ^n this is the case. For $\lambda_{(n)}$ it is the case, if τ contains a binary relation symbol which is interpreted as the natural linear ordering of $[n]$. For $\binom{\lambda}{n}$ it is not clear, how to represent it as an **SOL**(τ)-polynomial at all, although it is a generalised chromatic polynomial definable in **SOL**.

Motivated by this we define the *extended* **SOL**-*polynomials.*

Definition 4.5 (Extended **SOL**-polynomials)**.**

(i) *For every $\phi(\bar{v}) \in$ **SOL**(τ) we define the cardinality of the set defined by ϕ:*

$$card_{\mathcal{M},\bar{v}}(\phi(\bar{v})) =\mid \{\bar{a} \in M^m : \langle \mathcal{M},\bar{a}\rangle \models \phi(\bar{a})\}. \mid$$

(ii) *The* extended **SOL**(τ)-polynomials *are defined inductively by allowing in the Definition 4.1(ii) as monomials additionally: For every $\phi(\bar{v}) \in$ **SOL**(τ) and for every $X \in$ **X**, the polynomials*

$$X^{card_{\mathcal{M},\bar{v}}(\phi(\bar{v}))}, \quad X_{(card_{\mathcal{M},\bar{v}}(\phi(\bar{v})))}, \quad \binom{X}{card_{\mathcal{M},\bar{v}}(\phi(\bar{v}))}$$

are **SOL**-*definable $\mathcal{M}$-monomials.*

(iii) *Similarly, we define also* extended **MSOL**-polynomials.

Now we have:

Theorem 4.6. *Every extended* **SOL**(τ)*-polynomial over* τ*-structures* $\mathcal{A}$ *is a counting function of a generalised coloring definable in* **SOL**(τ) *in a suitable expansion of* $\mathcal{A}$.

Proof. We can literally repeat the proof of Theorem 4.4 with the modified definition. □

4.4. **Proof of Theorem E.** Now we can prove the converse of Theorem 4.6.

Theorem E. *Every counting function of a generalised coloring of graphs definable in* **SOL** *is an extended* **SOL***-polynomial of graphs.*

Proof. From Theorem 3.7 we know that for every $\mathcal{M}$ the number of elements given by $\chi_{\phi(R)}(\mathcal{M}, k)$ is a polynomial in k of the form

$$\sum_{j=0}^{d \cdot |M|^m} c_{\phi(R)}(\mathcal{M}, j)\binom{k}{j}$$

where $c_{\phi(R)}(\mathcal{M}, j)$ is the number of generalised $k - \phi$-colorings R with a fixed set of j colors. Furthermore the total number of colors used is bounded by $N = d \cdot \mid M \mid^m$. Using a relation variable of arity at most $d \cdot m$ (which is far too large), we interpret the set of colors used inside $\mathcal{M}$ by the set $M^{d \cdot m}$. Now the set generalised colorings with exactly the colors used from $A \subset M^{d \cdot m}$ is **SOL**-definable within $\mathcal{M}$ by a formula $\phi(A, F)$. We use a relation symbol $R_A(\bar{v})$ whose interpretation is A. So the extended **SOL**-polynomial is

$$\sum_{A,F:\phi(R_A,F)} \binom{\lambda}{card_{\mathcal{M},\bar{v}} R_A(\bar{v})}$$

□

4.5. **MSOL(τ)-polynomials.** In [Mak, Mak06] it is noted that most graph polynomials are actually **MSOL**-polynomials.

Clearly, every (extended) **MSOL**-polynomial is a generalised chromatic polynomial. Our proof of Theorems D and E do not reveal the exact relationship between **MSOL**-polynomials and generalised **MSOL**-definable chromatic polynomials.

We end this section with two questions, which remained unresolved.

Question 2. *What is the exact relationship between* **SOL**(τ)*-polynomials and extended* **SOL**(τ)*-polynomials?*

Question 3. *What is the exact relationship between* **MSOL***-polynomials and generalised* **MSOL***-definable chromatic polynomials?*

5. Enter categoricity

In this section we present an even more general approach to graph polynomials, using advanced model thoery, in particular the theory of categorical structures. We would like to remind the reader that in this section we require some background in model theory, which goes beyong what was needed in the previous sections. A good background reference is [Hod93]. A bit more elementary and still providing necessary background on categoricity is the monograph [Rot95].

We first describe a general method of attaching uniformly to the members G of a family of finite structures $\mathcal{G}$ an infinite structure $M(G)$. In the simplest case, the structure $M(G) = M(G, D)$ depends on an infinite set $D = \mathbb{N}$. The structure $M(G, \mathbb{N})$ encodes the family of structures $\langle G, [k] \rangle = G_k$ introduced in Section 3. The definable functions $f : G \to \mathbb{N}$ in $M(G, \mathbb{N})$ correspond to generalised colorings where $\mathbb{N}$ is an infinite set of colors. This approach is extended to definable sets in $M(G, \mathbb{N})$. Correspondingly, if instead of $\mathbb{N}$ we use α-many copies of $\mathbb{N}$ we get generalised multi-colorings. The novelty here is, that we allow D to carry more structure, giving rise to a richer class of generalised colorings.

5.1. Background on categoricity.

We quote from [Hod93, Rot95]. We assume that all vocabularies are countable or finite. A *theory* $T \subseteq \mathbf{FOL}(\tau)$ a consistent (satisfiable) set of first order sentences over the vocabulary τ. A *complete theory* $T \subseteq \mathbf{FOL}(\tau)$ is a maximally consistent set of first order sentences. For a τ-structure $\mathcal{M}$ we denote by $Th(\mathcal{M})$ the set of $\mathbf{FOL}(\tau)$-sentences true in $\mathcal{M}$.

Definition 5.1 (Background). *Let $T \subseteq \mathbf{FOL}(\tau)$ be a theory.*

(i) *T is* complete*, if it it is maximally consistent.*
(ii) *T has the* finite model property*, if each finite subset of T has a finite model.*
(iii) *Let κ be a cardinal (initial ordinal). T is κ if T has an infinite model and any two models of cardinality κ isomorphic.*
(iv) *If T is κ-categorical for some infinite κ, and has no finite models, then T is complete (Vaught's Test).*
(v) *If T is κ-categorical for some uncountable κ, then T is κ'-categorical for all uncountable κ' (Morley's Theorem).*
(vi) *Hence there are two cases which can occur independently in all combinations: T is (or is not) ω-categorical, or T is (or is not) ω_1-categorical. A complete theory which is categorical in all infinite powers is called* totally categorical.

(vii) *An element* $a \in M$ *is* algebraic over $C \subset M$ *if there is* τ-*formula* $\phi(x, \bar{c})$ *with one free variable* x *and parameters* $\bar{c}$ *from* C*, such that*

$$\{a \in M : \mathcal{M} \models \phi(a, \bar{c})\}$$

is finite.

(viii) *In a structure* $\mathcal{M}$ *we define an* algebraic closure of a set $C \subseteq M$*, denoted by* $acl(C)$*, as the set of elements in* M *which are algebraic over* C*.*

For the more complex notions related to the structure theory of totally categorical theories, such as *C-definable sets, minimal and strongly minimal sets rank, dimension, ω-stability, etc.*, we refer the reader to the standard texts, e.g. [Rot95, Hod93]. These notions are not used in our technical proofs, but they are mentioned in theorems needed in the proofs. The rank of a subset $a \subseteq MS$ of structure is denoted by $\mathrm{rk}\,(S)$.

5.2. **The Functor.** Let $\mathcal{G}$ be a class of finite structures of a finite language τ_0. Let $D_1, \ldots, D_k$ be countable infinite structures of finite languages $\tau_1, \ldots, \tau_k$, correspondingly.

For every $G \in \mathcal{G}$ we construct the structure $M(G, D_1, \ldots, D_k)$ of sorts $G, D_1, \ldots, D_k$ and F and the language $\tau = \tau_0 \cup \tau_1 \cdots \cup \tau_k$ and extra function symbol

$$\Phi : G \times F \to D_1 \times \ldots \times D_k.$$

The only condition on Φ is

$$\forall f, f' \in F([\forall g \in G\ \Phi(g, f) = \Phi(g, f')] \to f = f').$$

We write $f(g)$ instead of $\Phi(g, f)$ and so identify elements $f \in F$ with functions $G \to D_1 \times \cdots \times D_k$. In other words we have the canonical identification

$$\Phi^\star : F \leftrightarrow (D_1 \times \cdots \times D_k)^G,$$

and fixing an enumeration of G we may identify the right-hand-side with the cartesian power

$$(D_1 \times \cdots \times D_k)^{|G|}.$$

Remark 5.2. *By the virtue of the construction, given* $D_1, \ldots, D_k$*, the isomorphism type of* $M(G, D_1, \ldots, D_k)$ *depends only on* G*. Obviously,* G *can be recovered from* $M(G, D_1, \ldots, D_k)$*. So,* $M(G, D_1, \ldots, D_k)$ *can be seen as the complete invariant of* G*. In particular, every definable subset* S *of* F *is an invariant of* G*.*

Proposition 5.3. *$M(G, D_1, \dots, D_k)$ is definable using parameters in the disjoint union $D_1 \cup \dots \cup D_k$.*

Proof. Obviously $M(G, D_1, \dots, D_k)$ is definable in the disjoint union of $G, D_1, \dots, D_k$. But as G is finite, one can interprete this sort using $|G|$ constants. □

Corollary 5.4.

(i) *Assume that the theory of each D_i is ω-categorical. Then the theory* $\mathrm{Th}[M(G, D_1, \dots, D_k)]$ *is ω-categorical.*
(ii) *Assume that the theory of each D_i is strongly minimal. Then the theory* $\mathrm{Th}[M(G, D_1, \dots, D_k)]$ *is ω-stable with k independent dimensions. If $k = 1$ then the theory is categorical in uncountable cardinals.*

Theorem 5.5 (B. Zilber)**.** *Any theory satisfying (i) and (ii) has the finite model property. Moreover any countable model M can be represented as*

$$M = \bigcup_{i=1}^{\infty} M_i,$$

i.e., as a union of an increasing chain of finite substructures (logically) approximating M.

Proof. This follows from Theorem 7 of [CH03], where also more details may be found. □

Remark 5.6. *The finite model property takes a very simple form for a s.m. structure D. Namely, D has the finite model property if and only if* $\mathrm{acl}\,(X)$ *is finite for any finite $X \subseteq D$.*

5.3. **Counting functions for definable sets.** A very important consequence of the finite model property is the possibility to introduce a stronger *counting function* on definable sets.

We prove here the existence of the counting polynomials in a special case, for the theory $\mathrm{Th}[M(G, D_1, \dots, D_k)]$. This is a precise and detailed version of Theorem A. The more general case of ω-categorical ω-stable theories can be found in [CH03, Proposition 5.2.2.]. The more special case of theories categorical in all infinite cardinals has been proven in [Zil84a, Zil84b] and can be found [Zil93]. The proof under the special asumptions needed in this paper is really elementary and does not require any model-theoretic terminology if one assumes the D_i's to be just sets. It really is a slight generalization of the proof given for Theorem 3.7.

Theorem 5.7. *Let $M = M(G, D_1, \dots, D_k)$. Assume the finite model property holds in the strongly minimal structures $D_1, \dots D_k$. Then for every finite $C \subseteq M$ and any C-definable set $S \subseteq M^\ell$ there is a polynomial $p_S \in \mathbb{Q}[x]$ and a number n_S such that for every finite $X \subseteq M$ with $C \subseteq X$,*

(i) *letting $|D_i \cap \mathrm{acl}\,(X)| = x_i \geq n_S$, we have*

$$|S \cap \mathrm{acl}\, X| = p_S(x_1, \dots, x_k);$$

(ii) $\mathrm{rk}\,(S) = \deg(p_S)$*, the degree of the polynomial;*

(iii) *if g(S)=T for some automorphism g of M then $p_S = p_T$ and $n_S = n_T$.*

Furthermore, if $C = \emptyset$ we can take $n_S = 0$.

Proof. We construct the polynomial for a given S by induction on $\mathrm{rk}\,(S)$.

W.l.o.g. we may assume that S is an atom over C, that is defined by a principal type over C.

Let $f = \langle f_1, \dots, f_\ell \rangle \in S$. Recall that each f_i is determined by the values of $f_i(g) \in D_1 \times \cdots \times D_k$, for $g \in G$. Denote $f_{im}(g)$ the mth co-ordinate of $f_i(g)$, an element of D_m.

Suppose $f_{im}(g) \in \mathrm{acl}\,(C)$ for all $i \leq \ell$, $m \leq k$ and $g \in G$. Then $f \in \mathrm{acl}\,(C)$. Since S is an atom, $S \subseteq \mathrm{acl}\,(C)$ and hence

$$|S \cap \mathrm{acl}\,(X)| = |S \cap \mathrm{acl}\,(C)|$$

is a constant, independent of X. So, we are done in this case.

We may now assume that $f_{11}(g_0) \notin \mathrm{acl}\,(C)$. So, we have the partition

$$S = \bigcup_{a \in D_1 \setminus \mathrm{acl}\,(C)} S_a, \quad S_a = \{f \in S : f_{11}(g_0) = a\}.$$

Since $D_1 \setminus \mathrm{acl}\,(C)$ is an atom over C (use the strong minimality of D_1) and, of course $G \subseteq \mathrm{acl}\,(\emptyset)$, the subgroup of the automorphism group of M fixing C acts transitively on $D_1 \setminus \mathrm{acl}\,(C)$. Hence all the fibers S_a are conjugated by automorphisms over C and have the same Morley rank. The latter implies by the addition formula for ranks that

$$\mathrm{rk}\,(S_a) = \mathrm{rk}\,(S) - 1.$$

So, we may apply the induction hypothesis. By (iii) we get that

$$p_{S_a} = p_0, \text{ for all } a \in D_1 \setminus \mathrm{acl}\,(C),$$

for some polynomial p_0. By (ii) $\deg(p_0) = \mathrm{rk}\,(S_a) = \mathrm{rk}\,(S) - 1$.

Let $c_0 = |\mathrm{acl}\,(C)|$. So,

$$|(D_1 \setminus \mathrm{acl}\,(C)) \cap \mathrm{acl}\,(X)| = (x_1 - c_0).$$

We further calculate

$$S \cap \mathrm{acl}\,(X) = \bigcup_{a \in (D_1 \setminus \mathrm{acl}\,(C)) \cap \mathrm{acl}\,(X)} S_a \cap \mathrm{acl}\,(X) = (x_1 - c_0) \cdot p_0(x_1, \ldots, x_k).$$

□

5.4. **Generalised chromatic polynomials revisited.** In the light of Theorem 5.7 let us look first at the generalised colorings of Section 3.

We discuss it for the class $\mathcal{S}(\tau)$ of finite purely relational τ-structures. We denote by $\mathbf{SOL}^n(\tau)$ the set of $\mathbf{SOL}(\tau)$-formulas, where all second order variables are arity at most n. Let $\phi(\bar{R}, F) \in \mathbf{SOL}^n(\tau)$ define generalised coloring where $\bar{R}$ are relation parameters, and F denotes the coloring function. So the generalised chromatic polynomial on a τ-structure $\mathcal{A}$ is defined as

$$\chi_{\phi(\bar{R},F)}(\mathcal{A}, k) = \mid \{(\bar{r}, f) : \langle \mathcal{A}, \bar{r}, f, [k] \rangle \models \phi(\bar{r}, f)\} \mid$$

We first expand $\mathcal{A}$ such that quantification over relations becomes quantification over elements. So for each $\ell \leq n$ we add the set $\wp(A^\ell$ with the corresponding membership relation $\in_\ell$. We define the $\tau^\star$-structure

$$\mathcal{A}^\star = \langle \mathcal{A}, \wp(A^\ell \in_\ell, \ell \leq n \rangle$$

and apply our functor $M(\mathcal{A}^\star, \mathbb{N})$ to it with $D_1 = \mathbb{N}$. Let the $\tau^\sharp$ be the vocabulary of $M(\mathcal{A}^\star, \mathbb{N})$.

Now the formula $\phi(\bar{R}, F) \in \mathbf{SOL}^n(\tau)$ has a straighforward translation $\phi^\sharp(\bar{c}_{\bar{R}}, d_F) \in \mathbf{FOL}(\tau^\sharp)$, where the function symbol F becomes a variable d_F, de relation symbols $\bar{R}$ become variables $\bar{c}_{\bar{R}}$ of the appropriate sorts. Furthermore, it has no additional parameters, hence $C = \emptyset$.

Let $X \subseteq \mathbb{N}$ be finite. Due to Theorem 5.7(iii), w.l.o.g., $C = [k]$ for some $k \in \mathbb{N}$. Let $\mathcal{A}^\star_k$ be the substructure of $M(\mathcal{A}^\star, \mathbb{N})$ with universe $acl([k])$ and $F_k \subseteq F$ be its part of the sort F. We now easily verify that

(i) ${\mathcal{A}^\star}_k$ contains all of $\mathcal{A}^\star$.
(ii) F_k consists exactly of all functions f with range $Rg(f) \subseteq [k]$.
(iii) For $S = \{(\bar{c}, d) \in M(\mathcal{A}^\star, \mathbb{N}) : M(\mathcal{A}^\star, \mathbb{N}) \models \phi^\sharp(\bar{c}, d)\}$ we have that $\mid S \cap \mathrm{acl}\,(k) \mid = p_S(k)$ is a polynomial for every $k \geq n_S = 0$.
(iv) $\chi_{\phi(\bar{R},F)}(\mathcal{A}, k) = \mid \{(\bar{c}, d) \in \mathcal{A}^\star_k : M(\mathcal{A}^\star, \mathbb{N}) \models \phi^\sharp(\bar{c}, d)\} \mid = p_S(k)$

This proves Theorem B for the case of genralised chromatic polynomials in one variable.

5.5. **Proof of Theorem B.** To prove Theorem B in its full generality proceed as before. We obeserve the following points:

- For multi-colorings we use several copies of $\mathbb{N}$ as strongly minimal sets.
- If the generalised colorings are relations $r \subset G^{\alpha} \times \mathbb{N}^{\beta}$ the proof still works, provided $M(G, \mathbb{N}, \ldots, \mathbb{N})$ is ω-stable. This is, where we use in our definition of generalised multi-coloring, that for each $\bar{x} \in G^{\alpha}$ the set $r_{\bar{a}} = \{\bar{b} \in \mathbb{N}^{\beta} : r(\bar{a}, \bar{b})\}$ is bounded by fixed finite number d. Without this restriction ω-categoricity is violated.
- For $\mathbf{HOL}^m$-definable generalised colorings we use the appropriate expansions.

5.6. **The full generality.** The general theorem allows for more complicated stronly miniml structures to be used for D_1. A simple example would consist of a countable set of disjoint copies of a fixed finite structure such a finite field $GF(p^q)$. The colors then would be pairs (n, a) where $n \in \mathbb{N}$ and $a \in GF(p^q)$. We could request that a graph coloring f of a graph $G = (V, E)$ satisfies, say,

$$[((u, v) \in E \wedge f(u) = (n_u, a_u) \wedge f(v) = (n_v, a_v)) \rightarrow \\ (n_u \neq n_v \wedge a_u + a_v = 0)]$$

It seems possible that such colorings may be useful in modelling wiring conditions when labeled graphs model network devices.

6. Conclusions and open problems

Starting with the classical chromatic polynomial we have introduced generalised multi-colorings. We have shown that the corresponding counting functions are always polynomials, which we called generalised chromatic polynomials.

We have then shown that the class of generalised chromatic polynomials is very rich and covers virtually all examples of graph polynomials which have been studied in the literature. To make this precise, we used the notion of **SOL**-definable graph polynomials introduced in [Mak04], und extended it two-fold: allowing higher order logic, and by allowing polynomials with combinatorial counting functions as monomials.

Finally we have shown that the phenomenon that the counting functions of generalised **HOL**-definable colorings are polynomials has a purely model theoretic counterpart in the counting functions for definable sets in ω-categorical, ω-stable theories, and that all generalised extended chromatic polynomials definable in **HOL** can brought into this framework.

Indeed, the main contribution of this paper is that through model theory one sees a most general picture, in which graph polynomials are just part of it. One has, more generally, polynomial invariants of arbitrary finite structures and one associates with a general ω-categorical ω-stable structure the most general polynomial invariants. This complements one further combinatorial aspect in the structure theory of ω-categorical ω-stable structures. It also proved useful in the classification of finite homogeneous combinatorial geometries, [CH03].

References

[Big93] N. Biggs. *Algebraic Graph Theory, 2nd edition.* Cambridge University Press, 1993.

[Bol99] B. Bollobás. *Modern Graph Theory.* Springer, 1999.

[CGM0x] B. Courcelle, B. Godlin, and J.A. Makowsky. Towards a theory of graph polynomials, I: Second order definable polynomials. *XXX*, xx:xx–yy, 200x.

[CH03] G. Cherlin and E. Hrushovski. *Finite structures with few types*, volume xx of *Annals of Mathematics Studies.* Princeton University Press, 2003.

[CHL80] G. Cherlin, L. Harrington, and A. Lachlan. $\aleph_0$-categorica $\aleph_0$-stable structures. *Annals of Pure and Applied Logic*, 18:227–270, 1980.

[Die96] R. Diestel. *Graph Theory.* Graduate Texts in Mathematics. Springer, 1996.

[DKT05] F.M. Dong, K.M. Koh, and K.L. Teo. *Chromatic Polynomials and Chromaticity of Graphs.* World Scientific, 2005.

[EF95] H.D. Ebbinghaus and J. Flum. *Finite Model Theory.* Perspectives in Mathematical Logic. Springer, 1995.

[EFT94] H.D. Ebbinghaus, J. Flum, and W. Thomas. *Mathematical Logic, 2nd edition.* Undergraduate Texts in Mathematics. Springer-Verlag, 1994.

[GKP94] R. Graham, D. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison-Wesley, 2 edition, 1994.

[God93] C.D. Godsil. *Algebraic Combinatorics.* Chapman and Hall, 1993.

[HHR67] F. Harary, S. Hedetniemi, and G. Rins. An interpolation theorem for graphical homomorphisms. *Portugal. Math.*, 26:453–462, 1967.

[HM97] F. Hughes and G. MacGillivray. The achromatic number of graphs: a survey and some new results. *Bull. Inst. Combin. Appl.*, 19:27–56, 1997.

[Hod93] W. Hodges. *Model Theory*, volume 42 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, 1993.

[Mak] J.A. Makowsky. Lectures on graph polynomials. Lecture Notes 2005, Available at `www.cs.technion.ac.il/~janos/COURSES/238900`.

[Mak04] J.A. Makowsky. Algorithmic uses of the Feferman-Vaught theorem. *Annals of Pure and Applied Logic*, 126:1–3, 2004.

[Mak06] J.A. Makowsky. From a zoo to a zoology: Descriptive complexity for graph polynomials. In A. Beckmann, U. Berger, B. Löwe, and J.V. Tucker, editors, *Logical Approaches to Computational Barriers, Second Conference on Computability in Europe, CiE 2006, Swansea, UK, July 2006*, volume 3988 of *Lecture Notes in Computer Science*, pages 330–341. Springer, 2006.

[Rot95] P. Rothmaler. *Introduction to Model Theory*, volume 15 of *Algebra, Logic abd Applications Series*. Gordon and Breach Science Publishers, 1995.

[Sok05] A. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In *Survey in Combinatorics, 2005*, volume 327 of *London Mathematical Society Lecture Notes*, pages 173–226, 2005.

[Vol02] V. I. Voloshin. *Coloring Mixed Hypergraphs: Theory, Algorithms and Applications*, volume 17 of *Fields Institute Monographs*. American Mathematical Society, 2002.

[Zil84a] B. Zilber. Strongly minimal countably categorical theories, II. *Siberian Math.J.*, 25.3:71–88, 1984.

[Zil84b] B. Zilber. Strongly minimal countably categorical theories, III. *Siberian Math.J.*, 25.4:63–77, 1984.

[Zil93] B. Zilber. *Uncountably Categorical Theories*, volume 117 of *Transl.of Math.Monographs*. Amercan Mathematical Society, Providence, R.I., 1993.

E-mail address, J.A. Makowsky: `janos@cs.technion.ac.il`
E-mail address, B. Zilber: `zilber@maths.ox.ac.uk`

(J.A. Makowsky) Department of Computer Science,
Technion–Israel Institute of Technology,
32000 Haifa, Israel

(B. Zilber) Mathematical Institute,
University of Oxford,
Oxford, OX1 3LB, United Kingdom