STABLE DIVISION RINGS

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Abstract. It is shown that a stable division ring with positive characteristic has finite dimension over its centre. This is then extended to simple division rings.

Macintyre proved any ω-stable field to be either finite or algebraically closed [7]. This was generalised by Cherlin and Shelah to superstable fields [3]. It follows that a superstable division ring is a field [2]. The result was broadened to supersimple division rings by Pillay, Scanlon and Wagner in [8]. As for stable fields, infinite ones are conjectured to be separably closed. Scanlon proved that an infinite stable field has no Artin-Schreier extension [10]. Wagner adapted the argument to show that a simple field has only finitely many Artin-Schreier extensions [6]. Proving commutativity usually goes in two steps, showing first that the ring viewed as a vector space over its centre must have finite dimension, and proving that the centre cannot have skew extensions of finite degree. Concerning a stable division ring, at least can we show that in positive characteristic, it must have finite dimension over its centre. This also holds for a simple division ring.

1. One word on stable structures

In a given theory T, a formula $f(x, y)$ is said to have the *order property* if it totally orders an infinite sequence, i.e. if there exists an infinite sequence $a_1, a_2 \dots$ such that

$$T \models f(a_i, a_j) \text{ if and only if } i < j$$

The formula f has the *strict order property* if it defines a partial ordering with infinite chains, i.e. if there exists an infinite sequence $a_1, a_2 \dots$ such that

$$T \models \bigwedge_{i<j} f(a_i, a_j) \wedge a_i \neq a_j$$

If a formula has the strict order property, it has the order property.

Definition 1. A theory is *stable* if no formula has the order property. A structure is *stable* if its theory is so.

We refer to [9] and [12] for details about stable groups. We just recall that to any formula $f(x, y)$ in a group without the strict order property is associated an integer n, such that any strictly decreasing chain of subgroups defined by formulae $f(x, a_1), \dots, f(x, a_m)$ have no more than n elements. Moreover :

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Fact 1. (Baldwin-Saxl [1]) *In a stable group, to any formula $f(x,y)$ is associated an integer n, so that the intersection of any family of subgroups $H_1, \dots, H_m, \dots$ defined by formulae $f(x,a_1), \dots, f(x,a_m), \dots$ be the intersection of no more than n among them.*

Therefore, any strictly monotone chain of centralisers in a stable group is finite.

Proposition 2. *Let G be a group without the strict order property, and f a group homomorphism from G to G. If there is a fixed formula $f(x,y)$ so that each iterated image of f is definable by some formula $f(x,a_i)$, then G equals the product* $\mathrm{Ker} f^n \cdot \mathrm{Im} f^n$ *for some integer n. Consequently, if f is injective, it is onto.*

Proof. As the iterated images of f are uniformly definable, they become stationary at some rank n. □

2. Stable division rings

Theorem 3. *A stable division ring of positive characteristic must have finite dimension over its centre.*

Proof. Let D be this ring, p its characteristic, a an element outside the centre, and f_a the map mapping an element x of the ring to $x^a - x$.

(1) The iterated images and kernels of f become stationary : since

$$f_a^{p^n}(x) = \sum_{k=0}^{p^n} (-1)^{p^n-k} C_{p^n}^k x^{a^k} = x^{a^{p^n}} - x = f_{a^{p^n}}(x)$$

a sub-chain of the iterated images is uniformly definable : the iterated images become stationary by stability. The same argument holds for the iterated kernels.

(2) The map f is not onto : if it were, since the kernel is non-trivial, the sequence of iterated kernels would be properly ascending, a contradiction.

(3) D is a finite dimensional vector space over $C(a)$: after Proposition 2, there is an integer m such that

$$D = \mathrm{Ker} f^m + \mathrm{Im} f^m$$

Note that this is a direct sum. Let H be the image of f^m ; increasing m, we may assume the kernel of f^m to be $C(a^m)$. Let I be a minimal intersection of left translates of H by non-zero elements of D ; this is a proper left ideal of D and hence zero. However, by Fact 1 the intersection is a finite intersection, say of size n. After [4, Corollary 2 p. 49], the dimension of $C(a^m)$ over $C(a)$ is the same as the dimension of $Z(C(a^m))(a)$ over $Z(C(a^m))$, so H is a vector space over $C(a)$ having codimension at most m, and I has codimension at most $m \cdot n$.

(4) To conclude, let $D < D_1 < \cdots < D_n < D_{n+1}$ be a chain of centralisers, with D_n minimal non commutative. The ring D has finite dimension, say l, over the field D_{n+1}. According to [4, Corollary 2 p. 49], the dimension of D over its centre must be no greater than l^2. □

Remark 4. The centre of an infinite stable division ring must be infinite. In positive characteristic, it contains the algebraic closure of $\mathbf{F}_p$ according to [10] : every element of finite order lies in the centre.

3. Simple division rings

We do not define here what a simple theory is, but refer to [13] for more information. We shall just need the following facts. Recall that two subgroups of a given group are *commensurable* if the index of their intersection is finite in both of them.

Fact 2. (Schlichting [11, 13]) *Let G be a group and $\mathfrak{H}$ a family of uniformly commensurable subgroups. There is a subgroup N of G commensurable with members of $\mathfrak{H}$ and invariant under the action of the automorphisms group of G stabilising the family $\mathfrak{H}$ setwise. If the members of $\mathfrak{H}$ are definable, so is N.*

Fact 3. (Wagner [13]) *In a simple group, a descending chain of intersections of a family $H_1, H_2 \dots$ of subgroups defined respectively by formulae $f(x, a_1), f(x, a_2) \dots$ where $f(x, y)$ is a fixed formula, becomes stationary, up to finite index.*

Remark 5. If $D_1 < D_2$ are two infinite division rings, the additive index of D_1 in D_2 is infinite. As a consequence, in a simple division ring, any descending chain of centralisers becomes stationary.

Fact 4. *In a simple structure, no formula has the strict order property.*

Theorem 6. *A simple ring of positive characteristic must have finite dimension over its centre.*

Proof. Let D be this ring, p its characteristic, a an element outside the centre, and f_a mapping x to $x^a - x$.

(1) The iterated images and kernels of f become stationary, and f is not onto : as in the stable case by Fact 4.

(2) The centraliser of a is infinite : we may assume the order of a to be finite, and even a prime, say q. According to [5, Lemma 3.1.1], there is an element x of finite order such that xax^{-1} equals a^i but not a. Fermat's Theorem asserts that i^{q-1} equals one modulo q, so x^{q-1} and a commute : $C(a)$ is infinite, as it contains x^{q-1}.

(3) D is a vector space over $C(a)$ having finite dimension : according to Proposition 2, we get

$$D = \mathrm{Ker} f^m + \mathrm{Im} f^m$$

Let H stand for the image of f^m, and assume its kernel to be $C(a^m)$. Set N a minimal intersection up to finite index of non-zero left translates of H ; by Fact 3, it has finite size, say n. Consider the set $\mathfrak{H}$ of non-zero left translates of N. This is a uniformly commensurable invariant family ; by Fact 2, there is an additive invariant subgroup I commensurable with N. So I is a proper ideal, whence zero, and N must be finite. Since it is a right vector space over $C(a)$, it is actually zero. We conclude as in the stable case that D has finite dimension over $C(a)$, and over its centre by Remark 5. □

Proposition 7. *Let K be an infinite field, and f a field morphism of K. Let F be the set of points fixed by f. Let P be a polynomial splitting in F, and suppose that the iterated compositions $P(f)^n$ be uniformly definable. If K is simple, either K is an algebraic extension of F, or the image of $P(f)$ has finite index in K^+.*

Proof. We may assume K to be infinite. Let $(X - a_i)^{n_i}$ be the splitting factors of f. Note that $\mathrm{Ker} P(f)$ equals the sum $\bigoplus_i \mathrm{Ker}(f - a_i \cdot id)^{n_i}$, each factor $\mathrm{Ker}(f - a_i \cdot id)^{n_i}$

having dimension at most n_i over F. According to Proposition 2, the field K equals $\mathrm{Ker}P(f)^m + \mathrm{Im}P(f)^m$. Let H be the image of $P(f)^m$, and N a minimal intersection up to finite index of non-zero translates of H. Note that if N is finite, there is a minimal intersection which is a proper ideal, hence zero. By Fact 3, N is a finite intersection, say of size n. Write $\mathfrak{H}$ the set of non-zero translates of N. According to Fact 2, there is an additive invariant subgroup I of K, commensurable with N. So I is an ideal of K. If I is the whole of K, the image of $P(f)$ has finite index in K^+ ; should F be infinite, the map $P(f)$ would be onto as its image is a vector space over F. Otherwise, I is zero, and so is N. But H is a vector space over F having finite codimension, say r, so N has codimension at most $r \cdot n$. □

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