An affine definable C^rG manifold admits a unique affine definable $C^\infty G$ manifold structure

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Abstract

Let G be a compact subgroup of $GL_n(\mathbb{R})$. We prove that every affine definable C^rG manifold admits a unique affine definable $C^\infty G$ manifold structure up to definable $C^\infty G$ diffeomorphism ($1 \leqq r < \infty$). Moreover we prove that every strongly definable C^rG vector bundle over X admits a unique strongly definable $C^\infty G$ vector bundle structure up to definable $C^\infty G$ vector bundle isomorphism ($0 \leqq r < \infty$). Furthermore we consider raising differentiability of strong definable C^r fiber bundles ($0 \leqq r < \infty$).

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1 . Introduction.

By [15], if s is a non-negative integer, then every C^s Nash map between affine Nash manifolds is approximated in the definable C^s topology by Nash maps. This definable C^s topology is a new topology defined in [15]. There is a generalization of this result in the definable C^r category obtained by an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ on the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers, namely if $0 \leq s < r < \infty$, then every definable C^s map between affine definable C^r manifolds is approximated in the definable C^s topology by definable C^r maps (II.5.2 [16]).

In this paper, G denotes a compact subgroup of $GL_n(\mathbb{R})$, every definable map is continuous and any manifold does not have boundary, unless otherwise stated. Under our assumption, G is a compact algebraic subgroup of $GL_n(\mathbb{R})$ (e.g. 2.2 [13]). We consider an equivariant definable version of the above theorem in an o-minimal expansion $\mathcal{M}$ and an affine definable $C^\infty G$ manifold structure of an affine definable C^rG manifold. General references on o-minimal structures are [2], [4], see also [16]. Further properties and constructions of them are studied in [3], [5], [14].

We also consider strongly definable $C^\infty G$ vector bundle structures of strongly definable C^rG vector bundles ($0 \leqq r < \infty$). Moreover we consider raising differentiability of strong definable C^r fiber bundles ($0 \leqq r < \infty$).

Suppose that η is a definable C^rG vector bundle over an affine definable C^rG manifold X and $0 \leq r \leq \infty$. We say that η is *strongly definable* if there exist a representation Ω of G and a definable C^rG map $f : X \to G(\Omega, \alpha)$ such that η is definably C^rG vector bundle isomorphic to $f^*(\gamma(\Omega, \alpha))$, where α denotes the rank of η.

The following is the main result of this paper.

Theorem 1.1. *Let X be an affine definable C^rG manifold and $\mathcal{M}$ admits C^∞ cell decomposition and exponential.*

(1) If $1 \leqq r < \infty$ then, X admits a unique affine definable $C^\infty G$ manifold structure up to definable $C^\infty G$ diffeomorphism.

(2) If $0 \leqq r < \infty$, then every strongly definable C^rG vector bundle over X admits a unique strongly definable $C^\infty G$ vector bundle structure up to definable $C^\infty G$ vector bundle isomorphism.

(3) If $0 \leqq r < \infty$, each strongly definable C^r fiber bundle over an affine definable C^∞ manifold admits a unique strongly definable C^∞ fiber bundle structure up to definable C^∞ fiber bundle isomorphism.

Remark.

(1) If $1 \leqq s < r < \infty$, then definable C^r manifold structures of definable C^s manifolds are studied in [8].

(2) If $1 \leqq s < r < \infty$ and G is finite, then strongly definable C^r vector bundle structures of strongly definable C^sG vector bundles are studied in [9].

(3) If $1 \leqq s < r < \infty$, then strongly definable C^r fiber bundle structures of strongly definable C^s fiber bundles are studied in [6].

2 . Definable C^rG manifolds.

Recall the definition of definable C^rG manifolds ([11], [9]).

Definition 2.1 ([11], [9]). Let $0 \leq r \leq \infty$.

(1) A group homomorphism (resp. A group isomorphism) from G to $O_n(\mathbb{R})$ is a *definable group homomorphism* (resp. a *definable group isomorphism*) if it is a definable map (resp. a definable homeomorphism).

Note that a definable group homomorphism (resp. a definable group isomorphism) between G and $O_n(\mathbb{R})$ is a definable C^∞ map (resp. a definable C^∞ diffeomorphism) because G and $O_n(\mathbb{R})$ are Lie groups.

(2) An *n-dimensional representation* of G means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from G to $O_n(\mathbb{R})$. In this paper, we assume that every representation of G is orthogonal.

(3) A *definable C^rG manifold* is a pair (X, α) consisting of a definable C^r manifold X and a group action α of G on X such that $\alpha : G \times X \to X$ is a definable C^r map. For simplicity of notation, we write X instead of (X, α).

(4) A definable C^r submanifold of a definable C^rG manifold X is called a *definable C^rG submanifold* of X if it is G invariant.

(5) A definable C^r map (resp. A definable C^r diffeomorphism, A definable homeomorphism, A definable map) is a *definable C^rG map* (resp. a *definable C^rG diffeomorphism*, a *definable G homeomorphism*, a *definable G map*) if it is a G map.

(6) A definable C^rG manifold is called *affine* if it is definably C^rG diffeomorphic (definably G homeomorphic if $r = 0$) to a definable C^rG submanifold of some representation of G.

(7) A *definable C^rG manifold with boundary* is defined similarly.

If $0 \leq r < \infty$, then every definable C^r manifold is affine ([11], [10]) and if $\mathcal{M}$ is exponential, then each compact definable $C^\infty G$ manifold is affine [11].

Recall the definable C^s topology [9] and some results on it [9].

Let X and Y be definable C^s submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $0 \le s < \infty$. Let $C^s_{def}(X,Y)$ denote the set of definable C^s maps from X to Y. For $f \in C^s_{def}(X,Y)$ and $x \in X$, the differential df_x of f at x means a linear map from the tangent space T_xX of X at x to $\mathbb{R}^m$. Composing it with the orthogonal projection $\mathbb{R}^n \to T_xX$, one can extend df_x to a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Then $Df : X \to M(m,n;\mathbb{R}) = \mathbb{R}^{mn}$ is defined as the matrix representation of df. For each $1 \le k \le s$, we inductively define a C^{s-k} map

$$D^k f : X \to \mathbb{R}^{n^k m}, D^k f = D(D^{k-1} f).$$

Let $||f||_s$ denote the definable function on X defined by

$$||f||_s(x) = |f(x)| + |Df(x)| + \cdots + |D^s f(x)|.$$

For a positive definable function $\epsilon : X \to \mathbb{R}$, let

$$U_\epsilon = \{h \in C^s_{def}(X,Y) | ||h||_s < \epsilon\}.$$

We say that the *definable C^s topology* on $C^s_{def}(X,Y)$ is the topology defined by choosing $\{h+U_\epsilon\}_\epsilon$ as a fundamental neighborhood system of h in $C^s_{def}(X,Y)$. In the Nash category, we simply call it the *C^r topology*. If X is compact, then this topology coincides with the C^s Whitney topology (p 156 [16]).

Proposition 2.2 ([16], 4.9 [9])**.** *Let X, Y and Z be definable C^s submanifolds $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^l$, respectively, and $0 \le s < \infty$. Let $f \in C^s_{def}(X,Y)$ and $h \in C^s_{def}(Y,Z)$.*

(1) The map $h_* : C^s_{def}(X,Y) \to C^s_{def}(X,Z)$, $h_*(k) = h \circ k$ is continuous.

(2) The map $f^* : C^s_{def}(Y,Z) \to C^s_{def}(X,Z)$, $f^*(k) = k \circ f$ is continuous if and only if f is proper.

Proposition 2.3 ([16], 4.10 [9])**.** *Let X and Y be definable C^s submanifolds of $\mathbb{R}^n$ and $0 < s < \infty$. Let $f : X \to Y$ be a definable C^s map. If f is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image), then an approximation of f in the definable C^s topology is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image). Moreover if f is a diffeomorphism, then $h^{-1} \to f^{-1}$ as $h \to f$.*

Theorem 2.4 ([16], 4.11 [9])**.** *Let X and Y be affine definable C^r manifolds and $0 \le s < r < \infty$. Then every definable C^s map $f : X \to Y$ is approximated in the definable C^s topology by definable C^r maps.*

Theorem 2.5 ([8])**.** *If $0 \le s < r < \infty$, then every definable C^sG map between affine definable C^rG manifolds is approximated in the definable C^s topology by definable C^rG maps.*

Proposition 2.6 ([11])**.** *Every definable $C^\infty G$ submanifold X of a representation Ω of G has a definable $C^\infty G$ tubular neighborhood (U,θ) of X in Ω, namely U is a G invariant definable open neighborhood of X in Ω and $\theta : U \to X$ is a definable $C^\infty G$ map with $\theta|X = id_X$.*

Proposition 2.7 ([8])**.** *(Equivariant definable C^r partition of unity). Let X be a definable C^rG submanifold closed in a representation Ω of G and $\{U_i\}_{i=1}^l$ a finite G invariant definable open covering of X and $0 \le r < \infty$. Then there exist G invariant definable C^r functions $\lambda_1, \dots, \lambda_l : X \to \mathbb{R}$ such that $0 \le \lambda_i \le 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

By a way similar to the proof of the above proposition, we have the following.

Proposition 2.8 (Equivariant definable C^∞ partition of unity)**.** *Suppose that $\mathcal{M}$ admits C^∞ cell decomposition and exponential. Let X be a definable $C^\infty G$ submanifold closed in a representation Ω of G and $\{U_i\}_{i=1}^l$ a finite G invariant definable open covering of X. Then there exist G invariant definable C^∞ functions $\lambda_1, \dots, \lambda_l : X \to \mathbb{R}$ such that $0 \le \lambda_i \le 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

Proposition 2.9 ([9]). *Let X be a compact affine definable $C^\infty G$ manifold with boundary ∂X. Then X admits a definable $C^\infty G$ collar, that is , there exists a definable $C^\infty G$ imbedding $\phi : \partial X \times [0, 1) \to X$ such that $\phi|\partial X \times \{0\} = id_{\partial X}$, where the action on $[0, 1)$ is trivial.*

Theorem 2.10 ([9]). *Every definable $C^\infty G$ manifold is either compact or compactifiable.*

Let f be a map from a $C^r G$ manifold X to a representation Ω of G and $0 \leq r \leq \infty$. Denote the Haar measure of G by dg, and let x be a point in X. Recall the *averaging operator* A is defined by

$$A(f)(x) = \int_G g^{-1} f(gx) dg.$$

Proposition 2.11 (4.1 [1]). *Let G be a compact Lie group and $0 \leq r \leq \infty$. Suppose that $C^r(X, \Omega)$ denotes the set of C^r maps from a $C^r G$ submanifold X of a representation of G to a representation Ω of G.*

(1) The averaged map $A(f)$ of f is equivariant, and $A(f) = f$ if f is equivariant.

(2) If $f \in C^r(X, \Omega)$, then $A(f) \in C^r(X, \Omega)$.

(3) If f is a polynomial map, then so is $A(f)$.

(4) If X is compact and $r < \infty$, then $A : C^r(X, \Omega) \to C^r(X, \Omega)$ is continuous in the C^r Whitney topology.

Theorem 2.12. *If $0 \leq s < \infty$ and $\mathcal{M}$ admits C^∞ cell decomposition and exponential, then every definable $C^s G$ map between affine definable $C^\infty G$ manifolds is approximated in the definable C^s topology by definable $C^\infty G$ maps.*

Proof. Let $f : X \to Y$ be a definable $C^s G$ map. If X is compact, the proof is easy. We assume that X is noncompact.

Since $\mathcal{M}$ admits C^∞ cell decomposition, is exponential and by Theorem 2.10, X is definably $C^\infty G$ diffeomorphic to the interior of a compact definable $C^\infty G$ manifold Y with boundary ∂Y. By Proposition 2.9, we ca take the double W of Y. Then W is a compact definable $C^\infty G$ manifold. By [11], W is affine. Note that X is a definable $C^\infty G$ submanifold of W.

Since $\mathcal{M}$ admits C^∞ cell decomposition and by 2.3 [9], there exists a definable open subset Z of X such that $\dim(X - Z) < \dim X$ and $f|Z$ is a definable C^∞ map. Since X is a definable G set, we can take Z which is definable and G invariant. Since the action is orthogonal, the ϵ neighborhood $N(Z, \epsilon) = \{x \in X | d(x, Z) < \epsilon\}$ is a G invariant definable open G set. Since W is compact, $N(Z, \epsilon)$ is bounded and the closure N' of $N(Z, \epsilon)$ is compact. Applying Proposition 2.11, there exists a polynomial G map $F : N' \to \Xi$ such that F is an approximation of $f|N'$. By Proposition 2.8, gluing $f|X - Z$ and $f|N'$, we have a definable $C^\infty G$ map $h : X \to \Xi$. By Proposition 2.6, there exists a definable $C^\infty G$ tubular neighborhood (U, θ) of Y in Ξ. The map defined by $\theta \circ h$ is the required map. $\square$

By a way to a partial proof of equivariant Nash conjecture, we have the following theorem.

Theorem 2.13. *Let X be an affine definable $C^r G$ manifold and $1 \leqq r < \infty$. Then X admits an affine definable $C^\infty G$ manifold structure.*

Proof of Theorem 1.1 (1). By Theorem 2.13, X admits an affine definable $C^\infty G$ manifold structure. Uniqueness of affine definable $C^\infty G$ manifold structure follows from Theorem 2.12 and Proposition 2.3. $\square$

Remark that nonaffine definable $C^\infty G$ manifold structures of an affine definable C^r G manifold is not necessarily unique even if $\mathcal{M} = (\mathbb{R}, +, \cdot, <)$ ([12]). If the action on X is transitive, then definable $C^\infty G$ manifold structure is unique and there is no nonaffine definable $C^\infty G$ manifold structure ([12]).

3 . Definable C^rG vector bundles.

Recall the definition of definable C^rG vector bundles [9].

Definition 3.1 ([9])**.** Suppose that $0 \leq r \leq \infty$.

(1) A *definable C^rG vector bundle* is a definable C^r vector bundle $\eta = (E, p, X)$ satisfying the following three conditions.

(a) The total space E and the base space X are definable C^rG manifolds.

(b) The projection $p : E \to X$ is a definable C^rG map.

(c) For any $x \in X$ and $g \in G$, the map $p^{-1}(x) \to p^{-1}(gx)$ is linear.

(2) Let η and ζ be definable C^rG vector bundles over X. A definable C^r vector bundle morphism $\eta \to \zeta$ is called a *definable C^rG vector bundle morphism* if it is a G map. A definable C^rG vector bundle morphism $f : \eta \to \zeta$ is said to be a *definable C^rG vector bundle isomorphism* if there exists a definable C^rG vector bundle morphism $h : \zeta \to \eta$ such that $f \circ h = id$ and $h \circ f = id$.

(3) A definable C^r section of a definable C^rG vector bundle is a *definable C^rG section* if it is a G map.

(4) If $r = 0$, then a definable C^0G vector bundle (resp. a definable C^0G vector bundle morphism, a definable C^0G vector bundle isomorphism, a definable C^0G section) is simply called a *definable G vector bundle* (resp. a *definable G vector bundle morphism*, a *definable G vector bundle isomorphism*, a *definable G section*).

Recall universal G vector bundles (e.g. [9]) and existence of a Nash G tubular neighborhood of a Nash G submanifold of a representation of G (2.3 [12]).

Definition 3.2. Let Ω be an n-dimensional representation of G induced by a definable group homomorphism $B : G \to O_n(\mathbb{R})$. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \to B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer α, we define the vector bundle $\gamma(\Omega, \alpha) = (E(\Omega, \alpha), u, G(\Omega, \alpha))$ as follows: $G(\Omega, \alpha) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = \alpha\}$, $E(\Omega, \alpha) = \{(A, v) \in G(\Omega, \alpha) \times \Omega | Av = v\}$, $u : E(\Omega, \alpha) \to G(\Omega, \alpha)$, $u((A, v)) = A$, where A' denotes the transposed matrix of A and $Tr\ A$ stands for the trace of A. Then $\gamma(\Omega, \alpha)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, \alpha)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with Ω and α*. Remark that $G(\Omega, \alpha) \subset M(\Omega)$ and $E(\Omega, \alpha) \subset M(\Omega) \times \Omega$ are nonsingular algebraic G sets. In particular, they are Nash G submanifolds of $M(\Omega)$ and $M(\Omega) \times \Omega$, respectively.

Definition 3.3 ([9])**.** (1) Let G be a definable group. A definable G vector bundle $\eta = (E, p, X)$ over a definable G set X is called *strongly definable* if there exist a representation Ω of G and a definable G map $f : X \to G(\Omega, k)$ such that η is definably G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η.

(2) Let G be a definable C^r group and $0 \leq r \leq \infty$. A definable C^rG vector bundle $\eta = (E, p, X)$ over an affine definable C^rG manifold X is called *strongly definable* if there exist a representation Ω of G and a definable C^rG map $f : X \to G(\Omega, k)$ such that η is definably C^rG vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η.

Proposition 3.4 (2.3 [12])**.** *Every Nash G submanifold X of a representation Ω of G has a Nash G tubular neighborhood (U, θ) of X in Ω.*

Proof of Theorem 1.1 (2) Let η be a strongly definable C^rG vector bundle over X. Since η is strongly definable, there exists a definable C^rG map $f : X \to G(\Omega, \alpha)$

such that η is definably C^rG vector bundle isomorphic to $f^*(\gamma(\Omega, \alpha))$.

By Theorem 1.1 (1), f is approximated by a definable $C^\infty G$ map $h : X \to G(\Omega, \alpha)$. By [9], η is definably C^rG vector bundle isomorphic to a strongly definable $C^\infty G$ vector bundle $h^*(\gamma(\Omega, \alpha))$. By 1.7 [9] and Theorem 1.1 (1), uniqueness follows. $\square$

4 . Definable C^r fiber bundles.

By a way similar to the proof of 2.6 [6], we have the following.

Proposition 4.1. *Let $\mathcal{B}_K = (B_K, p_K, X_K)$ be the n-universal principal bundle relative to K, F an affine definable C^∞ manifold with an effective definable $C^\infty K$ action. Then the associated fiber bundle $\mathcal{B}_K[F] := (E, p, X_K, F, K)$ is a definable C^∞ fiber bundle.*

Proof of Theorem 1.1 (3). Let η be a strongly definable C^r fiber bundle over X. Then there exists the n-universal bundle $\mathcal{B}_K$ and a definable map $f : X \to X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to η. By Theorem 2.12, we have a definable C^∞ map $h : X \to X_K$ as an approximation of f. In particular h is definably homotopic to f. Thus by 1.1 [7], $\zeta := h^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$ and ζ is a strongly definable C^∞ fiber bundle.

(2) Let ζ' be another strongly definable C^∞ fiber bundle over X such that ζ' is definably fiber bundle isomorphic to η. Consider the strongly definable C^r fiber bundle (ζ, ζ', id_X) whose sections represent the fiber bundle isomorphisms between ζ and ζ' which is defined in 2.11 [6]. Then it has a continuous section. By a way similar to the proof of 2.12 [6], it admits a definable C^∞ section. This section gives a definable C^∞ fiber bundle isomorphism between ζ and ζ'. $\square$

References

[1] K.H. Dovermann, M. Masuda, and T. Petrie, *Fixed point free algebraic actions on varieties diffeomorphic to* $\mathbb{R}^n$, Progress in Math. **80**, Birkhäuser (1990), 49–80.

[2] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).

[3] L. van den Dries, A. Macintyre, and D. Marker, *The elementary theory of restricted analytic field with exponentiation*, Ann. of Math. **140** (1994), 183–205.

[4] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497-540.

[5] L. van den Dries and P. Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350**, (1998), 4377–4421.

[6] T. Kawakami, *Definable C^r fiber bundles and definable C^rG vector bundles*, Commun. Korean Math. Soc. **23** (2008), 257–268.

[7] T. Kawakami, *Definable fiber bundles in an o-minimal expansion of a real closed field*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. No. **60** (2010), 15–23.

[8] T. Kawakami, *Equivariant definable C^r approximation theorem, definable C^rG triviality of G invariant definable C^r functions and compactifications*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **55** (2005), 23–36

[9] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology and its appl. **123** (2002), 323-349.

[10] T. Kawakami, *Every definable C^r manifold is affine*, Bull. Korean Math. Soc. **42** (2005), 165–167.

[11] T. Kawakami, *Imbedding of manifolds defined on an o-minimal structures on* $(\mathbb{R}, +, \cdot, <)$, Bull. Korean Math. Soc. **36** (1999), 183–201.

[12] T. Kawakami, *Nash G manifold structures of compact or compactifiable $C^\infty G$ manifolds*, J. Math. Soc. Japan **48** (1996), 321–331.

[13] D.H. Park and D.Y. Suh, *Linear embeddings of semialgebraic G-spaces*, Math. Z. **242**, (2002), 725-742.

[14] Y. Peterzil, A. Pillay and S. Starchenko, *Definably simple groups in o-minimal structures*, Trans. Amer. Math. Soc. **352** (2000), 4397–4419.

[15] M. Shiota, *Approximation theorems for Nash mappings and Nash manifolds*, Trans. Amer. Math. Soc. **293** (1986), 319–337.

[16] M. Shiota, *Geometry of subanalyitc and semialgebraic sets*, Progress in Math. **150** (1997), Birkhäuser.