ORDERED FIELDS DENSE IN THEIR REAL CLOSURE AND DEFINABLE CONVEX VALUATIONS

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Abstract. In this paper, we undertake a systematic model and valuation theoretic study of the class of ordered fields which are dense in their real closure. We apply this study to determine definable henselian valuations on ordered fields, in the language of ordered rings. In light of our results, we re-examine recent conjectures in the context of strongly NIP ordered fields.

§1. Introduction. Let $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ be the language of rings and $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$ the language of ordered rings. There is a vast collection of results giving conditions on $\mathcal{L}_r$-definability[1] of henselian valuations in fields, many of which are from recent years (cf. e.g. [2, 9, 10, 19, 11]). A survey on $\mathcal{L}_r$-definability of henselian valuations is given in [7]. In this paper, we undertake a systematic study of definable henselian valuations in ordered fields, which enables us to consider definability in the richer language $\mathcal{L}_{or}$. It is natural to expect from such a study to obtain strenghtenings of existing $\mathcal{L}_r$-definability results. Ordered fields have been considered valuation theoretically mainly with respect to convex valuations. Note that due to [13, Lemma 2.1], henselian valuations in ordered fields are always convex. A general method for the construction of non-trivial $\mathcal{L}_{or}$-definable convex valuations in ordered fields which are not dense in their real closure is given in [12, Proposition 6.5]. By careful analysis of this method, we obtain sufficient conditions on the residue field and the value group of a henselian valuation v in order that v is $\mathcal{L}_{or}$-definable. These conditions motivate the study of the following two classes of structures: ordered abelian groups which are dense in their divisible hull, and ordered fields which are dense in their real closure. As it turns out, the former coincides with the elementary class of regular densely ordered abelian groups. We study the latter systematically, both from a model theoretic and an algebraic point of view.

[1]Throughout this paper, *definable* means *definable with parameters*.

The structure of this paper is as follows. In Section 2 we gather some basic preliminaries on ordered and valued abelian groups and fields. In Section 3 we study the class of ordered abelian groups which are dense in their divisible hull. We show that it coincides with the class of regular densely ordered abelian groups (see Proposition 3.2) and give a recursive elementary axiomatisation (see Corollary 3.3). Moreover, in Proposition 3.7 we obtain a characterisation of this class in terms of $\mathcal{L}_{\text{og}}$-definability of convex subgroups (where $\mathcal{L}_{\text{og}} = \{+, 0, <\}$ is the language of ordered groups). Analogously, in Section 4 we study the class of ordered fields which are dense in their real closure, first model theoretically in Subsection 4.1. Our main result is Theorem 4.1, from which we deduce that the property of an ordered field to be dense in its real closure is preserved under elementary equivalence (see Corollary 4.3). In Subsection 4.2 we briefly review the connections between dense and immediate extensions of ordered fields (see Corollary 4.8) and complements to the valuation ring (see Fact 4.12). In Subsection 4.3 we relate normal integer parts of ordered fields to dense subfields regular over $\mathbb{Q}$ (see Remark 4.14, Corollary 4.17). Finally, we obtain a characterisation of density in real closure in terms of absolute transcendence bases (see Corollary 4.18). In Section 5 we study $\mathcal{L}_{\text{or}}$-definable henselian valuations in ordered fields. In Subsection 5.1, we obtain our main result Theorem 5.3 and compare it to known conditions for $\mathcal{L}_{\text{r}}$-definability of specific henselian valuations. Special emphasis is put on definable valuations in almost real closed fields in Subsection 5.2. In Section 6, we relate our $\mathcal{L}_{\text{or}}$-definability results of henselian valuations in ordered fields to open questions regarding strongly NIP ordered fields.[2] We conclude by motivating and collecting open questions inspired by results throughout this paper in Section 7.

§2. Preliminaries on ordered abelian groups and valued fields. We denote by $\mathbb{N}_0$ the set of natural numbers with 0 and by $\mathbb{N}$ the set of natural numbers without 0. All notions on valued fields and groups can be found in [17, 4]. Throughout this work, we abbreviate the $\mathcal{L}_{\text{r}}$-structure of a field $(K, +, -, \cdot, 0, 1)$ by K, the $\mathcal{L}_{\text{or}}$-structure of an ordered field $(K, +, -, \cdot, 0, 1, <)$ by $(K, <)$ and the $\mathcal{L}_{\text{og}}$-structure of an ordered group $(G, +, 0, <)$ by G.

Let $G \subseteq H$ be an extension of ordered abelian groups. We say that G **is dense in** H or that the extension $G \subseteq H$ is dense if for any $a, b \in H$ there is $c \in G$ such that $a < c < b$. If $\mathbb{Z}$ is a convex subgroup of G, then we say that G is **discretely ordered**. Otherwise, G is **densely ordered**, i.e. G is dense in itself. Throughout this paper, we denote the **natural valuation** on a given ordered abelian group by v.[3] The value set vG is ordered by $v(g_1) < v(g_2)$ if $v(g_1) \neq v(g_2)$ and $|g_1| > |g_2|$. We say that an extension of ordered abelian groups $G \subseteq H$ is **immediate** if it is immediate with respect to the natural valuation.[4]

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[2]A preliminary version of this work is contained in our arXiv preprint [14], which contains also a systematic study of the class of strongly NIP almost real closed fields. This study, being of independent interest, will be the subject of the separate publication [15].

[3]See [17, p. 9] for a definition of the natural valuation, noting that an ordered abelian group is an ordered $\mathbb{Z}$-module.

[4]See [17, p. 3] for a definition of an immediate extension.

Let $\gamma \in vG$, and let G^γ and G_γ be the following convex subgroups of G:

$$G^\gamma = \{g \in G \mid v(g) \geq \gamma\} \text{ and } G_\gamma = \{g \in G \mid v(g) > \gamma\}.$$

The **archimedean component** $B(G, \gamma)$ of G corresponding to γ is given by $B(G, \gamma) = G^\gamma / G_\gamma$. If no confusion arises, we will only write B_γ instead of $B(G, \gamma)$. Note that B_γ is an ordered abelian group with the order induced by G. A valuation w on G is **convex** if for any $g_1, g_2 \in G$ with $0 < g_1 \leq g_2$, we have $w(g_1) \geq w(g_2)$. Note that there is a one-to-one correspondence between non-trivial convex subgroups of G and final segments of vG (cf. [17, p. 50 f.]).

We also consider G as a topological subspace of H under the order topology. We say that G has a **left-sided limit point** g_0 in H if for any $g_1 \in H$ with $g_1 > 0$ the intersection of $(g_0 - g_1, g_0)$ with G is non-empty. Similarly, g_0 is a **right-sided limit point** if for any $g_1 \in H^{>0}$ we have $(g_0, g_0 + g_1) \cap G \neq \emptyset$. A **limit point** is a point which is a left-sided or a right-sided limit point. The **divisible hull** of G is denoted by G^{div}. Note that G and G^{div} have the same value set under v, i.e. $vG = vG^{\text{div}}$. The **closure** of G in G^{div} with respect to the order topology is denoted by $\text{cl}(G)$. Note that G is dense in G^{div} if and only if $\text{cl}(G) = G^{\text{div}}$. Note further that G has a limit point in $G^{\text{div}} \setminus G$ (i.e. there is some $a \in G^{\text{div}} \setminus G$ such that a is a limit point of G in G^{div}) if and only if G is not closed in G^{div}.

For any ordered abelian groups G_1 and G_2, we denote the **lexicographic sum** of G_1 and G_2 by $G_1 \oplus G_2$. This is the abelian group $G_1 \times G_2$ with the lexicographic ordering $(a, b) < (c, d)$ if $a < c$, or $a = c$ and $b < d$. Let $(\Gamma, <)$ be an ordered set and for each $\gamma \in \Gamma$, let $A_\gamma \neq \{0\}$ be an archimedean ordered abelian group. For any element s in the product group $\prod_{\gamma \in \Gamma} A_\gamma$, define the **support of** s by $\text{supp}(s) = \{\gamma \in \Gamma \mid s(\gamma) \neq 0\}$. The **Hahn product** $\mathbb{H}_{\gamma \in \Gamma} A_\gamma$ is the subgroup of $\prod_{\gamma \in \Gamma} A_\gamma$ consisting of all elements with well-ordered support. Moreover, $\mathbb{H}_{\gamma \in \Gamma} A_\gamma$ becomes an ordered group under the order relation $s > 0 :\Leftrightarrow s(\min \text{supp}(s)) > 0$. We express elements s of $\mathbb{H}_{\gamma \in \Gamma} A_\gamma$ by $s = \sum_{\gamma \in \Gamma} s_\gamma \mathbb{1}_\gamma$, where $s_\gamma = s(\gamma)$ and $\mathbb{1}_\gamma$ is the characteristic function of γ mapping γ to 1 and everything else to 0. The **Hahn sum** $\coprod_{\gamma \in \Gamma} A_\gamma$ is the ordered subgroup of $\mathbb{H}_{\gamma \in \Gamma} A_\gamma$ consisting of all elements with finite support.

Let K be a field and v a valuation on K. We denote the **valuation ring** of v in K by $\mathcal{O}_v$, the **valuation ideal**, i.e. the maximal ideal of $\mathcal{O}_v$, by $\mathcal{M}_v$, the **ordered value group** by vK and the **residue field** $\mathcal{O}_v / \mathcal{M}_v$ by Kv. For $a \in \mathcal{O}_v$ we also denote $a + \mathcal{M}_v$ by $\bar{a}$. For an ordered field $(K, <)$, a valuation is called **convex** (in $(K, <)$) if the valuation ring $\mathcal{O}_v$ is a convex subset of K. In this case, the relation $\bar{a} < \bar{b} :\Leftrightarrow \bar{a} \neq \bar{b} \wedge a < b$ defines an order relation on Kv making it an ordered field.

Let $\mathcal{L}_{\text{vf}} = \mathcal{L}_{\text{r}} \cup \{\mathcal{O}_v\}$ be the **language of valued fields**, where $\mathcal{O}_v$ stands for a unary predicate. Let $(K, \mathcal{O}_v)$ be a valued field. An atomic formula of the form $v(t_1) \geq v(t_2)$, where t_1 and t_2 are $\mathcal{L}_{\text{r}}$-terms, stands for the $\mathcal{L}_{\text{vf}}$-formula $t_1 = t_2 = 0 \vee (t_2 \neq 0 \wedge \mathcal{O}_v(t_1/t_2))$. Thus, by abuse of notation, we also denote the $\mathcal{L}_{\text{vf}}$-structure $(K, \mathcal{O}_v)$ by (K, v). Similarly, we also call $(K, <, v)$ an ordered valued field. We say that a valuation v is $\mathcal{L}$-definable for some language $\mathcal{L} \in \{\mathcal{L}_{\text{r}}, \mathcal{L}_{\text{or}}\}$ if its valuation ring is an $\mathcal{L}$-definable subset of K.

Let K be a field and v, w be valuations on K. We write $v \leq w$ if and only if $\mathcal{O}_v \supseteq \mathcal{O}_w$. In this case we say that w is **finer than** v and v is **coarser than** v. If $\mathcal{O}_v \supsetneq \mathcal{O}_w$, we write $v < w$ and say that w is **strictly finer** than v and that v is **strictly coarser** than w. Note that $\leq$ defines an order relation on the set of convex valuations of an ordered field. We call two elements $a, b \in K$ **archimedean equivalent** (in symbols $a \sim b$) if there is some $n \in \mathbb{N}$ such that $|a| < n|b|$ and $|b| < n|a|$. Let $G = \{[a] \mid a \in K^\times\}$ the set of archimedean equivalence classes of $K^\times$. Equipped with addition $[a] + [b] = [ab]$ and the ordering $[a] < [b] :\Leftrightarrow a \not\sim b \wedge |b| < |a|$, the set G becomes an ordered abelian group. Then $v \colon K^\times \to G$ defines a convex valuation on K. This is called the **natural valuation** on K, denoted by v_{nat}. If not further specified, we say that an extension of ordered fields $(K, <) \subseteq (L, <)$ is **immediate** if it is immediate[5] with respect to the natural valuation. The extension is **dense** if K is dense in L.

Let $(k, <)$ be an ordered field and G an ordered abelian group. We denote the **ordered Hahn field** with coefficients in k and exponents in G by $k((G))$. We denote an element $s \in k((G))$ by $s = \sum_{g \in G} s_g t^g$, where $s_g = s(g)$ and t^g is the characteristic function on G mapping g to 1 and everything else to 0. The ordering on $k((G))$ is given by $s > 0 :\Leftrightarrow s(\min \operatorname{supp} s) > 0$, where $\operatorname{supp} s = \{g \in G \mid s(g) \neq 0\}$ is the **support of** s. Let $v_{\min}$ be the valuation on $k((G))$ given by $v_{\min}(s) = \min \operatorname{supp} s$ for $s \neq 0$. Note that $v_{\min}$ is convex and henselian. Note further that if k is archimedean, then $v_{\min}$ coincides with v_{nat}.

We will repeatedly use the Ax–Kochen–Ershov principle for ordered fields (cf. [6, Corollary 4.2(iii)]).

FACT 2.1 (Ax–Kochen–Ershov Principle). *Let $(K, <, v)$ and $(L, <, w)$ be two ordered henselian valued fields. Then $(Kv, <) \equiv (Lw, <)$ and $vK \equiv wL$ if and only if $(K, <, v) \equiv (L, <, w)$.*

§3. Regular densely ordered abelian groups. In this section, we study the class of ordered abelian groups which are dense in their divisible hull. In the light of ordered fields dense in their real closure and definable henselian valuations, this class is of interest for the following reasons:

• The divisible hull G^{div} of a non-trivial ordered abelian group G is the smallest extension of G which is a model of the theory of divisible ordered abelian groups T_{doag}. The real closure $(K^{\mathrm{rc}}, <)$ of an ordered field $(K, <)$ is the smallest extension of $(K, <)$ which is a model of the theory of real closed fields T_{rcf}. Thus, the divisible hull of an ordered abelian group can be seen as the group analogue to the real closure of an ordered field. Moreover, both T_{doag} and T_{rcf} share several model theoretic properties such as completeness, o-minimality and quantifier elimination. We are therefore interested in exploring algebraic and model theoretic similarities and differences between these two classes of ordered structures.

• The property of an ordered abelian group to be dense in its divisible hull can also be considered as a topological characteristic with respect to the order topology. To this end, for an ordered henselian valued field $(K, <, v)$, topological

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[5]For the definition of an immediate extension of valued fields, see [17, p. 27]

conditions on vK as a subspace of vK^{div} naturally arise in the context of the definability of v (see Theorem 5.3 and Corollary 5.7).

Throughout this section, we will prove properties of ordered abelian groups dense in their divisible hull and point out the analogues for ordered fields in Section 4.

Our first result, Proposition 3.2, shows that the class of ordered abelian groups which are dense in their divisible hull coincides with the well-known class of regular densely ordered abelian groups. Regular ordered abelian groups have been studied model theoretically in [20] and algebraically in [23]. An ordered abelian group is **regular** if it satisfies one of the equivalent conditions in Fact 3.1 (cf. [9, p. 14], [7, p. 137] and [2, p. 1148]).

FACT 3.1. *Let G be an ordered abelian group. Then the following are equivalent:*

1. For any prime $p \in \mathbb{N}$ and for any infinite convex subset $A \subseteq G$, there is a p-divisible element in A.

2. For any $n \in \mathbb{N}$ and any $a, b \in G$, if there are $g_1, \ldots, g_n \in G$ with $a \le g_1 < \ldots < g_n \le b$, then there is some $c \in G$ with $a \le nc \le b$.

3. For any non-trivial convex subgroup $H \subseteq G$, the quotient group G/H is divisible.

PROPOSITION 3.2. *Let G be an ordered abelian group. Then G is regular and densely ordered if and only if G is dense in G^{div}.*

PROOF. Since for divisible ordered abelian groups the conclusion in trivial, we assume that G is non-divisible. Suppose that G is regular and densely ordered. Let $a, b \in G$ and $n \in \mathbb{N}$ such that $\frac{a}{n} < \frac{b}{n}$. Since G is densely ordered, there are $g_1, \ldots, g_n \in G$ with $a \le g_1 < \ldots < g_n \le b$. By Fact 3.1 (2), there is some $c \in G$ with $a \le nc \le b$. Hence, $\frac{a}{n} < c < \frac{b}{n}$, as required.

Conversely, suppose that G is dense in G^{div}. Then G is densely ordered as G^{div} is densely ordered. Let $p \in \mathbb{N}$ be prime, $A \subseteq G$ an infinite convex subset of G and $a, b \in A$ with $a < b$. Then there is some $g \in G$ with $\frac{a}{p} < g < \frac{b}{p}$ and thus $a < pg < b$. Hence, A contains a p-divisible element, as required by Fact 3.1 (1). ⊣

Proposition 3.2 enables us to apply known results on regular densely ordered abelian groups to ordered abelian groups which are dense in their divisible hull.

COROLLARY 3.3. *There is a recursive $\mathcal{L}_{\mathrm{og}}$-theory Σ which axiomatises the class of non-trivial ordered abelian groups which are dense in their divisible hull.*

PROOF. Let Σ be the $\mathcal{L}_{\mathrm{og}}$-theory consisting of the axioms for non-trivial densely ordered abelian groups plus the collection of the following sentences (one for each $n \in \mathbb{N}$):

$$\forall a, b \; (\exists g_1, \ldots, g_n \; a \le g_1 < \ldots < g_n \le b \to \exists c \; a \le nc \le b).$$

By Fact 3.1 (2), Σ axiomatises the class of non-trivial regular densely ordered abelian groups, and thus also the class of non-trivial ordered abelian groups which are dense in their divisible hull by Proposition 3.2. ⊣

In Corollary 4.5 we will see the analogue of Corollary 3.3 for the class of ordered fields which are dense in their real closure. We deduce further properties of ordered abelian groups which are dense in their divisible hull from the following fact, which is due to [20] and [23].

FACT 3.4. *Let G be an ordered abelian group. Then the following hold:*

1. G is regular if and only if it has an archimedean model (i.e. there is some archimedean ordered abelian group H such that $H \equiv G$).

2. G is regular and discretely ordered if and only if it is a $\mathbb{Z}$-group (i.e. $G \equiv \mathbb{Z}$ as ordered groups).

COROLLARY 3.5. *Let G be an ordered abelian group. Then G is dense in G^{div} if and only if G has a densely ordered archimedean model.*

PROOF. Both directions follow immediately from Proposition 3.2 and Fact 3.4 (1). ⊣

We will see in Corollary 4.4 that the analogue of the backward direction of Corollary 3.5 also holds for ordered fields, i.e. that any ordered field which has an archimedean model is dense in its real closure. However, the analogue of the forward direction of Corollary 3.5 does not hold for ordered fields, as there is an ordered field which is dense in its real closure but has no archimedean model (see Proposition 4.6).

In Section 5, we address the question what convex valuations are $\mathcal{L}_{\mathrm{or}}$-definable in ordered fields. In analogy to Construction 5.2, we will show in Proposition 3.7 that for any densely ordered abelian group which is not dense in its divisible hull, there exists a proper non-trivial convex $\mathcal{L}_{\mathrm{og}}$-definable subgroup. The following lemma is a useful characterisation of densely ordered abelian groups and will be applied several times.

LEMMA 3.6. *Let G be an ordered abelian group. Then the following are equivalent:*

1. G is densely ordered.

2. 0 is a limit point of G in G.

3. 0 is a limit point of G in G^{div}.

4. Either vG has no last element, or vG has a last element γ and B_γ is densely ordered.

PROOF. We may assume that $G \neq \{0\}$, as in the case $G = \{0\}$ the equivalences are clear. The equivalence of (1) and (2) is an easy consequence of the definition of a dense order on an abelian group.

Obviously, (3) implies (2). For the converse, suppose that (1) holds. Let $g \in G \setminus \{0\}$ and $N \in \mathbb{N}$. We need to find $c \in G$ with $0 < c < \frac{|g|}{N}$. Since G is densely ordered, there are $c_1, \ldots, c_N \in G$ such that $0 < c_1 < \ldots < c_N < |g|$. Let $c = \min \{c_{i+1} - c_i \mid i \in \{0, \ldots, N\}\}$, where we set $c_0 = 0$ and $c_N = |g|$. We obtain $0 < c < 2c < \ldots < Nc < |g|$ and thus $0 < c < \frac{|g|}{N}$, as required.

It remains to show that (1) and (4) are equivalent. Suppose that vG has a last element γ and B_γ is not densely ordered. By (2) applied to B_γ, there is some $g \in G$ with $g > 0$ and $v(g) = \gamma$ such that there is no element in B_γ strictly between $0 + G_\gamma$ and $g + G_\gamma$. Let $h \in G$ such that $0 < h \leq g$. Then $v(h) \geq v(g) =$

γ. By maximality of γ, we have $v(h) = \gamma$. Thus, $0 + G_\gamma < h + G_\gamma \leq g + G_\gamma$. This implies $h + G_\gamma = g + G_\gamma$, i.e. $v(g - h) > \gamma$. Again, by maximality of γ, we obtain $g = h$. Hence, there is no element in G strictly between 0 and g, showing that G is not densely ordered. This shows that (1) implies (4). For the converse, suppose that (4) holds. We will show that 0 is a limit point of G in G. Let $g \in G$ with $g > 0$. If $v(g)$ is not maximal, then let $h \in G$ with $h > 0$ and $v(h) > v(g)$. Then $0 < h < g$, as required. Otherwise, $\gamma = v(g)$ is the maximum of vG, and by assumption, B_γ is densely ordered. Thus, there is some $h \in G$ with $v(h) = \gamma$ such that $0 + G_\gamma < h + G_\gamma < g + G_\gamma$, whence $0 < h < g$. ⊣

PROPOSITION 3.7. *Let G be a densely ordered abelian group. Then exactly one of the following holds:*

1. *G is dense in G^{div}.*
2. *G has a proper non-trivial $\mathcal{L}_{\mathrm{og}}$-definable convex subgroup.*

Moreover, if G is not dense in G^{div}, then the proper non-trivial $\mathcal{L}_{\mathrm{og}}$-definable convex subgroup of G can be defined by an $\mathcal{L}_{\mathrm{og}}$-formula using one parameter.

PROOF. First assume that G is not dense in G^{div}. In the following, we will construct a proper non-trivial convex subgroup of G.

Note that by Corollary 3.5, G is non-archimedean. Thus, vG has more than one element. Let $g_0 \in G^{\mathrm{div}} \setminus \mathrm{cl}(G)$ with $g_0 > 0$. If $\gamma = v(g_0)$ is the maximum of vG, then let $h \in G$ with $v(h) < \gamma$. Note that $v(g_0 + h) = v(h) < \gamma$ and that $g_0 + h$ is not a limit point of G in G^{div}, as g_0 is not a limit point of G in G^{div}. Hence, $g_0 + h \in G^{\mathrm{div}} \setminus \mathrm{cl}(G)$. Hence, by replacing g_0 by $g_0 + h$ if necessary, we may assume that $v(g_0)$ is not the maximum of vG.

Let $g_1 \in G$ with $g_1 > 0$ and $N \in \mathbb{N}$ such that $g_0 = \frac{g_1}{N}$. Consider the set

$$D = \left\{ g \in G^{\geq 0} \mid g < g_0 \right\} = \left\{ g \in G^{\geq 0} \mid Ng < g_1 \right\}.$$

This set is $\mathcal{L}_{\mathrm{og}}$-definable with the parameter g_1. Let

$$A = \left\{ g \in G^{\geq 0} \mid g + D \subseteq D \right\}.$$

Again, A is $\mathcal{L}_{\mathrm{og}}$-definable with the parameter g_1. Note that A is convex. Obviously, $A \neq G^{\geq 0}$, as for any $g \in D$ and $h \in G^{\geq 0}$ with $h > g_0 - g$ we have $g + h > g_0$ and thus $h \notin A$. Assume that $A = 0$. Then for any $g \in G^{>0}$, let $g' \in G^{\geq 0}$ with $g' < g_0$ and $g + g' > g_0$ and set $c_g = g + g'$. Then $g_0 < c_g < g_0 + g$. Since by Lemma 3.6 (3), 0 is a limit point of G in G^{div}, for any $h \in G^{\mathrm{div}}$ with $h > 0$, there exists some $g \in G$ such that $0 < g < h$. Thus, $g_0 < c_g < g_0 + g < g_0 + h$. This shows that g_0 is a limit point of G in G^{div}, contradicting the choice of $g_0 \notin \mathrm{cl}(G)$. Hence, $A \neq 0$.

Now for any $a, b \in A$ with $0 < a < b$, we have $(a + b) + D \subseteq a + D \subseteq D$, whence $a + b \in A$. Moreover, $0 < b - a < b$ and $-b < a - b < 0$. Thus, by convexity, $b - a \in A$ and $a - b \in -A$. Similarly, for any $a, b \in -A$ with $a < b < 0$, we have $a \pm b \in -A$ and $b - a \in A$. This shows that $H = -A \cup A$ is closed under addition and thus a $\mathcal{L}_{\mathrm{og}}$-definable convex subgroup of G. Since $0 \neq A \neq G^{\geq 0}$, we also have that H is a proper non-trivial subgroup of G.

Assume for the converse that G is dense in G^{div}. By Proposition 3.2, G is regular. Thus, by Fact 3.4 (1) G has an archimedean model G', say. Let $\varphi(x, \underline{y})$ be an $\mathcal{L}_{\mathrm{og}}$-formula and let ψ be the $\mathcal{L}_{\mathrm{og}}$-sentence stating that for any $\underline{y}$,

if $\varphi(x, \underline{y})$ defines a proper convex subgroup, then it is the trivial one. Since G' is archimedean, its only convex subgroups are $\{0\}$ and G'. Thus, ψ holds in G'. By elementary equivalence, ψ also holds in G. Hence, for any $\underline{b} \in G$, the formula $\varphi(x, \underline{b})$ does not define a proper non-trivial convex subgroup of G. $\dashv$

REMARK 3.8. Note that a divisible ordered abelian group G never admits a proper non-trivial $\mathcal{L}_{\mathrm{og}}$-definable convex subgroup. Indeed, since G is o-minimal, any $\mathcal{L}_{\mathrm{og}}$-definable convex subset of G is an interval with endpoints in $G \cup \{\pm\infty\}$, and thus the only $\mathcal{L}_{\mathrm{og}}$-definable convex subgroups of G are $\{0\}$ and G.

Next we want to study the extension of ordered abelian groups $G \subseteq G^{\mathrm{div}}$ valuation theoretically. It is known that any dense extension of ordered fields is immediate (cf. [17, p. 29 f.]). We will investigate when this is also the case for dense extensions of ordered abelian groups. Note that for any $\gamma \in vG$, we have $B(\gamma, G^{\mathrm{div}}) = B(\gamma, G)^{\mathrm{div}}$. Hence, the extension $G \subseteq G^{\mathrm{div}}$ is immediate if and only if all archimedean components of G are divisible.

PROPOSITION 3.9. *Let $G \subseteq H$ be a dense extension of ordered abelian groups. Suppose that vG has no last element. Then the extension $G \subseteq H$ is immediate.*

PROOF. In order to show that $G \subseteq H$ is immediate, we need to show that for any $a \in H$ there exists $b \in G$ such that $v(a - b) > v(a)$. Note that the density of G in H implies $vG = vH$. Let $a \in H$ with $a > 0$. Since vG has no last element, there is some $c \in G^{>0}$ such that $v(c) > v(a)$. By density of G in H, there is some $b \in G$ such that $a - c < b < a + c$. We obtain $v(a - b) \geq v(c) > v(a)$, as required. $\dashv$

COROLLARY 3.10. *Let $G \subseteq G^{\mathrm{div}}$ be a dense extension of ordered abelian groups such that vG has no last element. Then the extension $G \subseteq G^{\mathrm{div}}$ is immediate.*

PROOF. Apply Proposition 3.9 to $H = G^{\mathrm{div}}$. $\dashv$

Corollary 3.10 does not hold in general in the case where vG has a last element, as the following example will show.

EXAMPLE 3.11. Let $A = \left\{ \frac{a}{2^n} \,\middle|\, a, n \in \mathbb{Z} \right\}$. Note that A is dense in $A^{\mathrm{div}} = \mathbb{Q}$. For $G = \mathbb{Q} \oplus A$ we have $G^{\mathrm{div}} = \mathbb{Q} \oplus \mathbb{Q}$. Moreover, G is dense in G^{div}, as A is dense in $\mathbb{Q}$. However, the extension is not immediate, as the archimedean components do not coincide.

We will now consider the converse direction, that is, under what conditions an immediate extension of ordered abelian groups is dense.

LEMMA 3.12. *Let $G, A_1, A_2, \ldots$ be ordered abelian groups such that $G \subseteq H = \mathbb{H}_{n \in \omega} A_n$ is immediate. Let $a \in \mathbb{H}_{n \in \omega} A_n$. Then there exists a sequence $(d_n)_{n \in \omega}$ in G such that for any $k \in \omega$ and any $i \leq k$ we have $(d_0 + \ldots + d_k)(i) = a(i)$.*

PROOF. Since $G \subseteq H$ is immediate, there is some $d_0 \in G$ such that $v(a - d_0) > v(a)$. Then $d_0(0) = a_0$. Suppose that $d_0, \ldots, d_k \in G$ are already constructed such that $d' = d_0 + \ldots + d_k$ satisfies $d'(i) = a(i)$ for $i \leq k$. Again, since $G \subseteq H$ is immediate, there is some $d_{k+1} \in G$ such that $v((a - d') - d_{k+1}) > v(a - d') \geq k + 1$. Thus, $d_{k+1}(i) = (a - d')(i) = 0$ for $i \leq k$ and $d_{k+1}(k + 1) = (a - d')(k + 1)$. We obtain $(d_0 + \ldots + d_{k+1})(i) = a(i)$ for $i \leq k$ and $(d_0 + \ldots + d_{k+1})(k + 1) = (d' + d_{k+1})(k + 1) = a(k + 1)$, as required. $\dashv$

PROPOSITION 3.13. *Let $G \subseteq H$ be an immediate extension of ordered abelian groups such that H is densely ordered and $vH \subseteq \omega$. Then G is dense in H.*

PROOF. Let $\Gamma = vH$. By the Hahn Embedding Theorem (cf. [17, p. 14]), we may consider H as a subgroup of the Hahn product $\mathbb{H}_{n \in \omega} B_n$, where we set $B_n = \{0\}$ for $n \notin \Gamma$. Note that $H \subseteq \mathbb{H}_{n \in \omega} B_n$ is an immediate extension. Let $a, b \in H$ with $0 < a < b$. By Lemma 3.12, there exists a sequence $(d_n)_{n \in \omega}$ in G such that for any $k \in \omega$ and any $i \leq k$ we have $(d_0 + \ldots + d_k)(i) = a(i)$. Suppose that k is the last element of vG. Then by Lemma 3.6 (4), B_k is densely ordered. Let $c \in G$ with $v(c) = k$ and $a(k) < c(k) < b(k)$. Then $d' = d_1 + \ldots + d_{k-1} + c \in G$ and $a < d' < b$, as required. Now suppose that vG has no last element. Then let $c \in G$ with $v(c) = m > k$ and $c(m) > a(m)$. Then $d' = d_1 + \ldots + d_{m-1} + c \in G$ and $a < d' < b$, as required. $\dashv$

COROLLARY 3.14. *Let G be an ordered abelian group. Suppose that the extension $G \subseteq G^{\mathrm{div}}$ is immediate and that $vG \subseteq \omega$. Then G is dense in G^{div}.*

PROOF. Apply Proposition 3.13 to $H = G^{\mathrm{div}}$. $\dashv$

Note that withouth the condition $vG \subseteq \omega$ in Corollary 3.14, the conclusion that G is dense in G^{div} does not hold in general, as the following example will show.

EXAMPLE 3.15. Let H be the Hahn product $\mathbb{H}_{\gamma \in \omega+1} \mathbb{Q}$. Let H' be the Hahn sum $H' = \coprod_{\gamma \in \omega+1} \mathbb{Q} \subseteq H$. Note that $H' \subseteq H$ is an immediate extension (cf. [17, p. 3]). It follows that for any ordered abelian groups G_1 and G_2 with $H' \subseteq G_1 \subseteq G_2 \subseteq H$, also the extension $G_1 \subseteq G_2$ is immediate.

Let $G \subseteq H$ be given by

$$G = H' + a\mathbb{Z}, \text{ where } a = \sum_{\gamma \in \omega} \mathbb{1}_\gamma.$$

Now $G^{\mathrm{div}} = H' + a\mathbb{Q} \subseteq H$, and the extension $G \subseteq G^{\mathrm{div}}$ is immediate. Let $c = \frac{1}{2}a + \frac{1}{3}\mathbb{1}_\omega$ and $d = \frac{1}{2}a + \frac{2}{3}\mathbb{1}_\omega$. Then $c, d \in G^{\mathrm{div}}$ with $0 < c < d$. However, there is no element in G strictly between c and d. Thus G is not dense in G^{div}.

REMARK 3.16. By Corollary 3.10 and Corollary 3.14, we obtain the following: Let G be an ordered abelian group and $vG \cong \omega$. Then the extension $G \subseteq G^{\mathrm{div}}$ is dense if and only if it is immediate.

The final results of this section are on ordered abelian groups which are not dense in their divisible hull. These will be used in Subsection 5.2 to compare $\mathcal{L}_{\mathrm{r}}$- and $\mathcal{L}_{\mathrm{or}}$-definability of henselian valuations with real closed residue field in almost real closed fields. We start by giving some useful examples of ordered abelian groups which are not regular and thus not dense in their divisible hull.

EXAMPLE 3.17. 1. $\mathbb{Z} \oplus \mathbb{Z}$ is a discretely ordered group which is not regular.
2. Let A be as in Example 3.11. Consider $G = A \oplus A$. Since A is densely ordered, so is G by Lemma 3.6. Since there is no element in G between $\left(\frac{1}{3}, 0\right)$ and $\left(\frac{1}{3}, 1\right)$, which both lie in $G^{\mathrm{div}} = \mathbb{Q} \oplus \mathbb{Q}$, it follows that G is not dense in G^{div} and thus not regular. However, G has limit points in $G^{\mathrm{div}} \setminus G$, e.g. $\left(0, \frac{1}{3}\right)$.

Example 3.17 (2) shows that it is possible for an ordered abelian group to have limit points in its divisible hull but not to be dense in it. We can go even further and construct a non-divisible densely ordered abelian group G which does not have any limit points in $G^{\mathrm{div}} \setminus G$; or, in other words, G is closed in G^{div}. We conclude this section by proving results on non-divisible densely ordered abelian groups which are closed in their divisible hull. A natural example of these are non-divisible Hahn products.

PROPOSITION 3.18. *Let $\Gamma \neq \emptyset$ be a totally ordered set without a last element, and for any $\gamma \in \Gamma$, let $A_\gamma \neq \{0\}$ be an archimedean ordered abelian group. Suppose that $G = \mathbb{H}_{\gamma \in \Gamma} A_\gamma$ is non-divisible. Then G is closed in G^{div}.*

PROOF. Let $s \in G^{\mathrm{div}} \setminus G$, and let $\gamma_0 \in \Gamma$ with $\gamma_0 > \min\{\gamma \in \Gamma \mid s_\gamma \notin A_\gamma\}$. Such γ_0 exists, as Γ has no last element. Then the interval $(s - \mathbb{1}_{\gamma_0}, s + \mathbb{1}_{\gamma_0})$ does not contain any element of G. Hence, s is not a limit point of G.   ⊣

Proposition 3.18 shows in particular that any non-divisible Hahn product whose value set has no last element is not dense in its divisible hull. Thus, by Fact 3.4 (1), any such Hahn product has no archimedean model. A similar result holds for ordered Hahn fields which are not real closed (see Corollary 4.10). The final results of this section will give sufficient conditions for an ordered abelian group to be closed in its divisible hull.

PROPOSITION 3.19. *Let G be an ordered abelian group. Suppose that for every prime $p \in \mathbb{N}$, there is a p-divisible convex subgroup $G_p \neq \{0\}$ of G. Then G is closed in G^{div}.*

PROOF. If G is divisible, then the conclusion is trivial. Otherwise, let $a \in G$ and $N \in \mathbb{N}$ such that $\frac{a}{N} \in G^{\mathrm{div}} \setminus G$. Let $N = p_1 \ldots p_m$ be the prime factorisation of N. Consider the non-trivial N-divisible convex subgroup $H = \bigcap_{i=1}^m G_{p_i}$. Let $h \in H$ with $h > 0$. Consider the interval $I = \left(\frac{a}{N} - h, \frac{a}{N} + h\right)$ in G^{div}. Assume that $I \cap G \neq \emptyset$. Then for any $g \in I \cap G$ we have $v(Ng - a) = v\left(g - \frac{a}{N}\right) \geq v(h)$ and thus $Ng - a \in H$. Since H is N-divisible, we obtain $g - \frac{a}{N} \in H$ and thus $\frac{a}{N} \in G$, a contradiction. This gives us $I \cap G = \emptyset$, as required.   ⊣

COROLLARY 3.20. *Let G be an ordered abelian group. Suppose that G has a convex divisible subgroup $H \neq \{0\}$. Then G is closed in G^{div}.*

PROOF. For any prime p, let $G_p = H$. The conclusion follows from Proposition 3.19.   ⊣

The following example will show that the converse of Proposition 3.19 does not hold.

EXAMPLE 3.21. Let $2 = p_0 < p_1 < p_2 < \ldots$ be a complete list of all prime numbers. For $n \in \mathbb{N}$, let A_n be the following ordered abelian group

$$A_n = \left\{ \frac{a}{p_1^{m_1} \ldots p_n^{m_n}} \;\middle|\; a, m_1, \ldots, m_n \in \mathbb{Z} \right\}.$$

Then A_n is p_i-divisible for $i = 1, \ldots, n$. Let $G = \mathbb{H}_{n \in \mathbb{N}} A_n$. Note that $G^{\mathrm{div}} = \mathbb{H}_{n \in \mathbb{N}} \mathbb{Q}$. Since A_n is not 2-divisible for any $n \in \mathbb{N}$, there is no non-trivial 2-divisble convex subgroup of G. In particular G is not divisible and thus closed

in G^{div} by By Proposition 3.18. Moreover, for any prime $p_i \neq 2$, the maximal p_i-divisible subgroup of G is $\mathbb{H}_{n \in \mathbb{N}_{\geq i}} A_n$.

§4. Density in Real Closure. We now turn to ordered fields which are dense in their real closure.

4.1. Model theoretic properties. In this subsection, we study model theoretically the class of ordered fields which are dense in their real closure. At first, we change to a more general setting of complete theories which admit quantifier elimination. Since T_{doag} is an example of such a theory, we can consider these model theoretic results as a generalisation of some results in Section 3. We then also apply these generalisations to T_{rcf} to obtain properties of ordered fields dense in their real closure analogous to the ones in Section 3 of ordered abelian groups dense in their divisible hull. For a structure $\mathcal{M}$ and a subset $A \subseteq M$, denote the definable closure of A in $\mathcal{M}$ by $\mathrm{dcl}(A; \mathcal{M})$.

THEOREM 4.1. *Let $\mathcal{L}$ be a language expanding $\mathcal{L}_{\mathrm{og}}$, let $T \supseteq T_{\mathrm{doag}}$ be a complete $\mathcal{L}$-theory which admits quantifier elimination and let $\Sigma \subseteq T$ be a theory extending the theory of ordered abelian groups. Then there is a theory $\Sigma' \supseteq \Sigma$ such that for any $\mathcal{M}' \models T$ and any $\mathcal{M} \subseteq \mathcal{M}'$ with $\mathrm{dcl}(M; \mathcal{M}') = M'$ we have that $\mathcal{M} \models \Sigma'$ if and only if $\mathcal{M} \models \Sigma$ and M is dense in M'. Moreover, if Σ and T are recursive, then we can also choose Σ' to be recursive.*

PROOF. For any 0-definable function f in T, let $\varphi_f(\underline{x}, z)$ be a quantifier-free formula which is equivalent to $z < f(\underline{x})$ in T. Let A be the set of all pairs (f, g) of 0-definable functions in T with the same arity such that $T \models \forall \underline{x} \ (f(\underline{x}) = g(\underline{x}) \to \underline{x} = 0)$. In other words, A consists of all pairs of 0-definable functions which are distinct everywhere except possibly in 0. Set

$$\Sigma' = \Sigma \cup \{\forall \underline{x} \ (\underline{x} \neq 0 \to \neg \forall z \ (\varphi_f(\underline{x}, z) \leftrightarrow \varphi_g(\underline{x}, z))) \mid (f, g) \in A\}.$$

Note that if Σ and T are recursive, then so is Σ', as it is decidable whether a given $\mathcal{L}$-formula defines a function.

Let $\mathcal{M}' \models T$ and let $\mathcal{M} \subseteq \mathcal{M}'$ with $\mathrm{dcl}(M; \mathcal{M}') = M'$. Suppose that $\mathcal{M} \models \Sigma'$. We need to show that M is dense in M'. Let $\alpha, \beta \in M'$ such that $\alpha \neq \beta$. Let f and g be 0-definable functions and let $\underline{a} \in M$ such that $f(\underline{a}) = \alpha$ and $g(\underline{a}) = \beta$. Since also $\mathrm{dcl}(M \setminus \{0\}; \mathcal{M}') = M'$, we may assume that $\underline{a} \neq 0$. Hence, $\mathcal{M} \models \neg \forall z \ (\varphi_f(\underline{a}, z) \leftrightarrow \varphi_g(\underline{a}, z))$. This implies that for some $b \in M$ we have $\mathcal{M} \models \varphi_f(\underline{a}, b) \wedge \neg \varphi_g(\underline{a}, b)$ or $\mathcal{M} \models \neg \varphi_f(\underline{a}, b) \wedge \varphi_g(\underline{a}, b)$. Since φ_f and φ_g are quantifier-free, we obtain that either $\mathcal{M}' \models \varphi_f(\underline{a}, b) \wedge \neg \varphi_g(\underline{a}, b)$ or $\mathcal{M}' \models \neg \varphi_f(\underline{a}, b) \wedge \varphi_g(\underline{a}, b)$. Hence, either $\alpha = f(\underline{a}) \leq b < g(\underline{a}) = \beta$ or $\beta = g(\underline{a}) \leq b < f(\underline{a}) = \alpha$. This implies that M is dense in M'. Conversely, suppose that $\mathcal{M} \models \Sigma$ and M is dense in M'. Let $(f, g) \in A$ and $\underline{a} \in M$ with $\underline{a} \neq 0$. We need to show that $\mathcal{M} \models \neg \forall z \ (\varphi_f(\underline{a}, z) \leftrightarrow \varphi_g(\underline{a}, z))$. Let $\alpha = f(\underline{a})$ and $\beta = g(\underline{a})$. By assumption on A, we have $\alpha \neq \beta$, say $\alpha < \beta$. Since M is dense in M', there is $c \in M$ such that $\alpha < c < \beta$. Hence, $\mathcal{M}' \models \neg \varphi_f(\underline{a}, c)$ but $\mathcal{M}' \models \varphi_g(\underline{a}, c)$. Since φ_f and φ_g are quantifier-free, we obtain $\mathcal{M} \models \neg \varphi_f(\underline{a}, c)$ and $\mathcal{M} \models \varphi_g(\underline{a}, c)$, as required.   ⊣

COROLLARY 4.2. *Let $\mathcal{L}$ be a language expanding $\mathcal{L}_{\mathrm{og}}$ and let $\mathcal{M} = (M, +, 0, <, \dots)$ and $\mathcal{N} = (N, +, 0, <, \dots)$ be ordered $\mathcal{L}$-structures such that $(M, +, 0, <)$*

is a non-trivial ordered abelian group and $\mathcal{M} \equiv \mathcal{N}$. *Suppose that there exists a complete* $\mathcal{L}$-*theory* $T \supseteq T_{\mathrm{doag}}$ *admitting quantifier elimination such that there are* $\mathcal{M}', \mathcal{N}' \models T$ *with* $\mathcal{M} \subseteq \mathcal{M}'$, $\mathcal{N} \subseteq \mathcal{N}'$, $\mathrm{dcl}(M; \mathcal{M}') = M'$ *and* $\mathrm{dcl}(N; \mathcal{N}') = N'$. *Suppose further that* M *is dense* M'. *Then* N *is also dense in* N'.

PROOF. By applying Theorem 4.1 to $\Sigma = \mathrm{Th}(\mathcal{M})$, we obtain a theory Σ' such that $\mathcal{M} \models \Sigma'$. Moreover, since $\mathcal{M} \equiv \mathcal{N}$, we obtain $\mathcal{N} \models \Sigma'$. Hence, N is dense in N'.

⊣

We now apply the results above to the o-minimal complete theory of real closed fields T_{rcf}.

COROLLARY 4.3. *Let* $(K, <)$ *and* $(L, <)$ *be ordered fields such that* $(K, <) \equiv (L, <)$. *Suppose that* K *is dense in* K^{rc}. *Then* L *is dense in* L^{rc}.

PROOF. Note that due to o-minimality of T_{rcf}, for any ordered field $(K, <)$, the definable closure of K in K^{rc} is K^{rc}. Thus, the proof follows directly from Corollary 4.2 applied to $T = T_{\mathrm{rcf}}$.   ⊣

Let $(K, <)$ be an archimedean ordered field. Then $\mathbb{Q} \subseteq K \subseteq K^{\mathrm{rc}} \subseteq \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, also K is dense in K^{rc}. In other words, any archimedean ordered field is dense in its real closure. We thus obtain the following.

COROLLARY 4.4. *Let* $(K, <)$ *be an ordered field which has an archimedean model. Then* K *is dense in* K^{rc}.

PROOF. Since any archimedean ordered field is dense in its real closure, we obtain from Corollary 4.3 that K is dense in K^{rc}.   ⊣

The next result shows that the class of ordered fields which are dense in their real closure can be recursively axiomatised

COROLLARY 4.5. *There is a recursive* $\mathcal{L}_{\mathrm{or}}$-*theory* Σ' *which axiomatises the class of ordered fields which are dense in their real closure.*

PROOF. We apply Theorem 4.1 to the recursive $\mathcal{L}_{\mathrm{or}}$-theory $T = T_{\mathrm{rcf}}$ and the axiom system Σ for ordered fields in order to obtain a recursive $\mathcal{L}_{\mathrm{or}}$-theory Σ' with the required properties.   ⊣

By Corollary 4.4, any ordered field with an archimedean model is dense in its real closure. We will use Corollary 4.5 to show that the converse does not hold, i.e. that there exists an ordered field which is dense in its real closure but has no archimedean model. Let $(K, <)$ be an ordered field. An **integer part** of K is a discretely ordered subring $(Z, <) \subseteq (K, <)$ with 1 as least positive element such that for any $x \in K$ there exists $z \in Z$ with $z \leq x < z + 1$. The connections between integer parts and density in real closure are further explored in Subsection 4.3.

PROPOSITION 4.6. *There is an ordered field* $(K, <)$ *such that* K *is dense in* K^{rc} *but* $(K, <)$ *has no archimedean model.*

PROOF. Let $\varphi(x)$ be an $\mathcal{L}_{\mathrm{or}}$-formula defining $\mathbb{Z}$ in $(\mathbb{Q}, <)$ (cf. [21, Theorem 3.1]). Let Σ_1 be the $\mathcal{L}_{\mathrm{or}}$-theory Σ' from Corollary 4.5 and let Σ_2 be a set of axioms stating that $\varphi(x)$ defines an integer part. For any $\mathcal{L}_{\mathrm{or}}$-formula α, let $\widetilde{\alpha}$ be

the formula in which all quantifiers $\exists x$ and $\forall x$ are bounded by $\varphi(x)$, that is, all instances of subformulas of the form $\exists x \theta(x, \underline{y})$ are replaced by $\exists x(\varphi(x) \wedge \theta(x, \underline{y}))$ and all instances of subformulas of the form $\forall x \theta(x, \underline{y})$ are replaced by $\forall x(\varphi(x) \rightarrow \theta(x, \underline{y}))$. Let Σ be the deductive closure of $\Sigma_1 \cup \Sigma_2$. Note that Σ is recursive and $(\mathbb{Q}, <) \models \Sigma$. If for every $\mathcal{L}_{\mathrm{or}}$-sentence $\alpha \in \mathrm{Th}(\mathbb{Z}, +, -, \cdot, 0, 1, <)$, we had $\Sigma \vdash \widetilde{\alpha}$, then $\mathrm{Th}(\mathbb{Z}, +, -, \cdot, 0, 1, <)$ would be decidable. Hence, there is an $\mathcal{L}_{\mathrm{or}}$-sentence σ such that $\neg \sigma \in \mathrm{Th}(\mathbb{Z}, +, -, \cdot, 0, 1, <)$ and $\widetilde{\sigma}, \neg \widetilde{\sigma} \notin \Sigma$. Let $\widetilde{\Sigma}$ be the consistent $\mathcal{L}_{\mathrm{or}}$-theory $\Sigma \cup \{\widetilde{\sigma}\}$ and let $(K, <) \models \widetilde{\Sigma}$. Assume that $(K, <)$ is archimedean. Since $(K, <) \models \Sigma_2$, the formula $\varphi(x)$ defines the integer part $\mathbb{Z}$ of K. But then $(K, <) \models \neg \widetilde{\sigma}$, a contradiction. Hence, $(K, <)$ cannot have any archimedean models. However, since $(K, <) \models \Sigma_1$, we have that K is dense in K^{rc}. $\dashv$

4.2. Dense and immediate extensions. We now analyse algebraic properties of ordered fields which are dense in their real closure. As pointed out above, there is an intimate connection between integer parts and dense extensions of ordered fields. This connection is explored in [1] and will further be addressed here. Throughout this section, we denote the natural valuation v_{nat} simply by v.

We start by investigating the connection between dense and immediate extensions of ordered fields. Let $(K, <) \subseteq (L, <)$ be an extension of ordered fields and let w be a convex valuation on L. We say that K **is w-dense in** L if for any $a \in L$ and any $\alpha \in wL$, there exists $b \in K$ such that $w(a - b) > \alpha$.

PROPOSITION 4.7. *Let $(K, <) \subseteq (L, <)$ be an extension of ordered fields and let w be a convex valuation on L. Then K is w-dense in L if and only if K is dense in L.*

PROOF. Suppose that K is w-dense in L. Let $a, b \in L$ with $0 < a < b$. Let $\alpha = \max\{w(a), w(b), w(b - a)\}$ and let $c_1, c_2 \in K$ such that $w(c_1 - a) > \alpha$ and $w(c_2 - b) > \alpha$. Set $c = \frac{c_2 - c_1}{2}$. Then $w\left(\frac{b-a}{2} - c\right) = w((b - a) - (c_2 - c_1)) = w((c_1 - a) - (c_2 - b)) > \alpha$ and $w\left(b - \frac{b-a}{2}\right) = w\left(\frac{b-a}{2} - a\right) \leq \alpha$. Hence, either $a \leq c \leq \frac{b-a}{2} < b$ or $a < \frac{b-a}{2} \leq c \leq b$.

Conversely, suppose that K is dense in L. Let $a \in L$ and $\alpha \in wL$. Let $b \in L^{>0}$ with $w(b) > \alpha$. By density of K in L, there exists $c \in K$ with $a < c < a + b$. Thus, $w(c - a) \geq w(b) > \alpha$. $\dashv$

By [17, Lemma 1.31], any w-dense extension of ordered fields is also immediate with respect to w. We thus obtain the following corollary to Proposition 4.7.

COROLLARY 4.8. *Let $(K, <) \subseteq (L, <)$ be a dense extension of ordered fields. Then this extension is immediate with respect to any convex valuation on L.*

Corollary 4.8 can be applied to ordered Hahn fields as follows: Let $(k, <)$ be an ordered field and $G \neq \{0\}$ an ordered abelian group. Suppose that $k((G))$ is dense in $k((G))^{\mathrm{rc}}$. By Corollary 4.8, $k((G)) \subseteq k((G))^{\mathrm{rc}}$ is an immediate extension with respect to the valuation v_{min}. Hence, $k = k^{\mathrm{rc}}$ and $G = G^{\mathrm{div}}$. This implies that $k((G))$ is real closed. Hence, if $k((G))$ is not real closed, then it is not dense in its real closure. This can be strengthened with the following proposition.

PROPOSITION 4.9. *Let $(k, <)$ be an ordered field and $G \neq \{0\}$ an ordered abelian group. Then $k((G))$ is closed in $k((G))^{\mathrm{rc}}$.*

PROOF. Let $K = k((G))$ and $R = K^{\mathrm{rc}}$. We need to show that $R \setminus K$ is open in R. If $K = R$, then $R \setminus K = \emptyset$. Hence, assume that $K \subsetneq R$. Let at^{g_0} be the monomial of s of least exponent which is not contained in K, and let $g_1 \in G^{>g_0}$ be arbitrary. Then the open interval

$$I = (s - t^{g_1}, s + t^{g_1}) \subseteq R$$

contains s. However, any element in I contains the monomial at^{g_0} and is thus not contained in K. Hence, s is contained in an open neighbourhood in $R \setminus K$, as required.   ⊣

COROLLARY 4.10. *Let $(k, <)$ be an ordered field and let $G \neq \{0\}$ be an ordered abelian group. Suppose that $k((G))$ is not real closed. Then $(k((G)), <)$ has no archimedean model.*

PROOF. By Proposition 4.9, $k((G))$ is not dense in $k((G))^{\mathrm{rc}}$. Thus, by Corollary 4.4 $(k((G)), <)$ has no archimedean model.   ⊣

We obtain the following result regarding the existence of henselian valuations in ordered fields which are dense in their real closure.

COROLLARY 4.11. *Let $(K, <)$ be an ordered field which is not real closed but dense in K^{rc}. Then K does not admit a non-trivial henselian valuation.*

PROOF. If K is archimedean, then the only possible henselian valuation on K is the trivial one. If K is non-archimedean, let w be a henselian valuation on K. By Fact 2.1, we have $(K, <) \equiv (Kw((wK)), <)$. By Corollary 4.3, $Kw((wK))$ is dense in $Kw((wK))^{\mathrm{rc}}$. By Proposition 4.9, this is only possible if $wK = \{0\}$, i.e. w is the trivial valuation.   ⊣

We now focus on immediate extension with respect to the natural valuation v. For an ordered field K, let $\mathbf{A}_K$ be a group complement to $\mathcal{O}_v$ in K, i.e. $\mathbf{A}_K$ is an ordered subgroup of K such that $K = \mathbf{A}_K \oplus \mathcal{O}_v$. Note that for an extension of ordered field $(K, <) \subseteq (L, <)$, the group complement $\mathbf{A}_K$ can be extended to a group complementen $\mathbf{A}_L$. The following is a useful characterisation of dense extensions of ordered field (cf. [17, Lemma 1.32]).

FACT 4.12. *Let $(K, <) \subseteq (L, <)$ be an extension of ordered fields. Then K is dense in L if and only if $\mathbf{A}_K = \mathbf{A}_L$.*

Whenever the extension of ordered fields $(K, <) \subseteq (K^{\mathrm{rc}}, <)$ is dense and thus immediate, then Kv is real closed and vK is divisible. The converse holds under the additional assumption that vK is archimedean (cf. [16, Lemma 19]).

FACT 4.13. *Let $(K, <)$ be an ordered field such that Kv is real closed and vK is divisible and archimedean. Then K is dense in K^{rc}.*

[1, Remark 3.5] provides an example of an ordered field field $(K, <)$ such that Kv is real closed and vK can be chosen to be divisible and non-archimedean (i.e. $K \subseteq K^{\mathrm{rc}}$ is immediate) but $\mathbf{A}_K \neq \mathbf{A}_{K^{\mathrm{rc}}}$, whence K is not dense in K^{rc}.

4.3. Normal integer parts. We now turn to normal integer parts in connection to ordered fields dense in their real closure. Due to [18], every real closed field admits an integer part. In [1, Remark 3.3 (iii)] the question is asked whether every real closed fields admits a **normal** integer part, i.e. an integer part Z which is integrally closed in its field of fractions $\mathrm{ff}(Z)$.

REMARK 4.14. For an ordered field $(K, <)$, if Z is an integer part of K, then $\mathrm{ff}(Z)$ is dense in K. Moreover, if Z is normal, then $\mathbb{Q} \subseteq \mathrm{ff}(Z)$ is regular, i.e. $\mathbb{Q}$ is relatively algebraically closed in $\mathrm{ff}(Z)$. Hence, if an ordered field admits an normal integer part, then it has a dense subfield which is regular over $\mathbb{Q}$. Note that any archimedean ordered field has $\mathbb{Z}$ as a normal integer part. Hence, the following question is closely related to the one in [1, Remark 3.3 (iii)]: Given a non-archimedean real closed field $(K, <)$, does it contain a dense subfield $F \subseteq K$ such that F is regular over $\mathbb{Q}$? In the following, we will give a positive answer to this in the case that the absolute transcendence degree of K is infinite.

For an ordered field $(K, <)$, we denote its absolute transcendence degree, i.e. its transcendence degree over $\mathbb{Q}$, by $\mathrm{td}(K)$. We say that $T \subseteq K$ is a transcendence basis of K if it is a transcendence basis of K over $\mathbb{Q}$. The following result is due to [5, Lemma 2.3].

FACT 4.15. *Let $(K, <)$ be an uncountable ordered field. Then K admits a dense transcendence basis.*

The arguments in [5] can be generalised to countable ordered fields with countably infinite transcendence basis.

PROPOSITION 4.16. *Let $(K, <)$ be a countable ordered field with $\mathrm{td}(K) = \aleph_0$. Then K admits a dense transcendence basis.*

PROOF. Let $(I_n)_{n \in \mathbb{N}}$ be an enumeration of all intervals $(a, b) \subseteq K$. We construct a transcendence basis $\{t_1, t_2, \dots\}$ of K over $\mathbb{Q}$ such that $t_k \in I_k$ for any k. Let $t_1 \in I_1$ be an arbitrary element transcendental over $\mathbb{Q}$. Suppose that $t_1, \dots, t_n$ have already been chosen for some n. Assume, for a contradiciton, that all elements in I_{n+1} were algebraic over $\mathbb{Q}(t_1, \dots, t_n)$. Then by [5, Lemma 2.2], also K is algebraic over $\mathbb{Q}(t_1, \dots, t_n)$. This contradicts $\mathrm{td}(K) = \aleph_0$. Hence, we can choose $t_{n+1} \in I_{n+1}$ which is transcendental over $\mathbb{Q}(t_1, \dots, t_n)$. Now $T' = \{t_n \mid n \in \mathbb{N}\}$ is a dense subset of K and algebraically independent over $\mathbb{Q}$ and can thus be extended to a dense transcendence basis $T \supseteq T'$ of K. $\dashv$

COROLLARY 4.17. *Let $(K, <)$ be an ordered field with $\mathrm{td}(K) \geq \aleph_0$. Then there is a dense subfield F of K such that $\mathbb{Q} \subseteq F$ is regular.*

PROOF. By Fact 4.15 and Proposition 4.16, K has a dense transcendence basis T. Then $F = \mathbb{Q}(T)$ is dense in K and $\mathbb{Q} \subseteq F$ is regular. $\dashv$

Note that the conclusion of Corollary 4.17 also holds for ordered fields of transcendence degree 0. Indeed, if $(K, <)$ is an ordered field with $\mathrm{td}(K) = 0$, then K is algebraic over $\mathbb{Q}$ and thus archimedean. We can then set $F = \mathbb{Q}$. Thus, the question becomes whether the same conclusion can be made for non-archimedean ordered fields of non-zero finite transcendence degree (see Question 7.2).

As a final result of this section, we obtain the following characterisation of ordered fields dense in their real closure in terms of dense transcendence bases.

COROLLARY 4.18. *Let $(K, <)$ be an ordered field. Suppose that $\mathrm{td}(K) \geq \aleph_0$. Then K is dense in K^{rc} if and only if K has a transcendence basis T which is dense in K^{rc}.*

Proof. Note that any transcendence basis of K is a transcendence basis of K^{rc}. If K has a transcendence basis T which is dense in K^{rc}, then K has a proper subset dense in K^{rc} and thus K itself is dense in K^{rc}. Conversely, suppose that K is dense in K^{rc}. By Fact 4.15 and Proposition 4.16, K has a transcendence basis T which is dense in K. Since K is dense in K^{rc}, also T is dense in K^{rc}.   ⊣

§5. Definable Convex Valuations. In this section, we firstly investigate what convex valuations are $\mathcal{L}_{\mathrm{or}}$-definable in ordered fields and secondly we compare our results to known $\mathcal{L}_{\mathrm{r}}$-definability results of henselian valuations with special focus on almost real closed fields.

5.1. $\mathcal{L}_{\mathrm{or}}$-definability. We are going to analyse the construction method of $\mathcal{L}_{\mathrm{or}}$-definable convex valuations from [12, Proposition 6.5].

Fact 5.1. *Let $(K, <)$ be an ordered field. Then at least one of the following holds.*
 1. *K is dense in its real closure.*
 2. *K admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex[6] valuation.*

Note that a real closed field $(K, <)$ does not admit any non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation, as by o-minimality, any $\mathcal{L}_{\mathrm{or}}$-definable convex subset of K must be an interval with endpoints in $K \cup \{\pm\infty\}$. We will summarise the construction procedure of a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation ring of an ordered field which is not dense in its real closure given in [12, p. 163 f.]. Recall that for an ordered field $(K, <)$, we denote its topological closure in K^{rc} under the order topology by $\mathrm{cl}(K)$.

Construction 5.2. Let $(K, <)$ be an ordered field. Suppose that K is not dense in $R = K^{\mathrm{rc}}$. Let $s \in R \setminus \mathrm{cl}(K)$. Set $D_s := \{z \in K \mid z < s\}$ and $A_s := \{x \in K^{\geq 0} \mid x + D_s \subseteq D_s\}$. Set $\mathcal{O}_s := \{x \in K \mid |x|A_s \subseteq A_s\}$. Then $\mathcal{O}_s$ is a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation ring of K.

Theorem 5.3. *Let $(K, <)$ be an ordered field and v a henselian valuation on K. Suppose that at least one of the following holds.*
 1. *vK is discretely ordered.*
 2. *vK has a limit point in $vK^{\mathrm{div}} \setminus vK$.*
 3. *Kv has a limit point in $Kv^{\mathrm{rc}} \setminus Kv$.*
Then v is $\mathcal{L}_{\mathrm{or}}$-definable in K. Moreover, in the cases (1) and (2), v is definable by an $\mathcal{L}_{\mathrm{or}}$-formula with one parameter.

Proof. Let $k = Kv$ and $G = vK$. Note that K is not real closed, as in each case either k is not real closed or G is non-divisible. By Fact 2.1, $(K, <, v) \equiv (k((G)), <, v_{\min})$. Let $L = k((G))$, $w = v_{\min}$ and $R = L^{\mathrm{rc}}$. We will apply Construction 5.2 with some simplifications to define the valuation ring $k((G^{\geq 0}))$ of w in L. Proposition 4.9 shows that we can apply the construction procedure to any element in $s \in R \setminus L$.

First suppose that G is non-divisible. Let $g_0 \in G^{\mathrm{div}} \setminus G$ and $s = t^{g_0}$. Consider the $\mathcal{L}_{\mathrm{or}}$-definable set $D_s' = \{x \in L^{\geq 0} \mid x < t^{g_0}\}$. Since $g_0 \in G^{\mathrm{div}}$, there is some

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[6][12, Proposition 6.5] only states that K admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable valuation, but the proof indeed gives a construction method for a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable *convex* valuation.

$h \in G$ and $N \in \mathbb{N}$ such that $g_0 = \frac{h}{N}$. Thus, the set D'_s is defined by the $\mathcal{L}_{\mathrm{or}}$-formula with one parameter

$$x \geq 0 \wedge x^N < t^h.$$

Note that for any $x \in L^{\geq 0}$, we have $x \in D'_s$ if and only if $w(x) > g_0$. Thus, $D'_s = k((G^{>g_0}))^{\geq 0}$. Let $\mathcal{O}_s = \{x \in L \mid |x|D'_s \subseteq D'_s\}$. Note that this set is $\mathcal{L}_{\mathrm{or}}$-definable with one parameter. By definition, $\mathcal{O}_s$ contains exactly those elements in L such that for any $y \in L^{\geq 0}$ with $w(y) > g_0$ we have

(1) $$w(x) + w(y) = w(xy) > g_0.$$

In particular, for any $x \in L$ with $w(x) \geq 0$, condition (1) holds. Thus, $k((G^{\geq 0})) \subseteq \mathcal{O}_s$. To show the other set inclusion, we will make a case distinction, also specifying the element g_0 for the densely ordered case.

Suppose that G is discretely ordered. Let $g_1 \in G$ be the least element greater than g_0 and let $g_2 \in G$ be the least element greater than $g_0 - g_1$. Then $g_2 + g_1$ is the least element greater than g_0. By choice of g_1, this gives us $g_2 + g_1 = g_1$ and thus $g_2 = 0$. Let $x \in \mathcal{O}_s$. Since $t^{g_1} \in D'_s$, we have $w(xt^{g_1}) = w(x) + g_1 > g_0$. Hence, $w(x) > g_0 - g_1$. By choice of g_2 as the least element greater than $g_0 - g_1$, we obtain $w(x) \geq g_2 = 0$. This implies $\mathcal{O}_s \subseteq k((G^{\geq 0}))$, as required.

Suppose that G has a limit point in $G^{\mathrm{div}} \setminus G$. In this case, we choose $g_0 \in G^{\mathrm{div}} \setminus G$ such that g_0 is a limit point of G. We may assume that g_0 is a right-sided limit point, as otherwise we can replace it by $-g_0$. Let $x \in L \setminus k((G^{\geq 0}))$, i.e. $w(x) < 0$. Since g_0 is a right-sided limit point of G in G^{div}, the interval $(g_0, g_0 - w(x)) \subseteq G^{\mathrm{div}}$ contains some element $g_1 \in G$. Thus, $g_1 > g_0$ but $w(x) + w(t^{g_1}) = w(x) + g_1 < g_0$. This shows that x does not fulfil condition (1), whence $x \notin \mathcal{O}_s$. We thus obtain $\mathcal{O}_s \subseteq k((G^{\geq 0}))$.

Now suppose that k is not real closed and has a limit point a in $k^{\mathrm{rc}} \setminus k$. We may assume that a is a left-sided limit point, as otherwise we can replace it by $-a$. Then $D'_a = \{x \in L \mid a - 1 < x < a\}$ consists exactly of the elements of the form $b + r$, where $b \in k$ such that $a - 1 < b < a$ and $r \in k((G^{>0}))$. In other words, $D'_a = I + k((G^{>0}))$, where I is the convex set $(a - 1, a)$ in k. Note that I is non-empty, as a is a left-sided limit point of k. Let A'_a be the $\mathcal{L}_{\mathrm{or}}$-definable set $\{x \in L^{\geq 0} \mid x + D'_a \subseteq D'_a\}$. Since $k((G^{>0}))$ is closed under addition, we have $k((G^{>0})) + D'_a \subseteq D'_a$. Thus, $k((G^{>0}))^{\geq 0} \subseteq A'_a$. For the other inclusion, let $x \in L^{\geq 0} \setminus k((G^{>0}))$, i.e. $w(x) \leq 0$ and $x \geq 0$. If $w(x) > 0$, then $x + b \notin D'_a$ for any $b \in I$. Thus, $x \notin A'_a$. Suppose that $w(x) = 0$. Then x is of the form $c + r$ with $c \in k^{>0}$ and $r \in k((G^{>0}))$. If $c \geq 1$, then $x + b \notin D'_a$ for any $b \in I$, whence $x \notin A'_a$. If $c < 1$, let $b \in k \cap (a - c, a)$, which exists, as a is a left-sided limit point of k. Then $x + b = (c + b) + r > a + r$. Thus, $x + b \notin D'_a$ and $x \notin A'_a$. Hence, we have shown that $A'_a \subseteq k((G^{>0}))^{\geq 0}$.

Now $(-A'_a \cup A'_a) = k((G^{>0}))$ is the maximal ideal of the valuation ring $k((G^{\geq 0}))$. Thus, the valuation ring $k((G^{\geq 0})) = \{x \in L \mid x(-A'_a \cup A'_a) \subseteq (-A'_a \cup A'_a)\}$ is $\mathcal{L}_{\mathrm{or}}$-definable.

Now for any of the three cases, there is an $\mathcal{L}_{\mathrm{or}}$-formula $\varphi(x, \underline{y})$ (where in the cases (1) and (2) $\underline{y}$ is just one free variable) such that $(L, <, w) \models \exists \underline{y} \forall x \, (\varphi(x, \underline{y}) \leftrightarrow$

$w(x) \geq 0$). By elementary equivalence, there is some $\underline{b} \in K$ such that $(K, <, v) \models \forall x \, (\varphi(x, \underline{b}) \leftrightarrow v(x) \geq 0)$. In other words, $\varphi(x, \underline{b})$ defines v in K, as required. ⊣

Theorem 5.3 is not a full characterisation of all $\mathcal{L}_{\mathrm{or}}$-definable henselian valuations on an ordered field. Indeed, we can choose $K = \mathbb{R}((G))$ for G as in Example 3.21. Then $v_{\min} = v_0$ (see p. 19) satisfies neither of the conditions of Theorem 5.3, but v_0 is even $\mathcal{L}_{\mathrm{r}}$-definable by Fact 5.10 for $p = 2$. Moreover, not every ordered henselian valued field which is not dense in its real closure satisfies one of the conditions of Theorem 5.3. For instance, $(\mathbb{R}((G)), <, v_{\min})$, where G is as in Proposition 3.18, is not dense in its real closure by Proposition 4.9 and does not satisfy any of the conditions of Theorem 5.3.

COROLLARY 5.4. *Let $(K, <)$ be an ordered field and v a henselian valuation on K. Suppose that Kv is not real closed but dense in Kv^{rc}. Then v is $\mathcal{L}_{\mathrm{or}}$-definable in K.*

PROOF. This follows immediately from Theorem 5.3 (3), as any point in $Kv^{\mathrm{rc}} \setminus Kv$ is a limit point of Kv in Kv^{rc}. ⊣

REMARK 5.5. 1. Since any archimedean ordered field is dense in its real closure, a special case of Corollary 5.4 is the following: Let $(K, <)$ be a non-archimedean ordered field. Suppose that the natural valuation v_{nat} is henselian on K and that $(Kv_{\mathrm{nat}}, <)$ is not real closed. Then v_{nat} is $\mathcal{L}_{\mathrm{or}}$-definable in K.

2. Theorem 5.3 (2) implies a similar version of Corollary 5.4 if vK is non-divisible and dense in vK^{div}. However, we will see in Subsection 5.2 that under this condition we already have that v is $\mathcal{L}_{\mathrm{r}}$-definability without parameters.

3. The proof of Theorem 5.3 also shows the following: Let $(K, <)$ be an ordered field and v a non-trivial henselian valuation on K. Suppose that vK is non-divisible. Then there exists a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable coarsening of v which is definable by an $\mathcal{L}_{\mathrm{or}}$-formula with one parameter.

Comparison to $\mathcal{L}_{\mathrm{r}}$-definability. We will give a brief account of the known $\mathcal{L}_{\mathrm{r}}$-definability result of henselian valuations in the case that the value group is regular (cf. [9, Theorem 4]) and compare this to Theorem 5.3.

FACT 5.6. (See [9, Theorem 4].) *Let K be a field and v a henselian valuation on K. Suppose that vK is regular and non-divisible. Then v is parameter-free $\mathcal{L}_{\mathrm{r}}$-definable in K.*

By Proposition 3.2, we obtain the following.

COROLLARY 5.7. *Let $(K, <)$ be an ordered field and v a henselian valuation on K such that vK is non-divisible but dense in vK^{div}. Then v is parameter-free $\mathcal{L}_{\mathrm{r}}$-definable in K.*

Example 3.17 (1) shows that there are discretely ordered abelian groups which are not regular. Example 3.17 (2) exhibits a densely ordered abelian group G which is not regular but has limit points in $G^{\mathrm{div}} \setminus G$. This shows that there are ordered fields such that the cases (1) and (2) of Theorem 5.3 are not already covered by Fact 5.6.

5.2. Almost real closed fields. The class of almost real closed fields has first been studied systematically with respect to algebraic and model theoretic properties in [2]. Moreover, [2, Theorem 4.4] completely characterises all $\mathcal{L}_r$-definable henselian valuations in almost real closed fields. In the following, we will compare $\mathcal{L}_r$- and $\mathcal{L}_{or}$-definability of henselian valuations in almost real closed fields.

DEFINITION 5.8. Let $(K, <)$ be an ordered field, G an ordered abelian group and v a henselian valuation on K. We call K an **almost real closed field (with respect to v and G)** if Kv is real closed and $vK = G$.

Depending on the context, we may simply say that $(K, <)$ is an almost real closed field without specifying the henselian valuation v or the ordered abelian group $G = vK$. In [2], almost real closed fields are considered as pure fields, i.e. as structures in the language $\mathcal{L}_r$. Note that due to the Baer–Krull Representation Theorem (cf. [4, p. 37 f.]), any such field admits (possibly several distinct) orderings. We, however, consider almost real closed fields as ordered fields with a fixed order. By [2, Proposition 2.9], any convex valuation on an almost real closed field is already henselian. We thus make no distinction between convex and henselian valuations on almost real closed fields.

Let $(K, <)$ be an almost real closed field. We denote by $V(K)$ the set of all henselian valuations on K with real closed residue field, by v_1 the maximum of $V(K)$, i.e. the finest valuation in $V(K)$, and by v_0 be the minimum of $V(K)$, i.e. the coarsest valuation in $V(K)$. These exist by [2, Proposition 2.1]. By the remarks in [2, p. 1147 f.], v_0 is the only possible $\mathcal{L}_r$-definable henselian valuation in $V(K)$. Also in the language $\mathcal{L}_{or}$, there is at most one definable valuation in $V(K)$.

PROPOSITION 5.9. *Let $(K, <)$ be an almost real closed field and $v \in V(K)$. Suppose that v is $\mathcal{L}_{or}$-definable in K. Then v is the only $\mathcal{L}_{or}$-definable valuation in $V(K)$.*

PROOF. By Fact 2.1, we have

$$(K, <, v) \equiv (\mathbb{R}((vK)), <, v_{\min}).$$

Since v is $\mathcal{L}_{or}$-definable in K, there exists an $\mathcal{L}_{or}$-formula $\varphi(x, \underline{y})$ such that

$$K \models \exists \underline{y} \forall x \ (\varphi(x, \underline{y}) \leftrightarrow v(x) \geq 0).$$

By elementary equivalence, there exists $\underline{b} \in \mathbb{R}((vK))$ such that

$$\mathbb{R}((vK)) \models \forall x \ (\varphi(x, \underline{b}) \leftrightarrow v_{\min}(x) \geq 0).$$

Hence, $v_{\min}$ is $\mathcal{L}_{or}$-definable in $\mathbb{R}((vK))$.

Let $\psi(x, \underline{y})$ be an $\mathcal{L}_{or}$-formula and $\underline{c} \in K$ such that $\psi(x, \underline{c})$ defines a convex valuation w in K. Assume, for a contradiction, that w is strictly finer than v, i.e. $\mathcal{O}_w \subsetneq \mathcal{O}_v$. This implies

$$(K, <, v) \models \forall x \ (\psi(x, \underline{c}) \rightarrow v(x) \geq 0) \wedge \exists z \ (\neg\psi(z, \underline{c}) \wedge v(z) \geq 0).$$

By elementary equivalence, there is some $\underline{c}' \in \mathbb{R}((vK))$ such that $\psi(x, \underline{c}')$ defines a convex valuation w' in $\mathbb{R}((vK))$ with $\mathcal{O}_{w'} \subsetneq \mathcal{O}_{v_{\min}}$. This contradicts that fact

that $v_{\min}$ is the finest convex valuation on $\mathbb{R}((vK))$. Hence, v is the finest $\mathcal{L}_{\mathrm{or}}$-definable convex valuation in K.

Let $v' \in V(K)$ be $\mathcal{L}_{\mathrm{or}}$-definable. Arguing as above, v' is the finest $\mathcal{L}_{\mathrm{or}}$-definable convex valuation on K. This gives us $v' = v$, as required.   ⊣

Comparison of $\mathcal{L}_{\mathrm{r}}$- and $\mathcal{L}_{\mathrm{or}}$-definability of henselian valuations in almost real closed fields. Let p be a prime number. A valuation v on K is called p-Kummer henselian if Hensel's Lemma holds for polynomials of the form $x^p - a$ for $a \in \mathcal{O}_v$. A field L is called p-euclidean if $L = \pm L^p$. Let $V_p(K)$ be the set of all p-Kummer henselian valuations of K with p-euclidean residue field. Denote by v_p the minimum of $V_p(K)$ (cf. [2, p. 1126]).

FACT 5.10. (See [2, Theorem 4.4].) *Let $(K, <)$ be an almost real closed field and v a henselian valuation on K. Then v is $\mathcal{L}_{\mathrm{r}}$-definable in K if and only if vK is $\mathcal{L}_{\mathrm{og}}$-definable in v_1K and $v \le v_p$ for some prime p. Moreover, v_0 is $\mathcal{L}_{\mathrm{r}}$-definable if and only if there is a prime p such that v_0K has no non-trivial p-divisible convex subgroups.*

Recall that v_0 is the only possible $\mathcal{L}_{\mathrm{r}}$-definable valuation in $V(K)$. If the ordering on an almost real closed field $(K, <)$ is $\mathcal{L}_{\mathrm{vf}}$-definable for $v = v_0 \in V(K)$, then we obtain a complete characterisation of $\mathcal{L}_{\mathrm{or}}$-definable convex valuations in K.

LEMMA 5.11. *Let $(K, <)$ be an ordered field and let v be a henselian valuation on K such that Kv is 2-euclidean (i.e. root-closed for positive elements) and vK is 2-divisible. Then the ordering $<$ is parameter-free $\mathcal{L}_{\mathrm{vf}}$-definable in K. In particular, if v is $\mathcal{L}_{\mathrm{r}}$-definable in K, then any $\mathcal{L}_{\mathrm{or}}$-definable subset of K is already $\mathcal{L}_{\mathrm{r}}$-definable.*

PROOF. Let $k = Kv$ and $G = vK$. Consider the $\mathcal{L}_{\mathrm{vf}}$-formula $\varphi(x)$ given by

$$x = 0 \vee \exists y \; v(x - y^2) > v(x).$$

We will show that for any $a \in k((G))$, the formula $\varphi(a)$ holds if and only if $a \ge 0$. Let $a = a_g t^g + s \in k((G))^\times$, where $a_g \in k^\times$, $s \in k((G^{>g}))$ and $g = v_{\min}(a)$. Suppose that $\varphi(a)$ holds. Then there exists $y \in K^\times$ such that $v_{\min}(x - y^2) > g$. Hence, $a_g = y_g^2 > 0$, where y_g is the coefficient of the monomial t^g in y. Thus, $a > 0$. Now suppose that $a > 0$. Let $y = \sqrt{a_g} t^{g/2}$. Then $v_{\min}(a - y^2) = v_{\min}(s) > g = v_{\min}(a)$.

By Fact 2.1, $(K, <, v) \equiv (Kv((vK)), <, v_{\min})$. Hence, we obtain $K \models \forall x \; (x \ge 0 \leftrightarrow \varphi(x))$.   ⊣

PROPOSITION 5.12. *Let $(K, <)$ be an almost real closed field. Suppose that v_0 is $\mathcal{L}_{\mathrm{r}}$-definable and that v_0K is 2-divisible. Let w be a valuation on K. If w is $\mathcal{L}_{\mathrm{or}}$-definable, then it is $\mathcal{L}_{\mathrm{r}}$-definable.*

PROOF. Since v_0K is real closed, it is 2-euclidean. By Lemma 5.11, any $\mathcal{L}_{\mathrm{or}}$-definable valuation on K is already $\mathcal{L}_{\mathrm{r}}$-definable.   ⊣

REMARK 5.13. We obtain the following characterisation of $\mathcal{L}_{\mathrm{or}}$-definable convex valuations in almost real closed fields with respect to a 2-divisible value group: Let $(K, <)$ be an almost real closed field. Suppose that the value group

$v_0 K$ is 2-divisible and, for some prime p, it has no non-trivial p-divisible convex subgroup. Let v be a convex (and thus henselian) valuation on K. Hence, by Proposition 5.12 and Fact 5.10, v is $\mathcal{L}_{\mathrm{or}}$-definable in K if and only if vK is $\mathcal{L}_{\mathrm{og}}$-definable in $v_1 K$ and $v \le v_p$ for some prime p.

Proposition 5.12 shows in particular that for an almost real closed field, if v_0 is $\mathcal{L}_{\mathrm{r}}$-definable, then so is any $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation. The final result of this sections shows that any henselian valuation in an almost real closed field satisfying the hypothesis of Theorem 5.3 is already $\mathcal{L}_{\mathrm{r}}$-definable. Note that any discretely ordered abelian group does not have a non-trivial n-divisible convex subgroup for any $n \ge 2$.

PROPOSITION 5.14. *Let $(K, <)$ be an almost real closed field with respect to a henselian valuation v such that either vK is discretely ordered or vK is not closed in vK^{div}. Then $v = v_0$ and v_0 is $\mathcal{L}_{\mathrm{r}}$-definable.*

PROOF. Recall that v_0 is the unique henselian valuation on K such that $K v_0$ is real closed and $v_0 K$ has no non-trivial divisible convex subgroup. Moreover, v_0 is $\mathcal{L}_{\mathrm{r}}$-definable if and only if for some prime p, there is no p-divisible non-trivial convex subgroup of $v_0 K$. If $G = vK$ is discretely ordered, both conditions are satisfied, and thus $v = v_0$ is $\mathcal{L}_{\mathrm{r}}$-definable. If G is not closed in G^{div}, then by Corollary 3.20, G has no non-trivial convex subgroup, whence $v = v_0$. Moreover, by Proposition 3.19 there is a prime p such that G has no non-trivial p-divisible convex subgroup, whence v_0 is $\mathcal{L}_{\mathrm{r}}$-definable. $\dashv$

§6. Applications to definable valuations on strongly NIP ordered fields.
The study of $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation in strongly NIP fields has been motivated by a conjecture due to Shelah–Hasson that any strongly NIP field is either real closed, algebraically closed or admits a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation (cf. [3]). Specialised to ordered fields, this conjecture can be reformulated as follows: Any strongly NIP ordered field which is not real closed admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation. In [14], it is shown that this conjecture is equivalent to the following: Any strongly NIP ordered field is almost real closed.[7] The question becomes whether any strongly NIP ordered field is almost real closed with respect to an $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation. In this section, we will give a negative answer to this question.

All notions on strongly NIP theories can be found in [22]. For the purpose of understanding how Fact 5.10 can be applied to stronlgy NIP ordered fields, the following two results are helpful.

FACT 6.1. [8, Theorem 1] *Let G be an ordered abelian group. Then the following are equivalent:*

1. G is strongly NIP.

2. G is elementarily equivalent to a Hahn sum of archimedean ordered abelian groups $\coprod_{i \in I} G_i$, where for every prime p, we have $|\{i \in I \mid pG_i \ne G_i\}| < \infty$, and for any $i \in I$, we have $|\{p \text{ prime} \mid [G_i : pG_i] = \infty\}| < \infty$.

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[7]As mentioned in the introduction, this conjecture is the main subject of a separate publication [15].

LEMMA 6.2. [14, Lemma 6.11] *Let G be a strongly NIP ordered abelian group. Then the almost real closed field $(\mathbb{R}((G)), <)$ is strongly NIP.*

By Fact 6.1, the following are examples of strongly NIP non-divisible ordered abelian groups.

EXAMPLE 6.3.   1. Let
$$G_1 = \left\{ \frac{a}{p_1^{i_1} \dots p_m^{i_m}} \; \middle| \; m \in \mathbb{N}; a, i_1, \dots, i_m \in \mathbb{Z} \text{ and } p_1, \dots, p_m \geq 3 \text{ are prime} \right\}.$$

G_1 is p-divisble for any prime $p \geq 3$. Thus, $|\{p \text{ prime} \mid [G_1 : pG_1] = \infty\}| \leq 1$. This implies that G_1 is strongly NIP. Moreover, it dense in its divisible hull $\mathbb{Q}$ but not divisible, as $\frac{1}{2} \notin G_1$.

 2. Let $2 = p_0 < p_1 < \dots$ be a complete list of prime numbers. For every $n \in \mathbb{N}$, let
$$B_n = \left\{ \frac{a}{p_{i_1}^{m_1} \cdot \dots \cdot p_{i_k}^{m_k}} \; \middle| \; k \in \mathbb{N}; i_1, \dots, i_k \in \mathbb{N}_0 \setminus \{n\}; a, m_1, \dots, m_k \in \mathbb{Z} \right\}.$$

In other words, B_n is the smallest subgroup of $\mathbb{Q}$ which is p_i-divisible for any $i \in \mathbb{N}_0 \setminus \{n\}$ (so in particular 2-divisible) but not p_n-divisible. Then $G_2 = \coprod_{n \in \mathbb{N}} B_n$ is strongly NIP, as for any $i \in \mathbb{N}_0$, we have $|\{n \in \mathbb{N} \mid p_i B_n \neq B_n\}| = 1$, and for any $n \in \mathbb{N}$, we have $|\{p \text{ prime} \mid [B_n : pB_n] = \infty\}| \leq 1$. Note that G_2 is 2-divisible. Moreover, any convex subgroup of G_2 is of the form $\coprod_{n \geq k} B_n = H_k$ for some $k \in \mathbb{N}$. Note that H_k is non-divisible but p_i-divisible for any $i < k$.

Let G_1 and G_2 be as in Example 6.3. By Lemma 6.2, we obtain strongly NIP almost real closed fields $(\mathbb{R}((G_1)), <)$ and $(\mathbb{R}((G_2)), <)$. By Proposition 5.14, v_{nat} is $\mathcal{L}_{\mathrm{r}}$-definable in $(\mathbb{R}((G_1)), <)$.

Sinc G_2 has no non-trivial convex divisible subgroup, by Fact 5.10, the natural valuation v_{nat} in $\mathbb{R}((G_2))$ is $\mathcal{L}_{\mathrm{r}}$-definable if and only if there is a prime p such that G_2 has no non-trivial convex p-divisible subgroup. However, for any $i \in \mathbb{N}$, the non-trivial convex subgroup H_{i+1} of G_2 is p_i-divisible. This implies that v_{nat} is not $\mathcal{L}_{\mathrm{r}}$-definable. Moreover, since G_2 is 2-divisible, v_{nat} is also not $\mathcal{L}_{\mathrm{or}}$-definable by Proposition 5.12. As $v_{\mathrm{nat}} = v_0 = v_1$ (see page 19), it is the only valuation with respect to which $\mathbb{R}((G_2))$ is almost real closed. Thus, $(\mathbb{R}((G_2)), <)$ is a stronlgy NIP almost real closed field which is not almost real closed with respect to an $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation.

§7. **Open Questions.** We conclude with open questions connected to results throughout this work.

In Proposition 3.7 we have seen the following dichotomy for ordered abelian groups: Any ordered abelian group G is (exclusively) either dense in G^{div} or admits a proper non-trivial $\mathcal{L}_{\mathrm{og}}$-definable convex subgroup. For ordered fields, it is not known whether a similar dichotomy holds. Note that any archimedean ordered field $(K, <)$ does not admit a non-trivial valuation, as $\mathbb{Z}$ must be contained in any convex subring of K. Hence, any ordered field with an archimedean model does not admit a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation. However, we have seen in Proposition 4.6 that there are non-archimedean ordered fields which are dense in their real closure but do not have an archimedean model. In these

ordered fields, it may be possible to find non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuations. Note that these cannot be henselian by Corollary 4.11. In [12] it is not investigated whether the two cases in Fact 5.1 are exclusive. We pose this as the following question.

QUESTION 7.1. *Is there an ordered field which is dense in its real closure and admits an non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation?*

In Fact 4.15 and Proposition 4.16 we have seen that any ordered field K of infinite transcendence degree admits a dense transcendence basis T and thus a dense subfield $F = \mathbb{Q}(T)$ such that $\mathbb{Q} \subseteq F$ is regular. Note that a non-archimedean ordered field K with $\mathrm{td}(K) < \aleph_0$ cannot admit a transcendence basis dense in K. However, it is still possible that K has a dense subfield F such that $\mathbb{Q} \subseteq F$ is regular. We pose this as a question for a non-archimedean real closed field with transcendece degree 1.

QUESTION 7.2. *Let $K = \mathbb{Q}(t)^{\mathrm{rc}}$ ordered by $t > \mathbb{N}$. Is there a dense subfield $F \subseteq K$ such that $\mathbb{Q} \subseteq F$ is regular?*

We have seen in Proposition 5.14 that for any almost real closed field with respect to a henselian valuation v, if v satisfies the hypothesis of Theorem 5.3, then it is not only $\mathcal{L}_{\mathrm{or}}$- but already $\mathcal{L}_{\mathrm{r}}$-definable. Although we have shown that Theorem 5.3 generalises known $\mathcal{L}_{\mathrm{r}}$-definability results of henselian valuations (see p. 18), we have not provided an example of an ordered field $(K, <)$ and a henselian valuation v on K which is $\mathcal{L}_{\mathrm{or}}$- but not $\mathcal{L}_{\mathrm{r}}$-definable. Since Fact 5.10 provides a full characterisation of $\mathcal{L}_{\mathrm{r}}$-definable henselian valuations in almost real closed fields, we will pose the following more specific question.

QUESTION 7.3. *Is there an almost real closed field $(K, <)$ and a henselian valuation v on K such that v is $\mathcal{L}_{\mathrm{or}}$- but not $\mathcal{L}_{\mathrm{r}}$-definable?*

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