RANDOMIZATION OF DEFINABLE GROUPS

ALEXANDER BERENSTEIN AND JORGE DANIEL MUÑOZ

Abstract. We study randomizations of definable groups. Whenever the underlying theory is stable or NIP and the group is definably amenable, we show its randomization is definably connected.

1. Introduction

The randomization of a model M is a continuous structure build by taking random variables whose values belong to M. It was first introduced by Keisler in [9] and formalized in the continuous setting by Ben Yaacov and Keisler [7]. If we call $T = Th(M)$, this procedure generates a new theory T^R, the theory of the randomized structure, which does not depend on the underlying model M but only on its theory T. Randomizations are perhaps the most interesting way of transferring information from a discrete theory T (and even a continuous theory) to a new continuous theory T^R. Many desirable model-theoretic properties such as ω-categoricity, ω-stability, stability and NIP hold in T if and only if they hold in T^R ([7, 3]). Even a nice description of separable models can be achieved understanding the countable models of T [2]. On the other hand, tree properties are more difficult to control, it has been observed by Ben Yaacov that randomizing a simple unstable theory gives a theory with the TP_2 property. Also understanding indiscernible sequences and forking independence in the randomization can be quite hard, this being a problem that can not be described by pointwise assertions of the random variables.

This paper deals with randomizing groups, the main result that we prove is that if the underlying theory T is stable and G is a definable group, then its randomization G^R is a definable *connected* group. We prove a similar result when T is NIP and the group is definably amenable. In particular, we explicitly describe the generic elements in G^R in terms of the generics of G and do a basic study of stabilizers.

This paper is organized as follows. In section 2 we review some basic properties of definable groups in stable theories, NIP theories and amenable groups, both in the discrete and in the continuous setting. We expect the reader to be familiar with continuous logic and in particular definability, a good source for this material is [6]. We also provide a proof for the existence of the connected component G^{00} of G when G is a definable NIP group in the continuous setting. In section 3 we recall the basic theory of randomizations. In section 4 we show that certain structural properties transfer from G to G^R such as being abelian, divisible and definably nilpotent. We also prove the main set of theorems: a randomized stable group is connected as well as a the randomization of a NIP group which is definably amenable.

2. Preliminaries

2.1. **Definable groups.** In this section we recall some well known results on stable groups, in particular their connected components and the action of G on the space of types.

Theorem 2.1 (Corollary 7.1.6 in [10]). *Let T be stable and G a definable group. Then there exists $G^0 \leq G$ the smallest type-definable subgroup of bounded index. G^0 is normal and $\emptyset$-type-definable.*

2000 *Mathematics Subject Classification.* 03C45, 03C64, 03C90.

Key words and phrases. stable theories, NIP theories, metric structures, randomizations, connected groups, definably amenable groups.

We would like to thanks Itai Ben Yaacov for suggesting this problem and for helpful discussions.

We call G^0 the connected component of G and it is the intersection of all definable subgroups of finite index.

The group G acts on the space of types $S_1(G)$ in the following manner: for $a \in G$ and $p \in S_1(G)$

$$ap = \{\varphi(a^{-1}x)\ ;\ \varphi(x) \in p\}.$$

We will write Δ for a finite set of formulas $\varphi(x, y)$ where y denotes the tuple of variables that serves as parameters.

Definition 2.2. Let A be a set of parameters. A partial Δ-*type over* A is a set of formulas of the form $\varphi(x, a)$ or $\neg\varphi(x, a)$ where a is a tuple from A and $\varphi(x, y) \in \Delta$. If $p \in S_1(A)$ we denote by $p \upharpoonright \Delta$ its restriction to a Δ-type. A set of formulas Δ is said to be *invariant* if for any partial Δ-type p over G and $a \in G$, the set of formulas ap is again a partial Δ-type over G.

If $\varphi(x, y)$ is an $\mathcal{L}$-formula we denote by $\varphi^*(x, y, z)$ the formula $\varphi(zx, y)$. For Δ a set of formulas, define $\Delta^* = \{\varphi^*(x, y, z)\ ;\ \varphi(x, y) \in \Delta\}$. Δ^* is invariant and every partial Δ-type over G is also a partial Δ^*-type.

Definition 2.3. Let Δ be an invariant set of formulas and $p \in S_1(G)$. The Δ-*stabilizer of* p is

$$\mathrm{stab}(p, \Delta) = \{a \in G\ ;\ a(p \upharpoonright \Delta) = p \upharpoonright \Delta\}.$$

The *stabilizer of* p is

$$\mathrm{stab}(p) = \{a \in G\ ;\ ap = p\} = \bigcap\{\mathrm{stab}(p, \Delta)\ ;\ \Delta \text{ is invariant}\}.$$

Definition 2.4. A type $p \in S_1(G)$ is *generic* if for every $a \in G$, the type ap does not fork over A, the set of parameters over which G is definable.

Theorem 2.5 (Proposition 5.3.1 and Corollary 5.3.1 in [8])**.** *Let G be a stable group. Then:*

1. *$p \in S_1(G)$ is generic if and only if stab$(p) = G^0$.*
2. *There is a generic type in each coset modulo G^0.*
3. *The action of G restricted to the generic types is transitive.*
4. *If $a, b \in G$ belong to the same coset modulo G^0 and $p \in S_1(G)$ is generic then $ap = bp$.*

In the ω-stable case, G^0 is definable and of finite index, and there are finitely many generic types. Furthermore, a type is generic if it has maximal Morley rank.

Proposition 2.6. *Let G be a stable group and Δ a finite invariant set of formulas. Then:*

1. *The set $\{p \upharpoonright \Delta\ ;\ p \in S_1(G)$ is generic$\}$ is finite.*
2. *If Δ' is finite and invariant, and $\Delta \subseteq \Delta'$ then for every $\varphi(x, y) \in \Delta$ and every tuple c in G*

$$\frac{|\{p \upharpoonright \Delta\ ;\ p \in S_1(G)\ \textit{generic},\ \varphi(x, c) \in p\}|}{|\{p \upharpoonright \Delta\ ;\ p \in S_1(G)\ \textit{generic}\}|} = \frac{|\{p \upharpoonright \Delta'\ ;\ p \in S_1(G)\ \textit{generic},\ \varphi(x, c) \in p\}|}{|\{p \upharpoonright \Delta'\ ;\ p \in S_1(G)\ \textit{generic}\}|}.$$

Proof. The first numeral corresponds to Proposition 5.3.1 in [8]. For the second one, it is clear that $\mathrm{stab}(p, \Delta')$ is a finite index subgroup of $\mathrm{stab}(p, \Delta)$. Let $k = [\mathrm{stab}(p, \Delta) : \mathrm{stab}(p, \Delta')]$. Then, there is exactly one type of the form $(bp) \upharpoonright \Delta'$ in each coset of $\mathrm{stab}(p, \Delta')$ in G. Every coset of $\mathrm{stab}(p, \Delta)$ in G contains k cosets of $\mathrm{stab}(p, \Delta')$ in G. Hence,

$$|\{p \upharpoonright \Delta'\ ;\ p \in S_1(G) \text{ generic}, \varphi(x, c) \in p\}| = k|\{p \upharpoonright \Delta\ ;\ p \in S_1(G) \text{ generic}, \varphi(x, c) \in p\}|$$

Similarly, $|\{p \upharpoonright \Delta'\ ;\ p \in S_1(G) \text{ generic}\}| = k|\{p \upharpoonright \Delta\ ;\ p \in S_1(G) \text{ generic}\}|$ and the conclusion follows. □

Definable groups in the continuous setting

The results and definitions mentioned in this section are based on [4]. Let T be a continuous stable theory, G a type-definable group whose group operation is also type-definable. We identify G with its points in a κ-saturated, κ-strongly homogeneous model $\mathfrak{C}$, for κ large enough. We denote by $S_G(\mathfrak{C})$ the subspace of $S_1(\mathfrak{C})$ consisting of types that imply $x \in G$. This is a closed subspace of $S_1(\mathfrak{C})$.

Definition 2.7. We say that a type $p \in S_G(\mathfrak{C})$ is *generic* if every logical neighbourhood of p defines a generic set i.e. one for which finitely many translates cover the group.

As in the first order case, the group G acts on its space of types by left translation. The stabilizer is defined in the same way and once again we have the notion of connected component.

Proposition 2.8 (Theorem 6.14 in [4]). *Let G be a type-definable group over $\varnothing$ in a stable theory. Then there exists $G^0 \leq G$, the smallest type-definable subgroup of bounded index. G^0 is normal and type-definable over $\varnothing$.*

The next result is the analogous to Theorem 2.5 for continuous stable groups.

Theorem 2.9 (Proposition 6.9, Proposition 6.13 and Theorem 6.14 in [4]). *Let G be an $\varnothing$-definable group in a continuous stable theory. Then:*

(1) *$p \in S_G(\mathfrak{C})$ is generic if and only if $stab(p) = G^0$.*
(2) *The action of G restricted to the generic types is transitive.*
(3) *If $a, b \in G$ belong to the same coset modulo G^0 and $p \in S_G(\mathfrak{C})$ is generic then $ap = bp$.*
(4) *There is a generic type in each coset modulo G^0.*

2.2. **NIP theories.** Let T denote a complete continuous theory and $\mathfrak{C}$ a κ-saturated, κ-strongly homogeneous model of T, for κ large enough.

Definition 2.10. A formula $\varphi(x, y)$ has the *independence property* (IP) if there are $r, s \in [0, 1]$, $r < s$, sequences $\langle a_i\ ;\ i < \omega\rangle$, $\langle b_I\ ;\ I \subseteq \omega\rangle$ in $\mathfrak{C}$ such that $\varphi(a_i, b_I) \leq r$ for $i \in I$ and $\varphi(a_i, b_I) \geq s$ for $i \notin I$. A theory T is *NIP* if no formula has the independence property.

By compactness, a continuous formula $\varphi(x, y)$ has IP if we can find $r, s \in [0, 1]$, $r < s$, $\mathcal{N} \models T$ and sequences $\langle a_i\ ;\ i < \omega\rangle$, $\langle b_I\ ;\ I \subseteq \omega,\ |I| < \omega\rangle$ such that $\varphi(a_i, b_I) \leq r$ for $i \in I$ and $\varphi(a_i, b_I) \geq s$ for $i \notin I$.

Let T denote now a complete first order theory and $\mathfrak{C}$ a κ-saturated, κ-strongly homogeneous model of T, for κ large enough.

Definition 2.11. Let A be a subset of $\mathfrak{C}$, a *Keisler measure ν over A* is a finitely additive probability measure in the algebra of definable sets over A.

To every Keisler measure over A we can associate a regular probability measure in $S_1(A)$. The detailed construction is given in Chapter 7 of [12]. An important fact about NIP theories is that we can approximate Keisler measures by sampling on the space of types. For this purpose, for $p_1, \ldots, p_n \in S_1(A)$ and X Borel subset of $S_1(A)$ we will denote by $\mathrm{Av}(p_1, \ldots, p_n; X)$ the quotient $\frac{|\{k \in \{1,\ldots,n\}\ ;\ p_k \in X\}|}{n}$.

Proposition 2.12 (Proposition 7.11 in [12]). *Let ν be a Keisler measure over A, $\varphi(x, y)$ be a formula and $X_1, \ldots, X_m$ be Borel subsets of $S_1(A)$. Then, for any given $\epsilon > 0$ there are types $p_1, \ldots, p_n \in S_1(A)$ such that for every tuple b in A and every $k \leq m$*

$$|\nu(\varphi(x, b) \cap X_k) - \mathrm{Av}(p_1, \ldots, p_n; \varphi(x, b) \cap X_k)| < \epsilon.$$

An immediate consequence is the following corollary.

Corollary 2.13. *Assume that A has at least two different elements. Let ν be a Keisler measure over A and let $\varphi_1(x, y), \ldots, \varphi_m(x, y)$ be formulas. Then, for any given $\epsilon > 0$ there are types $p_1, \ldots, p_n \in S_1(A)$ such that for every tuple b in A and every $k \leq m$*

$$|\nu(\varphi_k(x, b)) - \mathrm{Av}(p_1, \ldots, p_n; \varphi_k(x, b))| < \epsilon.$$

Proof. Consider the formula

$$\psi(x, y, z_1, \ldots, z_{m+1}) : \bigwedge_{1 \leq k \leq m} (z_1 = z_{k+1} \Rightarrow \varphi_k(x, y)).$$

Take $\epsilon > 0$. By the previous proposition, there are types $p_1, \ldots, p_n \in S_1(A)$ such that for any tuples b, c in A

$$|\nu(\psi(x, b, c)) - \mathrm{Av}(p_1, \ldots, p_n; \psi(x, b, c))| < \epsilon.$$

Fix $k \le m$, take $a_1, a_2 \in A$ different from each other and define c as $c_1 = c_{k+1} = a_1$ y $c_j = a_2$ for $1 \le j \le m$ with $j \ne, 1, k+1$. Thus, for any tuple b in A, $\varphi_k(x,b)$ is equivalent to $\psi(x,b,c)$. Therefore,

$$|\nu(\varphi_k(x,b)) - \mathrm{Av}(p_1,\dots,p_n;\varphi_k(x,b))| = |\nu(\psi(x,b,c)) - \mathrm{Av}(p_1,\dots,p_n;\psi(x,b,c))| < \epsilon.$$

□

Definition 2.14. Let ν be a Keisler measure over $\mathfrak{C}$ and $A \subseteq \mathfrak{C}$. We say that ν is *invariant over* A if for any two tuples b, b' in $\mathfrak{C}$ with $tp(b/A) = tp(b'/A)$ and any formula $\varphi(x,y)$ we have that $\nu(\varphi(x,b)) = \nu(\varphi(x,b'))$.

Definition 2.15. Let $\mathfrak{M} = (M,\dots)$ be a model of T and let ν be a Keisler measure over $\mathfrak{C}$. We say that ν is *Borel-definable* over $\mathfrak{M}$ if it is invariant over M and for every formula $\varphi(x,y)$ and $r \in [0,1]$ the set

$$\{q \in S_{|y|}(M)\ ;\ \mu(\varphi(x,b)) < r \text{ for any } b \in \mathfrak{C},\ \text{with } b \vDash q\}$$

is Borel in $S_{|y|}(M)$.

Proposition 2.16 (Proposition 7.19 in [12])**.** *Let ν be a Keisler measure over $\mathfrak{C}$ invariant over $\mathfrak{M}$, then ν is Borel-definable over $\mathfrak{M}$.*

As usual, we identify a definable group G with its points in $\mathfrak{C}$. If ν is a Keisler measure over A we say that ν concentrates at G if $\nu(G) = 1$.

Definition 2.17. We say that G is *definably amenable* if there is a Keisler measure ν over $\mathfrak{C}$ that concentrates at G and is left-invariant i.e. for every $L(\mathfrak{C})$-formula $\varphi(x)$ that defines a set in G and $g \in G$ we have that $\nu(\varphi(x)) = \nu(g\varphi(x))$.

Definition 2.18. Given a small set of parameters A, we denote by G^{00}_A the intersection of all type-definable subgroups over A of bounded index. We say that G^{00} exists if G^{00}_A does not depend on A and it is equal to $G^{00}_{\varnothing}$.

We will show that in the case of a continuous NIP theory G^{00} always exists. The proof is a straightforward adaptation of the discrete case given in [12]:

Theorem 2.19. *Assume that T is a continuous NIP theory. Then G^{00} exists.*

Proof. Let $\mathfrak{C}$ be a monster model. Suppose on the contrary, that G^{00} does not exist. Then there is a collection $\{H_i\ ;\ i < \kappa\}$ of different type-definable subgroups of bounded index. We can assume that each H_i is the intersection of at most $\aleph_0$ conditions. By Ramsey's theorem and compactness there is a sequence $\langle H_i\ ;\ i < \omega\rangle$ that is indiscernible. This means that there is a type-definable set $\Phi(x,y)$ and an indiscernible sequence of tuples $\langle b_i\ ;\ i \in \omega\rangle$ such that H_i corresponds to the zeroset of $\Phi(x,b_i)$. Furthermore, by adding formulas to the type $\Phi(x,y)$ we can assume that $\Phi(x,b)$ defines a subgroup for every $b \in \mathfrak{C}$.

We will show that for every $i \in \omega$, H_i does not contain $\bigcap_{j\ne i} H_j$. Suppose that for some $i \in \omega$ it does. The intersection $\bigcap_{j\ne i} H_j$ has bounded index and so there are boundedly many subgroups containing it. Now insert in place of H_i a sufficiently long sequence $\langle H'_k\ ;\ k < \lambda\rangle$ such that the whole sequence is still indiscernible. Since each H'_k contains $\bigcap_{j\ne i} H_j$ we get a contradiction.

For each $i < \omega$, pick $a_i \in \bigcap_{j\ne i} H_j \setminus H_i$ such that the sequence $\langle a_i, b_i\ ; i \in \omega\rangle$ is indiscernible. Since $a_i \notin H_i$, there is $\phi(x,y) \in \Phi(x,y)$ such that $r = \phi(a_i,b_i) > 0$. In this way, $\phi(a_i,b_j) = 0$ if and only if $i \ne j$ and $\phi(a_i,b_i) = r$ for every $i \in \omega$.

Claim 1: For every $\epsilon > 0$ there are $\psi_1(x,y),\dots,\psi_n(x,y) \in \Phi(x,y)$ and there is $\delta > 0$ such that for every $x_1,x_2,x_3,y \in \mathfrak{C}$ if $\max_{i=1,2,3}\{\max\{\psi_1(x_i,y),\dots,\psi_n(x_i,y)\}\} < \delta$ then $\phi(x_1\cdot x_2\cdot x_3, y) \le \epsilon$.

Proof. Assume that the claim does not hold. Then there is $\epsilon > 0$ such that for every $n \in \mathbb{N}$, every $\psi_1(x,y), \ldots, \psi_n(x,y) \in \Phi(x,y)$ and every $m \in \mathbb{N}$ there are $a_1, a_2, a_3, b \in \mathfrak{C}$ such that

$$\max_{i=1,2,3}\{\max\{\psi_1(a_i,b),\ldots,\psi_n(a_i,b)\}\} < 1/m$$

and $\phi(a_1 \cdot a_2 \cdot a_3, b) > \epsilon$. This implies that the set

$$\{\psi(x_i,y) \le 1/m \ ; \ \psi(x,y) \in \Phi(x,y),\ i = 1,2,3,\ m \in \mathbb{N}\} \cup \{\phi(x_1 \cdot x_2 \cdot x_3, y) \ge \epsilon\}$$

is finitely satisfiable. By saturation there are $a_1, a_2, a_3, b \in \mathfrak{C}$ that satisfy all the conditions simultaneously. Therefore, $\Phi(a_i, b) = 0$, for $i = 1,2,3$. This set of conditions defines a subgroup, then $\Phi(a_1 \cdot a_2 \cdot a_3, b) = 0$. However, $\phi(a_1 \cdot a_2 \cdot a_3, b) \ge \epsilon$, a contradiction. □

Take $\epsilon = r/2$ and $\psi_1(x,y), \ldots, \psi_n(x,y), \delta$ as in the claim. We will show that $\max\{\psi_1(x,y), \ldots, \psi_n(x,y)\}$ has IP. For $I = \{i_1, \ldots, i_m\} \subset \omega$, define $a_I = a_{i_1} \cdots a_{i_m}$. If $i \notin I$ then $a_I \in H_i$. Hence,

$$\max\{\psi_1(a_I,b_i),\ldots,\psi_n(a_I,b_i)\} = 0.$$

In case $i = i_k \in I$, then $c_0 = a_{i_1} \cdots a_{i_{k-1}}$ and $c_1 = a_{i_{k+1}} \cdots a_{i_m}$ belong to H_i. Thus, $a_i = c_0^{-1} a_I c_1^{-1}$ and $\max\{\psi_1(c_0^{-1},b_i),\ldots,\psi_n(c_0^{-1},b_i)\} = \max\{\psi_1(c_1^{-1},b_i),\ldots,\psi_n(c_1^{-1},b_i)\} = 0$.

Claim 2: $\max\{\psi_1(a_I,b_i),\ldots,\psi_n(a_I,b_i)\} \ge \delta$.

Otherwise, $\psi_1(a_I,b_i) < \delta, \ldots, \psi_n(a_I,b_i) < \delta$ and by the Claim 1, $r = \phi(a_i,b_i) = \phi(c_0^{-1} a_I c_1^{-1}, b_i) \le r/2$. □

3. Randomizations

Given a first order language $\mathcal{L}$, the randomization language, denoted by $\mathcal{L}^R$, is a continuous, two-sorted language $(\mathbf{K}, \mathbf{B})$. The first sort, corresponding to random elements and the second one to events. The language consists of a map $[\![\varphi(\cdot)]\!] : \mathbf{K}^n \longrightarrow \mathbf{B}$ for each $\mathcal{L}$-formula φ with n free variables. It includes the Boolean operations $\top, \bot, \sqcup, \sqcap, \neg$ of sort $\mathbf{B}$ and a unary predicate μ in $\mathbf{B}$. We will review some basic properties of randomizations, for a complete presentation see [7] or [2].

Quantifiers in the $\mathcal{L}^R$-formulas must be understood in the approximate sense, i.e. $\forall x$ stands for $\sup x$ and $\exists x$ represents $\inf x$. The uppercase letters $X, Y, Z, \ldots$ will denote continuous variables of sort $\mathbf{K}$, while $A, B, C, \ldots$ will denote variables of sort $\mathbf{B}$. The lowercase letters $x, y, z, \ldots$ represent first order variables in the $\mathcal{L}$-formulas. We will also write $A = B$ instead of $d_{\mathbf{B}}(A,B) = 0$.

Definition 3.1. A *randomization* of $\mathfrak{M}$ is a pre-structure $(\mathcal{K}, \mathcal{B})$ in the language $\mathcal{L}^R$ satisfying:

1. $(\Omega, \mathcal{B}, \mu)$ is an atomless finitely additive probability space.
2. $\mathcal{K}$ is a set of functions from Ω to M.
3. For each $\mathcal{L}$-formula $\varphi(x)$ and each n-tuple f in $\mathcal{K}$,
$$[\![\varphi(f)]\!] = \{\omega \in \Omega \ ; \ \mathfrak{M} \models \varphi(f(\omega))\}.$$
4. $(\mathcal{K}, \mathcal{B}) \models \forall B \exists X \exists Y (B = [\![X = Y]\!])$.

   Namely, given $B \in \mathcal{B}$ and $\epsilon > 0$ there are $f, g \in \mathcal{K}$ such that $\mu([\![f = g]\!] \triangle B) < \epsilon$. So every event can be approximated by equality of functions.
5. For every $\mathcal{L}$-formula $\psi(x,y)$
$$(\mathcal{K}, \mathcal{B}) \models \forall Y \exists X ([\![\psi(X,Y)]\!] = [\![\exists x \psi(x,Y)]\!]).$$
   This means, that for any given tuple g in $\mathcal{K}^n$ and $\epsilon > 0$ there is $f \in \mathcal{K}$ such that $\mu([\![\psi(f,g)]\!] \triangle [\![\exists x \psi(x,g)]\!]) < \epsilon$. This guarantees the existence of approximate witnesses for the existential quantifier in the $\mathcal{L}$-formulas.
6. For $f, g \in \mathcal{K}$, $d_{\mathbf{K}}(f,g) = \mu[\![f \neq g]\!]$.
7. For $B, C \in \mathcal{B}$, $d_{\mathbf{B}}(B,C) = \mu(B \triangle C)$.

In this pre-structure $d_{\mathbf{K}}$ and $d_{\mathbf{B}}$ are pseudometrics. After taking the quotient and then the completion we obtain a continuous structure.

Definition 3.2. Let $([0,1],\mathcal{B}_\lambda,\lambda)$ be the usual Borel probability measure on $[0,1]$. If $\mathfrak{M}$ is a first order structure then $(\mathfrak{M}^{[0,1]},\mathcal{B}_\lambda)$ denotes the randomization whose universe in the sort $\mathbf{K}$ consists of the set of measurable functions from $[0,1]$ into M of countable range.

Definition 3.3. Let T^R be the common theory of all randomizations of models of T.

The following results are some important facts regarding the randomization theory T^R.

Theorem 3.4 (Theorem 2.1 in [7]). *T^R is complete.*

Theorem 3.5 (Theorem 2.3 in [7]). *If $(\mathcal{K},\mathcal{B})$ is a model of T^R then there is a model $\mathfrak{M}$ of T and a randomization $\mathfrak{M}^R$ of $\mathfrak{M}$ such that $(\mathcal{K},\mathcal{B})$ is isomorphic to $\mathfrak{M}^R$.*

Theorem 3.6 (Theorem 2.9 in [7]). *T^R admits strong quantifier elimination.*

An axiomatization of the randomization theory, T^R, is given in [7]. However, we will mention the following axiom which will be used throughout the paper.

Transfer Axiom: For every $\varphi \in T$

$$[\![\varphi]\!] = \top.$$

Theorem 3.7 (Theorem 2.7 in [7]). *Every model of T^R has perfect witnesses. This means that:*

- *For every $\mathcal{L}$-formula $\varphi(x,y)$ and every $g\in\mathcal{K}^{|y|}$ there is $f\in\mathcal{K}$ such that*
$$[\![\varphi(f,g)]\!] = [\![\exists x\varphi(x,g)]\!].$$
- *For every $B\in\mathcal{B}$ there are $f,g\in\mathcal{K}$ such that $B=[\![f=g]\!]$.*

Next, we introduce some properties that preserves the randomization.

Theorem 3.8 (Theorem 4.1 in [7]). *T is ω-stable if and only if T^R is ω-stable.*

Theorem 3.9 (Theorem 5.14 in [7]). *If T is stable then T^R is stable.*

Theorem 3.10 (Theorem 5.3 in [3]). *If T is an NIP theory then T^R is NIP too.*

The following construction will be useful later on. It allows one to expand the probability space by taking the product with the unit interval.

Lemma 3.11. *Let $(\mathcal{K},\mathcal{B})$ be a randomization of $\mathfrak{M}$, with probability space $(\Omega,\mathcal{B},\mu)$. Then there is $(\mathcal{K}_0,\mathcal{B}_0)$ an elementary extension of $(\mathcal{K},\mathcal{B})$, whose underlying probability space is $(\Omega\times[0,1],\mathcal{B}_0,\mu_0)$, where $\mathcal{B}_0$ is the product algebra and μ_0 the product measure (taking in $[0,1]$ the usual Borel algebra).*

Proof. Define $\mathcal{K}_0'$ as the set of functions $f:\Omega\times[0,1]\longrightarrow M$ such that there are $\{A_1,\dots,A_n\}$ partition of $[0,1]$ in measurable sets and $f_1,\dots,f_n\in\mathcal{K}$ satisfying $f(\omega,t)=f_i(\omega)$ for $(\omega,t)\in\Omega\times A_i$ and $i\le n$. We will show that $(\mathcal{K}_0',\mathcal{B}_0)$ is a randomization of $\mathfrak{M}$. So, the model will be the result of taking the quotient and then the completion. We will check the seven properties in Definition 3.1.

(1) $(\mathcal{B}_0,\mu_0)$ comes from $(\Omega\times[0,1],\mathcal{B}_0,\mu_0)$, which is an atomless finitely additive probability space.
(2) By definition $\mathcal{K}_0'\subseteq M^{\Omega\times[0,1]}$.
(3) Let $\psi(x)$ be an $\mathcal{L}$-formula and f a tuple in $\mathcal{K}_0'$. Then there are $A_1,\dots,A_m$ partition of $[0,1]$ and $g^1,\dots,g^m$ tuples in $\mathcal{K}$ such that $f(\omega,t)=g^i(\omega)$ for $(\omega,t)\in\Omega\times A_i$ e $i\le n$. Thus,
$$\begin{aligned}[\![\psi(f)]\!] &= \{(\omega,t)\in\Omega\times[0,1]\ ;\ \mathfrak{M}\vDash\psi(f(\omega,t))\}\\ &= \bigcup_{1\le i\le m}\{(\omega,t)\in\Omega\times A_i\ ;\ \mathfrak{M}\vDash\psi(f(\omega,t))\}\\ &= \bigcup_{1\le i\le m}[\![\psi(g^i)]\!]\times A_i.\end{aligned}$$
Clearly, this set belongs to $\mathcal{B}_0$.

(4) Take $B \in \mathcal{B}_0$ and $\epsilon > 0$. Then there are $B_1, \dots, B_n$ measurable subsets of Ω and $A_1, \dots, A_n$ measurable subsets of $[0,1]$ such that

$$\mu_0(B \,\triangle \bigcup_{1\le i\le n} B_i \times A_i) < \frac{\epsilon}{2}.$$

We can assume that $A_1, \dots, A_n$ are pairwise disjoint. Since $(\mathcal{K}, \mathcal{B})$ is a randomization there are $f_1, g_1, \dots, f_n, g_n \in \mathcal{K}$ satisfying $\mu(B_i \triangle [\![f_i = g_i]\!]) < \epsilon/(2n)$ for every $i \le n$. Next, take $f_0, g_0 \in \mathcal{K}$ such that $[\![f_0 \ne g_0]\!] = \top$ and define $f, g \in \mathcal{K}_0'$ as

$$f(\omega,t) = \begin{cases} f_i(\omega) & \text{if } t \in A_i \text{ for some } i \le n \\ f_0(\omega) & \text{if } t \in [0,1] \smallsetminus (A_1 \cup \cdots \cup A_n). \end{cases}$$

$$g(\omega,t) = \begin{cases} g_i(\omega) & \text{if } t \in A_i \text{ for some } i \le n \\ g_0(\omega) & \text{if } t \in [0,1] \smallsetminus (A_1 \cup \cdots \cup A_n). \end{cases}$$

We have that $[\![f = g]\!] = \bigcup_{1\le i\le n} [\![f_i = g_i]\!] \times A_i$. In this way,

$$\begin{aligned} \mu_0(B \triangle [\![f = g]\!]) &\le \mu_0(B \,\triangle \bigcup_{1\le i\le n} B_i \times A_i) + \mu_0(\bigcup_{1\le i\le n} B_i \times A_i \,\triangle \bigcup_{1\le i\le n} [\![f_i = g_i]\!] \times A_i) \\ &< \frac{\epsilon}{2} + \mu_0(\bigcup_{1\le i\le n} (B_i \triangle [\![f_i = g_i]\!]) \times A_i) \\ &\le \frac{\epsilon}{2} + \sum_{1\le i\le n} \mu_0((B_i \triangle [\![f_i = g_i]\!]) \times A_i) \\ &< \frac{\epsilon}{2} + n\left(\frac{\epsilon}{2n}\right) = \epsilon. \end{aligned}$$

(5) Let $\theta(x,y)$ be an $\mathcal{L}$-formula, $\epsilon > 0$ and let g be a tuple in $\mathcal{K}_0'$. There is a partition, $A_1, \dots, A_n$ of $[0,1]$ and tuples $g^1, \dots, g^n$ in $\mathcal{K}$ such that $g(w,t) = g^i(w)$ for $(\omega,t) \in \Omega \times A_i$ and $i \le n$. Since $(\mathcal{K}, \mathcal{B})$ is a randomization, for each $i \le n$, there is $f_i \in \mathcal{K}$ such that $\mu([\![\theta(f_i, g^i)]\!] \triangle [\![(\exists x)\theta(g^i)]\!]) < \epsilon/n$. Define $f \in \mathcal{K}_0'$ as $f(\omega,t) = f_i(\omega)$ for $(\omega,t) \in \Omega \times A_i$ with $i \le n$. Since,

$$[\![(\exists x)\theta(g)]\!] = \bigcup_{1\le i\le n} [\![(\exists x)\theta(g^i)]\!] \times A_i,$$

then

$$\begin{aligned} \mu_0([\![\theta(f,g)]\!] \triangle [\![(\exists x)\theta(g)]\!]) &= \mu_0(\bigcup_{1\le i\le n} [\![\theta(f_i,g^i)]\!] \times A_i \,\triangle \bigcup_{1\le i\le n} [\![(\exists x)\theta(g^i)]\!] \times A_i) \\ &\le \mu_0(\bigcup_{1\le i\le n} ([\![\theta(f_i,g^i)]\!] \triangle [\![(\exists x)\theta(g^i)]\!]) \times A_i) \\ &< n\left(\frac{\epsilon}{n}\right) = \epsilon. \end{aligned}$$

(6) and 7. It is immediate that $d_{\mathbf{K}}(f,g) = \mu_0[\![f \ne g]\!]$ and $d_{\mathbf{B}}(B,C) = \mu_0(B \triangle C)$ define pseudometrics.

Let $(\mathcal{K}_0, \mathcal{B}_0)$ the model obtained after taking the quotient and the completion. The elementary embedding of $(\mathcal{K}, \mathcal{B})$ into $(\mathcal{K}_0, \mathcal{B}_0)$ is given by: $B \longmapsto B \times [0,1]$ and $f \longmapsto f'$ where $f'(\omega,t) = f(\omega)$ for every $(\omega,t) \in \Omega \times [0,1]$. □

This extension of $(\mathcal{K}, \mathcal{B})$ will always be denoted by $(\mathcal{K}_0, \mathcal{B}_0)$ and we will refer to it as the *product extension*. Now we recall how definability works in the continuous setting.

Definition 3.12. Let $\mathfrak{U}$ be a continuous structure. A closed set $D \subseteq U^n$ is *definable* in $\mathfrak{U}$ over A if the distance predicate $\text{dist}(x,D) : U^n \longrightarrow [0,1]$ is definable in $\mathfrak{U}$ over A.

Let $f : U^n \longrightarrow U$ be a mapping. We say that f is *definable* in $\mathfrak{U}$ over A if the predicate $P : U^n \times U \longrightarrow [0,1]$ defined by

$$P(x,y) = d(f(x),y)$$

is definable in $\mathfrak{U}$ over A.

Lemma 3.13. *Let $D \subseteq M$ be $\varnothing$-definable. Then the set*

$$D^R = \{f \in \mathcal{K} \ ; \ \mu(\{\omega \in \Omega \ ; \ f(\omega) \in D\}) = 1\},$$

is an $\varnothing$-definable subset of $(\mathcal{K}, \mathcal{B})$. If D is definable with parameters $a_1, \dots, a_n$ then D^R is definable in every model of $Th(\mathfrak{M})^R$ that contains the constant functions $a_1, \dots, a_n$.

Proof. We will only proof the case with parameters. Let $a_1, \dots, a_n \in M$ be such that $D = \{b \in M \ ; \ \mathfrak{M} \models \varphi(b, a_1, \dots, a_n)\}$ for some $\mathcal{L}$-formula $\varphi(x, y_1, \dots, y_n)$. Take $(\mathcal{K}', \mathcal{B}')$ a model of $Th(\mathfrak{M})^R$ that contains the constant functions $a_1, \dots, a_n$. Let $\tilde{a}_1, \dots, \tilde{a}_n$ be such mappings. Then

$$D^R = \{f \in \mathcal{K} \ ; \ \mu[\![\varphi(f, \tilde{a}_1, \dots, \tilde{a}_n)]\!]) = 1\}.$$

Note that if D is not empty, then by perfect witnesses, D^R is not empty too. It is also true that

$$\operatorname{dist}(X, D^R) = 1 - \mu[\![\varphi(X, \tilde{a}_1, \dots, \tilde{a}_n)]\!].$$

Indeed, if $g \in \mathcal{K}'$ then for $f \in D^R$, $\{\omega \in \Omega \ ; \ f(\omega) \neq g(\omega)\} \supseteq \{\omega \in \Omega \ ; \ g(\omega) \notin D\}$. So, $d_{\mathbf{K}}(f, g) \geq 1 - \mu[\![\varphi(g, \tilde{a}_1, \dots, \tilde{a}_n)]\!]$. Therefore, $\operatorname{dist}(g, D^R) \geq 1 - \mu[\![\varphi(g, \tilde{a}_1, \dots, \tilde{a}_n)]\!]$. For the other side of the inequality, take $f \in \mathcal{K}'$ such that f agrees with g in $[\![\varphi(g, \tilde{a}_1, \dots, \tilde{a}_n)]\!]$ and agrees with some $f_0 \in D^R$ in $\neg[\![\varphi(g, \tilde{a}_1, \dots, \tilde{a}_n)]\!]$. Hence, $f \in D^R$ and $d_{\mathbf{K}}(f, g) = 1 - \mu[\![\varphi(g, \tilde{a}_1, \dots, \tilde{a}_n)]\!]$. □

4. Randomization of Definable Groups

Let L be a countable language, T a complete first order L-theory, $\mathfrak{M} \models T$ and G an $\varnothing$-definable group. Let $(\mathcal{K}, \mathcal{B})$ be a model of T^R which is κ-saturated for κ large enough. We know by Proposition 2.1.10 from [1] that $(\mathcal{K}, \mathcal{B})$ is isomorphic to a randomization of $\mathfrak{C}$, a κ-saturated model of T.

Proposition 4.1. *The set*

$$G^R = \{f \in \mathcal{K} \ ; \ \mu(\{\omega \in \Omega \ ; \ f(\omega) \in G\}) = 1\}$$

is a definable subgroup of $(\mathcal{K}, \mathcal{B})$ with the group operation defined pointwise i.e. for $f_1, f_2 \in G^R$, $f_1 \cdot f_2(\omega) = f_1(\omega) f_2(\omega)$.

Proof. By the Lemma 3.13, G^R is a definable subset of $\mathcal{K}$. In order to prove that the group operation is a definable map on G^R, first we must show that given $f_1, f_2 \in G^R$ the function $f_1 \cdot f_2$ defined in the statement belongs to $\mathcal{K}$. Let $\varphi(x)$ be the first order formula that defines G and let $\psi(x, y, z)$ be the formula that defines "$x \cdot y = z$". Then the sentence

$$\sigma : \forall x \forall y (\varphi(x) \wedge \varphi(y) \Rightarrow \exists z (\varphi(z) \wedge \psi(x, y, z) \wedge \forall w (\psi(x, y, w) \Rightarrow w = z)))$$

belongs to T. By the transfer axiom and the fact that $\mu[\![\varphi(f_1)]\!] = \mu[\![\varphi(f_2)]\!] = 1$ we obtain that there is a unique $f_3 \in \mathcal{K}$ such that $\mu[\![\varphi(f_3)]\!] = 1$ and $\mu[\![\psi(f_1, f_2, f_3)]\!] = 1$. This f_3 is the desired function. Finally, to show that $f_1 \cdot f_2$ is definable, we need to show that the distance to $f_1 \cdot f_2$ is a definable predicate. Indeed given $f_1, f_2, f \in G^R$ it is true that $d_{\mathbf{K}}(f_1 \cdot f_2, f) = 1 - \mu[\![\psi(f_1, f_2, f)]\!]$, which completes the proof. □

Proposition 4.2. *G^R is a topological group with the metric topology.*

Proof. Notice that if $d(f_1, f_1') < \epsilon/2$ and $d(f_2, f_2') < \epsilon/2$ then $d(f_1 \cdot f_1', f_2 \cdot f_2') < \epsilon$ and $d(f_1^{-1}, f_1'^{-1}) < \epsilon/2$. □

Now we present some group properties that are preserved by the randomization thanks to the transfer axiom.

Proposition 4.3. *If G is abelian then G^R is also abelian.*

Proof. Is an immediate consequence of the transfer axiom applied to the sentence

$$\forall x \forall y (xy = yx).$$

□

Proposition 4.4. *If G is divisible then G^R is divisible.*

Proof. Fix $n \geq 1$, then $G \models \forall x \exists y(n \cdot y = x)$. By the transfer axiom and perfect witnesses, for every $f \in G^R$ there is $g \in G^R$ such that $n \cdot g = f$. Since n was arbitrary, we get that G^R is divisible. □

Definition 4.5. A group G (classic or continuous) is said to be *definably nilpotent* if there is a sequence of normal definable subgroups $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that $[G, G_i] \subseteq G_{i-1}$ for $i = 1, \ldots, n$.

Proposition 4.6. *If G is definably nilpotent then G^R is definably nilpotent.*

Proof. Let $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ be a sequence of definable normal subgroups with $[G, G_i] \subseteq G_{i-1}$ for $i = 1, \ldots, n$. The groups G_i^R are definable subgroups. The transfer axiom applied to the sentence

$$\forall x \forall y((x \in G_i \wedge y \in G_{i-1}) \to x^{-1} y x \in G_{i-1}),$$

implies that G_{i-1}^R is a normal subgroup of G_i^R for $i = 1, \ldots, n$. Since $[G, G_i] \subseteq G_{i-1}$, then

$$G \models \forall x \forall y(x \in G_i \to x^{-1} y^{-1} x y \in G_{i-1}).$$

Applying the transfer axiom once again, we get that $[G^R, G_i^R] \subseteq G_{i-1}^R$ for $i = 1, \ldots, n$. □

Definition 4.7. A group G (classic or continuous) is said to be *definably solvable* if there is a sequence of normal definable subgroups $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that G_i / G_{i-1} is abelian for $i = 1, \ldots, n$.

Proposition 4.8. *If G is definably solvable then G^R is definably solvable.*

Proof. The proof is completely analogous to the nilpotent case, but considering the sentence

$$\forall x \forall y((x \in G_i \wedge y \in G_i) \to x^{-1} y^{-1} x y \in G_{i-1})$$

instead of $\forall x \forall y(x \in G_i \to x^{-1} y^{-1} x y \in G_{i-1})$. □

We will now consider a special kind of definable subgroups of G^R. As we will see later, some stabilizers will have this form.

Example 4.9. Fix a partition $\{A_i \ ; \ i \in \mathbb{N}\}$ of Ω satisfying $\mu(A_i) > 0$ for all $i \in \mathbb{N}$. Take $\{H_i \ ; \ i \in \mathbb{N}\}$ definable subgroups of G. The set

$$H = \{f \in G^R \ ; \ A_i \subseteq \{\omega \in \Omega \ ; \ f(\omega) \in H_i\}, \quad \text{for } i \in \mathbb{N}\}.$$

is a definable subgroup of G. Let $\varphi_i(x)$ define H_i for $i \in \mathbb{N}$, then the distance predicate is approximated by the predicates

$$\sigma_n(x) : 1 - \mu(A_1 \cap [\![\varphi_1(x)]\!]) - \cdots - \mu(A_n \cap [\![\varphi_n(x)]\!]).$$

Now suppose that H is a proper subgroup, we will show that the index of H in G^R is at least $2^{\aleph_0}$. Take $i \in \mathbb{N}$ such that $H_i \neq G$ (such index exists because $H \neq G^R$). Let $\{B_j \ ; \ j \in \mathbb{N}\}$ be a partition of A_i such that $\mu(B_j) > 0$ for every $j \in \mathbb{N}$. Since $[\![\exists x(x \in (G \setminus H_i))]\!] = \top$, we can find $f \in G^R$ such that $[\![f \in (G \setminus H_i)]\!] = \top$. Now, denote by S the set of functions that take value e outside A_i, and in each B_j take the constant value e or agrees with f. S has cardinality $2^{\aleph_0}$ and each of these functions is in a different coset of H in G^R, thus $[G : H] \geq 2^{\aleph_0}$.

Our next goal is to study the notion of connectedness in randomizations of definable groups.

Lemma 4.10. *If $(\Omega, \mathcal{B}, \mu)$ is a complete atomless measure algebra then there is a collection of measurable sets $\{A_t \ ; \ t \in [0,1]\}$ such that:*

- *$\mu(A_t) = t$ for $t \in [0,1]$,*
- *$A_s \subseteq A_t$ for $s < t$.*

Proof. First we construct the sets $\{A_r \ ; \ r \in \mathbb{Q} \cap [0,1]\}$. Fix $\{r_i \ ; \ i \in \mathbb{N}\}$ an enumeration of $\mathbb{Q} \cap [0,1]$, the construction will be done inductively. Let A_{r_0} be any set with $\mu(A_{r_0}) = r_0$. Suppose that the sets A_{r_j} are already defined for $j < i$, take A_{r_i} satisfying

$$\bigcup \{A_{r_j} \ ; \ j < i, \ r_j < r_i\} \subseteq A_{r_i} \subseteq \bigcap(\{A_{r_j} \ ; \ j < i, \ r_j > r_i\} \cup \{\Omega\})$$

and $\mu(A_{r_i}) = r_i$. This can be done since the algebra is atomless. Given the sets $\{A_r\ ;\ r \in \mathbb{Q} \cap [0,1]\}$ and given $t \in [0,1] \setminus \mathbb{Q}$ let $(r_n)_{n\in\mathbb{N}}$ be an increasing sequence of rationals converging to t and define $A_t = \bigcup_{n\in\mathbb{N}} A_{r_n}$. □

Proposition 4.11. *The group G^R has no proper definable subgroups of finite index.*

Proof. First we will prove that G^R is pathwise connected. Let $f_1, f_2 \in G^R$ and let $\{A_t\ ;\ t \in [0,1]\}$ be as in the previous lemma. For $t \in [0,1]$ define

$$h_t(\omega) = \begin{cases} f_1(\omega) & \text{if } \omega \in A_t \\ f_2(\omega) & \text{otherwise.} \end{cases}$$

We have that $h_0 = f_2$, $h_1 = f_1$ and the map $t \longmapsto h_t$ is continuous due to the fact that

$$\begin{aligned} d_{\mathbf{K}}(h_t, h_s) &= \mu[\![h_t \neq h_s]\!] \\ &= \mu(A_t \triangle A_s) \\ &= |t-s|. \end{aligned}$$

This shows that G^R is pathwise connected. Assume H is a definable subgroup of G^R of finite index. Since H is the zeroset of a definable predicate, it is H is closed. Also, each coset of H in G^R is closed, and G^R is covered by finitely many of these cosets. This implies that H is clopen. Hence, $H = G^R$. □

A definable group in a first order theory is said to be connected if it has no proper definable subgroups of finite index. However, this notion is not the right one in order to define connectedness in a continuous theory.

Definition 4.12. We say that a definable group in a continuous theory is *definably connected* if it has no proper type-definable subgroups of bounded index.

Our next goal is to show that G^R is definably connected for every ω-stable group G. We know by Theorem 3.8 that G^R is also ω-stable. The idea behind the proof is to find a type $p \in S_{G^R}(\mathcal{K})$ whose stabilizer is G^R and use Theorem 2.9 to conclude that the group es definably connected. For this purpose, we will use the generic types of G. Intuitively, for a function to be generic over G^R it is necessary that takes generic values. In addition, being a random function, it is natural to think that if we have n generic types the function should have measure $1/n$ on each of them. To guarantee independence from the elements of G^R this function will be constructed in the product extension.

Theorem 4.13. *If G is an ω-stable definable group then G^R is definably connected.*

Proof. Suppose that $[G : G^0] = n$ and let $p_1, \dots, p_n \in S(G)$ be a list of the generic types. Let $\eta_1(x, b_1), \dots, \eta_n(x, b_n)$ be the formulas that define the cosets of G^0 in G, where $b_1, \dots, b_n \in G$. Given $i \leq n$, take $\sigma_i \in S_n$ (the group of permutations of n elements) such that for all $g \in G$,

(**) $$\text{if } G \vDash \eta_i(g, b_i) \text{ then } gp_j = p_{\sigma_i(j)}.$$

Let $G^{\mathcal{U}}$ be an ultrapower of G that realizes the generic types. Take $g_1, \dots, g_n \in G^{\mathcal{U}}$ such that $g_i \vDash p_i$ for $1 \leq i \leq n$. Now, let $(\mathcal{K}', \mathcal{B}') = (\mathcal{K}, \mathcal{B})^{\mathcal{U}}$. The structure $(\mathcal{K}', \mathcal{B}')$ is an elementary extension of $(\mathcal{K}, \mathcal{B})$ and contains a copy of $G^{\mathcal{U}}$. Given $a \in G^{\mathcal{U}}$, let us denote by $\tilde{a}$ the corresponding element in $\mathcal{K}'$. Now, take $(\mathcal{K}'_0, \mathcal{B}'_0)$ the product extension, as in Lemma 3.11. For the sake of clarity, we will write $[\![\cdot]\!]_0$ for the events in $\mathcal{B}'_0$ and $[\![\cdot]\!]$ for the events in $\mathcal{B}'$. Since we can neglect null sets, we will work in $[0,1)$ instead of $[0,1]$.

Define $f \in \mathcal{K}'_0$ as follows:

$$f(\omega, t) = g_i \text{ for } (\omega, t) \in \Omega' \times \left[\frac{i-1}{n}, \frac{i}{n}\right) \text{ for } 1 \leq i \leq n.$$

Let $\mathbf{p} = tp(f/G^R)$, we will prove that $\operatorname{stab}(\mathbf{p}) = G^R$. It suffices to show that for any $h \in G^R$, $\varphi(x,y)$ an $\mathcal{L}$-formula and a tuple in $\mathcal{K}$

$$\mu[\![\varphi(f,a)]\!]_0 = \mu[\![\varphi(hf,a)]\!]_0.$$

We will need the following auxiliary construction. Given $\sigma \in S_n$ we define $T_\sigma : [0,1) \longrightarrow [0,1)$, a measure preserving transformation sending the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right)$ onto $\left[\frac{\sigma(i)-1}{n}, \frac{\sigma(i)}{n}\right)$ for each $i \leq n$. For $t = (k-1)\left(\frac{1}{n}\right) + s$ with $0 \leq s < \frac{1}{n}$ and $1 \leq k \leq n$, let $T_\sigma(t) = (\sigma(k)-1)\left(\frac{1}{n}\right) + s$. Defined in this manner, we also have that $T_{\sigma^{-1}} = (T_\sigma)^{-1}$.

Fix $h \in G^R$, $\varphi(x,y)$ an $\mathcal{L}$-formula and a tuple in $\mathcal{K}^n$. We will construct a measure preserving transformation

$$\hat{T} : [\![\varphi(hf,a)]\!]_0 \longrightarrow [\![\varphi(f,a)]\!]_0,$$

which completes the proof. We will use the following two facts.

Claim 1: The sets $[\![\eta_1(h,\tilde{b}_1)]\!], \ldots, [\![\eta_n(h,\tilde{b}_n)]\!]$ are a partition of Ω'.

Proof. We know that $G \vDash \forall x(\bigvee_{i\leq n} \eta_i(x,b_i))$. Therefore, $G^{\mathcal{U}}$ satisfies the same sentence and we have that $(\mathcal{K}',\mathcal{B}') \vDash [\![\forall x(\bigvee_{i\leq n} \eta_i(x,\tilde{b}_i))]\!] = \top$. In this way, $[\![\bigvee_{i\leq n} \eta_i(h,\tilde{b}_i)]\!] = \top$. Using a similar argument and the fact that for $i \neq j$, $G \vDash \neg\exists x(\eta_i(x,b_i) \wedge \eta_j(x,b_j))$ we conclude that $[\![\eta_i(h,\tilde{b}_1)]\!] \cap [\![\eta_j(h,\tilde{b}_1)]\!] = \bot$. □

Claim 2: For $i,j \leq n$, $[\![\eta_i(h,\tilde{b}_i)]\!] \cap [\![\varphi(h\tilde{g}_j,a)]\!] = [\![\eta_i(h,\tilde{b}_i)]\!] \cap [\![\varphi(\tilde{g}_{\sigma_i(j)},a)]\!]$.

Proof. Let $\psi(x,y,z) : \varphi(zx,y)$, then by (**), for all $i,j \leq n$,

$$G \vDash \forall y \forall z[(\eta_i(z,b_i) \wedge d_{p_j}x\varphi(x,y)) \leftrightarrow (\eta_i(z,b_i) \wedge d_{p_{\sigma_i(j)}}x\psi(x,y,z))].$$

By the transfer axiom,

$$(\mathcal{K}',\mathcal{B}') \vDash \forall Y \forall Z([\![(\eta_i(Z,\tilde{b}_i) \wedge d_{p_j}x\varphi(x,Y)) \leftrightarrow (\eta_i(Z,b_i) \wedge d_{p_{\sigma_i(j)}}x\psi(x,Y,Z))]\!] = \top).$$

Taking $Y = a$ and $Z = h$ we obtain the desired result. □

Given $(\omega,t) \in [\![\varphi(hf,a)]\!]_0$ define

$$\hat{T}(\omega,t) = (\omega, T_{\sigma_i}(t)) \text{ for } \omega \in [\![\eta_i(h,\tilde{b}_i)]\!] \text{ for } i \leq n.$$

- If $(\omega,t) \in [\![\varphi(hf,a)]\!]_0$ then $\hat{T}(\omega,t) \in [\![\varphi(f,a)]\!]_0$.

  Take $(\omega,t) \in [\![\varphi(hf,a)]\!]_0$. Suppose that $\omega \in [\![\eta_i(h,\tilde{b}_i)]\!]$ and that $\frac{j-1}{n} \leq t < \frac{j}{n}$ for some $j \leq n$. This means that $\omega \in [\![\eta_i(h,\tilde{b}_i)]\!] \cap [\![\varphi(h\tilde{g}_j,a)]\!]$, then by Claim 2, $\omega \in [\![\eta_i(h,\tilde{b}_i)]\!] \cap [\![\varphi(\tilde{g}_{\sigma_i(j)},a)]\!]$. Thus $(\omega, T_{\sigma_i}(t)) \in [\![\varphi(f,a)]\!]_0$.
- **$\hat{T}$ is one-to-one.**

  If $\hat{T}(\omega,t) = \hat{T}(\omega',t')$ then $\omega = \omega'$ and for some $i \in \{1,\ldots,n\}$ we have $\omega \in [\![\eta_i(h,\tilde{b}_i)]\!]$. Then $T_{\sigma_i^{-1}}(t) = T_{\sigma_i^{-1}}(t')$, hence $t = t'$.
- **$\hat{T}$ is surjective.**

  If $(\omega,t) \in [\![\varphi(f,a)]\!]_0$ with $\frac{j-1}{n} \leq t < \frac{j}{n}$ then $\omega \in [\![\eta_i(h,\tilde{b}_i)]\!] \cap [\![\varphi(\tilde{g}_j,a)]\!]$. Using again Claim 2, we obtain that $\omega \in [\![\eta_i(h,\tilde{b}_i)]\!] \cap [\![\varphi(h\tilde{g}_{\sigma_i^{-1}(j)},a)]\!]$. In this manner, $(\omega, T_{\sigma_i^{-1}}(t)) \in [\![\eta_i(h,\tilde{b}_i)]\!]_0 \cap [\![\varphi(hf,a)]\!]_0$ and $\hat{T}(\omega, T_{\sigma_i^{-1}}(t)) = (\omega,t)$.

- **$\hat{T}$ is measure preserving.**

  If we fix $1 \le i \le n$, $\hat{T}$ restricted to $[\![\eta_i(h,\tilde{b}_i)]\!]_0 \cap [\![\varphi(hf,a)]\!]_0$ has as image $[\![\eta_i(h,\tilde{b}_i)]\!]_0 \cap [\![\varphi(f,a)]\!]_0$ and corresponds to $Id_{\Omega'} \times T_{\sigma_i}$, a measure preserving transformation. Since the sets $[\![\eta_1(h,\tilde{b}_1)]\!], \dots, [\![\eta_n(h,\tilde{b}_n)]\!]$ are a partition of Ω', we conclude that $\hat{T}$ is measure preserving.

□

The following examples shows that for some types in an ω-stable theory their stabilizer is a definable subgroup of the form described in Example 4.9.

Example 4.14. Let T be the theory of vector spaces over $\mathbb{Q}$ and let V be a model of T of countable dimension and $W \succeq V$ of countable dimension over V. Consider $(V^{[0,1]}, \mathcal{B}_\lambda)$ and $(W^{[0,1]}, \mathcal{B}_\lambda)_0$. We can embed $(W^{[0,1]}, \mathcal{B}_\lambda)_0$ in a sufficiently saturated extension of $(V^{[0,1]}, \mathcal{B}_\lambda)$. We will study $\mathrm{stab}(p/V^{[0,1]})$ where p is realized in $(W^{[0,1]}, \mathcal{B}_\lambda)_0$.

Let us denote by χ_A the characteristic function of the set A and assume $f = \sum_{i,j\ge 0} w_{ij}\chi_{A_i\times B_j}$ where $\{A_i\ ;\ i \ge 0\}$ and $\{B_j\ ;\ j \ge 0\}$ are Borel partitions of $[0,1]$. Now let $g \in V^{[0,1]}$ and write $g = \sum_{k\ge 0} v_k\chi_{C_k}$, where as before $\{C_k\ ;\ k \ge 0\}$ are Borel and form a partition of $[0,1]$. We may also write $g = \sum_{k,i\ge 0} v_k\chi_{C_k\cap A_i}$. Assume $\mathrm{tp}(g + f/V^{[0,1]})$ $=\mathrm{tp}(f/V^{[0,1]})$ and consider the restriction to A_i (which belongs to $(V^{[0,1]}, \mathcal{B}_\lambda)$). Each A_i is definable from $V^{[0,1]}$. Then $\mu([\![f+g=v]\!] \cap A_i) = \mu([\![f=v]\!] \cap A_i)$ for every $v \in V$. Now, since V is countable then

$$\{(\omega,t) \in A_i \times [0,1]\ ;\ f(\omega,t) \in V\} = \bigcup_{v\in V} [\![f=v]\!] \cap (A_i \times [0,1])$$

is a Borel subset of $[0,1]\times[0,1]$. We study two cases.

Case 1. Assume that $\mu\{(\omega,t) \in A_i\times[0,1]\ ;\ f(\omega,t) \in V\} > 0$. We say such a set A_i is of type I. Now consider $s > 0$ and let $\{v_i\ ;\ i \le m\} \in V$ be the values on the range of f in $A_i \times [0,1]$ whose support has measure at least s. Then whenever $v \in V$ we have $\mathrm{tp}(\{v_i\ ;\ i \le m\}/V)$ $=\mathrm{tp}(\{v_i + v\ ;\ i \le m\}/V)$ if and only if $v = 0$. Thus we must have $g\restriction_{A_i} = 0$.

Case 2. Assume that $\mu\{(\omega,t) \in A_i \times [0,1]\ ;\ f(\omega,t) \in V\} = 0$ and call such sets of type II. Then the values of f resctricted to $A_i \times [0,1]$ belong to $W \smallsetminus V$ and for any $v \in V$ we have that $\mathrm{tp}(f(\omega,t)/V)$ $=\mathrm{tp}(f(\omega,t)+v/V)$. So for $(\omega,t) \in A_i\times[0,1]$ and any g, we have $\mathrm{tp}(f(\omega,t)/V)$ $=\mathrm{tp}(f(\omega,t)+g(\omega)/V)$.

We can conclude that $\mathrm{stab}(\mathrm{tp}(f/V^{[0,1]}) = \{g \in V^{[0,1]}\ ;\ g(\omega) = 0 \text{ for } \omega \in B\}$, where B is the union of the sets A_i of type I.

Example 4.15. Let T be the theory of vector spaces over $\mathbb{Q}$ expanded with a unary predicate for a vector subspace of infinite dimension and codimension. Repeat the previous analysis with (V,V_0) a countable model of T and (W,W_0) and elementary extension of countable dimension over (V,V_0). We will study $\mathrm{stab}(p/((V,V_0))^{[0,1]})$ where p is realized in $((W,W_0)^{[0,1]}, \mathcal{B}_\lambda)_0$.

Assume $f = \sum_{i,j\ge 0} w_{ij}\chi_{A_i\times B_j}$ where $\{A_i\ ;\ i \ge 0\}$ and $\{B_j\ ;\ j \ge 0\}$ are partitions of $[0,1]$ in terms of Borel sets. Now let $g \in (V,V_0)^{[0,1]}$ and write $g = \sum_{k,i\ge 0} v_k\chi_{C_k\cap A_i}$ where $\{C_k\ ;\ k \ge 0\}$ are Borel and form a partition of $[0,1]$. Suppose $\mathrm{tp}(g + f/(V,V_0)^{[0,1]})$ $=\mathrm{tp}(f/(V,V_0)^{[0,1]})$ and consider the restriction to A_i. Now, since V is countable then

$$\{(\omega,t) \in A_i \times [0,1]\ ;\ f(\omega,t) \in V\} = \bigcup_{v\in V} [\![f=v]\!] \cap (A_i \times [0,1])$$

is a Borel subset of $[0,1]\times[0,1]$. We have three cases.

Case 1. Assume that $\mu\{(\omega,t)\in A_i\times[0,1]\ ;\ f(\omega,t)\in V\}>0$. We say such a set A_i is of type I. If $s>0$ and $\{v_i\ ;\ i\le m\}\in V$ is the set of values on the range of f in $A_i\times[0,1]$ whose support has measure at least s. Then whenever $v\in V$ we have $\text{tp}(\{v_i\ ;\ i\le m\}/(V,V_0))=\text{tp}(\{v_i+v\ ;\ i\le m\}/(V,V_0))$ if and only if $v=0$. Thus we must have $g\restriction_{A_i}=0$.

Case 2. Assume that $\mu\{(\omega,t)\in A_i\times[0,1]\ ;\ f(\omega,t)\in V\}=0$ and $\mu(\llbracket f\in W_0\rrbracket\cap A_i)>0$ call such sets of type II. Then the values of f resctricted to $A_i\times[0,1]$ belong to $W\smallsetminus V$ and whenever $f(\omega,t)\in W_0$ we have that $\text{tp}(f(\omega,t)/(V,V_0))=\text{tp}(f(\omega,t)+v/(V,V_0))$ if and only if $v\in V_0$. Hence, $g\restriction_{A_i}$ should belong to V_0.

Case 3. If $\mu\{(\omega,t)\in A_i\times[0,1]\ ;\ f(\omega,t)\in V\}=0$ and $\mu(\llbracket f\in W_0\rrbracket\cap A_i)=0$ then we say A_i is of type III. We have that $\text{tp}(f(\omega,t)/(V,V_0))=\text{tp}(f(\omega,t)+v/(V,V_0))$ for any $v\in V$. Hence, for $(\omega,t)\in A_i\times[0,1]$ and any g, we have $\text{tp}(f(\omega,t)/(V,V_0))=\text{tp}(f(\omega,t)+g(\omega)/(V,V_0))$.

Therefore,

$$\text{stab}(\text{tp}(f/V^{[0,1]})=\{g\in V^{[0,1]}\ ;\ g(\omega)=0 \text{ for } \omega\in B \text{ and } g(\omega)\in V_0 \text{ for } \omega\in C\},$$

where B is the union of the sets A_i of type I and C is the union of the sets A_i of type II.

The proof of Theorem 4.13 can be generalized to the case of a stable group. In this case, we will not construct explicitly the element whose type over G^R is invariant. Instead, we will fix a finite set of formulas Δ, do a local argument for Δ-types and then construct a global invariant type.

Theorem 4.16. *If G is a stable definable group then G^R is definably connected.*

Proof. As in the proof of the previous proposition, let $G^{\mathcal{U}}$ be an ultrapower of G that realizes the generic types and let $(\mathcal{K}',\mathcal{B}')=(\mathcal{K},\mathcal{B})^{\mathcal{U}}$. For any given $a\in G^{\mathcal{U}}$ we denote the corresponding element of $\mathcal{K}'$ by $\tilde{a}$. Now, take $(\mathcal{K}_0',\mathcal{B}_0')$ the product extension.

Take $\varphi(x,y)$ an $\mathcal{L}$-formula and $\{p_1\restriction\varphi^*,\dots,p_n\restriction\varphi^*\}$ the generic types restricted to φ^* as in Proposition 2.6. For $i\le n$, let g_i be an element of $G^{\mathcal{U}}$ that realizes p_i. Define $f_\varphi\in\mathcal{K}_0'$ so that it agrees with $\tilde{g}_i$ in $\Omega'\times\left[\frac{i-1}{n},\frac{i}{n}\right)$ for $i\le n$. Define the set of formulas

$$\Sigma(X)=\{\mu\llbracket\varphi(X,a)\rrbracket=\mu\llbracket\varphi(f_\varphi,a)\rrbracket\ ;\ \varphi(x,y)\ \mathcal{L}\text{-formula},\ a\in\mathcal{K}^{|y|}\}.$$

First we will see that $\Sigma(X)$ is finitely satisfiable. In this way, by quantifier elimination it extends to a unique global type and then we will prove that the stabilizer of this type is G^R. The conclusion will follow from Theorem 2.9. Let Σ_0 be a finite subset of Σ. Without loss of generality we can assume that it is of the form $\{\mu\llbracket\varphi_1(X,a)\rrbracket=\mu\llbracket\varphi_1(f_{\varphi_1},a)\rrbracket,\dots,\mu\llbracket\varphi_n(X,a)\rrbracket=\mu\llbracket\varphi_n(f_{\varphi_n},a)\rrbracket\}$. Take $\Delta=\{\varphi_1(x,y),\dots,\varphi_n(x,y)\}$ and $\{p_1\restriction\Delta^*,\dots,p_m\restriction\Delta^*\}$ the restriction of the generic types to Δ^*. Let $g_j\in G^{\mathcal{U}}$ be such that $g_j\vDash p_j$ for $j\le m$. Define f_{Δ^*} in $\mathcal{K}_0'$ so that it agrees with $\tilde{g}_j$ in $\Omega'\times\left[\frac{j-1}{n},\frac{j}{n}\right)$ for $j\le m$. We will show that for $i\le n$, $\mu\llbracket\varphi_i(f_{\Delta^*},a)\rrbracket=\mu\llbracket\varphi_i(f_{\varphi_i},a)\rrbracket$. By Proposition 2.6 we have that

$$\begin{aligned}\mu\llbracket\varphi_i(f_{\Delta^*},a)\rrbracket&=\int\frac{|\{p\restriction\Delta^*\ ;\ p\in S_1(G)\text{ generic},\ \varphi(x,a(\omega))\in p\}|}{|\{p\restriction\Delta\ ;\ p\in S_1(G)\text{ generic}\}|}\,d\mu(\omega)\\&=\int\frac{|\{p\restriction\varphi^*\ ;\ p\in S_1(G)\text{ generic},\ \varphi(x,a(\omega))\in p\}|}{|\{p\restriction\varphi^*\ ;\ p\in S_1(G)\text{ generic}\}|}\,d\mu(\omega)\\&=\mu\llbracket\varphi_i(f_{\varphi_i},a)\rrbracket.\end{aligned}$$

This shows that Σ is finitely satisfiable and it determines a unique type $p\in S_{G^R}(\mathcal{K})$.

To show that the stabilizer of p is G^R, it is enough to see that $\mu\llbracket\varphi(hf_\varphi,a)\rrbracket=\mu\llbracket\varphi(f_\varphi,a)\rrbracket$ for every $\mathcal{L}$-formula $\varphi(x,y)$, $h\in G^R$ and $a\in\mathcal{K}^{|y|}$. Let $\{p_1\restriction\varphi^*,\dots,p_m\restriction\varphi^*\}$ be the generic types restricted to φ^*. Take $H=\bigcap_{i=1}^{m}\text{stab}(p_i,\varphi^*)$, then H is a definable subgroup of G of finite index. Note that if $c_1,c_2\in G$ belong to the same coset of H then $(c_1p_i)\restriction\varphi^*=(c_2p_i)\restriction\varphi^*$ far every $i \leq m$. Suppose that $\eta_1(x, b_1), \ldots, \eta_k(x, b_k)$ define the cosets of H in G. Now for each $i \leq k$ choose $\sigma_i \in S_n$ such that for every $g \in G$, if $G \vDash \eta_i(g, b_i)$ then $(gp_j) \upharpoonright \varphi^* = p_{\sigma_i(j)} \upharpoonright \varphi^*$ for every $j \leq m$. As in the previous proof, we can construct a measure preserving transformation $\hat{T} : [\![\varphi(hf_\varphi, a)]\!] \longrightarrow [\![\varphi(f_\varphi, a)]\!]$, which completes the proof. □

In these two proofs there is an essential component and it is the amenanability of the group G. If we fix $\omega \in \Omega'$, a formula $\varphi(x, y)$ and $a \in \mathcal{K}^{|y|}$, then the quotient

$$\frac{|\{p \upharpoonright \Delta^* \ ; \ p \in S_1(G) \text{ generic}, \ \varphi(x, a(\omega)) \in p\}|}{|\{p \upharpoonright \Delta \ ; \ p \in S_1(G) \text{ generic}\}|}$$

is the measure of $\varphi(x, a(\omega))$. In this way, $\mu[\![\varphi(f_\varphi, a)]\!]$ corresponds to integrating, as a function of ω, the measure of $\varphi(x, a(\omega))$. The fact that the measure is invariant by left translation guarantees that the stabilizer of the type is G^R. Having this in mind, we can generalize this result in order to find a global invariant type in the randomization of a particular class of NIP groups.

Lemma 4.17. *Let ν be a global M-invariant Keisler measure with $M \prec \mathfrak{C}$ small. Let $(\mathcal{K}', \mathcal{B}')$ be a model of T^R that contains $(\mathcal{K}, \mathcal{B})$ as an elementary substructure and that also contains the constant functions in M. Take $\varphi(x, y)$ an $\mathcal{L}$-formula and a a tuple in $\mathcal{K}$. Then the map*

$$\begin{aligned} r_{\varphi,a} : \Omega' &\longrightarrow [0,1] \\ \omega &\longmapsto \nu(\varphi(x, a(\omega))) \end{aligned}$$

is measurable.

Proof. Define the maps $\pi_a : \Omega' \longrightarrow S_n(M)$ and $\Theta_\varphi : S_n(M) \longrightarrow [0,1]$ by $\omega \longmapsto tp(a(\omega)/M)$ and $tp(b/M) \longmapsto \nu(\varphi(x, b))$ respectively. Note that Θ_φ is well-defined since ν is M-invariant. And by Proposition 2.16 it is also Borel. Now take c a tuple in M and $\psi(z, y)$ an $\mathcal{L}$-formula, we have that $(\pi_a)^{-1}([\psi(z, c)]) = [\![\psi(a, c)]\!]$. This shows that π_a is measurable. Since Θ_φ is Borel, $r_{\varphi,a} = \Theta_\varphi \circ \pi_a$ is also measurable. □

Proposition 4.18. *Let G be a definably amenable group in an NIP theory whose measure is M-invariant with $M \prec \mathfrak{C}$ small. Then there is a type $\mathbf{p} \in S_{G^R}(\mathcal{K})$ such that* $\text{stab}(\mathbf{p}) = G^R$.

Proof. Let $\mathfrak{C}^{\mathcal{U}}$ be an ultrapower of $\mathfrak{C}$ that realizes the generic types and let $(\mathcal{K}', \mathcal{B}') = (\mathcal{K}, \mathcal{B})^{\mathcal{U}}$. Given $a \in \mathfrak{C}^{\mathcal{U}}$, we denote the corresponding element in $\mathcal{K}'$ by $\tilde{a}$. Take $(\mathcal{K}_0', \mathcal{B}_0')$ the product extension. Given $\varphi(x, y)$ an $\mathcal{L}$-formula and a a tuple in $\mathcal{K}$ we define $\bar{r}_{\varphi,a} = \int r_{\varphi,a} \, d\mu$. We will prove that the set of $\mathcal{L}^R(\mathcal{K})$-conditions

$$\Sigma(X) = \{|\mu[\![\varphi(X, a)]\!] - \bar{r}_{\varphi,a}| \leq \epsilon \ ; \ \varphi(x, y) \ \mathcal{L}\text{-formula}, \ a \text{ a tuple in } \mathcal{K}, \ \epsilon > 0\}.$$

is finitely satisfiable in $(\mathcal{K}_0', \mathcal{B}_0')$. Take m conditions in $\Sigma(x)$, without loss of generality we can assume that they are of the form $|\mu[\![\varphi_k(x, a)]\!] - \bar{r}_{\varphi_k,a}| \leq \epsilon$ with $1 \leq k \leq m$. Given the formulas $\varphi_1(x, y), \ldots, \varphi_m(x, y)$ and this ϵ take $p_1, \ldots, p_n \in S_G(\mathfrak{C})$ as in Corollary 2.13. This means that for every tuple b in $\mathfrak{C}$ and every $k \leq m$

$$|\nu(\varphi_k(x, b)) - \text{Av}(p_1, \ldots, p_n; \varphi_k(x, b))| < \epsilon.$$

Let $g_1, \ldots, g_n \in \mathfrak{C}^{\mathcal{U}}$ be such that $g_i \vDash p_i$ for $1 \leq i \leq n$. Define $f \in \mathcal{K}_0'$ by

$$f(\omega, t) = g_i \text{ for } (\omega, t) \in \Omega' \times \left[\frac{i-1}{n}, \frac{i}{n}\right) \text{ with } 1 \leq i \leq n. \tag{***}$$

Fixing $k \leq m$ we obtain that

$$\begin{aligned} |\mu[\![\varphi_k(f, a)]\!] - \bar{r}_{\varphi_k,a}| &= \left| \int \lambda\{t \in [0,1] \ ; \ \mathfrak{C}' \vDash \varphi_k(f(\omega, t), a(\omega))\} \, d\mu - \int r_{\varphi_k,a} \, d\mu \right| \\ &\leq \int |\text{Av}(p_1, \ldots, p_n; \varphi_k(x, a(\omega))) - \nu(\varphi_k(x, a(\omega)))| \, d\mu \\ &< \epsilon. \end{aligned}$$

This shows that f realizes this set of conditions. In this way, $\Sigma(X)$ defines a partial type and by quantifier elimination it extends to a unique global type $\mathbf{p} \in S_{G^R}(\mathcal{K})$. We will prove that $\text{stab}(\mathbf{p}) = G^R$. Take $\hat{f}$ in some elementary extension of $(\mathcal{K}_0', \mathcal{B}_0')$ that realizes $\mathbf{p}$. It suffices to show that for every $\mathcal{L}$-formula $\varphi(x,y)$, tuple a in $\mathcal{K}$, $\epsilon > 0$ and $h \in G^R$

$$|\mu[\![\varphi(\hat{f},a)]\!] - \mu[\![\varphi(h\hat{f},a)]\!]| < \epsilon.$$

The conditions $\mu[\![\varphi(X,a)]\!] = \mu[\![\varphi(\hat{f},a)]\!]$ and $\mu[\![\varphi(hX,a)]\!] = \mu[\![\varphi(h\hat{f},a)]\!]$ belong to $\mathbf{p}$. Take $f \in \mathcal{K}_0'$ as in (***) so that $|\mu[\![\varphi(f,a)]\!] - \mu[\![\varphi(\hat{f},a)]\!]| < \epsilon/4$ and $|\mu[\![\varphi(hf,a)]\!] - \mu[\![\varphi(h\hat{f},a)]\!]| < \epsilon/4$. Using that G is definably amenable we obtain that $\nu(\varphi(x,a(\omega))) = \nu(\varphi(h(\omega)x,a(\omega)))$ for every $\omega \in \Omega'$. Hence,

$$\begin{aligned}
|\mu[\![\varphi(\hat{f},a)]\!] - \mu[\![\varphi(h\hat{f},a)]\!]| &\leq |\mu[\![\varphi(\hat{f},a)]\!] - \mu[\![\varphi(f,a)]\!]| + |\mu[\![\varphi(f,a)]\!] - \mu[\![\varphi(hf,a)]\!]| \\
&\quad + |\mu[\![\varphi(hf,a)]\!] - \mu[\![\varphi(h\hat{f},a)]\!]| \\
&< \frac{\epsilon}{2} + |\int \text{Av}(p_1,\dots,p_n;\varphi(x,a(\omega)))\, d\mu - \int \text{Av}(p_1,\dots,p_n;\varphi(hx,a(\omega)))\, d\mu| \\
&\leq \frac{\epsilon}{2} + \int |\text{Av}(p_1,\dots,p_n;\varphi(x,a(\omega))) - \text{Av}(p_1,\dots,p_n;\varphi(h(\omega)x,a(\omega)))|\, d\mu \\
&\leq \frac{\epsilon}{2} + \int |\nu(\varphi(x,a(\omega))) + \epsilon/4 - (\nu(\varphi(h(\omega)x,a(\omega))) - \epsilon/4)| d\mu \\
&= \epsilon.
\end{aligned}$$

□

For the next result we will need the notion of Lascar types.

Definition 4.19. Let a and b be two tuples in a continuous structure $\mathfrak{N}$ and $A \subseteq N$. We say that $a \sim_A b$ if there is an elementary extension $\mathfrak{N}_1$ of $\mathfrak{N}$ and an elementary substructure $\mathfrak{N}_0 \preceq \mathfrak{N}_1$ containing A such that $\text{tp}(a/N_0) = \text{tp}(b/N_0)$. Having the same *Lascar type over* A is the transitive closure of $\sim_A$, and in this case we write $\text{Lstp}(a/A) = \text{Lstp}(b/A)$.

From the definition follows immediately that types over models agree with Lascar types.

Proposition 4.20 (Fact 5.2 in [5])**.** *Having the same Lascar type over A is the finest bounded A-invariant equivalence relation.*

Corollary 4.21. *Let G be a definably amenable group in an NIP theory whose measure is M-invariant with $M \prec \mathfrak{C}$ small. Then G is definably connected.*

Proof. We will show that for any given type $\mathbf{p} \in S_{G^R}(\mathcal{K})$, we have $\text{stab}(\mathbf{p}) \subseteq G^{00}$. The result follows by taking $\mathbf{p}$ to be the global invariant type given by the previous proposition. Take $h \in \text{stab}(\mathbf{p})$ and let $\mathfrak{N} \preceq (\mathcal{K},\mathcal{B})$ be a small model containing h. If $g \models \mathbf{p}\restriction_{\mathfrak{N}}$ then also $hg \models \mathbf{p}\restriction_{\mathfrak{N}}$. Therefore, $\text{tp}(g/N) = \text{tp}(hg/N)$. This implies that $\text{Lstp}(g/N) = \text{Lstp}(hg/N)$. Since, the equivalence relation defined by $x \sim y$ if and only if $xy^{-1} \in G^{00}$ is bounded and N-invariant we conclude that $h = (hg)g^{-1} \in G^{00}$. □

The next example is the analogous to Example 4.14 for the NIP case.

Example 4.22. Let T be the theory of RCF and let $\tilde{\mathbb{R}}$ be a saturated extension of $\mathbb{R}$. As before, consider $(\mathbb{R}^{[0,1]}, \mathcal{B}_\lambda)$ and $(\tilde{\mathbb{R}}^{[0,1]}, \mathcal{B}_\lambda)_0$. We will study $\text{stab}(p/\mathbb{R}^{[0,1]})$ where p is realized in $(\tilde{\mathbb{R}}^{[0,1]}, \mathcal{B}_\lambda)_0$. Suppose $f = \sum_{i,j\geq 0} r_{ij}\chi_{A_i\times B_j}$ where $\{A_i\ ;\ i \geq 0\}$ and $\{B_j\ ;\ j \geq 0\}$ are Borel partitions of $[0,1]$. Take $g \in \mathbb{R}^{[0,1]}$ and write $g = \sum_{k,i\geq 0} s_k\chi_{C_k\cap A_i}$, where $\{C_k\ ;\ k \geq 0\}$ are Borel and form a partition of $[0,1]$. Assume $\text{tp}(g+f/\mathbb{R}^{[0,1]}) = \text{tp}(f/\mathbb{R}^{[0,1]})$ and consider the restriction to A_i. Now,

$$\{(\omega,t) \in A_i \times [0,1]\ ;\ f(\omega,t) \text{ is bounded}\} = \bigcup_{n\in\mathbb{N}} [\![-n < f < n]\!] \cap (A_i \times [0,1])$$

is a Borel subset of $[0,1]\times[0,1]$. We study two cases.

Case 1. Assume that $\mu\{(\omega,t) \in A_i \times [0,1]\ ;\ f(\omega,t)$ is bounded$\} > 0$. We say such a set A_i is of type I. Now consider $s > 0$ and let $\{r_i : i \leq m\} \in \tilde{\mathbb{R}}$ be the values on the range of f in $A_i \times [0,1]$ that are bounded and whose support has measure at least s. Then whenever $r \in \mathbb{R}$ we have $\text{tp}(\{r_i\ ;\ i \leq m\}/\mathbb{R}) = \text{tp}(\{r_i + r\ ;\ i \leq m\}/\mathbb{R})$ if and only if $r = 0$. Thus we must have $g \restriction_{A_i} = 0$.

Case 2. Assume that $\mu\{(\omega,t) \in A_i \times [0,1]\ ; f(\omega,t)$ is bounded$\} = 0$ and call such sets of type II. Then for any $r \in \mathbb{R}$ we have that $\text{tp}(f(\omega,t)/\mathbb{R}) = \text{tp}(f(\omega,t) + r/\mathbb{R})$. So for $(\omega,t) \in A_i \times [0,1]$ and any g, we have $\text{tp}(f(\omega,t)/\mathbb{R}) = \text{tp}(f(\omega,t) + g(\omega)/\mathbb{R})$.

We can conclude that $\text{stab}(\text{tp}(f/\mathbb{R}^{[0,1]}) = \{g \in \mathbb{R}^{[0,1]}\ ;\ g(\omega) = 0 \text{ for } \omega \in B\}$, where B is the union of the sets A_i of type I.

References

[1] Andrews, U., Goldbring, I. and Keisler, H. J. *Definable closure in randomizations*, Annals of Pure and Applied Logic 166 (2015), no. 3, pp. 325-342.

[2] Andrews, U. and Keisler, H. J. *Separable models of randomizations*, Journal of Symbolic Logic 80 (2015), no. 4, pp. 1149-1181.

[3] Ben Yaacov, I. *Continuous and Random Vapnik-Chervonenkis Classes*, Israel Journal of Mathematics 173 (2009), pp. 309-333.

[4] Ben Yaacov, I. *Stability and stable groups in continuous logic*, Journal of Symbolic Logic 75 (2010), no. 3, pp. 1111-1136.

[5] Ben Yaacov, I. *On theories of random variables*, Israel Journal of Mathematics 194 (2013), pp. 957-1012.

[6] Ben Yaacov, I., Berenstein, A., Henson, C. W. and Usvyatsov, A. *Model Theory for Metric Structures*, Model Theory with Applications to Algebra and Analysis, volume 2 (Chatzidakis, Z., Macpherson, D., Pillay, A. y Wilkie, A. eds.), London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315-427.

[7] Ben Yaacov, I. and Keisler, H. J. *Randomizations of Models as Metric Structures*, Confluentes Mathematici 1 (2009), no. 2, pp. 197-223.

[8] Buechler, S. *Essential Stability Theory*, Perspectives in Mathematical Logic, Springer-Verlag, 1996.

[9] Keisler, H. J., *Randomizing a model*, Advances in Mathematics 143 (1999), no. 1, pp. 124-158.

[10] Marker, D. *Model Theory: An Introduction*, Graduate Texts in Mathematics, Springer-Verlag, 2002.

[11] Poizat, B. *Stable groups*, Mathematical Surveys and Monographs, American Mathematical Society, 2001.

[12] Simon, P. *A guide to NIP theories*, Lecture Notes in Logic, Cambridge University Press, 2015.

[13] Wagner, F. *Stable groups*, London Mathematical society Lecture Note Series, Cambridge University Press, 1997.

Universidad de los Andes, Cra 1 No 18A-10, Bogotá, Colombia
URL: www.matematicas.uniandes.edu.co/~aberenst

Universidad de los Andes, Cra 1 No 18A-10, Bogotá, Colombia
E-mail address: jd.munoz@uniandes.edu.co