Notes on local o-minimality

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Abstract. We introduce and study some local versions of o-minimality, requiring that every definable set decomposes as the union of finitely many isolated points and intervals in a suitable neighborhood of every point. Motivating examples are the expansions of the order of reals by sine, cosine and other periodic functions.

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1 Introduction

O-minimal structures ([6], [3]) exclude certain "popular" and widely studied expansions of the reals, for instance those by trigonometric functions such as sine, cosine, and tangent. The obstruction is that these functions are periodical, which results in the definability of the integers (an infinite sequence of isolated points) in the corresponding structure. O-minimality is recovered if one restricts these functions to a suitable neighborhood of 0 [9], and in fact the expansion $\mathbf{R}_{an}$ of the real field by restricted analytic functions provides one of the most interesting examples of o-minimal structures.

One may wonder whether referring to "local" versions of o-minimality (that is, assuming o-minimality only in a suitable neighborhood of any given element) can ensure a reasonable model theoretic environment to approach sine and related functions.

These notes are devoted to discussing some of these local adaptations of o-minimality. We always deal with expansions $\mathcal{A}$ of dense linear orders $(A, \leq)$. In § 2 we introduce a weak notion of *local o-minimality* requiring that for every element $a \in A$, and every definable subset X of A, there is an interval around a on which X can be broken into a union of finitely many isolated points and intervals. A *stronger* form of local o-minimality is considered in § 3, dealing

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with structures $\mathcal{A}$ for which there is an interval around a where *every* definable set X decomposes in the way described above. We observe, that if $\mathcal{A}$ expands the ordered field of real numbers then these two notions coincide, but this is not true in general. We discuss both of these notions, listing in each case the elementary properties –such as Exchange Principle– they enjoy or lack. Actually one has to acknowledge that the positive features of these structures are not the dominant ones. In spite of this, it may be of interest to have a general picture of what occurs. It is natural to expect that strongly locally o-minimal structures exhibit better behaviour, and it turns out that this is the case, including for instance a correspondence between types and cuts, as well as a nice description of ω-categoricity (developed in § 4). The final sections of this paper are devoted to studying (strongly) locally o-minimal groups and rings.
We refer to the classical sources on o-minimality ([6] and [3]) and weak o-minimality ([4]). See also [2], [5] as general references about Model Theory.
The study of local o-minimality is part of the Ph.D. Thesis of the second author [10]. She would like to thank her Ph.D. advisor David Marker for introducing her to this notion, and for his helpful input during her initial investigation of the subject.
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2 Locally o-minimal structures

Definition 2.1 Let $\mathcal{A} = (A, \leq, \ldots)$ be a structure expanding a dense linear order without endpoints $(A, \leq)$. $\mathcal{A}$ is called *locally o-minimal* if and only if, for every $a \in A$ and every definable $X \subseteq A$, there is an interval I around a such that $X \cap I$ is a finite union of points and intervals.

Note that the previous definition makes sense even for expansions of arbitrary orderings. It is easily seen for instance, that an expansion of a discrete order without endpoints is locally o-minimal (for every element a just take the singleton $\{a\}$ itself as an open interval between the predecessor and the successor of a). So in this setting local o-minimality becomes trivial and loses its interest. Consequently we choose to restrict our attention to dense linear orders without endpoints.

Clearly, o-minimal structures are locally o-minimal. But even weakly o-minimal structures are so.

Proposition 2.2 *Every weakly o-minimal structure $\mathcal{A}$ is locally o-minimal.*

Proof. By definition of weak o-minimality, every subset of A definable in $\mathcal{A}$ is a finite union of convex subsets. Let $a \in A$ and X be a definable subset of A. We want to find an interval I containing a such that $X \cap I$ is a finite union of isolated points and intervals. We can assume that $a \in X$; otherwise we refer to $A - X$ and after finding for $a \notin X$, an interval I allowing a suitable decomposition of $I - X$ into points and intervals, we can deduce a similar representation of $X \cap I$.

Look to the left of a. Either there is some $\epsilon < a$ in A such that $]\epsilon, a] \cap X = \{a\}$, or for every $\epsilon < a$ in A there is some $c \in A$ such that $\epsilon < c < a$ and $c \in X$, in which case, due to weak o-minimality, there is some $\epsilon < a$ in A such that $]\epsilon, a] \subseteq X$. In conclusion, we can find $\epsilon < a$ in A such that either $]\epsilon, a] \cap X = \{a\}$ or $]\epsilon, a] \subseteq X$.
A similar interval can be found to the right of a. Let δ be its right endpoint and let $I =]\epsilon, \delta[$. Then $X \cap I$ is either $\{a\}$, or $]\epsilon, a]$, or $[a, \delta[$, or the whole I. ⊣

This provides many noteworthy examples of locally o-minimal structures, see [4], § 2.

Example 2.3 To get an expansion of the reals that is not locally o-minimal, just take $(\mathbf{R}, \leq, \mathbf{Q}, \ldots)$. As $\mathbf{Q}$ is dense and codense in $\mathbf{R}$, it overlaps every interval in $\mathbf{R}$ in a dense codense subset.

We will now provide a further characterization of local o-minimality.

Proposition 2.4 *A structure $\mathcal{A} = (A, \leq, \ldots)$ expanding a dense linear order without endpoints $(A, \leq)$ is locally o-minimal if and only if, for every $a \in A$ and every definable $X \subseteq A$, there are c, d in A such that $c < a < d$ and either $X \cap]c, d[$ or $]c, d[- X$ is equal to one of the following:*

(i) $\{a\}$,

(ii) $]c, a]$,

(iii) $[a, d[$,

(iv) the whole interval $]c, d[$.

Proof. ($\Leftarrow$) In every case $X \cap]c, d[$ is a finite union of isolated points and intervals, indeed either a singleton or the union of at most two (possibly half-closed) intervals.
($\Rightarrow$) Suppose that $\mathcal{A}$ is locally o-minimal. Take a, X and the corresponding interval I around a. First assume that a is in X, hence a is in the union of points and intervals decomposing X in $]c, d[$. If a is an isolated point in this union, then restrict I and get $c < a < d$ such that a is the only element of X between c and d. If a is in the interior of some interval of this union, then replace I by the interior of this interval. Finally, if a is the left or right endpoint of some interval, again restrict I to (ii) or (iii) respectively. If $a \notin X$, refer to $]c, d[- X$ instead of $X \cap]c, d[$. ⊣

Corollary 2.5 *Local o-minimality is preserved under elementary equivalence.*

Proof. Local o-minimality can be described as follows: For every formula $\varphi(v, \vec{w})$ of the language, *"for every a and $\vec{b}$ there are $c < a < d$ such that the elements x between c and d satisfying $\varphi(x, \vec{b})$ correspond to one of the 8 cases described in Proposition 2.4"*. ⊣

We will present some further examples of structures that are locally o-minimal but not weakly o-minimal. The following proposition provides useful criteria which can be applied to obtain such structures.

Proposition 2.6 *Let $\mathcal{A} = (A, \leq, \ldots)$ be a locally o-minimal structure, M a subset of A^2 such that, for every $a = (a_1, a_2) \in M$, $\{y_2 \in A : (a_1, y_2) \in M\}$ is definable in $\mathcal{A}$. Let $\mathcal{M}$ be the structure with domain M, totally ordered lexicographical by $\leq_{lex}$ (i.e., for every $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in M, $a \leq_{lex} b$ if and only if in $\mathcal{A}$ $a_1 \leq b_1$ and, if $a_1 = b_1$, then $a_2 \leq b_2$). Assume two further conditions on $\mathcal{M}$:*

1. *$\mathcal{M}$ is definable in $\mathcal{A}$, meaning that for every formula $\varphi(\vec{v})$ of the language of $\mathcal{M}$ there is a formula $\varphi'(\vec{v_1}, \vec{v_2})$ of the language of $\mathcal{A}$ such that, for every tuple $\vec{a}$ in M, $\mathcal{M} \models \varphi(\vec{a})$ if and only if $\mathcal{A} \models \varphi'(\vec{a_1}, \vec{a_2})$;*
2. *for every $a = (a_1, a_2) \in M$, there is an interval I around a_2 in A such that $\{a_1\} \times I \subseteq M$.*

Then $\mathcal{M}$ is locally o-minimal.

Note that Proposition 2.6 applies in particular to the case where M itself is definable in $\mathcal{A}$.

Proof. Let $X \subseteq M$ be definable in $\mathcal{M}$. Take $a \in X$, $a = (a_1, a_2)$. Then $\{y_2 \in A : (a_1, y_2) \in M\}$ is definable in $\mathcal{A}$, and by 1) $X' = \{y_2 \in A : (a_1, y_2) \in X\}$ is also definable in $\mathcal{A}$. By local o-minimality, there are $b_2 < c_2$ in A such that a_2 is in the (say open) interval $]b_2, c_2[$ and $]b_2, c_2[\cap X'$ can be written in $\mathcal{A}$ as a finite union $\cup_{j \leq k} I_j$ where each I_j ($j \leq k$) is a point or an interval. We refer here for simplicity to an open $]b_2, c_2[$, but it is easily seen that our argument works even with respect to closed, or half-closed intervals. By 2), we can assume that both $b = (a_1, b_2)$ and $c = (a_1, c_2)$ belong to M and that the whole interval $]b, c[$ is in M. For every $j \leq k$, put $I'_j = \{a_1\} \times I_j$. Then each I'_j is a point or an interval in $\mathcal{M}$ and $]b, c[\cap X = \cup_{j \leq k} I'_j$. In conclusion, $\mathcal{M}$ is locally o-minimal. $\dashv$

We will apply this criterion to prove the local o-minimality of a series of motivating examples, and most importantly of the expansion of the additive ordered group of reals $(\mathbf{R}, \leq, +)$ by the *sine* function; this was first observed by David Marker and Charles Steinhorn (personal communication).

Theorem 2.7 *(Marker-Steinhorn) The expansion of the additive ordered group of reals $(\mathbf{R}, \leq, +)$ by the* sine *function is locally o-minimal.*

Proof. The expansion of the real ordered field by the sine function restricted to the interval $[-\pi, \pi[$ is locally o-minimal, indeed o-minimal as a structure definable in the more general expansion of the real ordered field by restricted analytic functions, $\mathbf{R}_{an}$ [9]. Incidentally, observe that the ordered group of integers $(\mathbf{Z}, \leq, +)$ is also locally o-minimal provided that we enlarge our setting from the particular case of dense orders to discrete ones (however we will not need this fact here, and will simply refer to the ordered group of integers). Observe that $(\mathbf{R}, \leq, sin)$ can be viewed as the ordered sum of infinitely many copies of the interval $[-\pi, \pi[$, one copy $[-\pi + 2\pi n, \pi + 2\pi n[$ for every integer n. In other words, $(\mathbf{R}, \leq, sin)$ can be regarded as the direct product of the integers (more precisely of $(2\pi\mathbf{Z}, \leq)$ where sin acts as 0) and $[-\pi, \pi[$, with lexicographic ordering and the sine function restricted to $[-\pi, \pi[$. The elements of this structure can be identified with the pairs (n, x) where n is an integer and x a real number in $[-\pi, \pi[$; the correspondence with $\mathbf{R}$ being given by $(n, x) \to 2\pi n + x$ for every n and x. Thus we can apply Proposition 2.6 to $\mathbf{Z} \times [-\pi, \pi[$ viewed as a subset of the direct product of two copies of $(\mathbf{R}, \leq, +, sin)$ (where sin is meant restricted to $[-\pi, \pi[$), equipped with the structure given by $(n, x)+(n', x') = (n+n', x+x')$ if $-\pi \leq x+x' < \pi$, $(n, x)+(n', x') = (n+n'+1, x+x'-2\pi)$ if $x+x' \geq \pi$ and $(n, x)+(n', x') = (n+n'-1, x+x'+2\pi)$ if $x+x' < -\pi$, and of course $sin\ (n, x) = sin\ x$ for every choice of integers n, n' and real numbers x, x' in $[-\pi, \pi[$. Observe that $M = \mathbf{Z} \times [-\pi, \pi[$ satisfies the assumptions of Proposition 2.6 as a subset of the locally o-minimal structure $\mathcal{A} = (\mathbf{R}, \leq, +, sin)$ (where sine is restricted to $[-\pi, \pi[$). In fact, for every integer n, the set of the real numbers $x \in [-\pi, \pi[$ for which $(n, x) \in M$ is definable, is simply the interval $[-\pi, \pi[$. We have just seen that M satisfies Condition 1) in Proposition 2.6, and it is easily checked that 2) also holds. Thus the expansion of the additive group of reals by the sine function is locally o-minimal. $\dashv$

Observe that $(\mathbf{R}, \leq, +, sin)$ is not weakly o-minimal, as the integers form a definable subset consisting of infinitely many isolated points. Note also that if we involve multiplication we lose local o-minimality; in other words $(\mathbf{R}, \leq, +, \cdot, sin)$ is not locally o-minimal. In fact the integers are definable in this enlarged structure both as a set, via the formula $sin\, \pi v = 0$, and as a ring, via the restrictions of the addition and multiplication of $\mathbf{R}$; so even the rationals can be defined as the quotients $r \cdot s^{-1}$, where r and s range over the integers with $s \neq 0$. Therefore, due to what we observed in Example 2.3, the resulting structure cannot be locally o-minimal.
Additionally the argument in Theorem 2.7 can be used to produce further, similar locally o-minimal expansions of the ordered group of reals, for instance that by the cosine function, or by both sine and cosine, more generally by periodic analytic functions (admitting a common multiple of their periods).

We will once again apply Proposition 2.6 to obtain two more examples of locally o-minimal structures.

Examples 2.8 1. $(\mathbf{R}^n, \leq_{lex}, +, \cdot)$ with addition and multiplication defined componentwise and lexicographic ordering is locally o-minimal. This is clear when $n = 1$. For $n > 1$, just apply Proposition 2.6 suitably many times. Observe that the resulting structure is not weakly o-minimal when $n \geq 2$, because the set of tuples $\vec{a} = (a_1, \ldots, a_n)$ in $\mathbf{R}^n$ such that $a_2 = \ldots = a_n = 0$ is definable via $\vec{v} \cdot (1, 0, \ldots, 0) = \vec{v}$.

2. Consider the ordered group $(\mathbf{R}^{>0} \times \mathbf{R}, \leq_{lex}, *, (1, 0))$ where $\leq_{lex}$ denotes the lexicographic order and $*$ is defined in the following way: For a and a' positive reals and b, b' reals, $(a, b) * (a', b') = (a \cdot a', a \cdot b' + b)$.
Proposition 2.6 ensures that this ordered group is locally o-minimal because $\mathbf{R}^{>0}$, $\leq_{lex}$ and $*$ are definable in the o-minimal ordered real field,

and condition 2) is also easily satisfied.
However the resulting structure is not weakly o-minimal, because the set of ordered pairs $(a, 0)$ with a a positive real number is definable as the centralizer of $(2, 0)$.

Another direct consequence of Proposition 2.6 is the following:

Corollary 2.9 *The direct sum of finitely many locally o-minimal structures (ordered lexicographically by $\leq_{lex}$) is locally o-minimal.*

Note that this no longer holds if we refer to infinite direct sums.

Example 2.10 Let $\mathcal{R}$ be the expansion of the ordered field of reals by a 1-ary relation symbol P to be interpreted as the set of the roots of the polynomial $x \cdot (x^2 - 1)$, hence as $\{0, \pm 1\}$. So $\mathcal{R}$ is o-minimal. Consider the direct power $\mathcal{R}^{(\omega)}$ as an ordered structure with lexicographic ordering $\leq_{lex}$. Thus the elements of $\mathcal{R}^{(\omega)}$ form an ordered sequences of real numbers $r = (r_i)_{i \in \omega}$, such that $r_i = 0$ for every suitably large i. Furthermore, for r, s in $\mathbf{R}^{(\omega)}$, $r <_{lex} s$ means that if i is the least index such that $r_i \neq s_i$, then $r_i < s_i$. Addition and P are defined componentwise in $\mathcal{R}^{(\omega)}$. A given sequence r is in $P(\mathcal{R}^{(\omega)})$ if and only if $r_i \in P(\mathcal{R})$ for every $i \in \omega$. Take the sequence $0 = (0)_{i \in \omega}$ and any interval $]a, b[$ around it in $\mathcal{R}^{(\omega)}$. Without loss of generality one can assume

$$a = (0, 0, \ldots, a_i, 0, \ldots), \quad b = (0, 0, \ldots, b_i, 0, \ldots)$$

with $a_i = -1$ and $b_i = 1$ for some $i \in \omega$. $P(\mathcal{R}^{(\omega)})$ is a definable set but its intersection with $]a, b[$ consists of the elements

$$c = (0, 0, \ldots, c_i, c_{i+1}, c_{i+2}, \ldots)$$

where for all $j \geq i$ c_j is 0 or ± 1 and almost everywhere 0. So this intersection contains infinitely many points, but no intervals. Therefore $\mathcal{R}^{(\omega)}$ cannot be locally o-minimal.

On the other hand, the fact that weakly o-minimal structures are locally o-minimal, implies that local o-minimality inherits several negative features of the weakly o-minimal framework [4]; in particular we cannot expect a global monotonicity theorem, and prime models over subsets may sometimes not exist. One may wonder if locally o-minimal structures $\mathcal{A}$ satisfy a *local monotonicity* property, meaning that for any definable 1-ary function f of $\mathcal{A}$ and any point $a \in A$, there exists an interval I containing a that can be broken up into a finite union of points and intervals, on each of which f is monotone or constant. Recall that local monotonicity holds in weakly o-minimal structures [4]; though it sometimes fails in the wider framework of locally o-minimal structures.

Proposition 2.11 *There exist locally o-minimal structures without local monotonicity.*

This is proven by the following example.

Example 2.12 Consider the rational order $(\mathbf{Q}, \leq)$ with -1 and 0 as distinguished elements and the opposite function viewe as a map f from negative to positive rationals. Now surround every negative rational $a \neq -1$ by an "infinitesimal" neighbourhood, again isomorphic to the rational order. In this way the half line $\mathbf{Q}^{<-1}$ is replaced by the direct product $\mathbf{Q}^{<-1} \times \mathbf{Q}$ with the lexicographic order, and the same is true of the open interval of $\mathbf{Q}$ with endpoints -1 and 0. Extend f to this larger domain by assuming that, for every negative rational $a \neq -1$, f acts constantly on the neighbourhood around a (and takes it to $-a$). Let $\mathcal{A}$ be the structure obtained in this way. It is easily seen that $\mathcal{A}$ is locally o-minimal. Also, the function f is locally constant around every point $a < 0$ of A but -1. On the contrary, no interval around -1 can be broken into finitely many subintervals on which f is monotone or constant.
Observe that also the Exchange Principle fails in $\mathcal{A}$. In fact, if $a \in A$, $a < 0$ and $a \neq -1$, then a is not in the algebraic closure of the empty set, $f(a)$ is definable over a, but the converse is not true. $\dashv$

3 Strongly locally o-minimal structures

The conclusions of the previous section suggest that local o-minimality is too weak. Accordingly we explore here a more powerful variant.

Definition 3.1 An ordered structure $\mathcal{A}$ expanding a dense linear order without endpoints $(A, \leq)$ is said to be *strongly locally o-minimal* if and only if for every point $a \in A$ there exists an interval I around a, such that for every definable set X, $X \cap I$ is a finite union of intervals and points.

This means that the interval I endowed with the structure induced by $\mathcal{A}$ (that is, by taking the traces in I and its cartesian powers of the parametrically definable sets of $\mathcal{A}$ as the definable sets of I) is o-minimal. This easily implies that a strongly locally o-minimal structure $\mathcal{A}$ satisfies a weak form of local monotonicity, in the following sense.

Remark 3.2 As said, let $\mathcal{A}$ be any strongly locally o-minimal structure. Fix a point a in A and take an interval I such that every definable set intersects I in a finite union of intervals and points. Apply the proof of global monotonicity valid in the o-minimal setting [6] to I with its induced structure. Deduce that every function f from I to I definable in $\mathcal{A}$ has monotonicity in I.

However local monotonicity fails even in the strongly locally o-minimal framework. In fact, the structure $\mathcal{A}$ in Example 2.12 is strongly locally o-minimal.
Let us compare now local o-minimality and strong local o-minimality. It is clear that every strongly locally o-minimal structure is locally o-minimal. Actually the converse implication is also sometimes true.

Proposition 3.3 *If $\mathcal{A} = (A, \leq, \ldots)$ is locally o-minimal and for every $a \in A$ there exist $b, c \in A$ with $b < a < c$ such that the interval $[a, b]$ is compact with respect to the interval topology, then $\mathcal{A}$ is strongly locally o-minimal.*

Proof. Take any $a \in A$. Then there exists b, $c \in A$ with $b < a < c$ such that $[b, c]$ is compact. Choose any definable subset X of $\mathcal{A}$. We will show that $X \cap [b, c]$ is a finite union of points and intervals. For every x in $[b, c]$ take an interval I_x around x such that $X \cap I_x$ is a finite union of intervals and points. Then $\bigcup_{x \in [b,\, c]} I_x$ is an open cover of $[b, c]$. As $[b, c]$ is compact, there must exist a finite subcover $\bigcup_{j \in J} I_j$ of $\bigcup_{x \in [b,\, c]} I_x$ (for some finite index set J). As $X \cup I_j$ is a finite union of points and intervals for every j, it must be the case that $X \cap \bigcup_{j \in J} I_j$ is a finite union of points and intervals. $\dashv$

Corollary 3.4 *Every extension of* $(\mathbf{R}, \leq)$ *that is locally o-minimal, is strongly locally o-minimal.*

In fact any bounded interval in $\mathbf{R}$ is compact with respect to the interval topology. In particular, the expansion of the real order by addition and sine (as well as by other similar periodic analytic functions) is strongly locally o-minimal.

Remark 3.5 We will now emphasize another noteworthy property of the key motivating example of (strongly) locally o-minimal structures, i.e. $(\mathbf{R}, \leq, +, sin)$. This property strengthens the notion of strong local o-minimality itself. For simplicity let $\mathcal{R} = (\mathbf{R}, \leq, +, sin)$, and acl denote model theoretic algebraic closure. The additional condition $\mathcal{R}$ satisfies is the following:

(C) For every $a \in R$ there are b, $c \in acl(\emptyset)$ such that $b < a < c$ and $]b, c[$ intersects every definable subset X of $\mathbf{R}$ in finitely many isolated points and intervals.

In fact, take an integer n for which $\pi n < a < \pi(n + 2)$ and recall that $]\pi n, \pi(n+2)[$ with its induced structure is o-minimal. Observe that $\pi n \in acl(\emptyset)$ for every integer n. This is trivial when $n = 0$ (as 0 is the identity element of the group law), while, for n positive, πn can be defined as the n-th element $t > 0$ for which $sin\, t = 0$ and, for n negative, πn is the opposite of $\pi \cdot (-n)$ with respect to $+$.
Notice that (C) fails in arbitrary strongly locally o-minimal structures, see Example 2.12, or Example 3.9 below. On the other hand (C) implies strong local o-minimality.

If we enlarge our setting from expansions of the real order to arbitrary expansions of dense linear orderings without endpoints, then we encounter locally o-minimal non strongly locally o-minimal structures.

Proposition 3.6 *There exists a locally o-minimal structure that is not strongly locally o-minimal.*

This is witnessed by the following example.

Example 3.7 We consider a structure $\mathcal{A} = (A, \leq, \ldots)$ with a distinguished element 0 and 1-ary relations A_i, B_i, P_i and Q_i ($i \in \omega$). We require that the following conditions hold:

(i) $A = \cup_{i\in\omega}(A_i \cup B_i) \cup \{0\}$,

(ii) $A_i < A_{i+1}$ and $B_i > B_{i+1}$ for all $i \in \omega$,

(iii) $A_i < 0 < B_i$ for all $i \in \omega$,

(iv) for all $i \in \omega$ both A_i and B_i are isomorphic copies of $\mathbf{R}$ with the usual ordering,

(v) for all $i \in \omega$, P_i is a subset of A_i order isomorphic to the integers, and similarly Q_i is a subset of B_i again order isomorphic to the integers.

The definable sets in this structures are finite unions of copies of $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{R} - \mathbf{Z}$, intervals and points. In particular $\mathcal{A}$ is locally o-minimal.
To see that $\mathcal{A}$ is not strongly locally o-minimal take any interval around 0, say $]a, b[$ with $a \in A_i$ and $b \in B_j$ for some i and j in ω. Without loss of generality one can assume $i = j$. Then the definable set P_{i+1} is a copy of $\mathbf{Z}$ which intersects $]a, b[$ in infinitely many isolated points.

We will now alternate between describing basic abstract properties of strong local o-minimality, and introducing further examples that illustrate them. For instance, the structures $\mathcal{A}$ in the previous examples 2.12 and 3.7 are not weakly o-minimal. On the other hand there are weakly o-minimal structures which are not strongly locally o-minimal. In other words strong local o-minimality, unlike local o-minimality, is not implied by weak o-minimality.

Example 3.8 Consider the expansion of the ordered field of real algebraic numbers $\mathbf{R}_{alg}$ by a relation P for the subset $]-\pi, \pi[$. So P is a definable convex subset but is not an interval. It is known that this structure is weakly o-minimal, see [4], Proposition 2.1. Take a, ϵ in $\mathbf{R}_{alg}$, $0 < \epsilon < \frac{\pi}{2}$ and consider the interval $]a-\epsilon, a+\epsilon[$. Choose two rationals $r < r'$ such that $a-\epsilon < \pi+r < a < \pi+r' < a+\epsilon[$. Then the real algebraic numbers between $\pi+r$ and $\pi+r'$ form a convex subset of $]a-\epsilon, a+\epsilon[$, and in fact a definable subset, in particular the set of points x such that $x-r'$ is in P but $x-r$ is not. But X cannot be decomposed into a finite union of elements and intervals.

Observe that the strongly locally o-minimal version of Proposition 2.6 holds. In other words, if we replace in the statement of Proposition 2.6 *"locally o-minimal"* by *"strongly locally o-minimal"* everywhere, then the resulting statement is still true. Similarly, Corollary 2.9 remains valid if one replaces *"locally o-minimal"* by *"strongly locally o-minimal"* everywhere.
Furthermore, straightforward arguments using compactness and elementary chains show that every locally o-minimal structure $\mathcal{A}$ admits an elementary extension $\mathcal{A}'$, where for every a in A there are $c < a < d$ in A', such that for every subset X definable in $\mathcal{A}$, $X \cap]c, d[$ is as described in Proposition 2.4. Note that repeating this procedure cannot directly provide a strongly locally o-minimal elementary extension of $\mathcal{A}$. But it does suggest how to build a locally o-minimal non strongly locally o-minimal structure, and even shows that strong local o-minimality, unlike local o-minimality, is not preserved under elementary equivalence.

Example 3.9 Enlarge the ordered set of real algebraic numbers by countably many 1-ary relations P_n (n a positive integer), where for every n P_n is interpreted by the convex subset $]-\frac{\pi}{n}, \frac{\pi}{n}[$. This determines a sequence of nested convex subsets which are not intervals and whose intersection is the point 0. It is easily seen that this structure is not strongly locally o-minimal. Take $a = 0$ and observe that for every real algebraic $\epsilon > 0$, one can find n such that P_n is included in $]-\epsilon, \epsilon[$ which provides a convex subset that cannot be decomposed into a union of points and intervals. On the other hand, our structure is weakly o-minimal (as it is definable in the weakly o-minimal expansion of the ordered field of real algebraic numbers by P_1) and consequently locally o-minimal. Also, observe that a straightforward application of compactness provides an elementary extension where the intersection of the (interpretations of) P_n enlarges to a neighbourhood around 0. So one can define an open interval around 0 in this intersection. It follows that this elementary extension is strongly locally o-minimal.
Observe that the algebraic closure of $\emptyset$ is empty, so even in this strongly locally o-minimal elementary extension there is no interval around 0 with algebraic endpoints as in condition (C). In fact there is no interval with endpoints algebraic over 0 that works.

Corollary 3.10 *Strong local o-minimality is not preserved under elementary equivalence.*

Hence the following result is worthy of interest.

Theorem 3.11 *Every model of the first order theory of* $\mathcal{R} = (\mathbf{R}, \leq, +, sin)$ *is strongly locally o-minimal.*

Proof. Let us modify slightly what we observed in Remark 3.5 by adding to $\mathcal{R}$ two new $\emptyset$-definable 1-ary functions f and g as follows: for every $a \in \mathbf{R}$,

- ⋆ $f(a)$ is the maximal real $t < a$ such that $sin\, t = 0$ and sin is increasing around t,
- ⋆ $g(a)$ is the minimal real $t > a$ such that $sin\, t = 0$ and sin is increasing around t.

Then $f(a) < a < g(a)$. Also, $I(a) =]f(a), g(a)[$ is an interval of the form $]2\pi n, 2\pi(n+1)[$ for some integer n (hence of diameter 2π), unless a itself is in $2\pi\mathbf{Z}$, in which case $f(a) = a - 2\pi$, $g(a) = a + 2\pi$ and the diameter of $I(a)$ is 4π. For every a, $I(a)$ with the structure induced by $\mathcal{R}$ is o-minimal, which implies that for every formula $\varphi(v, \vec{w})$ there is a positive integer $k = k(\varphi, a)$ such that, for every tuple $\vec{m}$ in $\mathbf{R}$, $\varphi(\mathcal{R}, \vec{m}) \cap I(a)$ can be written as the union of at most k isolated points or intervals. This bound k depends a priori on φ but also on a. But for $a < a'$ in $\mathbf{R}$ – and $a, a' \notin 2\pi\mathbf{Z}$ for simplicity– the translation by $f(a') - f(a) = g(a') - g(a)$ determines a bijection between $I(a)$ and $I(a')$ taking the definable subsets of the former interval to the definable subsets of the latter, and conversely. Let

$$\varphi'(v, \vec{w}, z) \quad : \quad \exists u(\varphi(u, \vec{w}) \wedge v = u + z).$$

Then, for every $\vec{m}$ and q in $\mathbf{R}$, $\varphi'(\mathcal{R}, \vec{m}, q) = \varphi(\mathcal{R}, \vec{m}) + q$. In particular $\varphi(\mathcal{R}, \vec{m}) = \varphi'(\mathcal{R}, \vec{m}, 0)$. It follows that $k(\varphi, a) \leq k(\varphi', a)$ for every a. On the other hand it is easily seen that $k(\varphi', a) = k(\varphi', a')$ for every $a < a'$ in $\mathbf{R}$ –and a, a' out of $2\pi\mathbf{Z}$ for simplicity–. This provides a uniform bound $k(\varphi)$ (depending only on φ and hence valid for every $a \notin 2\pi\mathbf{Z}$) on the maximal number of isolated points and intervals necessary to decompose the various $\varphi(\mathcal{R}, \vec{m})$ in $I(a)$. It is easy to extend this bound to a $K(\varphi)$ which is also valid for $a \in 2\pi\mathbf{Z}$. In conclusion, $\mathcal{R}$ satisfies

- $\star$ $f(a) < a < g(a)$ for every a

and, for every formula $\varphi(v, \vec{w})$, the first order sentence saying that

- $\star$ for every a and $\vec{m}$, $\varphi(\mathcal{R}, \vec{m}) \cap I(a)$ can be written as the union of at most $K(\varphi)$ isolated points and intervals.

So these statements transfer to every model $\mathcal{A}$ of the first order theory of $\mathcal{R}$. Thus also $\mathcal{A}$ is strongly locally o-minimal. $\dashv$

Of course the previous theorem remains valid if *sin* is replaced or accompanied by the other elementary trigonometric (periodical) functions.

Example 3.12 Look at Example 2.6.3 in [4]. It provides an expansion $\mathcal{A}$ of the order of rationals by countably many equivalence relations E_n (n a natural number) such that the following conditions hold:

(i) E_0 has a unique equivalence class, that is, the whole domain of the structure,

(ii) for every natural number n, E_{n+1} refines the classes of E_n into infinitely many open convex classes that are again densely ordered without endpoints.

It is easily seen that one can arrange things in order to produce a structure $\mathcal{A}$ that is not strongly locally o-minimal. However, $\mathcal{A}$ is weakly o-minimal and hence locally o-minimal. Also, the first order theory of $\mathcal{A}$ does not have a prime model, as highlighted in [4].

Corollary 3.13 *Prime models over subsets may not exist in a first order theory of a strongly locally o-minimal structure.*

In conclusion, some crucial properties valid in the o-minimal case no longer hold in the strongly locally o-minimal setting. Nevertheless, an intriguing connection between types and cuts –resembling that of o-minimal models, see [6], § 3– can be established even for (certain) strongly locally o-minimal structures. We now give more details on this subject.

Remark 3.14 Let $\mathcal{A}$ be a strongly locally o-minimal structure. Suppose that $b < c$ are two elements in A and every definable subset of A intersects $]b, c[$ in finitely many points or intervals. Let $\Gamma(v)$ be a cut in $\mathcal{A}$ containing the formulas

$b < v$ and $v < c$. Then there is a unique 1-type over A extending $\Gamma(v)$.
This is due to the fact that for every definable subset X of A, both $]b, c[\cap X$ and $]b, c[-X$ can be expressed as unions of finitely many points and intervals, and of course, only one interval can be consistent with $\Gamma(v)$.

On the other hand, there do exist cuts in strongly locally o-minimal structures that do not satisfy the assumption in Remark 3.14, and can be enlarged to at least 2 1-types over A. Example 4.1 in the next section will confirm this.
Our results become sharper when we restrict our attention to strongly locally o-minimal expansions of the ordered reals. It turns out that strong local o-minimality has a nice characterization in this particular setting.

Proposition 3.15 *Let $\mathcal{A}$ be an expansion of $(\mathbf{R}, \leq)$. Then $\mathcal{A}$ is strongly locally o-minimal if and only if, for every cut $\Gamma(v)$ in $\mathcal{A}$ containing both a formula $b < v$ for some $b \in \mathbf{R}$ and a formula $v < c$ for some $c \in \mathbf{R}$, there is a unique 1-type over A extending $\Gamma(v)$.*

Proof. First assume $\mathcal{A}$ strongly locally o-minimal. Fix two reals b and c with $b < v < c \in \Gamma(v)$. For every $x \in [b, c]$, x is contained in a suitable open interval I_x intersecting every definable subset of A in the union of finitely many points and intervals. The I_x form an open cover of $[b, c]$ where x ranges over $[b, c]$. As $[b, c]$ is compact, a finite subcovering I_j ($j \leq t$, t a suitable positive integer) can be extracted. Then $]b, c[$ satisfies the assumption of Remark 3.14, which implies that $\Gamma(v)$ can be extended to a 1-type over A in a unique way.
To prove the converse, take any $b < c$ in $\mathbf{R}$. If cuts in $]b, c[$ extend uniquely to 1-types, then $]b, c[$ with the structure induced by $\mathcal{A}$ (i. e., assuming the traces of definable sets in $\mathcal{A}$ are the definable sets) is o-minimal. This shows that $\mathcal{A}$ is strongly locally o-minimal. $\dashv$

Continuing our investigation of strongly locally o-minimal expansions $\mathcal{A}$ of $(\mathbf{R}, \leq)$, it remains for us to show how the cuts corresponding to $\pm\infty$ extend to types.
Assume that $\mathcal{A}$ is not o-minimal. Then there exists a formula $\varphi(v)$ (possibly with parameters from $\mathbf{R}$) such that $\varphi(\mathcal{A})$ cannot be expressed as a union of k points and intervals for any positive integer k. But this decomposition is always possible after restricting to any interval $]b, c[$ with $b < c$. This means that both $\varphi(v)$ and its negation are consistent with the cut $\Gamma_{+\infty}(v) = \{v < c \, : \, c \in \mathbf{R}\}$ or with the cut $\Gamma_{-\infty}(v) = \{v > b \, : \, b \in \mathbf{R}\}$ (possibly with both). Hence $\Gamma_{+\infty}(v)$ or $\Gamma_{-\infty}(v)$ (possibly both) extend to at least 2 1-types over A.

Further assumptions on $\mathcal{A}$ suggested by the key motivating example $(\mathbf{R}, \leq, +, sin)$ –and, above all, condition (C) in Remark 3.5– imply some other noteworthy model theoretic properties in $\mathcal{A}$.

Proposition 3.16 *Let $\mathcal{A}$ be a (strongly) locally o-minimal structure satisfying (C). Then, for every subset B of A, the isolated types of the first theory of $(\mathcal{A}, b)_{b\in B}$ are dense.*

Proof. We have to show that for every formula $\varphi(v_1, \ldots, v_l)$ with parameters from B, if $\varphi(\mathcal{A}^l)$ is not empty, then there is some complete formula $\psi(v_1, \ldots, v_l)$ with parameters from B, such that $\psi(\mathcal{A}^l) \subseteq \varphi(\mathcal{A}^l)$. Arguing as in [6], Lemma 3.1, one sees that the crucial step in the proof is the case $l = 1$. So let us deal with this case. Take $a \in \varphi(\mathcal{A})$. By (C) there are b and c in $acl(\emptyset)$ such that $b < a < c$ and $]b, c[$ with the structure induced by $\mathcal{A}$ is o-minimal. Now proceed as in [6], Lemma 3.1, referring to $]b, c[\cap\varphi(\mathcal{A})$. $\dashv$

Proposition 3.17 *Let $\mathcal{A}$ be a (strongly) locally o-minimal structure satisfying (C). Then $\mathcal{A}$ satisfies the Exchange Principle: For every a, $a' \in A$ and finite subset B of A, if $a' \in acl(a, B) - acl(B)$, then $a \in acl(a', B)$.*

Proof. As (strong) local o-minimality and (C) are not affected by adding parameters, we can assume $B = \emptyset$. Refer to $b, c \in acl(\emptyset)$, $b < a < c$, $]b, c[$ o-minimal and proceed as in [6], Theorem 4.1. $\dashv$

Actually a weaker hypothesis on $\mathcal{A}$ is sufficient to ensure the last conclusion. Simply refer to b and c a'-definable rather than $\emptyset$-definable. In other words, assume that for every $a_1, a_2 \in A$, there are two $\emptyset$-definable functions f and g such that $f(a_1) < a_2 < g(a_1)$ and $]f(a_1), g(a_1)[$ is o-minimal with respect to the induced structure. Then replace b and c by $f(a')$ and $g(a')$ respectively.
Observe that the Exchange Principle may fail even in some strongly locally o-minimal elementary extensions of the structure $\mathcal{A}$ satisfying it. For instance, we know that $(\mathbf{R}, \leq, +, sin)$ fulfills condition (C), and so satisfies the Exchange Principle; also every model of the first order theory of this structure is strongly locally o-minimal. But take a real r such that $s = sin\, r$ is not in $acl(\emptyset)$. In a suitable elementary extension $\mathcal{A}'$ of $(\mathbf{R}, \leq, +, sin)$ one can find $t > \mathbf{R}$ such that $sin\, t = s$. Then $s \in acl(t) - acl(\emptyset)$ but $t \notin acl(s)$ (as s belongs to $\mathbf{R}$ and t does not).

4 ω-categorical structures

In this section we will investigate ω-categorical strongly locally o-minimal structures. Recall that ω-categorical o-minimal structures are classified in [6], Section 6, while ω-categorical weakly o-minimal structures are investigated in [1]. Here we obtain a nice, although partial, description in the strongly locally o-minimal setting. The following example will be useful to help illustrate what is going on.

Example 4.1 We will build an ω-categorical strongly locally o-minimal structure $\mathcal{A}$ that is not weakly o-minimal. To do so, consider an equivalence relation E on a dense linear order without endpoints $(A, \leq)$ such that

- each E-class is a isomorphic copy of $(\mathbf{Q}, \leq, -)$ where $-$ denotes opposite,
- each E-class is a convex subset of A,
- the quotient set of E (with respect to the order inherited by $\mathcal{A}$) is again a dense linear order without endpoints.

This ultimately provides a dense linear order without endpoints, expanded by the equivalence relation E, and a 1-ary function f admitting infinitely many fixed points (again ordered as the rationals), one in every E-class; on each E-class f is a mirror symmetry around this fixed point (just as $-$ is around 0 in $(\mathbf{Q}, \leq)$). It is easily seen that the structure $\mathcal{A}$ built in this way is strongly locally o-minimal; also, a back-and-forth argument shows that its theory is ω-categorical. However $\mathcal{A}$ is not weakly o-minimal, because the fixed points form an infinite definable non-convex set.
Observe that the equivalence relation E can be definably recovered from the remaining structure. In fact, let g denote the (definable) 1-ary function such that, for every a in A,

- if $f(a) = a$, then $g(a)$ is also a,
- if $f(a) > a$, then $g(a)$ is the least element $b > a$ satisfying $f(b) = b$,
- if $f(a) < a$, then $g(a)$ is the greatest element $b < a$ satisfying $f(b) = b$.

Then it is easily seen that two elements a, a' of A are equivalent in E if and only if $g(a) = g(a')$.
Let us refer to this example also to clarify what we claimed in the previous section, just after Remark 3.14, about types and cuts. Take any cut $\Gamma_0(v)$ in the quotient A/E (viewed as a copy of the order of rationals). Let $\Gamma_0(v)$ be of the form $B < v < C$ where $B \cup C$ is a partition of A/E and $B < C$.
Assume for simplicity that B has no maximum and C has no minimum. Form a cut $\Gamma(v)$ in A with the formulas $v > b$ for $E(b, \mathcal{A}) \in B$ and $v < c$ for $E(c, \mathcal{A}) \in C$. Observe that this cut extends to 3 different 1-types over A, according to whether $f(v) < v$, or $f(v) = v$, or $f(v) > v$ is satisfied.
When B has a maximum, or C has a minimum, we can also extend this cut to a new element in the E-class corresponding to this maximum or minimum.

With this example in mind, let us begin our analysis of strongly locally o-minimal ω-categorical structure.
For the moment we will work with an arbitrary structure $\mathcal{A}$ expanding a dense linear order without endpoints $(A, \leq)$. So $\mathcal{A}$ may not be ω-categorical or strongly locally o-minimal. Let L denote the language of $\mathcal{A}$. Define a binary relation E in $\mathcal{A}$ in the following way: For every x and y in A, $E(x, y)$ holds if and only if either $x = y$, or if I denotes the open interval with endpoints x and y, then for every formula $\varphi(v, \vec{w})$ of L there exists a positive integer k, such that for every tuple $\vec{a}$ in A, $\varphi(\mathcal{A}, \vec{a}) \cap I$ can be expressed as the union of at most k points or intervals of $\mathcal{A}$ (note that k is depending on $\varphi(v, \vec{w})$ but also on x and y).
In other words, given two elements $x < y$ in A, stating $E(x, y)$ is equivalent to saying that for every L-formula $\varphi(v, \vec{w})$ there exists some positive integer k such that the following first order condition $C(\varphi, k)$ holds :

$C(\varphi, k)$: *For every $\vec{a}$, $\varphi(\mathcal{A}, \vec{a}) \cap]x, y[$ can be written as the union of at most k points or intervals.*

This needs to be stated for all L-formulas $\varphi(v, \vec{w})$, where the tuple $\vec{w}$ and its length also vary.

Lemma 4.2 *E is an equivalence relation in $\mathcal{A}$, and every class of E is a convex set.*

Proof. Observe in particular, that for every $x < y < z$ in A, L-formula $\varphi(v, \vec{w})$ and tuple $\vec{a}$ in A, if both $]x, y[$ and $]y, z[$ intersect $\varphi(\mathcal{A}, \vec{a})$ in finitely many (and indeed boundedly many) points and intervals, then the same is true of $]x, z[$, and conversely. $\dashv$

Thus without any danger of confusion, for x and y in A one can write

$$E(x, \mathcal{A}) < E(y, \mathcal{A}) \iff x < y,$$

which also equips the quotient set A/E with a linear order relation.

Lemma 4.3 *Let $x < y$ be two elements in A equivalent with respect to E. Then $]x, y[$ with the structure induced by $\mathcal{A}$, is o-minimal.*

Proof. Just apply the definition of E. $\dashv$

Actually there are at least two ways a subset X of $\mathcal{A}$, and in particular an equivalence class of E, can inherit the structure of $\mathcal{A}$. The former is the one we already considered above, that is to take as definable sets of X the traces in X of parametrically definable sets of $\mathcal{A}$ (possibly involving parameters out of X); this will be called the structure induced by $\mathcal{A}$ on X. The latter is to take the traces in X of $\emptyset$-definable sets of $\mathcal{A}$ as relations in X; the resulting structure will be called the *pure* structure induced by $\mathcal{A}$ on X. Observe that X, if infinite, cannot be ω-categorical with respect to the (full) structure induced by $\mathcal{A}$; in fact, no infinite structure remains ω-categorical after naming its elements.

Remark 4.4 Observe that the conclusion of Lemma 4.3 may fail when we refer to E-classes, even if we work in a strongly locally o-minimal structure. For example, look at $\mathcal{R} = (\mathbf{R}, \leq, +, sin)$. This is a strongly locally o-minimal structure on which E acts trivially, defining a unique equivalence class coinciding with the whole domain $\mathbf{R}$ and then definable in a trivial way (but not as an interval with real endpoints). Moreover $\mathcal{R}$ is not o-minimal.

But things change if we assume ω-categoricity in addition to strong local o-minimality.

Theorem 4.5 *Let $\mathcal{A}$ be a strongly locally o-minimal ω-categorical structure. Assume that the language L of $\mathcal{A}$ is finite. Then the following statements hold.*

1. *E is $\emptyset$-definable in $\mathcal{A}$.*
2. *Every equivalence class of E in $\mathcal{A}$ is infinite (so a dense linear order); furthermore the class is o-minimal with respect to the structure induced by $\mathcal{A}$, and ω-categorical with respect to the pure structure induced by $\mathcal{A}$.*

3. *When x ranges over A there are only finitely many types of E-classes $E(x, \mathcal{A})$ (viewed as pure first order structures induced by $\mathcal{A}$) up to elementary equivalence.*

4. *The quotient set A/E, regarded as a coloured totally ordered set with finitely many colours with respect to $\leq$, and a colour for every type of E-class (viewed as pure structures) up to elementary equivalence, is ω-categorical.*

5. *Every model of the theory of $\mathcal{A}$ is strongly locally o-minimal.*

Proof. 1) By ω-categoricity for every formula $\varphi(v, \vec{w})$ there is a positive integer $k(\varphi)$ (depending only on φ) such that for every $\vec{a}$ and $x < y$ in A, if $E(x, y)$ holds then $C(\varphi, k(\varphi))$ also holds. Otherwise for every positive integer k one can find $\vec{a}$, x and y such that $x < y$, and $E(x, y)$ – consequently $\varphi(\mathcal{A}, \vec{a}) \cap]x, y[$, and $\varphi(\mathcal{A}, \vec{m}) \cap]x, y[$ for all tuples $\vec{m}$ of A, can be written as the union of boundedly many isolated points and intervals – but in the case of $\vec{a}$ this decomposition requires more that k pieces. This eventually produces infinitely many types of $(l + 2)$-tuples $(x, y, \vec{a})$ over $\emptyset$; each type corresponding to a positive integer h, such that for every $\vec{m}$, $\varphi(\mathcal{A}, \vec{m}) \cap]x, y[$ is the union of at most h isolated points or intervals, but $\vec{a}$ needs exactly h points or intervals. Furthermore the sequence of the h is increasing and unbounded. This contradicts ω-categoricity.
Again by ω-categoricity there must be a finite set Φ of formulas $\varphi(v, \vec{w})$ such that for $x < y$ in A, $E(x, y)$ holds if and only if x and y satisfy $C(\varphi, k(\varphi))$ for every formula φ in Φ. Otherwise, for every finite Φ one can find $x < y$ in A satisfying $C(\varphi, k(\varphi))$ for every φ in Φ but not equivalent with respect to E, so failing to satisfy $C(\psi, k(\psi))$ for some further formula ψ. This ultimately yields infinitely many 2-types over $\emptyset$ and contradicts ω-categoricity. So in conclusion, E is $\emptyset$-definable as claimed.
2) Let $x \in A$. Due to strong local o-minimality there is an interval $]b, c[$ around x in A, such that for every L-formula $\varphi(v, \vec{w})$, and tuple $\vec{a}$ from A, $\varphi(\mathcal{A}, \vec{a}) \cap]b, c[$ can be decomposed as the union of at most k points and intervals for some positive integer k. Let l denote the length of $\vec{w}$. As in 1) we can apply ω-categoricity and uniformly bound k when $\vec{a}$ ranges over A^l (otherwise, for every k, one can find $\vec{a}$ for which $\varphi(\mathcal{A}, \vec{a}) \cap]b, c[$ consists of finitely many points and intervals, but not less than $k+1$ points and intervals, and this eventually provides infinitely many l-types over $\{b, c\}$).
Now choose b', c' in A with $b < b' < x < c' < c$ and observe that the whole interval $]b', c'[$ is included in the equivalence class of x with respect to E.
The fact that the E-class of x, endowed with the structure induced by $\mathcal{A}$ is o-minimal, again depends on the hypotheses that $\mathcal{A}$ is strongly local o-minimal and ω-categorical. In fact, due to what we have just shown, there is an interval $]b', c'[\subseteq E(x, \mathcal{A})$ around x such that for every L-formula $\varphi(v, \vec{w})$ and tuple $\vec{a}$ in A, $\varphi(\mathcal{A}, \vec{a}) \cap]b', c'[$ is the union of at most $k(\varphi)$ points and intervals for a suitable common bound $k(\varphi)$. Even if we enlarge this interval $]b', c'[$ within $E(x, \mathcal{A})$, this value $k(\varphi)$ remains the same.
This implies that $k(\varphi)$ itself, or possibly $k(\varphi) + 1$ or $k(\varphi) + 2$ works as a bound

for the whole class $E(x, \mathcal{A})$. Let us explain why.
Choose $\varphi(v, \vec{w})$, $\vec{a}$ and $b' < c'$ in $E(x, \mathcal{A})$ such that $\varphi(\mathcal{A}, \vec{a}) \cap]b', c'[$ is the union of $k(\varphi)$, and not less than $k(\varphi)$ points and intervals (without loss of generality we can arrange things to obtain our result just for $k(\varphi)$). Fix such a decomposition of $\varphi(\mathcal{A}, \vec{a}) \cap]b', c'[$. Put

$$B' = \{r \in E(x, \mathcal{A}) : r \leq b'\}, \quad C' = \{r \in E(x, \mathcal{A}) : r \geq c'\}.$$

Let us check what happens when we involve B' in our considerations; C' can be treated in a similar way. Let I denote the leftmost connected component of the given decomposition of $\varphi(\mathcal{A}, \vec{a}) \cap]b', c'[$.
If I is a singleton or an interval whose left endpoint is $> b'$, then no element $r \in B'$ can satisfy $\varphi(v, \vec{a})$ unless r is the possible minimum of B'. Otherwise, if $r \in B$ is not this minimum and $s \in B'$ is any element $< r$, then $]s, c'[$ contradicts the choice of $k(\varphi)$. But also in the case when B' has a minimum, and r is this minimum, it is easily seen that $k(\varphi) + 1$ works as a bound in $E(x, \mathcal{A})$ instead of $k(\varphi)$.
Thus assume that I is an interval with b' as a left endpoint.
If B' is included in, or disjoint from $\varphi(\mathcal{A}, \vec{a})$ then we are done, as $k(\varphi)$ is a bound even in $E(x, \mathcal{A})$. If the only point of B' satisfying $\varphi(v, \vec{a})$ is the possible minimum of B', then $k(\varphi) + 1$ works.
If B' can be decomposed as the union of two non-empty sets $I_0 < I_1$, where I_1 is an interval extending I to the left and included in $\varphi(\mathcal{A}, \vec{a})$, and I_0 is disjoint from $\varphi(\mathcal{A}, \vec{a})$, then again we are done. If B' has a similar decomposition $I_0 \cup I_1$ where I_1 is a convex set, but not an interval, then for r any element in I_0 $]r, c'[$ contradicts the definition of E.
In the remaining cases one finds in B' three points $s < r_0 < r_1$ with $r_0 \in \varphi(\mathcal{A}, \vec{a})$ and $r_1 \notin \varphi(\mathcal{A}, \vec{a})$. But then $]s, c'[$ contradicts our assumptions.
This proves that the E-class of x with the structure induced by $\mathcal{A}$ is o-minimal. Of course this conclusion applies even to the pure structure induced by $\mathcal{A}$. But the E-class of x is also ω-categorical with respect to this pure structure, which is $\emptyset$-definable in the ω-categorical structure $\mathcal{A}$ (in the finite language L). This means, by [6], Theorem 6.1, that there exists a finite subset of elements $c_1, \ldots, c_m$ in $E(x, \mathcal{A})$ dividing it in intervals $I_j =]c_j, c_{j+1}[$ with $1 \leq j < m$ and possibly $I_0 =]-\infty, c_1[$, $I_m =]c_m, +\infty[$ (where $\pm\infty$ refer to $E(x, \mathcal{A})$), and an equivalence relation R among the j with $0 \leq j \leq m$, such that for each $(j, h) \in R$ there is a unique definable monotone bijection $f_{j,h}$ between I_j and I_h, so that $f_{j,j}$ is the identity of I_j for every j and $f_{j,q} = f_{h,q} f_{j,h}$ for all (h, q), $(j, h) \in R$. Also, as L is finite, the theory of $E(x, \mathcal{A})$, viewed as a pure structure induced by $\mathcal{A}$ is finitely axiomatizable (see [6], Corollary 6.2).
3) Otherwise infinitely many 1-types arise over $\emptyset$ in $\mathcal{A}$, and this contradicts the ω-categoricity of $\mathcal{A}$.
4) It is clear that A/E is interpretable in $\mathcal{A}$ without parameters as the quotient set of the $\emptyset$-definable equivalence relation E. Also its order and its colours can be first order defined in $\mathcal{A}$ without using any parameters. This implies that A/E, regarded as a coloured linearly ordered set is ω-categorical as claimed.
5) A consequence of what we have seen so far is that for every formula $\varphi(v, \vec{w})$ of

L, $\mathcal{A}$ satisfies the first order L-sentence stating that for every x, there exist b, c such that $b < x < c$ and for every $\vec{a}$ $\varphi(\mathcal{A}, \vec{a}) \cap]b, c[$ can be written as the union of at most $k(\varphi)$ points and intervals for some suitable $k(\varphi)$. Then the same holds in every structure $\mathcal{B}$ elementarily equivalent to $\mathcal{A}$. With this in mind, let us show that $\mathcal{B}$ is indeed strongly locally o-minimal. We proceed by contradiction. Let $a \in B$ witness that strong local o-minimality fails. Take a formula $\varphi_0(v, \vec{w_0})$ of L. Then there exist $b_0 < a < c_0$ in B such that for every $\vec{a_0}$, $\varphi_0(\mathcal{B}, \vec{a_0}) \cap]b_0, c_0[$ is the union of at most $k(\varphi_0)$ points or intervals. As a contradicts strong local o-minimality, the same interval $]b_0, c_0[$ cannot work for every formula of L, and so there are $\varphi_1(v, \vec{w_1})$ and $\vec{a_1}$ such that $\varphi_1(\mathcal{B}, \vec{a_1}) \cap]b_0, c_0[$ cannot be expressed as the union of at most $k(\varphi_1)$ points and intervals. However a smaller interval $]b_1, c_1[$ around a can satisfy this further condition (and still preserve the previous one). Repeating this procedure gives pairs (b_n, c_n) (with n ranging over the naturals numbers) with pairwise different 2-types over a. But this contradicts the ω-categoricity of $\mathcal{B}$ and $\mathcal{A}$, and hence proves strong local o-minimality. $\dashv$

Remarks 4.6 (i) For a classification of ω-categorical coloured linear orderings (possibly with infinitely many colours) see [7] and [8].

(ii) The points dividing a given E-class in the way described at the end of 2) can arise as interpretations of distinguished constants of L, but also as images of locally constant $\emptyset$-definable functions. For instance, in Example 4.1, the function f is symmetric on each E-class around a fixed point x, and x is not a constant in the language of the whole structure $\mathcal{A}$, but is the image of the $\emptyset$-definable function g, which is constant in the class.

(iii) Observe that Theorem 4.5 neglects the possible additional interactions $\mathcal{A}$ can establish within the various E-classes, for instance the further structure A/E can inherit. [1] witnesses the difficulties arising in the general analysis of ω-categoricity even in the (simpler) weakly o-minimal case.

However a partial converse of Theorem 4.5 can be stated, in the following terms.

Proposition 4.7 *Suppose that the following hypotheses hold:*

1. *Let $\mathcal{R} = (R, \leq, C_0, \ldots, C_m)$ be an ω-categorical coloured order with finitely many colours $C_0, \ldots, C_m$. Colours are regarded here as 1-ary relations partitioning R.*
2. *For every $i \leq m$ associate to C_i an o-minimal ω-categorical first order theory T_i such that different colours correspond to different theories.*
3. *For every $r \in R$ of colour C_i replace r with a (countable) model $\mathcal{A}_r$ of T_i.*
4. *Finally, let E be an equivalence relation with a class for every $r \in R$; the class is just the domain A_r of $\mathcal{A}_r$.*

Let $\mathcal{A} = (A, \leq, E, \ldots)$ be the structure built in this way. Then $\mathcal{A}$ is ω-categorical and strongly locally o-minimal.

Proof. Let $\mathcal{B}$ be a countable model of the theory of $\mathcal{A}$. Thus the quotient set B/E, regarded as a coloured linear order, is elementarily equivalent to $\mathcal{R}$, and as such, is isomorphic to $\mathcal{R}$. Let F be an isomorphism between $\mathcal{R}$ and the coloured linear order of domain B/E. Also, for every $r \in R$, let $B_{F(r)}$ denote the E-class corresponding to $F(r)$ in B; so $B_{F(r)}$ is the domain of a countable structure that is elementarily equivalent to $\mathcal{A}_r$ and as such, is isomorphic to $\mathcal{A}_r$, say by F_r. By combining F and the F_r in a suitable way one eventually builds an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. This proves the ω-categoricity of $\mathcal{A}$.
To prove strong local o-minimality, observe that the 1-type of an element x over $\mathcal{A}$ (as well as over any model of the theory of $\mathcal{A}$) is fully determined by the following:

- if x is already in the E-class of some element of A (so in the E-class corresponding to some $r \in R$, say of colour c_i), the 1-type of s over A_r in T_i;
- if x is in a "new" E-class (say of colour c_i), its type over $\emptyset$ in T_i and then the type of its E-class over R.

This clearly implies strong local o-minimality: For every $a \in A$, take $b < a < c$ in the E-class of a and use the fact that this E-class (as a model of some suitable T_i) is o-minimal. $\dashv$

5 Groups

We deal here with locally o-minimal expansions $\mathcal{A}$ of ordered groups (written additively) $(A, \leq, +, 0)$.
Let H be a subgroup of A definable in $\mathcal{A}$, $a \in H$. Then there is an interval I around a such that $H \cap I$ is a finite union of points and intervals. Thus either

1. there is an interval containing a, included in H, or
2. a is the only point of H in a suitably large interval around a.

The same holds for every element b in H: for instance, if $]a-h, a+h[$ is an interval in H, then $]b-h, b+h[$ is an interval in H containing b (just translate the former interval by adding $b-a$ to each endpoint). Then in order to examine H we can refer to the behaviour of 0 with respect to the previous two cases.
Case 1. There is an interval I around 0 such that $I \subseteq H$. We can assume H is closed under inverses, and hence is symmetric with respect to 0. Without loss of generality $H = [-a, a]$ for some $a > 0$. Observe that $2a \in H$, and indeed $[-2a, 2a] \subseteq H$ (in fact, for every b with $a < b \leq 2a$, $0 < b-a \leq a$, whence $b-a \in H$ and so $b = (b-a)+a$ is in H as well). By repeating this argument one ultimately proves that $[-na, na] \subseteq H$ for every positive integer n.
Assume now $(A, \leq, +, 0)$ is *archimedean*, i.e. for every a, b in A with $0 < a < b$ there is some positive integer n such that $na > b$. Then the previous analysis shows that $H = A$.

Without the archimedean assumption we can say that A has a big convex (possibly non-definable) subgroup $\{b \in H : -na \leq b \leq na$ for some positive integer $n\}$. Observe that in Example 1 of 2.8, the pairs $(0, b)$ with b a real number form a convex definable subgroup but do not exhaust the whole domain.

Example 5.1 To obtain an example of a dense definable subgroup that is not convex, refer to Proposition 2.6 and take the locally o-minimal structure $\mathcal{R} = (\mathbf{R}, \leq, +, 0, sin)$. Incidentally, observe that the discrete additive group of integers is definable in this structure via the formula $sin\, \pi v = 0$. By Proposition 2.6, the ordered group $\mathcal{M} = (\mathbf{Z} \times \mathbf{R}, \leq_{lex}, +, (0, 0))$ is locally o-minimal (and indeed strongly locally o-minimal) as a structure definable in $\mathcal{R}$ (as usual $\leq_{lex}$ denotes here the lexicographic order, while $+$ is defined componentwise). The formula $\exists w (v = w + w)$ defines in $\mathcal{M}$ the subgroup $(2\mathbf{Z}) \times \mathbf{R}$, which is dense with respect to $\leq_{lex}$ but not convex.

Case 2. There is $h \in A$, $h > 0$ such that for every $a \in H$, a is the only point of H in $]a - h, a + h[$.
Assume $(A, \leq, +, 0)$ is *archimedean*. Then for every $0 < a < b$ in H there is a positive integer n such that $b - a < nh$. So there are at most n elements of H between a and b. In other words H is discrete and indeed, again by the archimedean assumption, isomorphic to the ordered group of integers. Note that the (additive) subgroup defined by $sin\, \pi v = 0$ in Example 2.7 lies in case 2. Observe that if archimedeanity fails, then case 2 includes further examples of a different nature. For instance, in Example 2 in 2.8 the centralizer H of $(2, 0)$ is not discrete, but each point of H is the only representative of H in a suitably large neighbourhood.

Nevertheless, we can state the following at least in the archimedean case:

Lemma 5.2 *Let $\mathcal{A}$ be a locally o-minimal expansion of an archimedean (dense) ordered group $(A, \leq, +, 0)$, and H a non-zero definable subgroup of this group. Then either $H = A$ or H is isomorphic to the ordered group of integers.*

The comparison with the o-minimal case [6], Theorem 2.1, raises two further questions:

a) Is $\mathcal{A}$ abelian?

b) If yes, is $\mathcal{A}$ divisible?

We will first deal with question a). Clearly we cannot expect A to be abelian in the general (possibly non-archimedean) case, see Example 2 in 2.8. But we claim that commutativity holds at least when A is archimedean. In fact, we know that under this assumption, any nonzero subgroup - and in particular the centralizer $C(a)$ of a generic element $a \in A$- is either all of A or a discrete subgroup. We will exclude the latter possibility.
If $a = 0$, then its centralizer is A. So take $a \neq 0$. Without loss of generality

we can assume $a > 0$. Suppose that $C(a)$ is discrete. Again without loss of generality we can assume that a is the least positive element in $C(a)$, in other words $]0, a[\cap C(a) = \emptyset$. By local o-minimality, there is some $\epsilon > 0$ in A such that either for every x in the interval $]0, \epsilon]$ $a + x > x + a$ or for every x in this interval $a + x < x + a$. Assume for simplicity the former case (the latter can be treated in a similar way). Observe that the property $a + x > x + a$ is preserved under addition. In fact, if both x and y satisfy it, then

$$a+(x+y) = (a+x)+y > (x+a)+y = x+(a+y) > x+(y+a) = (x+y)+a.$$

Hence the points in $]\epsilon, 2\epsilon]$ also satisfy it, as such a point x can be written as $\epsilon + (-\epsilon + x)$ where $0 < -\epsilon + x \leq \epsilon$. Proceed inductively to obtain $a + x > x + a$ for every $x \in]0, n\epsilon]$ and for every positive integer n. Now use the archimedean assumption to find n such that $a < n\epsilon$; then the previous considerations lead to the contradiction $a + a > a + a$.

We will now treat question b). Assume A is abelian, whence nA is a subgroup of A for every integer $n > 1$. Observe that nA is dense: for $a < b$ in A (and hence $na < nb$ in nA), there is $c \in A$ between a and b so that $na < nc < nb$. This implies that for an archimedean $\mathcal{A}$, nA equals A for every n.
In the general case nA is cofinal as well: for every $a > 0$ in A, $na > a$. Moreover Example 5.1 provides a locally o-minimal ordered abelian group A such that $2A \neq A$.

Thus at least in the archimedean case we can state the following.

Theorem 5.3 *Let $\mathcal{A}$ be an archimedean locally o-minimal ordered group. Then $\mathcal{A}$ is abelian and divisible; in particular $\mathcal{A}$ is o-minimal.*

6 Rings

Here we deal with locally o-minimal expansions of ordered rings. We will use *ring* to mean an associative ring with identity 1.
Example 1 in 2.8 provides a ring of this kind which is not a field. But the following also holds:

Lemma 6.1 *Let $\mathcal{A} = (A, \leq, +, \cdot, 0, 1)$ be a locally o-minimal archimedean ordered ring. Then $\mathcal{A}$ is an ordered field.*

Proof. The center $C(\mathcal{A})$ of $\mathcal{A}$, i.e., the set of points of A commuting with every element of A with respect to $\cdot$, is a definable additive subgroup of $\mathcal{A}$ including the integers (that is, the multiples of 1 in A). It cannot be discrete because for $a < b$ in $C(\mathcal{A})$, $\frac{a+b}{2}$ is in $C(\mathcal{A})$ as well. Due to Lemma 5.2, $C(\mathcal{A}) = A$, whence $\mathcal{A}$ is commutative.
Now, for every non-zero $r \in A$ consider $r \cdot A$. This is again a definable additive subgroup of $\mathcal{A}$, and is dense because for $a < b$ in A, any element c between a and b satisfies $r \cdot a < r \cdot c < r \cdot b$. Thus $r \cdot A = A$, and there is some $s \in R$ for

which $r \cdot s = 1$.
In conclusion $\mathcal{A}$ is a field as claimed. $\dashv$

We can say even more.

Proposition 6.2 *Let $\mathcal{A}$ a locally o-minimal archimedean ordered field. Then every positive element of A is a square.*

Proof. The multiplicative ordered group $A^{>0}$ of positive elements of A is also archimedean. The squares in A form a multiplicative subgroup of $A^{>0}$, and this subgroup is dense (for $a < c < b$ in A, $a^2 < c^2 < b^2$). By Lemma 5.2, every positive element of A is a square. $\dashv$

Lemma 6.3 *A strongly locally o-minimal ordered field $\mathcal{A}$ is real closed.*

Note that we do not use an assumption of archimedeanity here.
Proof. Let $b < c$ be two elements in A, and $f(x)$ a polynomial in $A[x]$ such that $f(b) > 0$ and $f(c) < 0$. Choose any element $a \in A$. Use strong local o-minimality to obtain an interval I around a such that I with the structure induced by $\mathcal{A}$ is o-minimal. Take $b' < c'$ in I and form the polynomial $f'(x) = f(X)$ where

$$X = \frac{c-b}{c'-b'} \cdot (x - b') + b.$$

It is clear that $f'(b') = f(b) > 0$ and $f'(c') = f(c) < 0$. By the o-minimality of I (and by the same argument as in [6], Theorem 2.3) there is an intermediate root r' of $f'(x)$ between b' and c'. Let

$$r = \frac{c-b}{c'-b'} \cdot (r' - b') + b,$$

then $b < r < c$ and $f(r) = f'(r') = 0$, in other words $f(x)$ admits a root between b and c. In conclusion $\mathcal{A}$ satisfies the intermediate value property for polynomials and is therefore real closed. $\dashv$

By combining Lemmas 6.1 and 6.3 we obtain as an immediate consequence:

Theorem 6.4 *A strongly locally o-minimal archimedean ordered ring is a real closed field, and hence is o-minimal.*

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