A GROUP DEFINABLE IN AN O-MINIMAL STRUCTURE WHICH IS NOT AFFINE

PANTELIS E. ELEFTHERIOU

Abstract. In this short note we provide an example of a group G definable in an o-minimal structure $\mathcal{M}$ which does not admit an affine embedding; in other words, there is no definable isomorphism between topological groups $f : G \to G' \subseteq M^m$, such that the group topology on G' coincides with the subspace topology induced by M^m.

Let $\mathcal{M}$ be an o-minimal structure. By "definable" we mean "definable in $\mathcal{M}$" possibly with parameters. A group $G = \langle G, \oplus, e_G\rangle$ is said to be definable if both its domain and the graph of its group operation are definable subsets of M^n and M^{3n}, for some n, respectively.

By [Pi], we know that every definable group $G \subseteq M^n$ can be equipped with a unique definable manifold topology that makes it into a topological group. We refer to this topology as the *group topology of* G. It is shown in [Pi] that the group topology of G coincides with the subspace topology induced by M^n on a large subset V of G $\big(\dim(G \setminus V) < \dim(G)\big)$. We call G *affine* if the group topology of G coincides with the subspace topology on (the whole of) G.

Question. *Is every definable group affine (up to definable isomorphism)?*

Remark 0.1. (i) A definable isomorphism between two definable groups is a group isomorphism as well as a topological homeomorphism with respect to their group topologies.

(ii) By [ElSt, Remark 2.2], the Question can be restated as follows: *Given a definable group $G \subseteq M^n$, is there a definable injective map $\tau : G \to M^m$, $m \in \mathbb{N}$, such that the topology on $\tau(G)$ induced by the group topology of G via τ coincides with the subspace topology on $\tau(G)$ induced by M^m?* If yes, then such a τ is called an *affine embedding of* G.

The Question admits an affirmative answer in case $\mathcal{M}$ expands a real closed field, by [BO, Proof of Lemma 10.4] and [vdD, Chapter 10, Theorem (1.8)]. In fact, the above references concern affine embeddings of "abstract definable-manifolds", and the work in [BO] also yields affine embeddings which are moreover diffeomorphisms. The original proof of embedding semi-algebraic manifolds was given in [Ro].

We present here an example of a group definable in a linear o-minimal structure which is not affine. Linear o-minimal structures were studied in [LP], and groups definable in them were studied in [ElSt] and [El]. The main example of a linear

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o-minimal structure is that of an ordered vector space $\mathcal{M} = \langle M, +, <, 0, \{d\}_{d\in D}\rangle$ over an ordered division ring D. The main property of such an $\mathcal{M}$ that we use below is that every definable function $f : A \subseteq M^n \to M^m$ is piecewise linear, that is, there is a partition of A into finitely many definable sets A_i, $i = 1, \ldots, k$, such that for each $i = 1, \ldots, k$, the following holds: there is an $n \times m$ matrix λ with entries from D, and an element $a \in M^m$, such that for every $x \in A_i$, $f(x) = \lambda x + a$.

For $x, y \in M$, we define:

$$x \prec_D y \Leftrightarrow \forall d \in D, dx < y.$$

Example 0.2. Let $\mathcal{M} = \langle M, +, <, 0, \{d\}_{d\in D}\rangle$ be an ordered vector space over an ordered division ring D with the following property: there are $a, b, c > 0$ in M such that $b \prec_D c \prec_D a$. In particular, there is no definable function from $[0, b)$ onto $[0, c)$, and $\forall n \in \mathbb{N}$, $nc < a$.

Let $S = [0, a) \times [0, b)$ and $L = \mathbb{Z}(a, 0) + \mathbb{Z}(a - c, b)$ be the lattice in M^2 generated by the elements $(a, 0)$ and $(a - c, b)$. Define $G = \langle S, \oplus, 0\rangle$, where

$$x \oplus y = z \Leftrightarrow x + y - z \in L.$$

By [ElSt, Claim 2.7(ii)], G is definable.

Notation. By $\lim^G$ we denote a limit with respect to the group topology of G. A path is always a continuous map, and a G-path is a path which is continuous with respect to the group topology of G.

Claim. *There is no definable injective map $\tau : G \to M^m$, $m \in \mathbb{N}$, such that the induced topology on $\tau(G)$ coincides with the subspace topology.*

Proof. Assume, towards a contradiction, that there is such a τ. For every element $t \in [0, a)$, consider the one-to-one G-path

$$\phi_t : [0, b) \to \{t\} \times [0, b), \text{ with } \phi_t(x) = (t, x).$$

By definition of G, we see that for every $t \in [0, a - c]$, $\lim^G_{x\to b} \phi_t(x) = (t + c, 0)$. Therefore, for every $t \in [0, a - c]$, $\tau(\phi_t)$ is a path in M^m with the property:

$$\lim_{x\to b} \tau\big(\phi_t(x)\big) = \tau\big((t + c, 0)\big).$$

Consider now the image $\tau\left([0, a - c] \times \{0\}\right)$. It contains an infinite number of elements of the form $\tau\big((nc, 0)\big)$, $n \in \mathbb{N}$. Since τ is piecewise linear, there must exist some $n \in \mathbb{N}$ such that τ is linear on $\big[nc, (n + 1)c\big) \times \{0\}$. Hence the image $\tau\Big(\big[nc, (n + 1)c\big) \times \{0\}\Big)$ is in bijection with an interval $J \subseteq M$ via some projection onto one of the m coordinates. Since $\tau(\phi_{nc}) : [0, b) \to M^m$ is a path with

$$\tau\big(\phi_{nc}(0)\big) = \tau\big((nc, 0)\big) \text{ and } \lim_{x\to b} \tau\big(\phi_{nc}(x)\big) = \tau\Big(\big((n + 1)c, 0\big)\Big),$$

it is easy to see that there is a definable map from $\tau\Big(\phi_{nc}\big([0, b)\big)\Big)$ onto J, and, therefore, there is a definable map from $[0, b)$ onto J. It follows that there is a definable map from $[0, b)$ onto $\big[nc, (n + 1)c\big)$, a contradiction. $\square$

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Institut de Matemàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain

E-mail address: `pelefthe@gmail.com`