Superdecomposable pure injective modules over commutative Noetherian rings

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Abstract. We investigate width and Krull–Gabriel dimension over commutative Noetherian rings which are "tame" according to the Klingler–Levy analysis in [4], [5] and [6], in particular over Dedekind-like rings and their homomorphic images. We show that both are undefined in most cases.

1 Introduction

Let R be an associative ring with identity and let Mod_R denote the class of right modules over R. As Ringel observed in his [14], Model Theory and Algebra, when studying Mod_R, may sometimes use different languages, but often share common ideas and methods. For instance, they provide some notions of *dimension* aiming at measuring the complexity of Mod_R, and contributing in this way to its classification.
In the model theoretic approach, these "dimensions" are often introduced in terms of the lattice of *positive primitive formulas* (hereafter, pp-formulas) of R. They equip every pp-formula and the whole ring R with either an ordinal number, or ∞. An ordinal value estimates how simple the single formula and the whole ring are, while an "infinite" value ∞ means that the formula, or the ring, are too complicated to be measured in a satisfactory way. One of the most relevant complexity measures is the *Krull–Gabriel dimension* of R, that is, the m–dimension of the lattice of pp-formulas over R. This is related to the Cantor–Bendixson rank of the Ziegler space of indecomposable pure injective R-modules, as well as to the *width* of R, see [13] for a report about these concepts and their relationship. In particular rings with Krull–Gabriel dimension have width as well. Notably, both these notions, Krull–Gabriel dimension and width,

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are closely connected with the existence problem for *superdecomposable* pure injective R-modules, that is, pure injective R-modules M without any indecomposable nonzero direct summand. In fact a powerful criterion of Ziegler [15], Theorem 7.1, says that width excludes any superdecomposable pure injective modules over R; the converse is also true for a countable R, but is still open over uncountable rings. As a consequence, even the Krull–Gabriel dimension is incompatible with superdecomposable pure injectives.

It is still an open question which rings R possess superdecomposable pure injective modules. Both [3], Chapter 8, and [7], Chapter 10, deal with this matter. See also [9], p. 449 for a review of existing results. [10] deals with tame non-domestic string algebras over fields (in particular with the Gelfand-Ponomarev algebras $F[X, Y : X^n = Y^m = XY = 0]$ where F is a field, n, m are integers ≥ 2 and $n + m \geq 5$) ensuring existence of superdecomposable pure injective modules in the countable case, and in general lack of width and Krull–Gabriel dimension. [12] obtains the same conclusions over integral group rings $\mathbf{Z}G$ where G is a finite non-trivial group, while the case of group algebras FG, with F a field, is treated in [13].

In this paper we are going to examine

- width,
- Krull–Gabriel dimension,
- existence of superdecomposable pure injective modules

over commutative Noetherian rings R. We refer to the wild/tame dichotomy that Klingler and Levy recently developed in [4], [5] and [6] over these rings. This will be summarized in § 2 below. Anyway, one can advance here that Klinger and Levy single out two "opposite" notions of wildness and tameness, basically founded on the possibility of classifying finitely generated R-modules. In more detail they first introduce 4 capital subclasses of commutative Noetherian rings (Artinian triads, Drozd rings in the wild side, Klein rings and Dedekind-like rings in the tame one), and then show that a commutative Noetherian indecomposable ring R either

(i) projects itself onto an Artinian triad or onto a Drozd ring (and in this sense is "wild"), or

(ii) is a Klein ring or the homomorphic image of some Dedekind-like ring (and in this sense is "tame").

[4] and [5] treat these matters over *complete local* rings, while [6] drops this assumption and deals with arbitrary commutative Noetherian rings. See these papers for precise definitions and full details (or wait for the next § 2 for a short report about them).

Let us sketch now the plan of this paper. After summarizing in § 2 the Klingler–Levy analysis, we will discuss in § 3 how width, Krull–Gabriel dimension and

lack of superdecomposable pure injective modules are preserved under localization at maximal ideals. The relevance of this matter will be explained in § 2.
After these introductory sections we will tackle our main topic. In § 4 we will deal with Artinian triads and Drozd rings, and with the rings projecting themselves onto them, showing in this framework lack of Krull–Gabriel dimension and width, and existence of superdecomposable pure injectives. In § 5 we will treat the "easy" case of Klein rings, where the Krull–Gabriel dimension is defined.
The rest of the paper is devoted to Dedekind-like rings. § 6 deals shortly with discrete valuation domains (a comparatively trivial setting, where the Krull–Gabriel dimension is defined) and then with another more intriguing class of rings (the ones Klingler and Levy call *split*). Actually the analysis in the split case is almost completely, although implicitly, accomplished in the literature; it comes out that superdecomposable pure injective modules arise, and Krull-Gabriel dimension and width are "infinite". After that, we will turn our attention in § 7 to the crucial non-split case, again providing a negative solution in most cases (those the Klingler–Levy analysis applies to). Finally we will devote § 8 to homomorphic images of Dedekind-like rings.
The following criterion will be useful several times. It provides a condition sufficient to exclude Krull–Gabriel dimension and width over a given arbitrary ring R. It is formally stated and proved in [13] as Proposition 5.6, making some ideas of [10] and [12] explicit. Recall that a *pointed* module is a module with a distinguished element (or tuple of elements). In this framework *pointed* morphisms have to preserve, in addition to the module structure, this further element, or tuple of elements.

Theorem 1.1 *Let R be a ring. Let $(Q, \preceq_Q)$ and $(T, \preceq_T)$ denote two disjoint copies of the order $(\mathbf{Q}, \leq)$ of rationals, so two countable dense linear orderings without endpoints. Suppose that, for $q \in Q$ and $t \in T$, pointed R-modules $(M(q), m(q))$ and $(M(t), m(t))$ are given. Also, assume that the $M(q)$ and the $M(t)$ are finitely presented with a local endomorphism ring and that the following conditions hold:*

1. *For $q \prec_Q q'$ in Q, there is some pointed morphism of $(M(q), m(q))$ to $(M(q'), m(q'))$, but these (pointed) modules cannot be isomorphic.*
2. *Similarly, for $t \prec_T t'$ in T, there is a pointed morphism of $(M(t), M(t))$ to $(M(t'), m(t'))$, but these (pointed) modules cannot be isomorphic.*
3. *For t in T and q in Q, the pushout $(M(t.q), m(t.q))$ of $M(t)$ and $M(q)$ has a local endomorphism ring.*
4. *For $q \neq q'$ in Q and $t \neq t'$ in T, there is no isomorphism between the (pointed) modules $M(t.q)$ and $M(t.q')$, and between $M(t.q)$ and $M(t'.q)$.*
5. *For q in Q and t in T, there is no pointed morphism between the pointed modules $(M(q), m(q))$ and $(M(t), m(t))$ in either direction.*

Then R has no width, and consequently no Krull–Gabriel dimension. In particular, when R is countable, R possesses superdecomposable pure injective modules.

As said, we assume some familiarity with the Klingler–Levy analysis of modules over commutative Noetherian rings in [4], [5] and [6], as well as the notions of width and Krull–Gabriel dimension (see the references quoted at the beginning of this introduction). Good classical sources on the model theory of modules are [3], [7] and [15].

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The second author wishes to add the following statement: Vera Puninskaya died on October 15, 2007, when this paper was going to be accomplished. I would like to dedicate this article as a tribute to her memory. We cooperated in several joint papers. It was a wonderful experience to work with her in Mathematics for so many years.

2 Commutative Noetherian rings

Let us summarize here the Klingler-Levy tame/wild dichotomy for finitely generated modules over commutative Noetherian rings, as developed in the series of papers [4], [5] and [6]. First let us recall some preliminary notions and results from these papers.

Let R be a local commutative Noetherian ring. In the following J denotes the maximal ideal of R and $k = R/J$ its residue field; for a given R-module M, $\mu_R(M)$ is the minimal size of a set of generators of M over R. Klingler and Levy single out four capital classes of local commutative Noetherian rings R.

The first one is that of *Artinian triads*, meaning those rings R for which $\mu_R(J) = 3$ and $J^2 = 0$. So an example of Artinian triad is provided by the algebra $F[X, Y, Z]/\langle X, Y, Z\rangle^2$ where F is any field.

R is called a *Drozd ring* if $\mu_R(J) = \mu_R(J^2) = 2$, $J^3 = 0$ and there is some $x \in J - J^2$ for which $x^2 = 0$. Therefore the algebra $F[X, Y : X^2 = Y^3 = 0]$ (with F a field), as well as $\mathbf{Z}/p^3\mathbf{Z}\,[X : X^2 = p^2X = 0]$ (with p a prime), are examples of Drozd rings.

Notice that both Artinian triads and Drozd rings are Artinian. What is more relevant for our purposes, they are *"minimal wild"* among commutative Noetherian rings in a sense we are going to explain within a few lines. In this perspective their counterparts are the so called *"maximal tame"* R-modules, that is, Klein rings and Dedekind-like rings.

R is said to be a *Klein ring* if $\mu_R(J) = 2$, $\mu_R(J^2) = 1$, $J^3 = 0$ and, for every $x \in R$, $x^2 = 0$. Then it is easily seen that the residue field k of R has characteristic 2, while R has characteristic 2 or 4. Also, the socle of R is J^2. A basic example of Klein rings is the algebra $F[X, Y : X^2 = Y^2 = 0]$, in other words the group algebra $F\,C(2)^2$, where F is a field of characteristic 2 and $C(2)^2$ – the direct sum of two copies of the group with 2 elements – is the Klein group; and indeed Klein rings generalize this example. Anyway Klein rings also include

examples which are not algebras over a field, such as the ring (of characteristic 4) $\mathbf{Z}/4\mathbf{Z}\,[X : X^2 = 0]$ (see [4], Example 5.4).
Finally R is called a *Dedekind-like ring* when R is reduced and, if Γ is the normalization of R (i. e., its integral closure in the total ring of quotients), then Γ is the direct sum of (at most two) principal ideal domains, $\mu_R(\Gamma) = 2$ and the Jacobson radical of Γ is $J = \Gamma J$. One distinguishes here three cases: either

a) Γ/J is isomorphic to k (that is, $R = \Gamma$ is a discrete valuation domain), or

b) Γ/J is a (2-dimensional) extension field of k (in which case R is called *unsplit*), or

c) $\Gamma/J \simeq k \times k$ (in which case R is called *split*; R is said to be *strictly split* if a further condition holds, that is, J is the direct sum of two integral domains, necessarily discrete valuation domains; if R is split but not strictly split, then Γ itself is a principal ideal domain, and has exactly two maximal ideals).

Notably a *complete* split R has to be strictly split (see [5], p. 352). In particular the completion of a non–strictly split Dedekind-like ring R is strictly split.
A (possibly non-local) commutative Noetherian ring R is Dedekind-like if and only if every localization of R at a maximal ideal is Dedekind-like. Another intrinsic characterization of Dedekind-like rings, avoiding any explicit reference to localizations, is given in [6]. It is worth underlining that even non-local Dedekind-like rings are reduced.
Dedekind-like rings include several algebraically relevant examples. Let us mention among them integral group rings $\mathbf{Z}G$ for G a cyclic group of squarefree order –in particular a group of prime order– and algebras $F[X, Y : XY = 0]$ for F any field (observe that the latter class also involves Gelfand–Ponomarev algebras over F as homomorphic images). Actually both these examples refer to the strictly split setting. An unsplit example (which will be largely considered in § 7) is the ring $\mathbf{R} + X\mathbf{C}[[X]]$ of formal power series in X having complex coefficients but a real constant element.
A crucial result in [4]–[6] is a *dichotomy theorem* for *indecomposable* commutative Noetherian rings R saying that such a ring R satisfies *exactly* one of the following propositions: Either

- R projects itself onto an Artinian triad or a Drozd ring, or
- R is a Klein ring, or a homomorphic image of some complete split or unsplit Dedekind-like ring.

[4], [5], [6] also prove that the class of finite length modules over an Artinian triad or over a Drozd ring is *"wild"* in the sense that classifying all the finite length modules over such a ring R up to isomorphism is at least as difficult as classifying all the finitely generated modules over finite dimensional algebras over the residue field of R up to isomorphism and so can be believed unfeasible (see [4] and [6] for precise definitions). The dichotomy theorem transfers this

wildness result from Artinian triads and Drozd rings to any ring R (indecomposable or not) having an Artinian triad or a Drozd ring as a homomorphic image. In detail, it turns out that some localization of R at a maximal ideal has a wild class of finite length modules. It is in this sense that Artinian triads and Drozd rings are minimal wild. On the other side [5] and [6] provide a general description of finitely generated modules over Klein rings and Dedekind-like rings, with only one very partial characteristic 2 exception, concerning unsplit rings for which Γ/J is not a separable extension of k, see [5], [6]; in this sense, all the rings occurring in the second case of the dichotomy theorem can be viewed as *"tame"* and, in particular, Dedekind-like rings and Klein rings are maximal tame.
Anyway one has to be very careful here, because what the Klingler–Levy dichotomy says *about modules* is that, for R an indecomposable commutative Noetherian ring, *at least* one of the following possibilities occurs:

- for some maximal ideal P of R finite length R_P-modules are wild (in the sense explained before),
- R is a Klein ring or a homomorphic image of some Dedekind-like ring.

But it is still open if these two cases are compatible.

Observe that Artinian rings, Drozd rings and Klein rings are explicitly introduced as *local* rings, and even the approach to Dedekind-like rings in [4], [5] and [6] is through localization. Accordingly we will investigate in the next section how width, Krull–Gabriel dimension and superdecomposable pure injectives are preserved under localization. For instance, it comes out that, as proved in [3], Corollary 8.62 p. 213, if R is a commutative ring and some localization of R possesses superdecomposable pure injective modules, then R itself inherits this property. Since we expect negative answers –that is, existence of superdecomposable pure injective modules, and consequently lack of width and Krull–Gabriel dimension– in most cases, it is reasonable to devote a particular attention to local rings. This will be explained in more details in the next section.

3 Localization and width

In this short section we investigate the relationship between the main concepts we are interested in, that is, width, Krull–Gabriel dimension and superdecomposable pure injectives, and the localization process. All throughout this section R is any commutative ring and S is any multiplicative subset of R; R_S denotes the localization of R at S. Of course what we have in mind is the case when $S = R \setminus P$ for some maximal ideal P of R; in this particular setting we will denote R_S also by R_P.
Recall that R_S is built first by factoring R through the ideal of the elements $r \in R$ such that $rs = 0$ for some $s \in S$ –let $R(S)$ denote the corresponding quotient ring–, and then by forming in the usual way and with the usual rules

the fractions t/s with $t \in R(S)$ and $s \in S$. This defines a natural epimorphism in the category of rings from R to R_S, which inserts our setting in the general framework considered in [8].

Lemma 3.1 *If R_S admits superdecomposable pure injective modules, then R does.*

As already said, this is shown, for instance, in [3], Corollary 8.62. The point is that modules over R_S can be also viewed as modules over R (those where the scalars $r \in R$ annihilating some $s \in S$ act as 0 and each $s \in S$ acts automorphically).
Now let us deal with Krull–Gabriel dimension and width. For every ring $\overline{R}$ (R, or R_S, or anything else), let $L(\overline{R})$ denote the lattice of pp-formulas in 1 free variable over $\overline{R}$. Recall that the Krull–Gabriel dimension and the width of $\overline{R}$ are defined as the m–dimension and the width of $L(\overline{R})$ respectively.

Lemma 3.2 *If the Krull–Gabriel dimension, or the width, of R_S are undefined, then the same can be said of R.*

Proof. A common property of both m–dimension and width of modular lattices with top and bottom elements is that, if for some quotient lattice one of them is undefined, then the same is true for the whole lattice. On the other hand $L(R_S)$ can be viewed as a quotient lattice of $L(R)$, and hence transfers to $L(R)$ any possible negative feature about m–dimension and width. This follows from [8], Corollary 4, where it is explained how to translate an arbitrary pp-formula over R_S (and more generally over any ring $\overline{R}$ for which an epimorphism $R \to \overline{R}$ is given) to a pp-formula over R equivalent to it within R_S-modules. Moreover every pp-formula $\alpha(v)$ over R can be also viewed as a pp-formula over R_S via the epimorphism above, and the relation associating with every $\alpha(v)$ just $\alpha(v)$ considered over R_S is easily seen to define a map, and indeed a lattice morphism, from $L(R)$ onto $L(R_S)$ (surjectivity follows from the quoted results in [8]). Then we are done. $\dashv$

4 Wild rings

In this section we consider rings R having an Artinian triad or a Drozd ring as a homomorphic image and we show that, under this assumption, Mod_R includes superdecomposable pure injective modules (at least over a countable R) and lacks both width and Krull–Gabriel dimension. Of course, it suffices to restrict our analysis just to Artinian triads and Drozd rings. Accordingly we refer to the notation fixed in § 2, when introducing these kinds of rings. Let us start with Artinian triads.

Proposition 4.1 *Let R be an Artinian triad. Then R does not have Krull–Gabriel dimension and width and admits superdecomposable pure injective modules.*

Proof. This follows directly from [3], Corollary 8.65 p. 215. In fact, for any Artinian triad R, $J/J^2 \simeq J$ has dimension ≥ 3 over the residue field k; so R possesses a nonzero pure injective module with no indecomposable direct summands, in other words a superdecomposable pure injective module. Consequently, R has neither Krull–Gabriel dimension nor width. $\dashv$

Now let us treat Drozd rings.

Proposition 4.2 *Let R be a Drozd ring. Then R does not have Krull–Gabriel dimension and width. So, when countable, then R possesses superdecomposable pure injective modules.*

Proof. We need recall from [4] some information on the structure of a Drozd ring R. We already know that there is some element x in the maximal ideal J of R such that $x^2 = 0$; it follows that $J = \langle x, y\rangle$ for some y for which $J^2 = \langle xy, y^2\rangle$. Consequently any element $a \in J$ can be written (not uniquely) as

$$a = a_x x + a_y y + a_{xy} xy + a_{y^2} y^2$$

where a_x, a_y, a_{xy}, a_{y^2} are units in R or zero. At this point, form $R' = R/\langle xy\rangle$ and get a local commutative Artinian ring of length 4 whose maximal ideal J' has two generators x', y' (the images modulo $\langle xy\rangle$ of x, y respectively) satisfying $x'y' = 0$, $x'^2 = y'^3 = 0$. But this is just the setting investigated in [12] (which in its turn resembles the case of the Gelfand-Ponomarev algebras treated in [10]), so Proposition 5.6 and Theorem 5.8 of that paper apply and yield lack of width, and consequently existence of superdecomposable pure injective modules at least when R is countable. $\dashv$

At this point it is straightforward to deduce the following more general result:

Corollary 4.3 *Let R be any ring projecting itself onto an Artinian triad or onto a Drozd ring. Then R has neither width nor Krull–Gabriel dimension. Furthermore R, when countable, possesses superdecomposable pure injective modules.*

5 Klein rings

Now let us turn our attention to *"tame"* rings. The simplest examples here are *Klein rings*, and this section is just devoted to these rings. We again refer to the notation introduced in § 2.

Let R be a Klein ring, then the socle of R, $soc(R)$, equals J^2. Also, R is quasi-Frobenius (see [5], Theorem 11.3 p. 473), whence every R-module is the direct sum of a free R-module and a module over the quotient ring $R/soc(R)$, that is, over R/J^2.

Proposition 5.1 *No superdecomposable pure injective module exists over a Klein ring R. Also, R has both Krull–Gabriel dimension and width, and indeed the Krull–Gabriel dimension of R is 2.*

Proof. Free R-modules are easy to handle. In fact R has finite length. So the lattice of pp-subgroups of free R-modules has finite length as well, and consequently both width and Krull–Gabriel dimension are 0 in the free setting. Therefore without loss of generality we can restrict our investigation to modules over R/J^2. Accordingly let us change our notation and call R this quotient ring. So R is now an Artinian commutative local ring whose Jacobson radical J satisfies $J^2 = 0$; also, the residue field $R/J = k$ is the same as before and $\mu_R(J) = 2$; but now J, as a module over R, becomes a vectorspace over k (of dimension 2), just because $J^2 = 0$. In particular the analysis in [3], pp. 213–214, applies and reduces the existence problem of superdecomposable pure injective modules from R to the companion ring

$$\overline{R} = \begin{pmatrix} R/J & 0 \\ J & R/J \end{pmatrix},$$

which, due to the latest remarks, is easily seen to be isomorphic to the Kronecker algebra over k

$$k\,\tilde{A}_1 = \begin{pmatrix} k & 0 \\ k \oplus k & k \end{pmatrix}.$$

Accordingly $\overline{R}$ possesses no superdecomposable pure injective modules because the Cantor-Bendixson rank of the Ziegler spectrum $Zg_{\overline{R}}$ is defined, indeed it is 2. Then, due to the quoted reference in [3], R has no superdecomposable pure injective modules, and width and Krull–Gabriel dimension are defined.
Actually it is not difficult to deduce, on the basis of this connection with $k\,\tilde{A}_1$, that the Krull–Gabriel dimension of R is 2. In fact the reduction modulo the radical functor, as described in detail for instance in [3], Remark 8.68, sends, for every Klein ring R, finitely generated modules over R/J^2 to finite dimensional modules over $k\tilde{A}_1$, and provides in this way a stable equivalence between these two categories (see [1], Chapter X.2, in particular Theorem X.2.4). By [2], Corollary 3.9, this implies that the Krull-Gabriel dimension of R/J^2 (and R) is the same as $k\,\tilde{A}_1$, that is, 2. $\dashv$

6 Dedekind-like rings: Discrete valuations domains, and the split case

Let us begin now the analysis of Dedekind-like rings and their localizations. We are going first to investigate the behaviour of local Dedekind-like rings, and then to check how positive or negative answers to our main questions transfer to arbitrary Dedekind-like rings from their localizations. Actually we will deal in this section with local rings corresponding to the cases a), c) (according to the description in § 2). Consequently we will again refer in this section, and indeed in the remainder of this paper, to the notation introduced in § 2.

Discrete valuation domains and a) are easy to treat, and the following fact is well known (see the references quoted about this point in the introduction).

Fact 6.1 *Discrete valuation domains do not admit superdecomposable pure injective modules and have width and Krull–Gabriel dimension.*

It is worth mentioning here the Puninski result in [11] saying that the Krull–Gabriel dimension of a Noetherian (uni)serial ring R is 0 or 2 (and in any case different from 1) and equals the Cantor–Bendixson rank of the Ziegler spectrum of R.

When enlarging our perspective from the local setting to arbitrary rings, we can easily deduce what follows.

Corollary 6.2 *Let R be a Dedekind-like ring such that all the localizations of R at maximal ideals are discrete valuation domains. Then R has both width and Krull–Gabriel dimension, and no superdecomposable pure injective module.*

In fact R, as a Noetherian *reduced* ring, is nothing but a Dedekind domain, and everything relevant for our purposes about Dedekind domains is already said in [15].

So let us come back to local rings and in particular to c) and the split case. This is implicitly treated in [12]. Let us explain why.

Theorem 6.3 *Let R be a local split Dedekind-like ring. Then R has neither width nor Krull–Gabriel dimension. In particular, if R is countable, then R possesses superdecomposable pure injective modules.*

Proof. Under the *strictly* split assumption, Γ is the direct sum of two discrete valuation domains Γ_1 and Γ_2 with a common residue field k; the Jacobson radical of both Γ and R is the direct sum of the Jacobson radicals of Γ_1 and Γ_2; $\Gamma/J \simeq k \oplus k$ and k embeds itself into Γ/J via the diagonal map $x \to (x, x)$; finally, R is isomorphic to the pullback of Γ_1 and Γ_2 over k, so to the ring of those pairs $(x_1, x_2) \in \Gamma_1 \oplus \Gamma_2$ such that x_1 and x_2 have the same canonical projection onto k. This is just the framework of Remark 5.7 and Theorem 5.8 in [12]. So those results apply and show our claim in the strictly split case.

But actually the proof of [12] refers to a homomorphic image of length 4 of R and so works even in the *non-strictly* split case. The reason is that, if R is not strictly split, then its completion is split, and we can still refer to a ring of length 4 as a common homomorphic image of R and its completion $\tilde{R}$. In fact R and $\tilde{R}$ admit the "same" Artinian proper homomorphic images. Let us explain why (we thank Larry Levy for suggesting this argument). R is a local domain of Krull dimension 1, and so a proper homomorphic image of R has Krull dimension 0 and is Artinian, in particular, if viewed as a module over R, has finite length. Now observe that, for every R-module A of finite length, $J^n A = 0$ for some n (otherwise, by Nakayama's Lemma, $A \supset JA \supset J^2 A \supset \ldots$, which contradicts the finite length assumption). It follows that the natural map from A to its J-adic completion $\tilde{A}$ is an isomorphism because nothing changes when one takes the completion. On the other hand every $\tilde{R}$-module A of finite length has finite length also over R (see [6], Lemma 6.5), and then the natural map $A \to \tilde{A}$ is an isomorphism. In conclusion the completion process carries isomorphically the proper homomorphic images of R onto the proper Artinian homomorphic images of $\tilde{R}$. $\dashv$

In particular the Artinian homomorphic image of $\tilde{R}$ involved in the proof of [12] can be viewed as a proper homomorphic image of R. Notice that R can admit some further non-Artinian proper homomorphic image, for instance the pullback of a discrete valuation domain and an Artinian valuation ring (meaning an Artinian local principal ideal ring, see [5], 11.5, or also our § 8 below).

Corollary 6.4 *Let R be a Dedekind-like ring such that some localization of R at a maximal ideal is split. Then R has neither width nor Krull–Gabriel dimension. In particular, if R is countable, then R possesses superdecomposable pure injective modules.*

7 Dedekind-like rings: The unsplit case

We deal here with local unsplit Dedekind-like rings R. We still refer to the notation introduced in § 2. Due to what we said in that section, we also assume that Γ/J is a separable extension of k. Our aim is to show the following negative result.

Theorem 7.1 *The lattice of pp-formulas over a local unsplit Dedekind-like ring R such that Γ/J is a separable extension of k has no width and no Krull–Gabriel dimension. In particular R, if countable, possesses superdecomposable pure injective modules.*

In fact, we will see that Theorem 1.1 (the criterion stated at the end of our introduction) applies to our unsplit Dedekind-like rings R, which clearly implies our claim. Hence our strategy will be to single out two suitable sequences of pointed R-modules $(M(q), m(q))$ and $(M(t), m(t))$ (with q, t in two disjoint copies $(Q, \preceq_Q)$ and $(T, \preceq_T)$ of the order of rationals), satisfying all the conditions 1-5 in 1.1. The construction of these sequences and the check of 1-5 will be the core of this section.
Anyway, before beginning the proof of Theorem 7.1, let us underline the following relevant consequence for arbitrary (possibly non-local) Dedekind-like rings.

Corollary 7.2 *Let R be a Dedekind-like ring. Assume that, for some maximal ideal P of R, the localization of R at P satisfies the assumptions of Theorem 7.1. Then R has neither width nor Krull–Gabriel dimension. In particular, if R is countable, then R possesses superdecomposable pure injective modules.*

Now let us start the proof of Theorem 7.1. As an initial step, let us remember and discuss once again the description of an unsplit Dedekind-like ring R. Γ –the normalization of R– is a principal ideal domain and J is the common Jacobson radical of both R and Γ. As already recalled, we assume that the quotient field $F = \Gamma/J$ is a separable 2-dimensional extension of $k = R/J$. Let ρ denote the canonical projection of Γ onto F (and at the same time its restriction to R, that is, the canonical projection of R onto k), $\overline{\rho}$ be the conjugate of ρ with respect to k. Then R can be also introduced as

$$R = \{a \in \Gamma \,:\, \rho(a) = \overline{\rho}(a)\}.$$

Example 7.3 The basic example of this framework is $R = \mathbf{R} + X\mathbf{C}[[X]]$, i. e. the ring of formal power series $a(X) = \sum_{i=0}^{+\infty} a_i X^i$ in the indeterminate X having coefficients a_i in $\mathbf{C}$ and a constant element a_0 in $\mathbf{R}$ (see [5], Examples 2.18 p. 355). Then $\Gamma = \mathbf{C}[[X]]$, $J = \langle X \rangle = X\mathbf{C}[[X]]$, $F = \mathbf{C} = \mathbf{R}(i)$ is the complex field, $k = \mathbf{R}$ is the real field, $\rho(a(X)) = a_0$ for every $a(X)$, hence $\overline{\rho}(a(X)) = \overline{a_0}$ = the conjugate of a_0 with respect to the reals. We will often refer to this example below. Actually [5] involves power series to ensure locality and completeness, but the polynomial ring $\mathbf{R} + X\mathbf{C}[X]$ works as well and, if we like a countable framework, we could even replace $\mathbf{C}$ by the field of complex algebraic numbers, and $\mathbf{R}$ by the subfield of real algebraic numbers.

Klingler and Levy introduce in [5], 2.2, two basic constructions to produce new indecomposable R-modules. Both of them regard the quotient R-modules Γ/J^h where h is a positive integer or $h = \infty$ (in which case, Γ/J^∞ just means Γ viewed as an R-module). Note that, if $h \neq 1, \infty$, then J^{h-1}/J^h can be naturally identified with F –as a module over R, or over k–. Actually we will be concerned below for our purposes only with the values $2 \leq h \leq 3$. Also, let us introduce for simplicity the following notation: For $a \in \Gamma$, let $a(h)$ abbreviate the class of a modulo J^h. For technical reasons it is convenient for us the extend this notation to $h = 1$. So for every $a \in \Gamma$ $a(1)$ just means $a + J$.
In the basic Example 7.3, for $h \neq 1, \infty$, $\Gamma/J^h \simeq \mathbf{C}[[X]]/\langle X^h \rangle$ and hence its elements can be regarded as polynomials in X with complex coefficients and degree $< h$; in particular J^{h-1}/J^h is consisting of the polynomials of the form aX^{h-1} with $a \in \mathbf{C}$ and hence is a vectorspace isomorphic to $\mathbf{C}$ over $\mathbf{R}$. Also, for $a \in \mathbf{C}[[X]]$, $a(h)$ abbreviates $a + \langle X^h \rangle$.
Now let us describe the two constructions in [5], as promised.
The former is called *top gluing* and involves two modules Γ/J^h and Γ/J^k with $h, k \neq 1, \infty$ (actually $h, k \in \{2, 3\}$ for our purposes). Both Γ/J^h and Γ/J^k project themselves as R-modules onto Γ/J in the natural way, through ρ and $\overline{\rho}$. Let us denote by ρ, $\overline{\rho}$ respectively these morphisms. Then the *top glue* of Γ/J^h and Γ/J^k is introduced as

$$\{(a, b) \in \Gamma/J^h \oplus \Gamma/J^k \,:\, \rho(a) = \overline{\rho}(b)\}.$$

In the basic Example 7.3, this means to consider those pairs of formal series $(a(X), b(X))$ in $\mathbf{C}[[X]]$ for which b_0 is the conjugate of a_0.
The latter operation in [5] is called *bottom gluing*. It again deals with two modules Γ/J^k and Γ/J^h (with $2 \leq k, h \leq 3$ for our purposes) and with "their" F, J^{k-1}/J^k, J^{h-1}/J^h respectively, and singles out the pairs consisting of an element in J^{k-1}/J^k –their left F– and its conjugate in J^{h-1}/J^h –their right F–. See [5] for more details, and observe that factoring the direct sum of Γ/J^k and Γ/J^h through their bottom glue just identifies elements in the left F and in the

right F via conjugacy. In Example 7.3 bottom gluing two modules $\mathbf{C}[[X]]/\langle X^k\rangle$ and $\mathbf{C}[[X]]/\langle X^h\rangle$ means to single out the ordered pairs of classes of polynomials bX^{k-1}, $\bar{b}X^{h-1}$ where b ranges over the complex field. Factoring through this bottom glue has the effect to identify the two elements of these pairs.
As said, [5] proposes several ways to build a deal of indecomposable R-modules via repeated top or bottom gluing. For instance, [5], Definition 2.4, introduces *nonreduced diagrams* as follows. Take a string $q = h_1 k_1 \ldots h_d k_d$ where d is a positive integer, $(h_1, \ldots, h_d) \neq (k_d, \ldots, k_1)$ and only h_1 and k_d can equal ∞ or 1. Then

(i) form the direct sum of the R-modules (indeed, Γ-modules) Γ/J^{h_s}, Γ/J^{k_s} for $1 \leq s \leq d$,

(ii) for every $1 \leq s \leq d$, replace the direct sum of Γ/J^{h_s}, Γ/J^{k_s} by their top glue,

(iii) for $1 \leq s < d$, bottom glue Γ/J^{k_s} and $\Gamma/J^{h_{s+1}}$,

(iv) finally, factor out the module built in (ii) through its submodule in (iii).

Let $M(q)$ denote the module constructed in this way from q. Actually we are interested, as said, in strings q on 2 and 3. More precisely we deal with the case where d is an integer ≥ 2, so the length of q is even and ≥ 4, and furthermore

$$h_1 = 2,\ h_2 = \ldots = h_d = 3,$$

$$k_1 = k_d = 2,\ 2 \leq k_2, \ldots, k_{d-1} \leq 3.$$

Observe that under these assumptions

$$(h_1, \ldots, h_d) = (2, 3, \ldots, 3), \quad (k_1, \ldots, k_d) = (2, \ldots, 2)$$

are neither equal nor symmetrical (in the sense that the former sequence does not coincide with the mirror image of the latter).
Let Q denotes the set of these strings. Hence 22 32, or 22 32 33 32, or 22 33 32, are in Q. In general, a string in Q has even length ≥ 4, and indeed is an ordered sequence of pairs 22, 32 and 33 having 22 as prefix, 32 as suffix and 32 or 33 as intermediate elements.
As pointed out in [5], Theorems 2.7 and 2.8, when q ranges over Q, the modules $M(q)$ built in the way described above are

- indecomposable
- pairwise nonisomorphic.

The latter claim also depends on our restrictions on the h_s and the k_s; in fact they ensure that the strings $q \in Q$ are unsymmetrical, as required in [5], Definition 2.4.
Also, observe that each $M(q)$ is pure injective, as a module of finite length over the commutative ring R.

Furthermore each $M(q)$ is finitely presented as a finitely generated module over a Noetherian ring. Let us illustrate and comment this in more detail.
First note that Γ itself is finitely generated as a module over R, and indeed $\mu_R(\Gamma) = 2$. A possible set of generators of Γ over R consists of 1 and β where $\beta(1)$ solves a suitable irreducible polynomial of degree 2 over k, say $X^2+a(1)X+b(1)$ with a, $b \in R$; in particular $F = k(\beta(1))$.
The relations on 1 and β defining Γ over R translate (or try to translate) this in the language of modules. Actually, as $1(1)$ and $\beta(1)$ are linearly independent over k, the only relations we can expect are of the form $s_0 + s_1\beta = 0$ with s_0 and s_1 in J. By the way, observe that J is principal as an ideal of Γ but has at least 2 generators over R; indeed, if $\pi \in J$ and $\pi\Gamma = J$, then $J = \langle \pi, \pi\beta \rangle$ over R. In particular s_0 and s_1 themselves can be written as linear combinations of π, $\pi\beta$ with coefficients in R.
Let us give an example of a possible relation defining Γ over R in terms of the generators 1 and β. As said, $\beta(1)$ is a root of $X^2 + a(1)X + b(1)$ over k, and hence

$$\beta^2 + a\beta + b = c\pi + d\pi\beta$$

for some suitable c and d in R. We cannot admit this equality as a relation of Γ over R because $\beta \notin R$ and hence there is no way to involve β^2. Anyway we can add among our relations the weaker condition we get by multiplying the two members of the previous equality by π, so

$$(\pi\beta)\beta + (\pi a)\beta + (\pi b)1 = (c\pi^2)1 + (d\pi^2)\beta$$

or also

$$(-c\pi^2 + \pi b)1 + (-d\pi^2 + \pi\beta + \pi a)\beta = 0.$$

Now let us consider any quotient Γ/J^h with $h \geq 1$, again viewed as a module over R. $1(h)$, $\beta(h)$ still are a set of generators, and relations have to include those saying that $\langle \pi, \pi\beta \rangle^h$ becomes 0, hence $\pi^h 1(h) = \pi^h \beta(h)$ and so on.

Both top gluing and factoring through a bottom glue preserve the property of being finitely generated. For instance consider the top glue between Γ/J^h and Γ/J^k for h, $k \geq 2$. Now a set of possible generators consists of the four pairs $(1(h), 1(k))$, $(\beta(h), \overline{\beta}(k))$ (where $\overline{\beta}(1)$ is the conjugate of $\beta(1)$ over k, so $\overline{\beta} \in R$ satisfies $\rho(\overline{\beta}) = \overline{\rho}(\beta)$) $(\pi(h), 0(k))$ and $((\pi\beta)(h), 0(k))$. Note that the old generators of Γ/J^h and Γ/J^k, hence $1(h)$, $\beta(h)$ and $1(k)$, $\beta(k)$, do not determine any element in the top glue as pairs like $(1(h), 0(k))$ and so on; this is because, for instance, $\rho(1(h)) = 1(1)$ while $\overline{\rho}(0(k)) = 0(1)$. Also observe that we could have chosen $(0(h), \pi(k))$ and $(0(h), (\pi\beta)(k))$ instead of $(\pi(h), 0(k))$, $((\pi\beta)(h), 0(k))$ as generators (this is easy to check and will be implicitly proved in the next lines).
Relations explain the connections between these generators on the basis of the algebraic properties of $\beta(1)$ and $\overline{\beta}(1)$. For instance, it is easily seen that

- $-(\pi(h), 0(k)) + \pi(1(h), 1(k)) = (0(h), \pi(k))$,
- $-((\pi\beta)(h), 0(k)) + \pi(\beta(h), \overline{\beta}(k)) = (0(h), (\pi\overline{\beta})(k))$,

- $-((\pi\beta)(h), 0(k)) + \pi\beta(1(h), 1(k)) = (0(h), (\pi\beta)(k))$.

Combining this and the elementary law $\beta(1) + \overline{\beta}(1) = a(1)$, that is,

$$\beta + \overline{\beta} + a = e\pi + f\pi\beta$$

for some suitable e, $f \in R$, one gets via some straightforward substitutions a non trivial relation connecting our 4 generators

$$(a\pi - e\pi^2 + \pi\beta - f\pi^2\beta)(1(h), 1(k)) + \pi(\beta(h), \overline{\beta}(k)) +$$

$$+(e\pi - a)(\pi(h), 0(k)) + (f\pi - 2)((\pi\beta)(h), 0(k)) = (0(h), 0(k)).$$

Finally, when factoring out through a bottom glue of Γ/J^k and Γ/J^h, one has to add further relations, just explaining the bottom glue procedure.

Let us illustrate all this in more detail; we refer to Example 7.3 and describe what happens in that framework.

Example 7.4 $\Gamma = \mathbf{C}[[X]]$ has two generators 1, i over $R = \mathbf{R} + X\mathbf{C}[[X]]$. The basic relation between them, translating in the framework of R-modules the key equality $i^2 + 1 = 0$ is $X\,1 + (iX)\,i = 0$.
For $h \geq 2$ (and $h \leq 3$ if you like), Γ/J^h (that is, $\mathbf{C}[[X]]/\langle X^h \rangle$) as a module over R has

- two generators $1(h)$, $i(h)$,
- new relations such as $X^h\,1(h) = X^h\,i(h) = 0(h)$.

Now let us describe the top glue of $\mathbf{C}[[X]]/\langle X^2 \rangle$ and $\mathbf{C}[[X]]/\langle X^3 \rangle$.

- Here a possible set of generators consists of $(1(2),\, 1(3))$, $(i(2),\, -i(3))$, $(X(2),\, 0(3))$, $((iX)(2),\, 0(3))$. Observe that the old generators of $\mathbf{C}[[X]]/\langle X^2 \rangle$ and $\mathbf{C}[[X]]/\langle X^3 \rangle$, hence $1(2)$, $i(2)$ and $1(3)$, $i(3)$ respectively, cannot define directly any element in the top glue. Also, notice that $(0(2),\, X(3))$ can be obtained as $X(1(2),\, 1(3)) - (X(2),\, 0(3))$ and something similar can be calculated for $(0(2),\, (iX)(3)))$;
- relations are

$$-(X(2),\, 0(3)) + X(1(2),\, 1(3)) = (X(2),\, 0(3)) + iX(i(2),\, -i(3))$$

$$(= (0(2),\, X(3))),$$

$$-((iX)(2),\, 0(3)) + iX(1(2),\, 1(3)) = ((iX)(2),\, 0(3)) - X(i(2),\, -i(3))$$

$$(= (0(2),\, (iX)(3))),$$

moreover

$$X(X(2),\, 0(3)) = -iX((iX)(2),\, 0(3)) = (0(2),\, 0(3)),$$

$$iX(X(2),\, 0(3)) = X((iX)(2),\, 0(3)) = (0(2),\, 0(3)),$$

$$X^3(1(2),\, 1(3)) = iX^3(1(2),\, 1(3)) = (0(2),\, 0(3)),$$

$$X^3(i(2),\, -i(3)) = iX^3(i(2),\, -i(3)) = (0(2),\, 0(3)),$$

and so on.

Finally let us deal with a bottom glue, for instance between $\mathbf{C}[[X]]/\langle X^3\rangle$ and $\mathbf{C}[[X]]/\langle X^2\rangle$. The corresponding quotient requires new relations on the pairs in $\mathbf{C}[[X]]/\langle X^3\rangle \oplus \mathbf{C}[[X]]/\langle X^2\rangle$, such as

$$X(X(3), 0(2)) = (0(3),\ X(2))$$

(when the latter element is defined as before) and

$$X(iX(3), 0(2)) = -(0(3),\ (iX)(2)).$$

Now let us come back to the general setting. We introduce an order relation $\preceq_Q$ in Q. $\preceq_Q$ is defined as follows (here $q \prec_Q q'$ abbreviates, as usual, $q \preceq_Q q'$ and $q \neq q'$, while $\emptyset$ denotes the empty word). Basically we put

$$\emptyset \prec_Q 33 \prec_Q 32.$$

In more detail, for q and q' in Q we define $q \preceq_Q q'$ if and only if either

- q is an initial segment of q', or
- the leftmost pair where q and q' differ is 33 in q and 32 in q'.

For instance, $22\,33\,32 \prec_Q 22\,32 \prec_Q 22\,32\,32$.

Lemma 7.5 *$\preceq_Q$ is a dense linear order without endpoints in Q. In particular $(Q, \preceq_Q)$ is isomorphic to the order of rationals.*

Proof. The latter assertion is a direct consequence of the former because Q is countable. So let us check the first statement about $\preceq_Q$. Reflexivity is trivial. If q, $q' \in Q$ satisfy both $q \preceq_Q q'$ and $q' \preceq_Q q$, then q and q' cannot differ at any place, whence each string is an initial part of the other and in conclusion $q = q'$. Now take q, q', $q'' \in Q$ such that $q \preceq_Q q'$ and $q' \preceq_Q q''$, we claim that $q \preceq_Q q''$. This is clear when $q = q'$ or $q' = q''$. So let us assume $q \prec_Q q'$, hence either

a) q is a *proper* initial segment of q', or

b) the leftmost different pair in q and q' is 33 in q and 32 in q'.

In a similar way we can assume $q' \prec_Q q''$, which means that even q' and q'' satisfy the same dichotomy as q and q', let a$'$), b$'$) denote the corresponding cases. If a) and a$'$) hold, then clearly $q \preceq_Q q''$ because q is shorter. Assume a) and b$'$); if the leftmost difference between q' and q'' arises before the end of q, then 33 occurs in q and q' in a place where q'' has 32, whence $q \preceq_Q q''$; otherwise q is an initial segment of q'' and, again, $q \preceq_Q q''$. The conjunction of a$'$) and b), or of a$'$) and b$'$) can be treated in a similar way.
So $\preceq_Q$ is a partial order, and it is easily seen to be linear, too.
Observe that every string $q = 22 \ldots 32$ in Q is preceded by $22 \ldots 33\,32$ and followed by $22 \ldots 32\,32$. Thus $(Q, \preceq_Q)$ has no endpoints.
Finally let us prove density. Suppose $q \prec_Q q'$ in Q. If q has the form $22 \ldots 32$

and q' is $22\ldots 32\ldots 32$ and lengthens q, then $22\ldots 32\ldots 33\,32$ can be inserted between them. On the other side, if q is of the form $22\ldots 33\,q(1)\,32$ and q' is $22\ldots 32\,q'(1)\,32$ for some suitable $q(1)$ and $q'(1)$, then $22\ldots 32\,q'(1)\,33\,32$ is an intermediate element between them. Hence $\preceq_Q$ is dense, as claimed. $\dashv$

Every $q \in Q$ was associated with a module $M(q)$ over R. Let us also introduce a distinguished tuple $m(q)$ of elements in $M(q)$: $m(q)$ is just the 4-uple of generators of the leftmost top glue in $M(q)$ (that corresponding to the pair 22). So in Examples 7.3 and 7.4 $m(q)$ is consisting of the pairs $(1(2), 1(3))$, $(i(2), i(3))$, $X(2), 0(3))$, $((iX)(2), 0(3))$ in the leftmost top glue.
As $M(q)$ is finitely presented, there is a pp-formula φ_q in four free variables over R generating the pp-type of $m(q)$ in $M(q)$. Basically φ_q asserts that there are new elements –the generators of the top glues on the right in $M(q)$– satisfying all the relations that these top glues and the bottom glues in $M(q)$ require.
Now we show that the pointed R-modules $(M(q), m(q))$ $(q \in Q)$ satisfy the condition 1 in Theorem 1.1.

Lemma 7.6 *For $q \prec_q q'$ in Q, there is a pointed morphism of $(M(q), m(q))$ to $(M(q'), m(q'))$. Anyway $(M(q), m(q))$ and $(M(q'), m(q'))$ are not isomorphic.*

Proof. We already excluded any possible isomorphism (pointed or not) between $M(q)$ and $M(q')$ for $q \neq q'$ in Q. So it remains to prove the existence of a pointed morphism of $(M(q), m(q))$ to $(M(q'), m(q'))$ when $q \prec_Q q'$. Let us distinguish two cases.
Case 1: q is an initial segment of q'. More generally, let us assume that $M(q')$ is obtained by $M(q)$ via some bottom glue on the right (apart from the fact that the rightmost pair in q is 32). Then a pointed morphism of $(M(q), m(q))$ to $(M(q'), m(q'))$ is defined by sending of course $m(q)$ to $m(q')$ and the other generators of $M(q)$ to themselves (regarded as elements in $M(q')$); finally in the rightmost top glue of $M(q)$ every generator a is sent to the class of the pair $(a, 0)$ with respect to the bottom glue following on the right in $M(q')$. One checks that the relations of $M(q)$ are preserved in $M(q')$ and hence a pointed morphism is really determined.
Case 2: Let $q = 22\ldots 33\,q(1)$ and $q' = 22\ldots 32\,q'(1)$ for some suitable $q(1)$ and $q'(1)$. Indeed we can assume that q' is just $22\ldots 32$, hence that $q'(1)$ is empty. In fact, if q'' is the string built in this way and a pointed morphism f of $(M(q), m(q))$ into $(M(q''), m(q''))$ can be found, then a pointed morphism of $(M(q), m(q))$ into $(M(q'), m(q'))$ is easily obtained as the composition of f and the pointed morphism of $(M(q''), m(q''))$ into $(M(q'), m(q'))$ given in Case 1. So we can assume that $M(q)$ replaces on the right hand of the rightmost top glue of $M(q')$ a copy of Γ/J^2 by Γ/J^3 and possibly adds something more on the right. As Γ/J^2 is a homomorphic image of Γ/J^3, f is easily determined: just project Γ/J^3 onto Γ/J^2 in the natural way, and send the generators of $M(q)$ on the left of Γ/J^3 to themselves and those on the right to 0. $\dashv$

At this point, we introduce a new countable dense linear order without endpoints $(T, \preceq_T)$ among the strings $h_1k_1\ldots h_dk_d$ and we associate with every $t \in T$ a pointed module $(M(t), m(t))$ over R. Needless to say, $(T, \preceq_T)$ is expected to be isomorphic to the order of rationals. To do this, we proceed as for Q with the following changes.

- First, we consider the strings t of the form $33\,23\ \ldots\ 22$. Hence 22 is the *rightmost* pair in t, and is preceded on the left by pairs of the form 23 or 33, while the leftmost pairs are 33 23. So the strings in T can be viewed as the mirror images of the ones of Q with an additional prefix 33. Formally speaking, what we get in this way are strings $h_1k_1\ \ldots\ h_dk_d$ of even length $2d \geq 6$ (so $d \geq 3$) on 2 and 3, also satisfying

$$2 \leq\ h_1, \ldots, h_d \leq 3, \quad h_1 = 3, \quad h_2 = 2, \quad h_d = 2,$$

$$k_1 = \ldots = k_{d-1} = 3, \quad k_d = 2.$$

  In particular, observe that $(h_1,\ \ldots,\ h_d) = (3,\ 2,\ \ldots,\ 2)$ and $(k_1,\ \ldots,\ k_d) = (3,\ 3,\ \ldots,\ 2)$ are not symmetrical.

- The order relation $\preceq_T$ is defined by putting $\emptyset \prec_T 33 \prec_T 23$ and then by extending in the obvious way. So, for $q,\ q' \in Q$, $q \preceq_Q q'$ if and only if $33\,\mu(q) \preceq_T 33\,\mu(q')$ where μ denotes the mirror image. It is easily seen that even $(T,\ \preceq_T)$ is a countable dense linear order without endpoints, so is isomorphic to the order of rationals.

- For $t \in T$, $M(t)$ and $m(t)$ are defined as in the case of Q with the only difference that now we move from right to left, so $m(t)$ is the pair of generators of the rightmost top glue in $M(t)$, that regarding 22. Note that T and Q are disjoint, so there is no ambiguity between $M(q)$ and $M(t)$ for $q \in Q$ and $t \in T$ in our notation.

What was observed about $(M(q), m(q))$ for $q \in Q$ applies also to the new pointed modules $(M(t), m(t))$ for $t \in T$ via the obvious adaptations. In particular the hypothesis 2 in Theorem 1.1 holds. So let us check that the $(M(q), m(q))$ and the $(M(t), m(t))$, with $q \in Q$ and $t \in T$, satisfy even the additional conditions 3-5 (which concludes our proof of Theorem 7.1).
Accordingly, for $t \in T$ and $q \in Q$, let us consider the pushout $M(t.q)$ of $(M(t), m(t))$ and $(M(q), m(q))$. Roughly speaking, this means

- to take the leftmost top glue in $M(t)$ –where the 4-uple $m(t)$ lies– and the morphism sending it to $M(t)$ first by embedding it as a direct summand and then by factoring out through the leftmost bottom glue, so "fixing" $m(t)$ pointwise;

- similarly to take the rightmost top glue in $M(q)$ –where the 4-uple $m(q)$ lies– and the specular "natural" morphism sending it to $M(q)$;

and then to identify in the direct sum of $M(t)$ and $M(q)$ the images of these top glues (in particular $m(t)$ and $m(q)$. So the structure of $M(t.q)$ is similar to that of $M(t)$ and $M(q)$, as $M(t.q)$ can be regarded as the R-module corresponding to the string obtained by juxtaposing t and q and then identifying

the rightmost 22 in t and the leftmost 22 in q (by the way, let $t.q$ denote the result of this operation; for instance, if $t = 33\,23\,33\,22$ and $q = 22\,33\,32$, then $t.q = 33\,23\,33\,22\,33\,32$). Hence $M(t.q)$ has a *central* top glue that corresponds to the pair 22 in both $M(t)$ and $M(q)$ and where the sequences $m(t)$ and $m(q)$ are identified; $M(t)$ follows on the left, and $M(q)$ on the right.
Each $M(t.q)$ is still finitely presented and indecomposable pure injective –just as $M(t)$ and $M(q)$–, in particular its endomorphism ring is local, which ensures 3.
Also, for $t \neq t'$ in T and $q \neq q'$ in Q, there is no (pointed) isomorphism between either $M(t.q)$ and $M(t'.q)$, or $M(t.q)$ and $M(t.q')$. This follows from Theorem 2.8 (i) in [5] because, for $t \neq t'$, $t.q$ and $t'.q$ are not equal and the sequence of elements of odd place $3\ldots$ in $t.q$ cannot be the mirror image of the sequence of elements of even place $\ldots 2$ in $t'.q$. This shows 4.
It remains to check 5, so to exclude for $t \in T$ and $q \in Q$ any pointed morphism between $(M(t), m(t))$ and $(M(q), m(q))$ in either direction. Recall that both $m(t)$ and $m(q)$ coincide with the generators of the top glue corresponding to 22 in $M(t)$ and $M(q)$. In particular, in the case of Examples 7.3 and 7.4 this regards $(1(2),\, 1(2))$, $(i(2),\, i(2))$, $(X(2),\, 0(2))$, $((iX)(2),\, 0(2))$. Still refer to these examples and observe that the sentence saying that X divides $(X(2),\, 0(2))$ is

> true in $M(t)$, where $(X(2),\, 0(2))$ equals the element $X(0(\ldots),\, X(3))$ in its bottom glue with the top glue on the left (dots refer here to the first summand Γ/J^h in this top glue),
>
> but false in $M(q)$.

So no pointed morphism is possible from $(M(t), m(t))$ to $(M(q), m(q))$. In the same way, the sentence saying that X divides $(0(2),\, X(2))$ (introduced as $X(1(2),\, 1(2)) - (X(2),\, 0(2))$) is

> true in $M(q)$, where $(0(2),\, X(2))$ equals the element $X(X(3),\, 0(\ldots))$ in its bottom glue with the top glue on the right (dots refer here to the second summand of this top glue),
>
> but false in $M(t)$.

This excludes any pointed morphism from $(M(q), m(q))$ to $(M(t), m(t))$, too.
The argument over an arbitrary R is basically the same. Just use $(\pi(2),\, 0(2))$ for $M(t)$ and $(0(2),\, \pi(2)) = \pi(1(2),\, 1(2)) - (\pi(2),\, 0(2))$ for $M(q)$. Let us exclude, for instance, any pointed morphism from $(M(t), m(t))$ to $(M(q), m(q))$. It suffices to notice that the sentence saying that π divides $(\pi(2),\, 0(2))$ is true in $M(t)$ as $(\pi(2),\, 0(2))$ equals $\pi(0(\ldots),\, \pi(3))$ in its bottom glue on the left, but is false in $M(q)$. Otherwise there are r_0, r_1, r_2, $r_3 \in R$ such that $(\pi(2),\, 0(2))$ and $\pi(r_0(1(2),\, 1(2)) + r_1(\beta(2), \overline{\beta}(2)) + r_2(\pi(2),\, 0(2)) + r_3((\pi\beta)(2),\, 0(2)))$ coincide in $M(q)$ and consequently in its leftmost top glue (as it is straightforward to check). This establishes the equality

$$(\pi(2),\, 0(2)) =$$

$$= \pi(r_0(1(2),\, 1(2)) + r_1(\beta(2), \overline{\beta}(2)) + r_2(\pi(2),\, 0(2)) + r_3((\pi\beta)(2),\, 0(2))) =$$
$$= \pi((r_0 + r_1\beta)(2),\, (r_0 + r_1\overline{\beta})(2))).$$

Equalizing the second coordinates we get

$$0(2) = \pi(r_0 + r_1\overline{\beta})(2).$$

It follows that $r_0 + r_1\overline{\beta} \in J$, whence r_0, $r_1 \in J$ because $1(1)$ and $\overline{\beta}(1)$ are linearly independent over k. Thus $\pi(2) = \pi(r_0 + r_1\beta)(2) = 0(2)$ –a contradiction–.

This concludes the proof of Theorem 7.1 and the whole section. ⊣

8 Dedekind-like rings: Homomorphic images

We deal here with modules over homomorphic images of Dedekind-like rings R. We wish to investigate superdecomposables, width and Krull–Gabriel dimension in this enlarged setting.
Of course, we may expect that width and Krull–Gabriel dimension are sometimes defined. For instance, this is trivially true when we refer to k as a homomorphic image of R and so we treat k-vectorspaces. Also, discrete valuation domains are homomorphic images of strictly split Dedekind-like rings ([4], Lemma 2.19), and we saw in Fact 6.1 that they have width and Krull–Gabriel dimension. Moreover, [4], Theorem 5.2, points out that Klein rings may be homomorphic images of (local) Dedekind-like rings (when the residue field is imperfect), and we know that width and Krull–Gabriel dimension are defined for every Klein ring.
Anyway, it should be also expected that width and Krull–Gabriel dimension are lacking in most cases. Let us discuss this in more detail.
Due to what we saw in § 2, restricting our attention to the local framework makes sense also when dealing with homomorphic images of Dedekind-like rings. In fact, let R be a commutative ring and $I \neq R$ be a proper ideal of R, and take any maximal ideal P of R extending I (then P/I is a maximal ideal in R/I). Under these assumptions the localization of R/I at P/I is a homomorphic image of R_P and, if R is Dedekind-like, then also this localization is. In this way, an arbitrary localization of a homomorphic image R/I of a Dedekind-like ring R can be also seen as a homomorphic image of a *local* Dedekind-like ring. Hence negative results about width and Krull–Gabriel dimension, as well as positive results about superdecomposables, transfer to the original rings from their localizations. So let us provide in the remainder of this section an overview of what happens under the locality assumption.
So, fix a local Dedekind-like ring R (and still refer to the notation introduced in § 2). Discrete valuation domains were already treated at the very beginning of this section, so we can assume from now on that R is either unsplit or split. In the former case, we keep the assumption that Γ/J is a separable extension of k. Recall that a non strictly split R possesses the same proper Artinian homomorphic images as its (strictly split) completion. So we can assume without

loss of generality that R is unsplit or *strictly* split. Let $I \neq 0$, $I \subseteq J$ be an ideal of R. According to [5], Theorem 11.7, either

- $I = \Gamma I$ and $R/I = \{a \in \Gamma/I \,:\, \rho(a) \in k\}$ is the pullback of a commutative square just as R, but referring to Γ/I and, via ρ, to $k \subset F$, or
- $I \neq \Gamma I$, in which case every R/I-module is the direct sum of a free R/I-module and a $R/\Gamma I$-module, and $R/\Gamma I$ is a pullback as in the previous case.

The case of free R/I-modules is easy to treat. In fact, due to [5], Lemma 11.4, if $I \neq \Gamma I$, then the R-module R/I has finite length. As R is commutative and hence pp-subgroups are R-submodules, even the lattice of pp-subgroups of R/I has finite length. This lattice is isomorphic to that of pp-subgroups of any direct power of R/I, and consequently to that of any free R/I-module. Thus all these lattices are of finite length, and so have m–dimension 0 and width 0.
In this way we are led to consider modules over R/I for $0 \neq I \subseteq J$ an ideal of Γ. So, all throughout the remainder of this section, let I denote an ideal with these properties. By [5], Lemma 11.6, Γ/I is either

- an *Artinian valuation ring*, meaning an Artinian local principal ideal ring, or
- a direct sum of two Artinian valuation rings, or
- a direct sum of an Artinian valuation ring and a discrete valuation ring.

Now assume that R is unpslit. Then Γ/I is an Artinian valuation ring. Due to [5], Theorem 11.9, (i), a module as built in the proof of Theorem 7.1 can be seen as a module over R/I if and only if Γ/I has length at least 3 (as a module over itself).
Something similar holds even in the strictly split case, again due to [5], Theorem 11.9. In fact, statement (ii) in this reference ensures that, if R is strictly split and consequently Γ is the direct sum of two discrete valuations domains Γ_1 and Γ_2, $J = J_1 \oplus J_2$ where, for every $t = 1,\, 2$, J_t is the Jacobson radical of Γ_t and I itself decomposes as $I_1 \oplus I_2$ with I_t an ideal of Γ_t ($t = 1,\, 2$), then an R-module as in [12], Theorem 5.8, and as in our Theorem 6.3 before, is a module over R/I if and only if, for every $t = 1,\, 2$, the maximal length of the modules Γ_t/I_t building it is less than or equal to the length of Γ_t/I_t.
So the approaches used in Theorem 7.1 in the unsplit case and in Theorem 6.3 and ultimately in [12], Theorem 5.8, in the strictly split one actually apply to a larger setting and ensure the following.

Theorem 8.1 *Let R be local Dedekind-like ring, $I \subseteq J$ be a non-zero proper ideal of Γ. For R unsplit, assume that Γ/J is a separable extension of k.*

1. *If R is unsplit and Γ/I has length ≥ 3 as a module over itself, then R/I has neither width nor Krull–Gabriel dimension. Furthermore, at least when R/I is countable, R/I possesses superdecomposable pure injective modules.*

2. *Let R be strictly split, $I = I_1 \oplus I_2$ with I_t an ideal of Γ_t ($t = 1, 2$). Assume that each Γ_t/I_t ($t = 1, 2$) has length ≥ 2 and at least one of them has length ≥ 3. Then R/I has neither width nor Krull–Gabriel dimension. Furthermore, at least when R/I is countable, R/I possesses superdecomposable pure injective modules.*

This answers negatively our questions for most local R, as expected. Moreover an arbitrary (possibly non-local) Dedekind-like ring R having some maximal ideal P for which R_P corresponds to the description in the statement of Theorem 8.1 inherits from this localization lack of width and Krull–Gabriel dimension and, at least in the countable case, superdecomposable pure injectives.

What can we say in the remaining cases? Strictly split rings are easy to handle. In fact the following holds.

Proposition 8.2 *Let R be a local Dedekind-like unsplit ring, $I \subseteq J$ be a non-zero proper ideal of Γ, $I = I_1 \oplus I_2$ with I_t an ideal of Γ_t ($t = 1, 2$). Assume that each Γ_t/I_t ($t = 1, 2$) has length ≤ 2. Then R/I has both width and Krull–Gabriel dimension (and indeed its Krull–Gabriel dimension is ≤ 2).*

Proof. It suffices to show our thesis when each Γ_t/I_t has length exactly 2. Actually the Krull–Gabriel dimension of R/I is 2 in this case. In fact every Γ_t/I_t has radical square 0, whence $J^2 = J_1^2 \oplus J_2^2$ is 0 modulo I. Moreover $\mu(R/I)$ is 2. Thus the same argument given at the end of § 5 about Klein rings applies to R/I. ⊣

Before concluding let us state the only case of homomorphic images of local R to be solved yet. We express it again in terms and R and I (the kernel of the corresponding homomorphism from R).

> R is unsplit, and Γ is not a separable extension of k or the length of Γ/I is at most 2.

One may conjecture here that both width and Krull–Gabriel dimension are defined at least under the separability assumption.

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