Lie groups and o-minimality

A guide on groups definable in o-minimal structures

Annalisa Conversano and Alf Onshuus

Contents

Preface

This book is about groups definable in o-minimal expansions of real closed fields through the lens of Lie theory. In the seminal work [**61**] from 1988, Anand Pillay proved that every group definable in an o-minimal expansion of the real field admits a manifold structure compatible with the group operation which makes it a finite-dimensional Lie group. This result prompted two big directions of research in the field. One direction studies which structural properties of Lie groups extend to groups definable in an o-minimal expansion of a real closed field. The other investigates the limitations that definability entails in a Lie group (for example, classifying which Lie groups are definable in an o-minimal expansion of the real field).

Our aim is to provide an exposition of the structure of groups definable in o-minimal theories, including the proofs that many features and structure decompositions that are known for Lie groups are also true for groups definable in o-minimal structures. This indicates that groups definable in o-minimal structures are a model theoretic counterpart of Lie groups, echoing the fact that o-minimality represents a generalization of semialgebraic and subanalytic geometry as proposed by Lou van den Dries' "Tame topology and o-minimal structures," ([**72**]).

This book is a natural follow-up to van den Dries' book focusing on groups, and we view Tame Topology as a starting point for the studies we undertake. We do recall the results we need from van den Dries' book, but we do not include any of the proofs. Other than that, we provide, as much as possible, a self-contained text providing many of the results which have been developed in the area for almost forty years. Accordingly, we have a threefold goal for this text. That it can be used as a textbook for graduate courses for students familiar with o-minimality, that it can serve as a reference for model theorists working in the field, but also as a good source for mathematicians working in areas concerning Lie groups who are interested in using o-minimality in their respective fields.

Because we intend this book to be a valuable reference for mathematicians interested in applying o-minimality, there is a focus on groups definable in o-minimal expansions of the real field. For example, we provide a complete characterization of the structure of Lie groups which are Lie isomorphic to groups definable in an o-minimal expansion of the real field, solving the problem mentioned above. Nevertheless, the book is not restricted to this setting. We do define and develop concepts in the context of o-minimal expansions of an arbitrary real closed field, because we feel that it is the right approach. Furthermore, even for those interested only in using o-minimality on expansions of the real field, the proofs in this general context offer valuable insights into the tools that o-minimality can (and cannot) provide. The focus on the real field did, however, influence our choice of topics. Although quite central and important in the study of groups definable in o-minimal theories, we will not be covering compact domination, the infinitesimal subgroup G^{00}, Keisler measures and many other influential themes. We recommend the excellent surveys [**48**] and [**50**] by Margarita Otero and Kobi Peterzil for an overview of some of these themes.

Our goal of self containment was limited because of the difficulty and length of some of the proofs of results we needed since some of them go beyond what we thought should be the scope of the book:

- Although we do provide a summary and include some of the proofs on classic Lie algebras and Lie groups, including proofs of the more fundamental theorems such Lie's Theorems or Ado's Theorem went beyond what we could include in this text and we had to refer the reader to [**42**], [**44**] and [**30**] for the proofs of some of the deeper and more intricate results on the area. Similarly, for the theory of linear algebraic groups, for which we use [**6**] as reference.
- We were also not able to include the proofs of o-minimality of the exponential field [**76**], of the restricted analytic fields ([**73**]) and of $\mathcal{R}_{\mathrm{Pfaff}}$.
- Probably the most glaring omission that we were not able to include in the text is a proof of the Torsion Theorem (Fact 6.1) which is central to our results but that we found that including it would imply extending this text beyond what we found reasonable. We included a section (Section 6.5 of Chapter 6) discussing different proofs and the theory that one needs to develop for each of them.

Chapters 2–4 are the foundational chapters, devoted to developing some aspects of differential geometry and algebraic topology (such as Lie algebras and universal covers) in the setting of definable groups. Together with the material from o-minimal topology developed in *Tame Topology* and summarized in Chapter 1 (such as dimension, connected components and Euler Characteristic) they provide the framework that will allow us to understand the structural theory of groups definable in o-minimal structures.

The structural Chapters 5, 6 and 7 focus on particular cases of definable groups (semisimple, definably compact and torsion-free respectively). The first sections of each of these chapters develop the key properties and characteristics of definable groups, which we find can be particularly interesting for people coming from o-minimality. In the later sections we study both the linear case (usually over arbitrary real closed fields) and what definability implies for a real Lie group, which may be especially relevant for people looking to apply o-minimality to other branches of mathematics.

The various definable aspects come together in Chapter 8, where we describe the structure of a definable group and give sufficient conditions for a continuous homomorphism between definable groups to be definable. In Chapter 9 we give a complete characterization of when a Lie group is Lie isomorphic to a group definable in an o-minimal structure, and give necessary conditions for a Lie homomorphism between definable groups to be definable in some o-minimal expansion of the real field.

Prerequisites

As mentioned above, we assume familiarity with [**72**]. We also need to assume knowledge in linear algebra, general topology, and differential geometry at an advanced undergraduate/basic graduate course level.

Acknowledgments

This book would not have been possible without the work and help of many people. First, we would like to thank our coauthors Sacha Post and Sergei Starchenko, who were a key part of the research that concluded in the characterization of Lie groups which can be represented in an o-minimal theory which is presented in the final chapters. This book also includes work of many people who developed the theory of groups definable in o-minimal structures; they not only contributed with the research that made this book possible, but are also responsible to have created a wholesome and welcoming community of which we are proud to be a part of. We acknowledged many of them in the Historical Notes at the end of each chapter, but we are sure to have missed some. We would also like to give a special mention to Alessandro Berarducci and Charles Steinhorn who introduced us to o-minimality, as well as Margarita Otero, Kobi Peterzil, Anand Pillay.

The idea of this book came to be after a class taught in the subject at Universidad de los Andes. We are grateful to the students Rivaldo Cifuentes, Felipe Estrada, Jesús Ospino, Gloria Pinilla, Santiago Pinzón, and David Rincón whose attention, comments and insights motivated us to undertake this endeavor. We would also like to thank Chris Miller, who encouraged us to write this book when we were just starting to considet the possibility.

CHAPTER 1

Introduction to Model Theory and o-minimality

Although we assume familiarity with o-minimal theory as covered in *Tame Topology* ([**72**]), for the sake of completeness we will recall the definitions and results we will use throughout the book. Except for the last section, this chapter is mostly a recollection of definitions and results from [**72**]. The second section of the chapter defines manifolds in the o-minimal context, which builds on the smoothness results from *Tame Topology*, but we need to go beyond what is covered there for our purposes. However, since it is a general setting (meaning we are not studying definable groups but general definable sets) we felt this chapter was the best place for it.

We will therefore divide this chapter in two sections. Section 1.1 recalls definitions and results from *Tame Topology* that will be used throughout this book. Section 1.2 develops o-minimal differential geometry beyond what is covered in *Tame Topology.*

1.1. Basic notions and topology

We will present the results and definitions in the same order as they appear in *Tame Topology.* All references in this section are references to *Tame Topology*, quoted by stating the chapter in roman numerals together with the number of the statement.

The notion of o-minimality is based on the notion of definability. For this we fix a language which will always include $\{<, +, \cdot, 0, 1\}$. We will work in a structure $\mathcal{R}$ interpreting this symbols and assume that $<$ is a dense linear order. By *definable* we mean definable in first order logic using the symbols in the language.

Alternatively (as presented in *Tame Topology*), we can define a family $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ where each $\mathcal{B}_n$ is a family of subsets of $\mathcal{R}^n$ closed under boolean operations (union, intersection and complement) and such that:

(1) $A \in \mathcal{B}_n \Rightarrow A \times \mathcal{R}, \mathcal{R} \times A \in \mathcal{B}_{n+1}$.
(2) $\{(x_1, \dots, x_n) \mid x_i = x_j\} \in \mathcal{B}_n$
(3) $A \in \mathcal{B}_n \Rightarrow \pi(A) \in \mathcal{B}_n$ where π is the projection from $\mathcal{R}^{n+1}$ to $\mathcal{R}^n$ forgetting the last coordinate.
(4) Any singleton $\{p\} \in \mathcal{B}_1$, $\{(x, y) \mid x < y\} \in \mathcal{B}_2$ and both $\{(x, y, z) \mid x + y = z\}$ and $\{(x, y, z) \mid x \cdot y = z\}$ are sets in $\mathcal{B}_3$.

Definition 1.1. The structure $\mathcal{R}$ is *o-minimal* if every definable subset of $\mathcal{R}$ is a finite union of intervals and points.

We will equip $\mathcal{R}$ with the interval topology, and $\mathcal{R}^m$ with the corresponding product topology. Unless specifically mentioned otherwise, the terms *continuity*, *limit*, and for any set X the terms *closure* $\overline{X} = cl(X)$ of X and *interior* X° of X will be assumed to be with respect to this topology. Similarly, the *boundary* $bd(X)$ will be defined as $cl(X) \setminus X$ and we will say that A is *open in* B is $A \subset B$ and $A = B \cap U$ for some open U.

In general, except when $\mathcal{R} = \mathbb{R}$ the set $\mathcal{R}$ will not be connected. However, the following notion provides a very adequate substitute.

Definition 1.2. A definable set X is *definably connected* if it is not the union of two disjoint *definable* open sets.

EXERCISE 1.1 (Intermediate Value Theorem). *Let $f : [a, b] \to \mathcal{R}$ be a definable continuous function with $a, b \in \mathcal{R}$ and $a < b$. Assume that for some s we have $f(a) \leq s \leq f(b)$. Then for some $c \in (a, b)$ we have $f(c) = s$.*

The following are Propositions I 4.2 and I 4.6.

Fact 1.3. *If $(G, \odot, <)$ is an ordered group whose theory is o-minimal, then G is abelian and divisible.*

If $(G, +, \cdot, <)$ is an ordered field whose theory is o-minimal, then G is real closed.

In light of the previous results, we will assume from now on that $\mathcal{R}$ is a real closed field with field operations $+, \cdot$ and compatible order $<$.

Tarski-Seidenberg Theorem (Theorem II, 2.7 is the special case when $\mathcal{R} = \mathbb{R}$) states that all definable sets in a real closed field are in fact definable without quantifiers, showing that the real field is o-minimal. This also shows that the theory of real closed fields is complete, which will allow us to do some *transfer theorems:* Any first order statement is true in $(\mathbb{R}, +, \cdot, <)$ if and only if it is true in $(\mathcal{R}, +, \cdot, <)$ for any real closed field $\mathcal{R}$.

1.1.1. Cell Decomposition Theorem and some applications. The most important structure theorem for definable subsets in an o-minimal theory is the Cell Decomposition Theorem.

Definition 1.4. For any sequence $(i_1, \ldots, i_n)$ of 0's and 1's a $(i_1, \ldots, i_n)$-cell will be a subset of $\mathcal{R}^n$. We define the notion of cell inductively (on n) as as follows:

A (0)-cell is a point in $\mathcal{R}$. a (1)-cell is an interval in $\mathcal{R}$.

If D is a $(i_1, \ldots, i_n)$-cell and f, g are definable continuous functions from D to $\mathcal{R}$ and such that $f(x) < g(x)$ for all x, then

$$(f, g) := \{(\bar{x}, x_{n+1}) \mid \bar{x} \in D \text{ and } f(\bar{x}) < x_{n+1} < g(\bar{x})\}$$

is a $(i_1, \ldots, i_n, 1)$-cell, and

$$\Gamma(f) := \{(\bar{x}, x_{n+1}) \mid \bar{x} \in D \text{ and } f(\bar{x}) = x_{n+1}\}$$

is a $(i_1, \ldots, i_n, 0)$-cell.

The following is Proposition III 2.9.

Fact 1.5. *All cells are definably connected.*

EXERCISE 1.2. *Prove that all $(1, 1, 1, \ldots, 1)$-cells are open.*

EXERCISE 1.3. *Prove that all cells are in definable bijection with $(0, 1)^k$ for some k. (This is not true without assuming that $\mathcal{R}$ is a field).*

Definition 1.6. A cell decomposition if $\mathcal{R}^m$ is a partition of $\mathcal{R}^m$ into cells with the following property defined inductively on m:

- For $m = 1$ a cell decomposition is just a partition of the space into finitely many intervals and points.
- A cell decomposition of $\mathcal{R}^{m+1}$ is a finite partition $\mathcal{D}$ of $\mathcal{R}^{m+1}$ into cells such that, if $\pi : \mathcal{R}^{m+1} \to \mathcal{R}^m$ is the projection into the first m coordinates, the set of projections $\{\pi(A) \mid A \in \mathcal{D}\}$ is a cell decomposition of $\mathcal{R}^m$.

If $A_1, \dots, A_k$ are subsets of $\mathcal{R}^n$, we say that a cell decomposition $\mathcal{D}$ of $\mathcal{R}^n$ *partitions* A if for any $C \in \mathcal{D}$ and any i either $C \cap A_i = \emptyset$ or $C \subseteq A_i$.

The Cell Decomposition Theorem (Theorem III 2.10) states:

Fact 1.7.

- *Let $A_1, \dots, A_k$ be definable subsets of $\mathcal{R}^n$. Then there is a cell decomposition of $\mathcal{R}^n$ partitioning $A_1, \dots, A_k$.*
- *Let $f : A \to \mathcal{R}$ be a definable function, with $A \subset \mathcal{R}^m$. Then there is a cell decomposition $\mathcal{D}$ of $\mathcal{R}^m$ partitioning A and such that for any cell $C \in \mathcal{D}$ with $C \subset A$ the restriction $f|_C$ of f to the cell C is continuous.*

The Cell Decomposition Theorem has many easy consequences. The following is Proposition III 2.18.

Fact 1.8. *Let $X \subset \mathcal{R}^m$. Then X has finitely many definably connected components.*

By definition, any cell is a $(i_1, \dots, i_n)$-cell for some sequence $(i_1, \dots, i_n)$. We define the *dimension* of C to be the number of 1's in this sequence. The *dimension of a definable set* X is the maximum dimension of a cell appearing in a cell decomposition of X.

The following are Propositions IV 1.3 and IV 1.5:

Fact 1.9. *The following hold:*

- $\dim(X \cup Y) = \max(\dim(X), \dim(Y))$.
- $\dim\left(\overline{X} \setminus X\right) < \dim(X)$.
- *If $\{F_a\}_{a \in X}$ is a definable family of disjoint sets and $D := \{a \in X \mid \dim(F_a) = d\}$, then D is definable and*
$$\dim\left(\bigcup_{a \in D} F_a\right) = dim(D) + d.$$

Besides the dimension, the other key invariant for sets definable in o-minimal theories is the Euler Characteristic.

Definition 1.10. We define the Euler Characteristic of a definable set as follows: For any cell C, let $E(C) = (-1)^{\dim(C)}$. Given a cell decomposition $\mathcal{P}$ of a definable set X we define $E_{\mathcal{P}}(X) := \sum_{C \in \mathcal{P},\ C \cap X \neq \emptyset} E(C)$.

In light of the following result $E_{\mathcal{P}}(X)$ is independent of the partition $\mathcal{P}$, so we drop it.

The following are Proposition IV 2.2 and Corollary IV 2.11.

Fact 1.11. *The following hold:*

- *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are cell decompositions of a definable set X, then*
$$E_{\mathcal{P}_1}(X) = E_{\mathcal{P}_2}(X).$$
- *If X and Y are definably isomorphic definable sets, then $E(X) = E(Y)$.*
- *If X, Y are definable disjoint sets, then $E(X \cup Y) = E(X) + E(Y)$.*
- *If $f : X \to Y$ is a definable surjection from the definable set X into the definable set Y and $E(f^{-1}(y)) = k$ for all $y \in Y$, then $E(X) = kE(Y)$.*

Although we will not use it, the following is a remarkable result (the corollary following VIII 2.11):

Fact 1.12. *Let X, Y be definable set and assume that $\mathcal{R}$ expands a real closed field. Then X and Y are definably isomorphic if and only if* $\dim(X) = \dim(Y)$ *and* $E(X) = E(Y)$.

The following is *Definable Choice*, which is proved from Cell Decomposition and which we will use many times to treat quotients of definable groups as definable groups. It is Proposition VI 1.2.

Fact 1.13. *The following hold.*

- *If $S \subset \mathcal{R}^{m+n}$ and $\pi : \mathcal{R}^{m+n} \to \mathcal{R}^m$ is the projection to the first m coordinates then there is a definable map $f : \pi(S) \to \mathcal{R}^n$ such that the graph $\Gamma(f) \subseteq S$.*
- *If E is a definable equivalence relation on a set X, there is a definable set $L \subseteq X$ of E-representatives: This is, every E-equivalence class has one and only one element in L.*

1.1.2. Point-set topology. Proposition VI 1.9 states:

Fact 1.14. *Let $f : X \to \mathcal{R}^n$ be a definable continuous function and let X be closed and bounded. Then $f(X)$ is closed and bounded.*

Definition 1.15. A *path* in X is a smooth map from the interval $[a, b]$ in $\mathcal{R}$ to X.

Proposition VI 3.2 and Corollary VI 3,3 state:

Fact 1.16. *Is a definable set X is definably connected, then any two points in X can be connected by a definable path.*

If X and Y are definable sets in $\mathcal{R}^n$ with X definably connected and $X \cap bd(Y) = \emptyset$, then either $X \subseteq X$ or $X \cap Y = \emptyset$.

Definition 1.17. A *definable curve* in X is a definable continuous map from the interval $(-1, 1)$ in $\mathcal{R}$ to X.
A curve f is *completable* if there is a point $p \in X$ such that $\lim_{x \to 1} f(x) = p$.
A curve f in X is *completable* in X if it is completable to a point $p \in X$.

Corollary III 1.6 implies the following.

Fact 1.18. *If X is bounded, any curve in X is completable. Therefore, if X is closed and bounded any curve in X is completable in X.*

The euclidean distance is definable, and continuous. Which implies that the distance between two disjoint closed definable sets is greater than zero. So any two disjoint definable closed subsets of a definable B can be separated by disjoint definable open subsets of B. An easy induction proves the following which is Lemma VI 3.6.

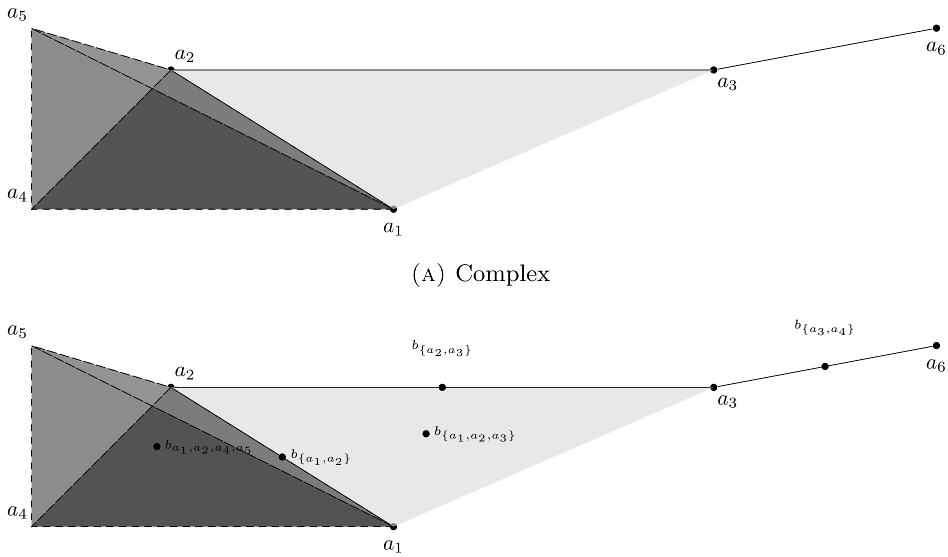
Fact 1.19. *Let $B \subset \mathcal{R}^n$ be a definable set and let $U_1, \ldots, U_n$ be open subsets such that $B \subseteq \bigcup_i U_i$. Then there are definable open sets $V!, \ldots, V_n$ with $\overline{V_i} \subset U_i$ such that $B \subseteq \bigcup_i V_i$.*

1.1.3. Triangulation. We define a *simplex* in $\mathcal{R}^m$ as follows: Let $p_0, \ldots, p_k$ be points in $\mathcal{R}^m$ such that the dimension of the affine span is k, so

$$\dim\left(\left\{\sum_{i=0}^{k} \lambda_i p_i \mid \lambda_i \in \mathcal{R}\right\}\right) = k.$$

Whenever this holds, we will define *the simplex generated by* $p_1, \ldots, p_k$

$$(p_0, \ldots, p_k) := \left\{\sum_{i=0}^{k} \lambda_i p_i \mid \text{ all } \lambda_i > 0 \text{ and } \sum_{i=0}^{k} \lambda_i = 1\right\}.$$



(A) Complex

(B) Barycenters

FIGURE 1. A definable complex and its barycenters.

A simplex generated by k points is a k-simplex. One can check that the topological closure of $(p_0, \ldots, p_k)$ is

$$[p_0, \ldots, p_k] := \left\{ \sum_{i=0}^{k} \lambda_i p_i \mid \text{ all } \lambda_i \geq 0 \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

The points p_i are the *vertices* of $(p_0, \ldots, p_k)$ and $[p_0, \ldots, p_k]$.

A face of $(p_0, \ldots, p_k)$ is a simplex spanned by a subset of $\{p_0, \ldots, p_k\}$. Distinct subsets of $\{p_0, \ldots, p_k\}$ span disjoint faces, all contained in $[p_0, \ldots, p_k]$. In fact, $[p_0, \ldots, p_k]$ is the union of all the disjoint faces of $(p_0, \ldots, p_k)$.

The *barycenter* of $(p_0, \ldots, p_k)$ is the point $\frac{\sum_{i=0}^{k} p_i}{k+1}$.

Definition 1.20. A *definable simplicial complex* is a finite collection K of simplices in $\mathcal{R}^n$ such that for all simplices C_1, C_2 in K either $\overline{C_1} \cap \overline{C_2} = \emptyset$ or $\overline{C_1} \cap \overline{C_2} = \overline{D}$ for some common face D of C_1 and C_2.

Figure 1 depicts an example of a definable complex and its barycenters, which will be used as an example in Chapter 4.

Remark 1.21. Notice that definable simplicial complexes don't need to be closed. This is a major difference with the standard notions as presented in most texts.

Definition 1.22. Given a definable simplicial complex K we define $|K|$ to be the union of points in all simplices of K (a semilinear definable set) and $Vert(K)$ to be the set of vertices of simplices in K, a finite set of points.

Any set which is equal to $|K|$ for some definable simplicial complex K will be called *a polyhedron*.

The following is Theorem VIII 1-7:

Fact 1.23. *Every definable set $S \subseteq \mathcal{R}^m$ is definably homeomorphic to (i.e. there is a definable homeomorphism between S and) a polyhedron $|K|$ for some complex K in $\mathcal{R}^m$.*

1.1.4. Definable spaces. We will recall some of the results of Chapter X.

Let X be a definable set.

A *definable atlas* on X is a finite family $\{(U_i, \phi_i)\}_{i \in I}$ such that $U_i \subseteq X$, $\bigcup_i U_i = X$, $\phi_i : U_i \to \mathcal{R}^{n_i}$ for some n_i, and such that if $U_i \cap U_j \neq \emptyset$ then the transition map $\phi_i \circ (\phi_j)^{-1}$ from $\phi_j(U_i \cap U_j)$ to $\phi_i(U_i \cap U_j)$ is continuous with respect to the topologies inherited as subsets of $\mathcal{R}^{n_j}$ and $\mathcal{R}^{n_i}$, respectively.

Any definable atlas on a definable set X endows X with a topology, where $W \subseteq X$ is open if $\phi_i(W \cap U_i)$ is an open subset of $\phi_i(U_i)$ for any $i \in I$.

Definition 1.24. A *definable space* is a definable set X which allows a definable atlas.

A morphism between spaces X and Y is a definable map from X to Y which is an homeomorphism of the respective manifold topologies.

Remark 1.25. In the original version, X did not need to be a definable space. The definable structure on X was defined by stating that $Y \subseteq X$ is definable if and only if $\phi_\alpha(Y \cap U_\alpha)$ is a definable subset of $\mathcal{R}^n$.

In this case the definable structure on any such X depended on the choice of atlases. To make sense of this, van den Dries defines the sheaf of definable functions $(DC(W))_{W \in DO(X)}$ where $DO(X)$ is the set of definable subsets of X which are open in the manifold topology, and for each such W we have $DC(W)$ is the $\mathcal{R}$-algebra of definable continuous functions from W to $\mathcal{R}$.

A morphism F between spaces X and Z is then a map from X to Z such that $F^{-1}(V)$ is a definable open subset of X for any definable open set V of Z and such that F preserves the sheaf structure: $h \circ F|_{F^{-1}(V)}$ is in $DC(F^{-1}(V))$ whenever $h \in DC(V)$. Since definable pre-images of definable sets are definable and composition of definable functions are definable, our definition is a particular case of the original definition.

Definition 1.26. A subset X of $\mathcal{R}^n$ has a natural definable space structure, consisting of taking the identity map as the only chart in a definable atlas. In this case, the topology induced in X is just the subspace topology inherited from $\mathcal{R}^n$.

An *affine* definable space is a definable space isomorphic to a definable space resulting of taking the natural definable space structure on a definable set.

Theorem X 1.18 states:

Fact 1.27. *Every definable space with a regular induced topology is affine.*

1.2. Differential o-minimal geometry

We will need to develop and understand the Lie structure of definable groups which is of course tied to the differential structure. In this next section we will prove some results not included in *Tame Topology*, but before we get there we will review the differentiability of o-minimal expansions of real closed fields developed in Chapter VII.

1.2.1. Preliminaries: Smoothness results from *Tame Topology*. The following are Definitions VII 1.2 and VII 1.5.

Definition 1.28. Let $I \subset \mathcal{R}$ be an open interval and let $f(x)$ be a function $f_I \to \mathcal{R}^n$.

We say that $f(x)$ is *differentiable at a point* $p \in I$ *with derivative* $a \in \mathcal{R}^n$ if

$$\lim_{t \to 0} \frac{f(a+t) - f(a)}{t} = a.$$

The *derivative* $f'(x)$ *of* $f(x)$ is the function from I to $\mathcal{R}^n$ such that $f'(p)$ is the derivative of $f(x)$ at p.

Let $f : U \to \mathcal{R}^n$ be a function with U an open subset of $\mathcal{R}^m$. We say that f *is differentiable at* $p \in U$ *with differential* T if T is a linear function from $\mathcal{R}^n$ to $\mathcal{R}^m$ and for each $\epsilon > 0$ we have

$$|f(p+v) - f(p) - T(v)| < \epsilon|v|$$

for any sufficiently small $v \in \mathcal{R}^m$. If this is the case, we denote T by $\mathrm{d}_p f$.

If $f = f_1, \ldots, f_n$ and we define partial derivatives in the standard way, then the matrix $\left(\frac{\partial f_j}{\partial x_j}\right)_{(i,j)}$ is exactly the matrix $\mathrm{d}_p f$ and is called the *Jacobian matrix of* f *at* p.

The *Mean Value Theorem* is VII 2.3 in *Tame Topology.*

Fact 1.29. Mean Value Theorem *Let* $f : [a, b] \to \mathcal{R}$ *be a function which is continuous at every point in the interval* $[a, b]$ *and differentiable at any point in* (a, b). *Then there is some* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The differential function behaves as in the classical case. The following is VII 1.6 in *Tame Topology.*

Fact 1.30. *Let* $f, g : U \to \mathcal{R}^n$ *are differentiable at* $p \in U$ *and* U *open in* $\mathcal{R}^m$.

Then $f + g$, *and* cf *are differentiable at* p *and*

$$\mathrm{d}_p(f+g) = \mathrm{d}_p f + \mathrm{d}_p g, \quad \textit{and}$$
$$\mathrm{d}_p cf = c\mathrm{d}_p f.$$

If f *is continuous in* U, V *is an open subset of* $\mathcal{R}^n$ *with* $f(p) \in V$ *and* $h : V \to \mathcal{R}^k$ *is differentiable at* $f(p)$ *then* $h \circ f$ *is differentiable at* p *and*

$$\mathrm{d}_p(h \circ f) = \big(\mathrm{d}_{f(p)}(h)\big) \cdot \mathrm{d}_p(f).$$

Where the multiplication in the left is matrix multiplication (or, equivalently, the composition of the linear transformations).

This result is the standard Chain Rule *of multivariable differentiation.*

Definition 1.31. We call $f : U \to \mathcal{R}^n$ a C^1-map if the partial derivatives $\left(\frac{\partial f_i}{\partial x_j}\right)$ are all defined as $\mathcal{R}$-valued functions on U and are continuous.

The following is Theorem VII 2.11.

Fact 1.32. Inverse Function Theorem: *Let* $f : U \to \mathcal{R}^m$ *be a definable* C^1-*map on a definable open set* $U \subset \mathcal{R}^m$ *and* $p \in U$ *a point where* $\mathrm{d}_p f : \mathcal{R}^m \to \mathcal{R}^m$ *is invertible. Then there are definable open neighborhoods* $U' \subseteq U$ *of* p *and* V' *of* $f(p)$ *such that* f *maps* U' *homeomorphically onto* V' *and* $f^{-1} : V' \to U'$ *is* C^1.

Finally, we will need to define higher degrees of smoothness.

Definition 1.33. We define, inductively (in k) a function $f : U \to \mathcal{R}^n$ to be a C^{k+1}-map if the partial derivatives $(\frac{\partial f_i}{\partial x_j})$ are all defined as $\mathcal{R}$-valued functions on U and are C^k.

We will say that a cell C is a C^k-cell if all the functions involved in the definition of C as a cell are C^k (this is, in the inductive definition of a cell, when forming $\Gamma(f)$ and (f, g) we require f and g to be C^k).

The following is Theorem VII 3.2 in *Tame Topology.*

Theorem 1.34. *The following hold.*

- *For any definable sets $A_1, \dots, A_k \subset \mathcal{R}^m$ there is a decomposition of $\mathcal{R}^m$ into C^k-cells partitioning $A_1, \dots, A_k$.*
- *For every definable funciton $f : A \to \mathcal{R}$ with A definable, there is a cell decomposition partitioning of A into C^k-cells such that the restriction of f to every cell contained in A is C^k.*

1.2.2. Manifolds. We will now define a n-manifold structure on any definable set X.

A *definable coordinate chart* (U, ϕ) is a definable map ϕ between a definable subset U of X and an open subset of $\mathcal{R}^n$. A *definable C^k-atlas* on X is a definable family $\{(U_\alpha, \phi_\alpha)\}$ such that $\bigcup_\alpha U_\alpha = X$, each (U_α, ϕ_α) is a definable coordinate chart on X and if $U_\alpha \cap U_\beta \neq \emptyset$ then the transition map $\phi_\alpha \circ (\phi_\beta)^{-1}$ from $\phi_\beta(U_\alpha \cap U_\beta)$ to $\phi_\alpha(U_\alpha \cap U_\beta)$ is a C^k-diffeomorphism. Any such map $\phi_\alpha \circ (\phi_\beta)^{-1}$ will be called a *transition map.* X together with a definable C^k-atlas will be a *definable C^k-manifold.*

Any reference to a definable manifold will assume that a fixed definable atlas is given. Any definable manifold X has therefore an induced topology where a set $W \subset X$ is open whenever $\phi_\alpha(U_\alpha \cap W)$ is an open subset of $\mathcal{R}^n$ for any (U_α, ϕ_α) in the atlas of X. This topology is the *manifold topology* on X.

A topology τ on a definable manifold X is *compatible with the manifold structure on X* if any subset W of X is open in the manifold topology if and only if it is τ-open.

Similarly, two definable C^k-atlases on X are defined to be *compatible* if the union is a definable C^k-atlas.

Given a fixed definable manifold X, by a *definable chart of X* we will mean pairs (U, ϕ) with U an open subset of X in the manifold topology and such that $\{(U, \phi)\}$ is compatible with the atlas of X.

A definable map $f : X \to Y$ between C^k-manifolds X, Y with atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ will be defined to be a C^k-map (we will sometimes refer to it as *smooth* if k is clear from the context) if it is a continuous map and for every α, β if $f(U_\alpha) \cap V_\beta \neq \emptyset$ then the map from the open subset of $\mathcal{R}^k$ $\phi_\alpha\left(f^{-1}(V_\beta) \cap U_\alpha\right)$ to $\mathcal{R}^k$ given by $(\phi_\alpha)^{-1} \circ f \circ \psi_\beta$ is C^k.

EXERCISE 1.4. *Prove that the product of definable C^k-manifolds is a definable C^k-manifold such that the projection on each coordinate is smooth.*

EXERCISE 1.5. *Prove that if we assume $\mathcal{R} = \mathbb{R}$, then if $X \subset \mathbb{R}^n$ is definable and compact then X is a manifold in the standard sense if and only if it is a definable manifold.*

EXERCISE 1.6. *Prove that if we assume $\mathcal{R} = \mathbb{R}$, is any manifold a definable manifold?*

1.2.3. Tangent space. We will fix some level of differentiability $k \geq 1$ so that by *definable manifolds* we mean C^k-definable manifolds and by *smooth definable maps* we mean definable C^k-differentiable maps.

Let X be a definable manifold of dimension n and let $p \in X$ and let (U, ϕ) be a coordinate chart in the definable atlas of X and $p \in U$.

Definition 1.35. Given any two definable curves $f, g : \mathcal{R} \mapsto X$ with $f(0) = g(0) = p$, we say that f *is tangent to g at* 0 if

$$\mathrm{d}_0(\phi \circ f) = \mathrm{d}_0(\phi \circ g).$$

EXERCISE 1.7. *Let (U_1, ϕ_1) and (U_2, ϕ_2) be two coordinate charts of a definable atlas on X with $p \in U_1 \cap U_2$ and let f and g be definable curves in X with $f(0) = g(0) = p$. Prove that*

$$\mathrm{d}_0(\phi_1 \circ f) = \mathrm{d}_0(\phi_1 \circ g)$$

if and only if

$$\mathrm{d}_0(\phi_2 \circ f) = \mathrm{d}_0(\phi_2 \circ g).$$

EXERCISE 1.8. *Let $\mathcal{F}$ be the set of definable curves f in X such that $f(0) = p$. Prove that tangency is an equivalence relation on $\mathcal{F}$ which is linear: If $f, f', g, g' \in \mathcal{F}$ and f is tangent to f' and g is tangent to g' then $\lambda f + \mu g$ is tangent to $\lambda f' + \mu g'$ for any scalars $\lambda, \mu \in \mathcal{R}$.*

Definition 1.36. The *tangent space $T_p(X)$ of X at p* is defined to be the class of all definable curves f in X with $f(0) = p$ quotient by the equivalence relation of being tangent.

Given a definable manifold X the class of definable curves f in X with $f(0) = p$ is not definable. However, by definition, if f is a curve in X with $f(p) = 0$ and (U, ϕ) be a coordinate chart in the definable atlas of X and $p \in U$, then if we define $l : \mathcal{R} : \mathcal{R}^n$ by

$$l(\lambda) := \lambda \cdot (d_{\phi \circ f}(p)) + \phi(p)$$

then l is a linear function from $\mathcal{R}$ to $\mathcal{R}^n$, and the restriction of $\phi^{-1} \circ l$ to the interval $(-1, 1)$ is a curve with $\phi^{-1} \circ l = p$ which is tangent to f.

This means that the tangent space $T_p(X)$ is isomorphic to the definable set

$$\left\{\phi^{-1} \circ l \mid l : \mathcal{R} \to \mathcal{R}^n \text{ is a linear map with } l(0) = p\right\}$$

so that we can treat it as a definable object. Although technically the choice of representatives depends on the chart (U, ϕ) that we choose, Exercise 1.8 implies that the vector space structure doesn't. In fact, as we will soon see, the right choice of ϕ can be a powerful tool to understand the structure of the tangent spaces and the maps between them.

Let X be a definable n-dimensional manifold and Y a definable m-dimensional manifold amd $p \in X$. Let $F : X \to Y$ be definable map which is smooth at p. Let (U, ϕ) and (V, ψ) be coordinate charts of X and Y with $p \in U$ and $F(p) \in V$. By the Chain Rule, if g is a curve in X with $g(0) = p$ then

$$\psi \circ F \circ g = \psi \circ F \circ \phi^{-1} \phi \circ g$$

and by the Chain Rule

$$\mathrm{d}_0(\psi \circ F \circ g) = \left(\mathrm{d}_{\phi(p)} \left(\psi \circ F \circ \phi^{-1}\right)\right) (\mathrm{d}_0) \left(\phi \circ g\right)$$

so that if g_1 and g_2 are curves in X with $g_1(0) = g_2(0) = p$, then $F \circ g_1$ and $F \circ g_2$ are curves in Y with $F \circ g_1(0) = F \circ g_2(0) = F(p)$ are curves in Y which are tangent at $F(p)$ whenever g_1 and g_2 are tangent at p. This implies that F induces a map from $T_X(p)$ to $T_Y(F(p))$ sending the tangency class of a curve g in X (with $g(0) = p$) to the tangency class of $F \circ g$. This map is called $\mathrm{d}_p(F)$. It is a linear map (this is left as an exercise) so after a choice of basis which, as explained in the previous section can be done in a natural way after fixing charts (U, ϕ) and (V, ψ) with $p \in U$ and $F(p) \in V$, is represented by an m by n matrix.

EXERCISE 1.9. *Prove that $\mathrm{d}_p(F)$ is a linear map.*

Definition 1.37. Let X and Y be definable manifolds, let $p \in X$ and let $F : X \to Y$ be a map from X to Y which is smooth at p. We define the *rank of F at p* to be the rank of the matrix $\mathrm{d}_p(F)$.

EXERCISE 1.10. *Prove that the rank of F does not depend on the choice of charts.*

Definition 1.38. Let X and Y be definable manifolds and $F : X \to Y$ a smooth map between them.

We will say that F *is a definable submersion* if it is surjective and the rank of F is equal to the dimension of Y at every point. Equivalently (using linear algebra) because the tangent space has the same dimension as the manifold, F is a submersion if and only if at every point $p \in X$ the map $\mathrm{d}_p(F)$ is a surjection from $T_p(X)$ to $T_{f(p)}(Y)$.

We will say that F *is a definable immersion* if it is injective and the rank of F is equal to the dimension of X at every point. Equivalently, F is an immersion if and only if $\mathrm{d}_p(F)$ is a surjection from $T_p(X)$ to $T_{f(p)}(Y)$ at every point $p \in X$.

An *embedded submanifold* of Y is a submanifold X such that the identity map is an immersion.

Example 1.39. Fix $n - m \in \mathbb{N}$ and some $k \leq m, n$ Let $f : \mathcal{R}^n \to \mathcal{R}^m$ be the function

$$f(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0)$$

Let e_i be the vector with 1 in the ith coordinate and zeroes everywhere else.

The tangent space $T_p(\mathcal{R}^n)$ is generated by the derivatives of the functions $p+te_i$ with $0 \leq i \leq n$.

The tangent space $T_{f(p)}(\mathcal{R}^m)$ is generated by the derivatives of the functions $p + te_i$ with $0 \leq i \leq m$.

Since $\frac{\partial f_i}{\partial x_j}(p)$ is 1 if $i = j \leq k$ and 0 otherwise, we have that at any point p

$$\mathrm{d}_p(f) := (m_{i,j})_{1 \leq i \leq n,\ 1 \leq i \leq m}$$

where $m_{i,j} = 1$ whenever $i = j \leq k$ and 0's everywhere else and the rank of f at any p is equal to k.

This implies that f is an immersion if and only if $k = n$ and a submersion if $k = m$.

$c = (c_1, \dots, c_m) \in \mathcal{R}^m$ is in the image of f if and only if coordinates $c_{k+1}, \dots, c_m$ are zero. For any such c

$$f^{-1}(c) := \{(c_1, \dots, c_k, x_{k+1}, \dots, x_n) \mid x_i \in \mathcal{R}\}$$

so it is a $n - k$ dimensional submanifold of $\mathcal{R}^n$ isomorphic to $\mathcal{R}^{n-k}$. If we identify $f^{-1}(c)$ with $\mathcal{R}^{n-k}$, the injection sending $f^{-1}(c)$ to its image in $\mathcal{R}^n$ sends $(x_{k+1}, \dots, x_n)$ to $(c_1, \dots, c_k, x_{k+1}, \dots, x_n)$.

As such, for any $p \in \mathcal{R}^{n-k}$ we have that if i is the injection map described above, $d_p(i)$ is the matrix

$$\begin{pmatrix} \mathbf{I}_{n-k} & \mathbf{0} \end{pmatrix}$$

which has rank $n - k$ so that $f^{-1}(c)$ is an immersed submanifold if $\mathcal{R}^n$.

1.2.4. Constant rank.

We will prove in Chapter 3 that all definable groups admit a definable manifold structure which will make the group a Lie group. This will imply in turn that any definable homomorphism between definable groups will be a differential map of constant rank.

Maps with constant rank have many important properties, which we will now study.

Lemma 1.40. *Let $f : U \to \mathcal{R}^m$ be a definable smooth map with $U \subseteq \mathcal{R}^n$ and assume that f has constant rank k for every point in U. Let $p \in U$.*

Then there are definable neighborhoods $W \subseteq U$ of p and $V \subseteq \mathcal{R}^m$ of $f(p)$ with $f^{-1}(V) \subseteq U$, and smooth definable diffeomorphisms $F : W \to \mathcal{R}^n$ and $G : W \to \mathcal{R}^n$ with $F(p) = \overline{0}$, $G(f(p)) = \overline{0}$ and such that on $F(f^{-1}(V) \cap W)$ we have

$$G \circ f \circ F^{-1}(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0).$$

Proof. Since all translations in $\mathcal{R}^n$ and $\mathcal{R}^m$ are definable diffeomorphisms, we may assume that $f\left(\overline{0}\right) = \overline{0}$: this just changes the F and G by composing them with a translation, but the notation gets messy if we don't assume this.

We will fix some notation. Given two disjoint sets of variables $\overline{x}, \overline{y}$ and a function $Q(\overline{x}, \overline{y})$ we will refer to $\frac{\partial Q}{\partial \overline{x}}$ to be the matrix of partial derivatives of Q only with respect to the variables in $\overline{x}$. In particular, if $\overline{x}$ is $x_1, \dots x_k$ and $\overline{y}$ is $(x_{k+1}, \dots, x_n)$ then the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial \overline{x}} & \frac{\partial f}{\partial \overline{y}} \end{pmatrix}$$

is the Jacobian of f.

Since the rank of f is k at every point, by definition $\mathrm{d}_{\overline{0}} f$ has rank k so that it has a $k \times k$ invertible minor. We can change variables so that we may assume that the top left $k \times k$ minor of $\mathrm{d}_{\overline{0}} f$ is invertible. By continuity of the determinant, this assumption implies that the top left $k \times k$ minor of $\mathrm{d}_p f$ is invertible for every p in a neighborhood U_0 of U.

Let $\overline{x} := x_1, \dots x_k$ and $\overline{y} := (x_{k+1}, \dots, x_n)$. We will also split the coordinates in the image space $\mathcal{R}^m$ and assume that $f(\overline{x}, \overline{y}) = (f_1(\overline{x}, \overline{y}), f_2(\overline{x}, \overline{y}))$ where $f_1(\overline{x}, \overline{y})$ are the first k coordinates of $f(\overline{x}, \overline{y})$ and $f_2(\overline{x}, \overline{y})$ are the last $m - k$ coordinates of $f(\overline{x}, \overline{y})$.

Let F be the function from $\mathcal{R}^n$ to itself with $F(\overline{x}, \overline{y}) = (f_1(\overline{x}, \overline{y}), \overline{y})$.

Claim 1.41. *F is invertible around a definable neighborhood of $\overline{0}$ and $f \circ F^{-1}(\overline{x}, \overline{y}) = (\overline{x}, s(\overline{x}))$ for some smooth function $s : \mathcal{R}^k \to \mathcal{R}^{m-k}$.*

Proof: For any m let $\mathbf{I}_m$ be the identity $m \times m$ matrix. The Jacobian of F is

$$\begin{pmatrix} \frac{\partial f_1}{\partial \overline{x}} & \frac{\partial f_1}{\partial \overline{y}} \\ \mathbf{0} & \mathbf{I}_{n-k} \end{pmatrix}$$

so at any point $p = (\overline{p}_1, \overline{p}_2)$ (again splitting the first k coordinates from the rest) the determinant of $\mathrm{d}_p F$ is equal to the determinant of $\mathrm{d}_{p_1} f_1$ which by hypothesis is non zero for any $p \in U_1$. Since $F(\overline{0}) = \left(f_1(\overline{0}), \overline{0}\right) = \overline{0}$, this implies by the Inverse Function Theorem that there is a neighborhood U_1 of $\overline{0}$ contained in $F(U_0)$ where the inverse of F is defined and smooth.

Now, if we split $F^{-1}(\overline{x}, \overline{y})$ as $(r(\overline{x}, \overline{y}), t(\overline{x}, \overline{y}))$, then in $W := U_1 \cap U_0$

$$\begin{aligned} \overline{x}, \overline{y} &= F(F^{-1}(\overline{x}, \overline{y})) \\ &= F(r(\overline{x}, \overline{y}), t(\overline{x}, \overline{y})) \\ &= f_1\left(r\left(\overline{x}, \overline{y}\right), t\left(\overline{x}, \overline{y}\right)\right), t\left(\overline{x}, \overline{y}\right) \end{aligned}$$

which implies of course that $t(\overline{x}, \overline{y}) = \overline{y}$ and $f_1(r(\overline{x}, \overline{y}), t(\overline{x}, \overline{y})) = f_1(r(\overline{x}, \overline{y}), \overline{y}) = \overline{x}$.

So on a definable neighborhood of the origin

$$\begin{aligned} f \circ F^{-1}(\overline{x}, \overline{y}) &= f_1(F^{-1}(\overline{x}, \overline{y})), f_2(F^{-1}(\overline{x}, \overline{y})) \\ &= f_1\left(r\left(\overline{x}, \overline{y}\right), t\left(\overline{x}, \overline{y}\right)\right), f_2\left(r\left(\overline{x}, \overline{y}\right), t\left(\overline{x}, \overline{y}\right)\right) \\ &= \overline{x}, f_2\left(r\left(\overline{x}, \overline{y}\right), \overline{y}\right). \end{aligned}$$

Let $s(\overline{x}, \overline{y}) := f_2\left(r\left(\overline{x}, \overline{y}\right), \overline{y}\right)$. So the Jacobian of $f \circ F^{-1}$ (in W) will be given by

$$\begin{pmatrix} \mathbf{I}_k & \mathbf{0} \\ \frac{\partial s}{\partial \overline{x}} & \frac{\partial s}{\partial \overline{y}} \end{pmatrix}.$$

The rank of $f \circ F^{-1}$ is at most k (in fact it must be k) which implies that $\frac{\partial s}{\partial \overline{y}}(p) = 0$ for any point $p \in W$. By the Mean Value Theorem, the value of s is constant in W when we vary the last $n-k$ coordinates, so that actually have $s(\overline{x}, \overline{y}) = s(\overline{x})$ is a function from $\mathcal{R}^k$ to $\mathcal{R}^{m-k}$. This completes the proof of the claim. $\blacksquare_{Claim}$

We can now complete the proof of the lemma. Let $G : \mathcal{R}^m \to \mathcal{R}^m$ be the function such that $s(q) = (q_1, q_2 - s(q_1))$ where we are splitting $q = (q_1, q_2)$ with q_1 are the first k coordinates and q_2 the last $m-k$. G is a diffeomorphism (the Jacobian is a lower triangular matrix with ones in the diagonal).

So

$$G \circ f \circ F^{-1}(\overline{x}, \overline{y}) = G \circ (f \circ F^{-1})(\overline{x}, \overline{y}) = G(\overline{x}, s(\overline{x})) = (\overline{x}, s(\overline{x}) - s(\overline{x})) = \left(\overline{x}, \overline{0}\right),$$

as required. $\square$

Theorem 1.42. *Let N and M be definable manifolds of dimension n and m, respectively. Let $f : U \to M$ be a definable smooth map with constant rank k with $U \subseteq N$. Let $p \in U$.*

Then there is a definable chart (ϕ, W) of N centered at p and a definable chart (ψ, V) of M centered at $f(p)$ such that $\phi(p) = \overline{0}$, $\psi(f(p)) = \overline{0}$ and such that on $\phi(f^{-1}(V) \cap W)$ we have

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

Proof. This follows directly from the lemma. Let (ϕ', W') be any chart of N that contains p with $W' \subset U$ and let (ψ', V') be a chart in M containing $f(p)$.

Let F, G be the definable diffeomorphisms obtained by applying Lemma 1.40 to $f' := \psi' \circ f \circ \phi'^{-1}$. Then, if we define $\phi := F \circ \phi'$ and $\psi := G \circ \psi'$ we get

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

$\square$

This theorem has some very important consequences. In particular, the following follows as a corollary using the computations in Example 1.39.

Theorem 1.43. *Let N and M be definable manifolds of dimension n and m, respectively. Let $U \subseteq N$ be open and let $f : U \to N$ be a definable smooth map with constant rank k in U. Let $p \in U$ and assume $f(p) = c$.*

Then the following hold:

- *The set $f^{-1}(c)$ is an embedded submanifold of N of dimension $n-k$ and $T_p(f^{-1}(c)) = \ker(\mathrm{d}_p f)$.*
- *Every $p \in N$ has a neighborhood $W \subset U$ such that its image is a submanifold of M of dimension k and $T_c(f(W)) = (\mathrm{d}_p f)(T_p(N))$.*

Historical Notes

All of Section 1.1 are direct quotations from [**72**].

The definable manifold structure on definable groups was originally defined by Pillay in [**61**]. Van den Dries and Miller developed a lot of the definable manifold structures for the case $\mathcal{R} = \mathbb{R}$, a development that was continued for the general case by Peterzil, Pillay and Starchenko in [**51**]. However, most of the results of Section 1.2 are adaptations of results in Spivak's [**68**] to the o-minimal case.

CHAPTER 2

The Group Topology

A *Lie group* is a real or complex smooth manifold with a smooth group operation such that the inverse map is smooth as well.

We will see that every group definable in an o-minimal expansion of the real field is a real Lie group and, more generally, every group definable in an o-minimal expansion of a real closed field can be endowed with a differentiable manifold structure that is compatible with the group operation and with many properties in common with Lie groups. In this chapter we start presenting this manifold structure by showing that every definable group admits a definable topology making it a topological group, and proving several consequences of this topology on the group.

2.1. The t-topology

Every group definable in an o-minimal structure $\mathcal{M}$ can be equipped with the topology induced by the order of $\mathcal{M}$. However, the group operation does not need to be continuous with the respect to this topology. For example, suppose $\mathcal{M} = (M, +, \cdot, 0, 1)$ is a real closed field and set $G = [0, 1)$ with the group operation given by

$$x \oplus y = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{otherwise.} \end{cases}$$

The group operation $\oplus$ is not continuous at 0 with respect to the order topology. However, it is continuous at any point of the cofinite set $V = (0, 1)$, and if we set $\mathcal{B}_0 = \{[0, \varepsilon) \bigcup (1 - \varepsilon, 1) : \varepsilon \in V\}$ to be a basis for the neighborhoods of 0, then it becomes continuous at 0 too. In this section we show how this example can be generalized to every definable group.

Fix $\mathcal{R}$ to be an o-minimal expansion of a real closed field. With an abuse of notation we will write $\mathcal{R}^k$ to denote the k-th power of the universe of $\mathcal{R}$.

Theorem 2.1. *Let $(G, \cdot)$ be a group definable in $\mathcal{R}$ of dimension n. Then there is a definable open subset V of G of co-dimension less than n and a definable topology t on G such that:*

- *$(G, \cdot, t)$ is a topological group.*
- *The t-topology coincides with the order topology on V.*

The proof is based in the piecewise continuity implied by definability.

We are working on an expansion of a field where all intervals are definably isomorphic, which implies that all cells are definably isomorphic to products of bounded intervals. As such, any definable group G of dimension n can be embedded into $\mathcal{R}^n$. This allows us to assume throughout this chapter that we are given a definable group $(G, \cdot)$ where G is a subset of $\mathcal{R}^n$ of dimension n. This is not necessary, but does simplify notation. As such, we will use standard notation in $\mathcal{R}$, for elements in G: $|x|$ sill denote the euclidean norm and $d(x, y)$ will be the euclidean distance for $x, y \in G$.

Once we have proved Theorem 2.1 we will be able to use the fact that G is a topological group. In the mean time (throughout this section), we will work with the topology we have: the order topology that G has as a subset of $\mathcal{R}^n$. In this section *continuous* will mean continuity with respect to the order topology and *differential* or *smooth* will refer to the differential structure that G inherits as a subset of $\mathcal{R}^n$ with respect to the standard differential structure.

Definition 2.2. Let X be a definable set. A set $Y \subseteq X$ will be said to have small co-dimension if $\dim(X \setminus Y) < \dim(X)$.

The following will be quite useful.

Fact 2.3. *Let $X \subset \mathcal{R}^k$ be a definable set of dimension k. Then the following hold:*

- *There is an open $U \subset X$ of small co-dimension.*
- *Let f be a definable function with domain X. Then there is an open $U \subset X$ of small codimension such that the restriction of f to U is C^k.*

Proof. The union of the open cells in any cell decomposition of X is an open subset of X of small co-dimension. □

Exercise 2.1. *Let X be a definable set and $Y \subseteq X$ be a definable subset of small co-dimension.*

Let $\{X_a\}_{a\in D}$ be fibration *of X: This is, a definable family of disjoint subsets of X whose union is X. Then the set*

$$\{a \in D \mid \dim(X_a \setminus Y) < \dim(X_a)\}$$

is a subset of D of small co-dimension.

Hint: Use the definition of small co-dimension and Fact 2.3.

We will now start the proof of Theorem 2.1. The main idea is to isolate a large set V where the group operations are well behaved and use it to define the t-topology (Definition 2.4 below).

Let $W \subset G \times G$ be an open set of small co-dimension where the multiplication is C^k. For any $a \in W$ let $W_a := \{x \in G \mid (x, a) \in W\}$, ${}_aW := \{x \in G \mid (a, x) \in W\}$ and $W^\rho_a := \{x \in G \mid (ax, x^{-1}) \in W\}$.

Let:

$$\begin{aligned} G^R &:= \{a \in G \mid \dim(G \setminus W_a) < \dim(G)\}, \\ G^L &:= \{a \in G \mid \dim(G \setminus {}_aW) < \dim(G)\}, \\ G^\rho &:= \{a \in G \mid \dim(G \setminus W^\rho_a) < \dim(G)\}, \end{aligned}$$

and G^I be an open subset of G of small co-dimension where the inverse is C^k.

Dimension is preserved by definable isomorphisms, and it is additive. It follows that both $G^R \cap G^L \cap G^\rho \cap G^I$ and its inverse have small co-dimension. Let V be the interior of

$$(G^R \cap G^L \cap G^\rho \cap G^I) \cap (G^R \cap G^L \cap G^\rho \cap G^I)^{-1}$$

of small co-dimension.

Remark. *This construction yields an open subset V of G of small co-dimension which is closed under inverses, and such that the sets of points for which right (and left) multiplication by a is continuous has small co-dimension for any $a \in V$. The use of G^ρ is more subtle, but it has an important roll in proving that the t-topology coincides with the order topology in the open set V.*

We will now define the t-topology in G.

Definition 2.4. A subset $\mathcal{O}$ of G is *t-open* if given any $v \in V$ the set $(v^{-1}\mathcal{O}) \cap V$ is an open subset of V.

EXERCISE 2.2. *Show that Definition 2.4 defines a topology on G.*

EXERCISE 2.3. *Show that t-openness of a definable set is a definable condition.*

EXERCISE 2.4. *If $\mathcal{O}$ is a non empty t-open set, then* $\dim(\mathcal{O}) = n$.

The proofs of the following lemmas rely in the following easy observation.

Observation 2.5. Let $X \subset \mathcal{R}^k, Y \subset \mathcal{R}^l$ be sets, let $x \in X$ and $f : X \to Y$ be a function which is continuous at x. If $f(x)$ is an interior point of Y, then x is an interior point of X.

Lemma 2.6. *The following hold:*

- *The inverse is a t-homeomorphism on $(G, \cdot)$.*
- *For any $g \in G$, the right multiplication by G is t-homeomorphism.*
- *For any $g \in G$, the left multiplication by G is t-homeomorphism.*

PROOF. We start with the inverse. Since the inverse operation is its own dual, it is enough to show that if $\mathcal{O}$ is t-open, then $(x^{-1}\mathcal{O}^{-1}) \cap V$ is open for any $x \in V$.

Let $y \in (x^{-1}\mathcal{O}^{-1}) \cap V$, and let $v \in W_{y^{-1}} \cap (W_{x^{-1}})y \cap V$.

Inversion is continuous at y ($y \in V$) so the maps

$$y \xrightarrow{(\ )^{-1}} y^{-1} \xrightarrow{(v^{-1}\cdot\ )} v^{-1}y^{-1} \xrightarrow{(\ \cdot x^{-1})} v^{-1}y^{-1}x^{-1}$$

are all continuous (at y, y^{-1} and $v^{-1}y^{-1}$, respectively) by choice of v. So the composition is a map from G to G which is continuous at y.

As $xy \in \mathcal{O}^{-1}$ we have $v^{-1}y^{-1}x^{-1} \in (v\mathcal{O}) \cap V$ which is open by t-openness of $\mathcal{O}$. So y is an interior point of $(x^{-1}\mathcal{O}^{-1}) \cap V$ as required.

For continuity of right multiplication, we again will assume $\mathcal{O}$ is t-open and we will prove that $g^{-1}\mathcal{O}$ is t-open, so that for any fixed $x \in V$ the set $x^{-1}g^{-1}\mathcal{O} \cap V$ is open. Let $y \in x^{-1}g^{-1}\mathcal{O} \cap V$.

Choose v in the intersection of $W_y x^{-1}$ (so $vx \in W_y$) and Vg, both sets of small co-dimension. The element vg^{-1} is in V by choice of v, so by t-openness of $\mathcal{O}$ $vg^{-1}\mathcal{O} \cap V$ is open.

The element $vxy \in vg^{-1}\mathcal{O} \cap V$ is the image of y by translation by vx and translation by vx is continuous at y since $(vx, y) \in W$ by choice of v. By Observation 2.5 y is an interior point of $x^{-1}g^{-1}\mathcal{O} \cap V$, as required.

That the third item follows from the previous two is left as an exercise. □

Lemma 2.7. *Any open $U \subseteq V$ is t-open.*

PROOF. Let x be any element in V. We need to show that $x^{-1}V \cap V$ is open. Let $y \in x^{-1}V \cap V$, so that $xy = a \in V$ for some a.

Let v be such that $v \in W_y, v \in y^{-1}W_x$ and $x \in W_a^{\rho}$, so that $(y, v), (x, yv)$ and (av, v^{-1}) are all in W.

It follows that the map sending y to av given by

$$y \xrightarrow{(\ \cdot v)} yv \xrightarrow{(x\cdot\ )} x(yv) = av \xrightarrow{(\ \cdot v^{-1})} a$$

is continuous at y. Since a is in V and V is open, Observation 2.5 implies y is an interior point of $x^{-1}V \cap V$, as required. □

Corollary 2.8. *G endowed with the t-topology is a topological group.*

PROOF. If we choose any point a in V and $b \in W_a \cap a^{-1}V \cap V$, we have $(a, b) \in W$ and $a, b, ab \in V$. By Lemma 2.7 the t-topology and the order topology coincide in $V \times V \times V$ so that multiplication is t-continuous at (a, b).

This, together with Lemma 2.6 is enough to get that the group $(G, \cdot)$ endowed with the t-topology is a continuous group:

For any $x, y \in G$, multiplication is equivalent to the composition of the functions in the diagram

$$(x,y) \xrightarrow{(ax^{-1}\cdot\ \ )\times id} (a,y) \xrightarrow{id\times(\ \ \cdot y^{-1}b)} (a,b) \xrightarrow{(\ \cdot\ )} (ab) \xrightarrow{(\ \ \cdot b^{-1}y)} ay \xrightarrow{(xa^{-1}\cdot\ \ )} xy$$

so it is a composition of t-continuous maps, and therefore t-continuous. $\square$

2.2. First Topological properties

The t-topology on G is a definable topology that makes G a topological group which comes together with a definable subset V of small co-dimension which is definably diffeomorphic to an open subset of $\mathcal{R}$ and such that on V the order topology and the t-topology coincide. Throughout this section we will fix a definable $(G, \cdot)$ and an open subset V of G witnessing the t-topology of G.

We start with some exercises exhibiting the behavior of t-closed and t-open subsets of the topology.

We will first start with some definitions.

Definition 2.9. Let X be a definable set endowed with either the order topology or the t-topology. A definable set $Y \subseteq X$ will be *definably connected* if given any two definable open subsets U_1, U_2 of X, if $Y \subset U_1 \cup U_2$ then either $Y \subset U_1$ or $Y \subset U_2$.

When working with the t-topology of a definable group, we will sometimes abuse notation and refer to t-definably connected as *t-connected.*

Exercise 2.5. *For any X and t-open U the set $X \cdot U := \bigcup_{x \in X} x \cdot U$ is t-open.*

Exercise 2.6. *If U_1, U_2 are t-open and t-connected, then $U_1 \cdot U_2$ are t-open and t-connected.*

Exercise 2.7. *If U is a t-neighborhood of the identity, then for any set X the t-closure $\overline{X}^t$ is contained in both $U \cdot X$ and $X \cdot U$.*

Exercise 2.8. *If H is a subgroup of $(G, \cdot)$ and $\overline{H}^t$ is the closure of H with respect to the t-topology, then $\overline{H}^t$ is a subgroup of G.*

We finish this introduction of consequences of the t-topology with a couple of technical lemmas, which will allow us to move between the $\mathcal{R}$-topology and the t-topology.

Lemma 2.10. *Assume that* $\dim(G) = n$. *Let Y be a definable subset of G. Then the following hold:*

(1) $\dim(Y) = \dim(gY \cap V)$ *for some $g \in G$. In fact,*
$$\dim(\{g \in G \mid \dim(Y) = \dim(gY \cap V)\}) = n.$$
(2) *If $X \subset Y$ is a definable subset of Y, then* $\dim(Y \setminus X) = \dim(Y)$ *if and only if there is some $y \in Y \setminus X$ and a t-neighborhood U of y such that $U \cap X = \emptyset$.*
(3) *If $\overline{Y}^t$ is the closure of Y with respect to the t-topology, then*
$$\dim\left(\overline{Y}^t\right) = \dim(Y).$$
Even more,
$$\dim\left(\overline{Y}^t \setminus Y\right) < \dim(Y).$$

Proof. Let $k = \dim(Y)$.

Consider the map ϕ from $G \times Y$ to G given by $\phi(g, y) = g \cdot y$. Notice that $\phi^{-1}(g) = \{(g \cdot y^{-1}, y) \mid y \in Y\}$ so $\dim(\phi^{-1}(g)) = \dim(Y) = k$. By Fact 1.9 $\dim(\phi^{-1}(V)) = \dim(V) + k = n + k$.

On the other hand,

$$\phi^{-1}(V) := \bigcup_{g\in G} gY \cap V$$

so

$$\dim\left(\phi^{-1}(V)\right) \le n + \max_{g\in G}\{\dim(gY \cap V)\}.$$

This proves the first item.

Let $X \subset Y$ be definable. By the previous lemma there is some g such that $\dim(g(Y \setminus X) \cap V) = \dim(Y \setminus X) = \dim(Y)$. In V the topology and the t-topology coincide, so $\dim(\overline{Z} \setminus Z) < \dim(Z)$ for all definable Z by Fact 2.3. It follows that

$$(gY \setminus gX) \cap V \not\subseteq \overline{gY \cap V}$$

and by definition there is some $y \in Y \setminus X$ and an open (t-open) subset V_{gy} of V containing gy such that $V_{gy} \cap gX = \emptyset$. The t-neighborhood $g^{-1}V_{gy}$ of y does not intersect gX, which proves (2).

By the definition of closure, the third item follows from the second one. □

Lemma 2.11. *Assume that* $\dim(G) = n$. *Let* Y, W *be definable subsets of* G *with* $\dim(W) < n$. *Then*

$$\dim(\{g \in G \mid \dim(Y) = \dim(gY \cap W)\}) < n.$$

In particular, for any Y *there are infinitely many* g*'s such that* $gY \cap V$ *has small co-dimension in* gY.

Proof. As in the previous lemma let $\phi(g, y) = gy$. Then $\dim\left(\phi^{-1}(h)\right) = \dim(Y)$ for any $h \in G$ so

$$\dim(\{(g, y) \in G \times Y \mid gy \in W\}) = \dim(W) + \dim(Y).$$

On the other hand,

$$\{(g, y) \in G \times Y \mid gy \in W\} = \bigcup_{g\in G}\{(g, y) \in G \times Y \mid gy \in W\}$$

and since for any fixed g the function $\phi(g, \ )$ is an isomorphism, Fact 2.3 implies that if

$$\dim(g \in G \mid \dim(\{\, y \in Y \mid gy \in W\,\}) = \dim(Y)) = n$$

then

$$\dim(W) + \dim(Y) = \dim(\{(g, y) \in G \times Y \mid gy \in W\}) \ge n + \dim(Y)$$

contradicting $\dim(W) < n$.

So

$$\dim(g \in G \mid \dim(\{\, y \in Y \mid gy \in W\,\}) = \dim(Y)) < n,$$

as required.

For the "in particular" statement, take $W := G \setminus V$.

□

Exercise 2.9. *Let* $(G, \cdot)$ *be a definable group, and let* $X \subset G$ *be a definable subset of* G *of small co-dimension. Then there is some* $a \in G$ *such that the co-dimension of* $X \bigcup aX$ *is smaller than the co-dimension of* X. *Even more,* a *can be taken to be inside any definable subset of* G *of small co-dimension.*

Exercise 2.10. *Let* $(G, \cdot)$ *be a definable group, and let* $V \subset G$ *the set in the construction of the* t*-topology. Assume that the codimension of* G *is* $k < n$. *Then* G *can be covered by* 2^k *translates of* V.

Proposition 2.12. *Let G be a definable group. Let $f : (0,1) \to G$ be a definable function. Endow $(0,1)$ with the order topology and G with the t-topology. Then for some $t \in (0,1)$ the function f restricted to $(t,1)$ is a continuous function.*

Proof. Consider the set $Y = \{f(t) \mid t \in (0,1)\}$. By Lemma 2.11 for some g the set $gY \cap V$ has codimension 0, so it is finite, and $\{g \cdot f(t) \mid t \in (t_0,1)\}$ is contained in V for some t_0.

By Cell Decomposition, for some $t_1 > t_0$ the function $g \cdot f(t)$ is continuous in $(t_1,1)$. Since $t_1 > t_0$ we have that $g \cdot f(t)$ is contained in V for t in $(t_1,1)$ so t-continuity and continuity coincide and $g \cdot f$ is t-continuous in $(t_1,1)$, which implies that $f = (g^{-1}g) \cdot f$ is t-continuous in $(t_1,1)$. □

2.2.1. Definable subgroups.

Theorem 2.13. *Let H be a definable subgroup of a definable group G. Then the following hold:*

(1) H is a closed subset of G in the t-topology of G.
(2) The t-topology of H is the subset topology of the t-topology of G.

Proof. Assume that H is a definable subgroup. The closure $\overline{H}$ of H with respect to the t-topology is a definable subgroup of G which by Lemma 2.10 has the same dimension as H. It follows that there is no nontrivial coclass of H in $\overline{H}$ which implies $H = \overline{H}$ which proves (1).

To prove (2), let n be the dimension of G and d be the dimension of H. We will be working with many topologies at the same time: the G-topology and the H-topology will be the t-topologies of G and H, respectively. Within a cell C (or a union of disjoint cells) of dimension m, the topology (or the cell topology) will be the one inherited with the diffeomorphism with $\mathcal{R}^n$. Subspace topologies will be denoted as such.

Let U be a G-open set of codimension smaller than n where the G-topology coincides with the cell topology. For simplicity, we will assume that $U \subseteq \mathcal{R}^n$.

$U \subseteq \bigcup_{g \in G} gH$, and by definable choice G/H has dimension $n - d$ so by Fact 2.3 for some g the dimension of $U \cap gH$ is d. Since gH and H are G-topologically equivalent, we can prove the statement for gH (with the topology inherited from the H-topology in H via multiplication by g) instead of H. So we may assume without loss of generality that $H \cap U$ has dimension d.

Let V_H be the H-open subset of small codimension such that the H-topology of V_H coincides with the topology inherited from a diffeomorphism with $\mathcal{R}^d$.

Let $H_1, \ldots, H_k$ be the d-dimensional cells of V_H given by Cell Decomposition. So each H_i is an open subset of V_H in H (so H-open).

Claim 2.14. *For some $h \in H_1$ there is a neighborhood $W \subseteq U$ of h such that $W \cap H = W \cap H_1$.*

Proof: For any $X \subset G$, let $\overline{X}$ denote the closure of X in the $\mathcal{R}$-topology. By Lemma 2.10 $\dim(\overline{H} \setminus H) < d$ so $H_1 \not\subseteq \overline{H} \setminus H$ and there is some $h \in H_1$ such that $h \notin \overline{H} \setminus H$. Since $h \notin (H \setminus H_1)$ we have $h \notin \overline{H} \setminus H_1 \subseteq \overline{H \setminus H_1}$. So $W \cap (H \setminus H_1) = \emptyset$ for some neighborhood W of h and $W \cap H = W \cap H_1$ as required. $\blacksquare_{Claim}$

Since any open subset of W is a subset of U which is G-open by hypothesis, and any open subset of V_H is H-open, we have that the H-topology around h coincides with the subset topology inherited from the G-topology.

Multiplication by any $g \in H$ is a diffeomorphism both in the H and in the G topologies, so the same is true for any element in H and (2) holds. □

2.2.2. Definable compactness. In $\aleph_0$-saturated ordered structures only sequences that are eventually constant converge, so the notion of compactness by sequences is not meaningful in a model-theoretic context. The standard notion of compactness does not seem to be very useful either, as closed bounded intervals are not compact. A different definition is needed to capture the notion of compactness for definable sets.

Definition 2.15. Let X be a definable topological space. By *an open definable curve* in X we mean a continuous definable map $f : (0,1) \to X$, where $(0,1)$ endowed with the order topology and G with the t-topology.

We will say that an open definable curve f in X is *completable* if $\lim_{t\to 1^-} f(t) = g$ for some $g \in X$.

Finally, we say that a definable $Y \subset X$ is *definably compact* if any open definable curve whose image is contained in Y is completable in Y.

From now on, we will assume the t-topology on definable groups. A subset X of a definable G is therefore definably compact if for any definable map definable map $f : (0,1) \to X$ which is continuous when G is endowed with the t-topology there is some $g \in X$ such that $\lim_{t\to 1^-} f(t) = g$.

Proposition 2.16. *Let X, Y be definably compact definable topological spaces. Then the following hold.*

(1) $X \times Y$ endowed with the product topology is definably compact.
(2) If $Z \subset X \subseteq G$ is definably compact, then Z is closed.
(3) If $(G, \cdot)$ is a definable group and $Z \subset X \subset G$ are definable sets with Z closed and X is definably compact, then Z is definably compact.

Proof. Let $\phi : (0,1) \to X \times Y$ be an open definable curve in $X \times Y$. By definition of the product space and the product topology, $\phi = \phi_X \times \phi_Y$ with $\phi_X : (0,1) \to X$ and $\phi_Y : (0,1) \to Y$ open definable curves. By hypothesis, there are $x \in X$ and $y \in Y$ such that $\lim_{t\to 1^-} f(t) = x$ and $\lim_{t\to 1^-} \phi_Y(t) = y$. Then $\lim_{t\to 1^-} f(t) = (x, y)$, so ϕ is completable as required in (1).

For (2), let $v \in V$ where V is the open set in the construction of the t-topology of G, and let $B_{<\epsilon}(v) := \{x \in G \mid d(x,v) < \epsilon\} \cap V$. Let z in the closure of Z and consider the t-neighborhoods $U_\epsilon := (z \cdot v^{-1})B_{<\epsilon}(v)$ of z. By definition of closure $U_\epsilon \cap Z \neq \emptyset$ and by definable choice there is a definable function $f : (0,1) \to G$ such that $f(1-\epsilon) \in U_\epsilon \cap Z$. By Proposition 2.12, f is continuous in an interval $(t, 1)$. So the restriction $f\mid_{(t,1)}$ of f to $(t,1)$ is an end segment of an open curve in Z which by definable compactness is completable to some point $z' \in Z$. But then $z' \in (z \cdot v^{-1}) \odot B_{<\epsilon}(v)$ so $v \cdot z^{-1} \cdot z' \in B_{<\epsilon}(v)$ for all $\epsilon < 0$. The t-topology coincides with the order topology in V, the order topology is Hausdorff in $\mathcal{R}$ and $B_{<\epsilon}(v)$ is a neighborhood basis for v, so $v \cdot z^{-1} \cdot z' = v$ and $z' = z$, as required.

Let Z be closed. If f is an open curve into Z, it is an open curve into X. If X is definably compact then there is some $x \in X$ such that $\lim_{t\to 1} f(t) = x$. But Z is closed so $x \in Z$ and f is completable in Z. $\square$

Proposition 2.17. *Let G be a definable group, and let E be a definable equivalence relation on G. Endow G/E with the quotient topology.*

If G is definably compact then G/E is definably compact.

Proof. Let $f : (0,1) \to G/E$ be an open definable curve. By definable choice, there is $\rho : Y \to X$ with $\pi(\rho(y)) = y$. By Proposition 2.12, there is some $t \in (0,1)$ such that $\rho \circ f : (0,1) \to X$ is t-continuous in $(t,1)$. By definable compactness,

$\lim_{t\to 1^-}\rho\circ f(t) = g$ with $g\in G$. By definition of the quotient topology the projection $\pi : G\to G/E$ is continuous so

$$\lim_{t\to 1^-} f(t) = \lim_{t\to 1^-}\pi\circ\rho\circ f(t) = \pi\left(\lim_{t\to 1^-}\rho\circ f(t)\right) = \pi(g) = [g]_E.$$

So f is completable, as required. □

Proposition 2.18. *Assume we are working with the standard topology (the one inherited from the order topology in $\mathcal{R}$) for definable subsets of $\mathcal{R}^k$. The the following hold:*

- *Definable compactness of a definable set X is equivalent to X being closed and bounded.*
- *Definable compactness is preserved under projections: If $\pi : X\to Y$ is a surjective continuous function onto a definable set Y and X is definably compact, then Y is definably compact.*

Proof. Assume that $X\subseteq\mathcal{R}^k$ is closed and bounded. Let

$$f(x) = (f_1(x), f_2(x), \dots f_k(x))$$

be a continuous function from $(0,1)$ to X. By Cell Decomposition, for any i there is an element $t_i\in(0,1)$ such that f_i is either increasing or decreasing in $(t_i,1)$. If it is increasing, let $a_i = \sup\{f(x)\mid x\in(t_i,1)\}$, and if it is decreasing let $\inf\{f(x)\mid x\in(t_i,1)\}$. By boundedness $a_i\in\mathcal{R}$. Since X is closed in the product topology, $\overline{a} := (a_1,a_2,\dots,a_k)\in X$. By construction, $\lim_{t\to 1^-} f(t) = \overline{a}$.

For the converse, assume that X is definably compact. By Proposition 2.16 X is closed. Let $\overline{a}\in X$. Assume towards a contradiction that X was not bounded, then

$$X\not\subseteq B_{<1/(1-t)}(\overline{a}) := \{\overline{x}\in\mathcal{R}^k \mid\ d(\overline{x},\overline{a}) < 1/(1-t)\}$$

for $t\in(0,1)$, so that if $X\setminus B_{<1/(1-t)}(\overline{a})\neq\emptyset$. By definable choice, let $f:(0,1)\to X$ be a definable function with $f(x)\in X\setminus B_{<1/(1-x)}(\overline{a})$. By Cell Decomposition f is continuous in an interval $(t,1)$ so by definable compactness $\lim_{t\to 1^-} f(t) = \overline{b}$ with $\overline{b}\in X$. By triangular inequality and continuity we have $d(\overline{b},\overline{a}) > 2/(1-x)$ for all $x\in(0,1)$, a contradiction.

To prove that definable compactness is preserved under projections in the $\mathcal{R}$-topology, let $\pi : X\to Y$ be a continuous definable surjective function, and let $f:(0,1)\to Y$ be an open definable curve. By definable choice, there is $\rho : Y\to X$ with $\pi(\rho(y)) = y$. By Cell Decomposition, there is some $t\in(0,1)$ such that $\rho\circ f:(0,1)\to X$ is continuous on $(t,1)$ so that if X is definably compact $\lim_{t\to 1^-}\rho\circ f(t) = \overline{a}$ with $a\in X$. So

$$\pi(\overline{a}) = \pi\left(\lim_{t\to 1^-}\rho\circ f(t)\right) = \lim_{t\to 1^-}\pi\circ\rho\circ f(t) = \lim_{t\to 1^-} f(t)$$

by construction and continuity of π. Since $\pi(\overline{a})\in Z$, f is completable in Z as required. □

2.3. t-Connectedness and the Descending Chain Condition

Another key concept in the study of groups definable in o-minimal structures will be the t-connected component of the identity. Given any definable group G we will denote this component throughout the book by G^0.

We start with the following exercises.

Exercise 2.11. *Show that G^0 is the only open and closed definably connected set containing the identity and that G^0 is a t-open normal subgroup of G.*

Proposition 2.19. *The t-connected component G^0 of G is a definable subgroup.*

PROOF. Let $\{V_i\}$ be the (definably connected open) cells in V given by a cell decomposition of V. There are finitely many and each is t-open and t-connected by Lemma 2.7. The set

$$U := \bigcup_i (V_i)^{-1} \cdot V_i.$$

is a union of t-connected t-open subsets of G each of which contains the identity, so U is a t-open t-connected (definable) subset of G.

There is only a finite number of V_i's, so we can find some l such that if for some $n \in \mathbb{N}$ $V_i \cap U^n \neq \emptyset$ then $U^l \cap V_i \neq \emptyset$.

Claim 2.20. *If $V_i \cap U^n \neq \emptyset$ for some $n \in \mathbb{N}$, then $V_i \subset U^{l+1}$.*

Proof: By choice of l, $V_i \cap U^n \neq \emptyset$ for some $n \in \mathbb{N}$ implies $V_i \cap U^l \neq \emptyset$; let $x \in V_i \cap U^l$. Since $x \in V_i$, $x^{-1}V_i \subset U$ by definition of U and $x \in U^l$ implies $V_i \subset U^l \cdot U$. $\blacksquare_{Claim}$

In particular, $U^n \cap V = U^{l+1} \cap V$ for any natural number n greater than $l+1$. Now, V has small co-dimension and U^n has dimension n, so $U^n \cap V$ is t-dense in U^n. In particular, for any $n \geq l+2$,

$$\overline{U^n}^t = \overline{U^n \cap V}^t = \overline{U^{l+1} \cap V}^t \subset U \cdot U^{l+1} = U^{l+2}.$$

So U^{l+2} is a subgroup and it is a clopen definably connected set containing the identity. U^{l+2} is the connected component of the identity in G. □

Corollary 2.21. *G^0 is a normal subgroup of finite index.*

PROOF. The set V in the definition of the t-topology is t-dense and the set $\{aG^0\}_{a \in G}$ contains t-open disjoint sets. This implies that $V \cap aG^0 \neq \emptyset$ for any $a \in G$ and $\{V \cap aG^0\}_{a \in G}$ is a partition of V into t-open subsets of V. By Lemma 2.7 $V \cap aG^0$ is open with respect to the original topology, so that implies that $\{V \cap aG^0\}_{a \in G}$ is a partition of V into subsets of V which are open with respect to the $\mathcal{R}$-topology. Cell decomposition implies there can be only finitely many different sets $\{V \cap aG^0\}_{a \in G}$, which implies that $[G : G^0]$ is finite, as required. □

Proposition 2.22. *Let $H \leq G$ be a non-trivial subgroup of a definable group G. Then the following are equivalent.*

(1) H contains a t-open subset.
(2) H contains a t-open neighborhood of the identity.
(3) H is t-open.
(4) H contains the t-connected component G^0 of G.

If in addition we require H to be definable, then the following are also equivalent.

(5) Dimension of H is n.
(6) $[G : H]$ is finite.
(7) H contains an open neighborhood of V.

PROOF. (1) $\Rightarrow$ (2) is immediate.

For (2) $\Rightarrow$ (3), let $a \in H$ and $U \subset H$ open neighborhood of the identity. The $a \in aU \subset H$ is an open neighborhood of a.

(3) $\Rightarrow$ (1) follows from definition of t-openness and invariance of dimensions under definable translations.

Assume (3). Any open subgroup is closed, so that H is clopen. The t-connected component H^0 of H is therefore a clopen t-connected subset of G, so by Exercise 2.11 it must be equal to G^0. This implies (4).

(4) $\Rightarrow$ (2) is trivial.

Now, assume that H is definable.

The equivalence of (5) and (6) follows from definable choice for the relation $x \sim y$ if and only if $xy^{-1} \in H$ and Fact 1.9.

If (4) holds then H contains an the t-connected component G^0 then $(G : H) < [G : G^0]$ which is finite so (5) holds.

If (5) holds H has dimension n then there is an open cell C in its cell decomposition. Since V is dense, $V \cap C$ is open and (7) holds.

Finally, the order topology and the t-topology coincide in V so that (1) follows from (7). $\square$

Corollary 2.23. *Let G be a definable group. If H is a t-open subgroup of G, then H is definable.*

PROOF. If H is an open t-subgroup then it contains G^0 and $[H : G^0] \leq [G : G^0]$, so it is finite. So H can be recovered by finitely many translations of the definable group G^0. Since multiplication is definable, H is definable. $\square$

Proposition 2.24 (DCC on definable subgroups)**.** *Let G be a definable group and let $G = G_0 \geq G_1 \geq \dots$ be an infinite sequence of definable subgroups. Then there is $n \in \mathbb{N}$ such that $G_i = G_j$ for all $i, j \geq n$.*

PROOF. For any definable group H, let $(d, r)(H) = (\dim(H^0), [H : H_0])$ which by Corollary 2.21 is an element in $\mathbb{N} \times \mathbb{N}$.

By Proposition 2.22, $H_1 \geq H_2$ implies that either $\dim((H_1)^0) > \dim((H_2)^0)$, or $(H_1)^0 = (H_2)^0$ and $[H_1 : (H_1)^0] > [H_2 : (H_2)^0]$. So $(d, r)(G_0), (d, r)(G_1) \dots$ is a descending chain in the lexicographic order of $\mathbb{N} \times \mathbb{N}$, so it must become a constant sequence at some point. $\square$

2.3.1. The definable Hull. The Descending Chain Condition allows us to define a "Definable hull" of a subset of a group, which will be very useful throughout this book.

Definition 2.25. For any subset X of G, the *definable subgroup generated by* X, denoted $\langle X \rangle_{\mathrm{def}}$, is the smallest definable subgroup of G containing X. Sometimes $\langle X \rangle_{\mathrm{def}}$ is called the *definable hull of* X.

As usual, for a group G and $X \subset G$ we define the *centralizer* $C_G(X)$ of X in G to be the set $\{g \in G \mid gx = xg \ \forall x \in X\}$ and the *center* $Z(G) = C_G(G)$.

Lemma 2.26. *Let X be a subset of a definable group G. Then*

(i) *$C_G(X)$ is definable.*
(ii) *If the elements of X commute pairwise, $\langle X \rangle_{\mathrm{def}}$ is abelian.*
(iii) *If H is a normal subgroup of G, so is $\langle H \rangle_{\mathrm{def}}$.*
(iv) *If $L \lhd H$ are subgroups of G, then H normalizes $\langle L \rangle_{\mathrm{def}}$ and $\langle L \rangle_{\mathrm{def}} \lhd \langle H \rangle_{\mathrm{def}}$.*

PROOF. (i) $C_G(X) = \bigcap_{x \in X} C_G(x)$, and $C_G(x)$ is a definable subgroup of G for any x. By DCC this must be a finite intersection and a finite intersection of definable sets is definable.

(ii) Because the elements of X commute pairwise, $X \subset Z(C_G(X))$ and the latter is definable by (i). It follows that $\langle X \rangle_{\mathrm{def}} \subset Z(C_G(X))$, which is abelian.

(iii) For $g \in G$, and any subgroup K, we write K^g for the conjugate $g^{-1}Kg$. Note that for each $g \in G$, $K^g \subset \langle K \rangle_{\mathrm{def}}^g$, and the latter is a definable subgroup. If H is a normal subgroup, we deduce that

$$\langle H \rangle_{\mathrm{def}} = \langle H^g \rangle_{\mathrm{def}} \subseteq \langle H \rangle_{\mathrm{def}}^g.$$

Conjugating, $\langle H \rangle_{\mathrm{def}}^{g^{-1}} \subseteq \langle H \rangle_{\mathrm{def}}$. Because g is an arbitrary element, $\langle H \rangle_{\mathrm{def}}$ is normal, as required.

(iv) Let $L \lhd H$. As in the proof of (3), $\langle L\rangle_{\mathrm{def}} = \langle H^h\rangle_{\mathrm{def}}$ for any $h \in H$ so that $\langle L\rangle_{\mathrm{def}}$ is invariant under conjugation by elements of H and H normalizes $\langle L\rangle_{\mathrm{def}}$.

The set $\{g \in G \mid g\langle L\rangle_{\mathrm{def}}g^{-1} = \langle L\rangle_{\mathrm{def}}\}$ is a definable subgroup of G containing H so by definition $\langle H\rangle_{\mathrm{def}}$ must be contained in it.

□

EXERCISE 2.12. *Let H, G be definable groups and let $L \geq H$ be a subgroup of G. Let $\pi : G \to G/H$ be the natural projection. Let $\langle \pi(L)\rangle_{\mathrm{def}}^{G/H}$ be the definable hull of $\pi(L)$ in G/H, and $\langle L\rangle_{\mathrm{def}}$ be the definable hull of L in G.*

Show that $\pi(\langle L\rangle_{\mathrm{def}}) = \langle \pi(L)\rangle_{\mathrm{def}}^{G/H}$.

Proposition 2.27. *Let G be a solvable definable group. Then G is definably solvable: This is, there is a subnormal series*

$$\{e\} \leq G_1 \leq \cdots \leq G_n = G$$

where G_i is definable and G_{i+1}/G_i is abelian.

PROOF. Let G be a definable solvable group. All subgroups of G are solvable, so they admit a subnormal series. We will refer to the smallest length of a subnormal series as the "depth" of the subgroup.

We will prove the following by induction: If H is any subgroup of depth smaller than or equal to l, then $\langle H\rangle_{\mathrm{def}}$ has a subnormal series of *definable* groups of length smaller than or equal to l.

If the depth of H is one, then Lemma 2.26(2) implies that $\langle H\rangle_{\mathrm{def}}$ is abelian so the induction statement holds.

Assume now that the statement holds for $l < k$ and assume that H has depth k. By definition, there is a normal subgroup N with $N \lhd H$ with depth $k-1$ such that H/N is abelian. By Lemma 2.26(4) we have $\langle N\rangle_{\mathrm{def}} \lhd \langle H\rangle_{\mathrm{def}}$ and by induction hypothesis $\langle N\rangle_{\mathrm{def}}$ has a subnormal series of *definable* subgroups of depth $k-1$.

Let $H_{\langle L\rangle_{\mathrm{def}}} := \{h\langle L\rangle_{\mathrm{def}} \mid h \in H\}$. Since H/L is abelian, $[H,H] \subseteq L$ which implies that $H_{\langle L\rangle_{\mathrm{def}}}$ is an abelian subgroup of $\langle H\rangle_{\mathrm{def}}/\langle L\rangle_{\mathrm{def}}$. By Exercise 2.12

$$\langle H_{\langle L\rangle_{\mathrm{def}}}\rangle_{\mathrm{def}} = \langle H\rangle_{\mathrm{def}}/\langle L\rangle_{\mathrm{def}}$$

so by Lemma 2.26(2) $\langle H\rangle_{\mathrm{def}}/\langle L\rangle_{\mathrm{def}}$ is abelian. So we can add $\langle H\rangle_{\mathrm{def}}$ to the subnormal series of definable subgroups of $\langle L\rangle_{\mathrm{def}}$ of depth $k-1$ we had by induction hypothesis, which completes the induction.

The proposition follows by applying the induction statement to G. □

Recall that a subgroup H of a group G is *characteristic* if any automorphisms of G fixes H setwise.

EXERCISE 2.13. *Let G be a definable group. Prove that $H_1 = \langle [G,G]\rangle_{\mathrm{def}}$ is a characteristic subgroup of G (closed under automorphisms).*

Proposition 2.28. *Let G be any infinite definable solvable group. Then there is a characteristic (normal) definable infinite abelian subgroup H of G.*

PROOF. We will prove the statement by induction on G. Since G^0 is a characteristic subgroup, any characteristic (normal) definable infinite abelian subgroup of G^0 is characteristic in G, so we may assume that G is connected.

We will prove the statement by induction on the dimension of G. By Proposition 2.27, let $H \lhd G$ be a definable subgroup with G/H infinite abelian. This implies that $[G,G] \lhd H$ and since H is definable, $H_1 = \langle [G,G]\rangle_{\mathrm{def}}$ is a definable subgroup of G contained in H so it must have dimension smaller than $\dim(G)$.

Now we have two cases. If H_1 is finite, then $[G, G]$ is finite, which implies the generating set $\{xyx^{-1}y^{-1} \mid x, y \in G$ is finite. But the latter is a continuous image of the connected group $G \times G$, so it must be connected. If it is finite, it must be the identity, which implies G is abelian and we are done..

If H_1 is infinite, by induction hypothesis it contains an abelian characteristic group A. Since H_1 is fixed under automorphisms (Exercise 2.13) A is fixed under automorphisms which completes the proof. □

2.4. Manifold structure and one dimensional groups.

Recall from Chapter 1 that a definable set X will be an n-dimensional $\mathcal{R}$-manifold if there is a definable family $\{U_i\}_{i\in I}$ of open subsets U_i of $\mathcal{R}^n$, definable injections $f_i : U_i \to X$ such that $\bigcup_{i\in I} f_i(U_i) = X$, and such that the transition maps are continuous: If $W := f_i(U_i) \cap f_j(U_j)$ then restriction of $f_j^{-1} \circ f_i$ is a definable homeomorphism from $f_i^{-1}(W)$ to $f_j^{-1}(W)$.

Lemma 2.29. *Let $(G, \cdot)$ be a definable group endowed with the t-topology and let V be the definable subset of $\mathcal{R}^n$ in the construction of the t-topology. Then, if we define the function $f_g : V \to gV$ given by multiplication by g on the left for any $g \in G$, the set $\{f_g(V)\}_{g\in G}$ is a covering of G which together with the maps f_g define a manifold structure on G.*

Proof. That each f_g is injective and that $\{f_g(V)\}_{g\in G}$ is a covering of G both follow from the group structure on G. So we only need to prove that the transition maps are $\mathcal{R}$-continuous.

Let $W = aV \cap bV$ for $a, b \in G$. Then $a^{-1}W, b^{-1}W$ are subsets of V and the transition map from $a^{-1}W$ to $b^{-1}W$ is given by multiplication by $b^{-1}a$. This map is t-continuous (is multiplication on the left by an element of G) and since we are working with open subsets of V, it is $\mathcal{R}$-continuous. □

The following theorem is an easy corollary of the lemma together with Fact 1.27.

Theorem 2.30. *Let G be a definable group, endowed with the t-topology. Then G is definably homeomorphic to a definable group G' with $G' \subset \mathcal{R}^m$ for some n and such that the t-topology of G' coincides with the topology on the definable set G' inherited from $\mathcal{R}^m$.*

Proof. By Exercise 2.10, $G = \bigcup_i g_iV$ for finitely many g_i's and $\phi_i(x) := g_i^{-1}x$ maps g_iV to $V \subset \mathcal{R}^n$, which by Theorem 2.1 is an homeomorphism from g_iV with the t-topology inherited from G to V with the topology inherited from $\mathcal{R}^n$.

It follows from Lemma 2.29 that $\{(g_iV, \phi_i)\}_i$ is a definable atlas on G so that G is a definable space in the sense of Definition 1.24. The definable space topology on G is by construction the t-topology, so by Fact 1.27 and definition 1.26 G there is a definable map from G to some $G' \subset \mathcal{R}^m$ for some m which is an homeomorphism to G' endowed with the topology inherited from $\mathcal{R}^m$, the order topology. □

Theorem 2.31. *Let $(G, \cdot)$ be a one dimensional t-connected definable group. Then either*

1. *G endowed with the t-topology is definably compact (definably homeomorphic to the circle $\{(x, y) \mid x^2 + y^2 = 1\}$), or*
2. *G is definably isomorphic to an interval I and the order in G inherited from the $\mathcal{R}$-order on I makes G an ordered group.*

Proof. We will prove that we can always cover G with a continuous image of one or two intervals in $\mathcal{R}^n$. If there are two intervals, then G will be definably compact.

Again, Exercise 2.10 implies that G is contained in the continuous image of finitely many translates of V. Since V is an open subset of $\mathcal{R}$ it is the union of finitely many intervals in $\mathcal{R}$. So we can view G as a manifold where the open subsets of $\mathcal{R}$ are intervals and finitely many of them cover G.

Let $(I_1, f_1), \dots (I_k, f_k)$ be a finite covering of G by intervals $I_i \subset \mathcal{R}$ and continuous maps $f_i : I_i \to G$, such that the transition maps are continuous and such that k is the smallest natural number that allows such a covering.

Since G is t-connected, if $k \geq 2$, there are two intervals I_i, I_j such that $f_i(I_i) \cap f_j(I_j) \neq \emptyset$. For notation purposes let $I = I_i, J = I_j, f = f_i$ and $g = f_j$. Furthermore, after composing by a linear map we may (and will) assume $I = (0,1)$ and $J = (0,1)$.

Now $W = f_i(I_i) \cap f_j(I_j)$ is an open subset of G so $f^{-1}(W)$ and $g^{-1}(W)$ are open subsets of I, J, respectively, so they are a finite union of intervals. Let $f^{-1}(W) = \bigcup_{i<l}(a_i, b_i)$ and $g^{-1}(W) = \bigcup_{i<l}(c_i, d_i)$ be such that $f((a_i, b_i)) = g((c_i, d_i))$.

We will assume the map $f^{-1}g$ is order preserving on (c_i, d_i) (the order reversing case is analogous). The set $\Gamma := \{(s,t) \in I \times J \mid f(s) = g(t)\}$ is a closed subset of $I \times J$. Since $(a_i, c_i) \in \overline{\Gamma}$ it must be that $(a_i, c_i) \notin I \times J$. This can only happen if $a_i = 0$ or $c_i = 0$. Similarly, either $b_i = 1$ or $d_i = 1$.

Notice also that if $a_i = 0$ and $b_i = 1$ then $f(I) \subseteq f(I_i) \subseteq g(J)$ contradicting minimality of k. So there are only two possibilities: $a_i = 0$ and $d_i = 1$, or $c_i = 0$ and $b_i = 1$.

Both sets $\{(a_i, b_i)\}_{i<l}$ and $\{(c_i, d_i)\}_{i<l}$ were sets of pairwise disjoint intervals, so it can't happen that two different a_i's are 0, nor can it happen that two different b_j's are one. Therefore $l \leq 2$. We now analyze each of the cases $l = 1$ and $l = 2$ to complete the proof.

If $l = 2$ then $f^{-1}(f(I) \cap g(J)) = (0,b) \cup (a,1)$, $g^{-1}(f(I) \cap g(J)) = (0,d) \cup (c,1)$ with $f((0,b)) = g((c,1))$ and $g((0,d)) = f((a,1))$. Notice that $\lim_{t\to 0^+} f(t) = \lim_{s\to c^+} g(s) = g(c)$, $\lim_{t\to 1^-} f(t) = g(d)$, $\lim_{t\to 0^+} g(t) = f(a)$ and $\lim_{t\to 1^-} g(t) = f(b)$. We can therefore extend f and g continuously to $\overline{I}$ and $\overline{J}$, and $f\left(\overline{I}\right) \cup g\left(\overline{J}\right) = \overline{f(I) \cup g(J)}$ is closed, so clopen (it is the union of two open sets) and definable. Since $\overline{I}$ and $\overline{J}$ are definably compact, G is definably compact. The homeomorphism to the circle follows from the construction and is left as an exercise.

If $l = 1$, then either $a_1 = 0, d_1 = 1$, or $c_1 = 0, b_1 = 1$. These cases are analogous, so we assume the former. Let $b = b_1 = 1, c = c_1 = 0, a = a_1, d = d_1$ and $f^{-1} \circ g : (0,d) \to (a,1)$ is a homeomorphism. Let θ be a polynomial such that $\theta(-1) = 0, \theta(0) = a$ and the restriction of h to $(-1,0)$ is increasing, so that

$$\phi(x) = \begin{cases} \theta(x) & \text{if } -1 < x \leq 0, \\ f^{-1} \circ g & \text{if } x \in [0,d] \end{cases}$$

is an homeomorphism from $(-1,d)$ to $(0,1)$. This implies that

$$h(x) = \begin{cases} f(\phi(x)) & \text{if } -1 < x \leq d, \\ g(x) & \text{if } x \in (d,1) \end{cases}$$

we have a continuous injection h from $(-1,1)$ to G such that $h((-1,1)) = f(I) \cup g(J)$, contradicting minimality of k.

Assume that G is not definably compact, and let $h : I \to G$ be an homeomorphism. To complete the proof of the theorem, we need to show that $h^{-1}(g \cdot h(s)) < h^{-1}(g \cdot h(t))$ for any $s < t$ in I and any $g \in G$.

For any fixed $g \in G$ the map $\phi_g : x \mapsto h^{-1}(g \cdot h(x))$ is an homeomorphism from I to I, so it must be either strictly increasing or strictly decreasing. Assume towards a contradiction that ϕ_g is strictly decreasing for some g. In this case, $\lim_{s \to 0^+} \phi_g(s) = 1$ and $\lim_{s \to 1^-} \phi_g(s) = 0$ so that by the Intermediate Value Theorem there is some $t \in I$ with $\phi_g(t) = t$. By definition of ϕ_g we would have $h(t) = g \cdot h(t)$ so $g = e$. But ϕ_e is the identity in I so it can't be order reversing, a contradiction. $\square$

In the second case, one can endow I with the group structure inherited from G making $(I, \cdot, <)$ an o-minimal group and, as mentioned in Chapter 1, it is divisible and abelian.

For the definably compact case we will also prove abelianity, but we need some results before proving this.

Theorem 2.31 has the following very nice consequence.

Theorem 2.32. *Let G be a definably connected one dimensional group. Then G is abelian and divisible.*

Proof. By Theorem 2.31, either G is an ordered o-minimal group, or G is t-definably compact. In the former case the result follows from Chapter 1. In the definably compact case, that G is abelian follows using t-connectedness and one dimension from the following lemma:

Lemma 2.33. *Let G be an infinite t-definably compact definable group. Then G has an infinite abelian definable subgroup.*

Proof. By DCC, it is enough to show that any group G that doesn't have any infinite definable subgroup is abelian. So assume towards a contradiction that G doesn't have any infinite definable subgroup and it is not abelian. So in particular it has finite center Z.

Let $a \notin Z$. So the centralizer of a $C_G(a) = \{g \in G \mid ga = ag\}$ is not G so by hypothesis it must be finite. Let $a^G := \{gag^{-1} \mid g \in G\}$. The map $\psi_a : g \mapsto gag^{-1}$ is a continuous finite to one map which implies that $\dim(a^G) = \dim(G)$ and by Lemma 2.10 a^G contains a t-open set. But conjugation is t-continuous and transitive in a^G, which implies that a^G is open. But a^G is also the image of G under the t-continuous map ψ_a, so Proposition 2.17 implies that a^G is t-definably compact, so t-closed. This contradicts t-connectedness of G. $\square$

$\square$

Remark. *The statement of Lemma 2.33 holds in fact for all infinite definable groups, not just definably compact. In the non definably compact case the result will follow since we will prove in Chapter 7 that any non definably compact definable group has a definable one dimensional torsion free subgroup. This is one of the technical results from where much of our understanding of torsion free definable groups is based on, but we will need to leave the proof for later in the book.*

In light of Theorem 2.30, given any definable group G we can find a definably isomorphic copy which is a topological group with the induced topology. So for any statement which is invariant under isomorphisms we can apply all the results from o-minimality (Cell Decomposition and in general all results in Chapter 1) without needing to worry about t-continuity versus (order/subspace) continuity.

2.5. Examples

We end the chapter with a few examples of definable groups. Suppose $\mathcal{M} = (M, <, +, \cdot, 0, 1, \dots)$ is a an o-minimal expansion of a real closed field.

Example 2.34. Every finite group G is definable in $\mathcal{M}$. G is definably compact and $G^0 = V = \{e\}$.

Example 2.35. Consider the group $(G, \oplus)$ with universe $[0,1)$ where $\oplus$ is addition modulo 1 described at the beginning of the chapter. It is a 1-dimensional definably compact definably connected group. As already mentioned, $V = (0,1)$. The set $\{V, a \oplus V\}$ is a definable atlas of G for any $a \in V$.

Example 2.36. The group of $n \times n$ invertible matrices $G = \mathrm{GL}_n(M)$ with the operation of matrix multiplication is already a topological (semialgebraic) group with the ambient topology induced by M^{n^2}. Therefore, every definable subgroup of G can be endowed with the induced topology.

G is not definably compact because it is not bounded in M^{n^2}. It is not definably connected either, as it has two definably connected components, and G^0 is the subgroup of matrices with positive determinant.

Example 2.37. By a theorem of Chevalley, every compact Lie group is Lie isomorphic to a algebraic (i.e. a Zariski closed) subgroup of $\mathrm{GL}_n(\mathbb{R})$, for some positive integer n. Therefore, by Proposition 2.18, every compact Lie group is Lie isomorphic to a definably compact semialgebraic linear group. Conversely, every definably compact group that is definable in an o-minimal expansion of the real field is a compact Lie group (this will follow from the results of the next chapter).

If a Lie group is not compact it may not be isomorphic to any group definable in any o-minimal expansion of the real group. Throughout the book we will establish precisely which Lie groups are.

Example 2.38. Every algebraic group G over an algebraically closed field $\mathcal{K} = (K, +, \cdot)$ of characteristic 0 is definable in $\mathcal{K}$, and therefore it is definable in any maximal real closed field of $\mathcal{K}$.

Historical notes and comments

The t-topology on topological groups was defined by A. Pillay in [**61**], although our approach is closer to the construction in [**72**] which was used in the o-minimal context in [**63**]. All these proofs are based in Hrushovski's adaptation ([**37**]) of Weil's theorem for algebraic groups ([**75**]).

DCC on definable subgroups was proved by Pillay in [**61**] and [**60**]. Another proof of it was given by A. Strzebonski in [**69**]. The rest of the results in Section 2.3 are corollaries of DCC and appear as such in [**60**], although, as mentioned there, the use of DCC for definable groups already appeared in the context of groups of finite Morley rank (see for example [**10**]).

Definable compactness was introduced by Y. Peterzil and C. Steinhorn in [**59**], where the results of Section 2.2.2 were proved.

One dimensional groups were studied by V. Razenj and A. Strzebonski in [**64**] and [**70**].

CHAPTER 3

The Lie structure.

A *Lie group* is a real or complex smooth manifold with a smooth group operation such that the inverse map is smooth as well. Every complex Lie group becomes a real Lie group by identifying $\mathbb{C}$ with $\mathbb{R}^2$. We assume all Lie groups are over $\mathbb{R}$. We also assume all Lie groups are *finite-dimensional*.

By analogy, a *$\mathcal{R}$-definable C^k-Lie group* is a definable group G which is a C^k-manifold and such that the group operation and inverse functions are C^k-maps. An $\mathbb{R}$-definable Lie group G is therefore a definable group in an o-minimal expansion of the real field which is a finite-dimensional Lie group in the classical sense. In this case, we will just say that G is a *definable Lie group*.

Given $\mathcal{R}$-definable C^k-Lie groups G and H, a map $f : G \to H$ will be a *definable C^k-Lie homomorphism* if it is a definable C^k group homomorphism. When the k is fixed, we will refer to C^k maps as *smooth maps*.

In this chapter we will prove that, endowed with the t-topology defined in the previous chapter, G admits a C^k-manifold structure that makes it a C^k-Lie group. We will then show that the Lie algebra is definable and prove analogues of theorems from the classical Lie context. We will prove, for example, that a centerless definable Lie group is definably isomorphic to its adjoint representation.

3.1. Lie group structure on G

Theorem 3.1. *Any definable group G can be endowed with a manifold structure compatible with the group operation that makes G a C^k-Lie group for any $k \in \mathbb{N}$.*

Proof. The Lie group structure is a local property, so we can identify G with its definably connected component. Assume throughout this chapter that G is definably connected.

We will refer to the construction in the proof of Theorem 2.1. In particular, W will be a subset of $G \times G$ where multiplication is C^k, V is a t-open subset of G of small codimension homeomorphic to an open subset of $\mathcal{R}^n$, and for any $a \in G$ the sets ${}_aW, a^{-1}({}_aW)$ and W_a^ρ are all sets of small codimension defined as in the aforementioned proof.

Claim 3.2. *If a, b, c and abc are all elements in V, then $(a, b, c) \mapsto abc$ is C^k.*

Proof: The sets V, ${}_cW, c^{-1}({}_bW), b^{-1}c^{-1}({}_aW)$ and W_{abc}^ρ all have finite codimension by construction of V, so the intersection is not empty. Let v be an element in the intersection.

By construction $(c, v), (b, cv), (a, bcv)$ and $(abcv, v^{-1})$ are all in W, so multiplication is C^k by definition of W. But $abc = ((a(b(cv)))v^{-1})$ so the claim follows. $\blacksquare_{Claim}$

Fix $v \in V$. Multiplication and inverse are t-continuous and $vv^{-1}v = v$, so there is a neighborhood $U_v \subseteq V$ of v such that $U_v(U_v)^{-1}U_v \subset V$. Let $U := v^{-1}U_v$ the t-open neighborhood of the identity, with the differential structure inherited from U_v as an open subset of $\mathcal{R}^n$. By Lemma 2.7 multiplication by v is an homeomorphism between U with the t-topology and U_v with the topology of $\mathcal{R}^n$ so we can use this map as a chart making U a manifold. From now on, we will fix this manifold structure on U.

Lemma 3.3. *The group multiplication and inverse are C^k in $U \times U$ and U, respectively.*

PROOF. Since the coordinate chart is translation by v, the inverse function in U corresponds to the map

$$u \mapsto v\left(v^{-1}u\right)^{-1} = vu^{-1}v$$

for $u \in U_v$. Claim 3.2 implies that the map is C^k.

Similarly, by definition, multiplication in $U \times U$ is C^k if and only if the map $(u_1, u_2) \mapsto u_1 v^{-1} v_2$ is C^k, which again follows from Claim 3.2. □

The implication that a *local lie group structure* given by Lemma 3.3 implies the Lie group structure stated in Theorem 3.1 is standard, but we include the proof nonetheless.

For notation purposes, we will assume that U is itself a subset of $\mathcal{R}^n$ so that the differential structure in U is the one inherited from the real field. The manifold structure will be given by the atlas $\{gU_0\}$, where $U_0 \subset U$ is an open neighborhood of the identity such that $U_0U_0 \subseteq U$ and $(U_0)^{-1} = U_0$ (both exist by t-continuity).

One first shows that for any $g \in G$ conjugation by g is C^k in a neighborhood of the identity and use this to prove that the translations maps are C^k and that the inverse and multiplication are smooth with this structure.

Claim 3.4. *For any $g \in G$, the conjugation map $x \mapsto x^g = g^{-1}xg$ is C^k in a neighborhood of the identity.*

Proof: By continuity, let $U_e \subseteq U$ be a t-open neighborhood of e such that $U_e(U_e)^{-1}U_e \subseteq U$ and $U_e = U_e^{-1}$.

Multiplication restricted to $U \times U$ is C^k, for any elements $x, g \in U$ the conjugation map $x \mapsto x^g = g^{-1}xg$ is C^k in U.

The set

$$\left\{ g \in G \mid U \xrightarrow{(\ )^g} U^g \text{ is } C^k\text{-around } e \right\}$$

is a set containing U_e. But conjugation by a product is the composition of conjugations and the identity is always sent to the identity, so that the set above is a subgroup of G. Proposition 2.22 implies that the group must contain the definably connected component, which by hypothesis is G. $\blacksquare_{Claim}$

Claim 3.5. *The atlas $\{gU_0\}g \in G$ defines a manifold structure on G.*

Proof: We need to show that the transition maps are C^k.

If $x \in gU_0 \cap hU_0$, then $x = gu_1 = hu_2$ with $u_1, u_2 \in U_0$. So the transition map around x is given by the map $u_1 \mapsto (h^{-1}g)u_1$. Since $h^{-1}g = u_2u_1^{-1}$ we know by construction that $h^{-1}g \in U$ and the transition map is C^k by Lemma 3.3. $\blacksquare_{Claim}$

To complete the proof of Theorem 3.1, we need to show that the inverse is C^k around any $g \in G$ and that multiplication is C^k around (g, h) for any $g, h \in G$. Since we already have a manifold structure, we can prove smoothness relative to any choice of charts.

If we take the chart gU_g around g and $g^{-1}U_{g^{-1}}$ around g^{-1}, then, as

$$gu_1 \mapsto (gu_1)^{-1} = (g^{-1}g)(u_1g^{-1}) = g^{-1}u_1^g.$$

So smoothness of the inverse around g is implied by the smoothness of conjugation by g around the identity, which is precisely Claim 3.4.

As for multiplication, if we take elements g, h, gh and the projections given by the charts gU_g, hU_h and $(gh)U_{gh}$, then since

$$(gu_1)(hu_2) = (gh)\left((gh)^{-1}(gu_1hu_2)\right) = (gh)(h^{-1}u_1hu_2) = (gh)(u_1^h u_2),$$

smoothness of multiplication is equivalent to proving that the map from $U_g \times U_h$ sending (u_1, u_2) into $(u_1)^h u_2$ is smooth around (e, e). This follows from Claim 3.4 and Lemma 3.3. □

3.1.1. Homomorphisms, subgroups and quotients. In light of the previous section, we will fix $k \geq 1$ and assume that any definable group is endowed with the C^k-Lie group implied by Theorem 3.1.

We begin the study of homomorphisms with the following proposition.

Proposition 3.6. *Let $(G, \odot_G)$ and $(H, \odot_H)$ be definable groups, endowed with some definable C^k-Lie group structure as in Theorem 3.1 and let $f : H \to G$ be a definable group homomorphism. Then f is a C^k-Lie homomorphism. In particular, any two definable atlases are compatible and the C^k-Lie structure on G given in Theorem 3.1 is unique.*

PROOF. By the C^k-cell decomposition, there is a definable open subset U of G of small codimension where f is C^k. Let $a \in U$.

Let $b \in G$. Since f is a group homomorphism, the map

$$x \xrightarrow{(\quad \odot_G(b^{-1}a))} x(b^{-1}a) \xrightarrow{f(\ )} f(x(b^{-1}a)) \xrightarrow{(\quad \odot_H(f(a^{-1}b)))} f(x).$$

Since f is C^k at a and multiplication on G and H are C^k everywhere, $f(x)$ is C^k at any point $b \in G$ and it is C^k.

Finally, if $\{(U_\alpha, \phi_\alpha)\}_\alpha$ and $\{(V_\beta, \psi_\beta)\}_\beta$ are two definable C^k-manifold structures making G a C^k-Lie groups, then the identity map is smooth which implies that $\{(U_\alpha, \phi_\alpha)\}_\alpha$ and $\{(V_\beta, \psi_\beta)\}_\beta$ are compatible atlases. □

We now turn to some more technical results. We will use Theorem 1.43 extensively, and in order to do so we need the following lemma.

Lemma 3.7. *Let G and H be definable Lie groups and let f be a definable homomorphism from G to H. Then the rank of f is constant.*

PROOF. By Proposition 3.6, f is smooth so the rank is defined for every point in G.

Because multiplication by constants in both G and H are invertible diffeomorphisms, if $m_c(x)$ is left multiplication by c in G then $\mathrm{d}_x m_c$ is invertible for all $x \in G$. A similar thing happens in H so the Chain Rule implies that the rank of $\mathrm{d}_x f$ is constant for all x. □

Theorem 1.43 thus applies to all definable group homomorphisms:

Theorem 3.8. *Let G and H be definable groups with their Lie structure. Let f be a definable homomorphism from G to H with differential rank l.*

Let $g \in G$ and $h = f(g)$.

Then the following hold:

(1) *The set $f^{-1}(h)$ is a definable submanifold of G of dimension $n - l$ and $T_g(f^{-1}(c)) = \ker(\mathrm{d}_g f)$.*
(2) *$f(G)$ is an immersed submanifold of H of dimension l and $T_h(f(G)) = (\mathrm{d}_g f)(T_g(H))$.*
(3) *If f is surjective, then l is equal to the dimension of H for all $x \in G$. This is, f is a* definable submersion.

(4) *If f is injective, then l is equal to the dimension of G for all $x \in G$. This is, f is a* definable immersion.

PROOF. The first two items follow immediately from Theorem 1.43 and Lemma 3.7. This Lemma also implies that for the last two statements it is enough to show that the assertion holds for *some* $x \in G$.

To prove (3), Assume that f is surjetive. Assume $\dim(G) = n$ and $\dim(H) = m$. By Definable Choice, let $\rho : H \to G$ be a definable function such that $f \circ \rho$ is the identity in H. Restricting to open sets in the covering atlas containing c and $\rho(c)$ and composing with the coordinate charts we may assume that f goes from an open subset of $\mathcal{R}^n$ to $\mathcal{R}^m$. By C^k-Cell Decomposition there is some point $c \in H$ where ρ is C^k. Restricted to said open sets $\mathrm{d}_{\rho(c)} f \circ \mathrm{d}_c \rho$ is the identity so the Jacobian under this chart is the identity in $GL_m(\mathcal{R})$, which by the Chain Rule implies that the Jacobian representing $\mathrm{d}_{\rho(c)} f$ is invertible and by definition the rank of $\mathrm{d}_{\rho(c)} f$ is m.

Item (4) is proved in a similar way. Let ρ to be the inverse of f (with domain $f(G)$) so that $f \circ \rho$ is the identity map in $f(G)$. Since $\dim(G) = \dim(f(G)) = n$, Cell Decomposition implies there is $U \subset f(G)$ definably isomorphic to an open subset of $\mathcal{R}^n$ and C^k-Cell Decomposition implies that there is some subset $V \subset U$ where ρ is C^k. Under the chart sending V to $\mathcal{R}^n$, for any $c \in V$ the Jacobian of the map $\mathrm{d}_{\rho(c)}(f) \circ \mathrm{d}_c(\rho)$ is the identity matrix in $GL_n(\mathcal{R})$ so by the Chain Rule the rank of $\mathrm{d}_{\rho(c)} f$ is n. □

In Lie groups, the definition of *Lie subgroup* is not universal. Some authors just ask for a subgroup which is an immersed submanifold while others (see [**44**] for example) ask also that the manifold topology on the subgroup is the subspace topology. However, in the definable case there is no distinction since by Theorem 2.13 the group topology of any definable subgroup H of a definable group G is the subspace topology of the group topology of G.

We therefore get the following corollaries of Theorem 3.8.

Theorem 3.9. *If H is a definable subgroup of a definable Lie group G, then H is a closed subset, the identity map from H to G is an immersion, and the topology on the definable Lie group H is the subspace topology inherited from G.*

This is, H is a closed definable Lie subgroup *of G.*

Something similar holds for quotients, and every definable surjective homomorphism will be submersion and a topological quotient map.

Theorem 3.10. *Let G be a definable group, and let H be a normal definable subgroup of G. Let f be a definable surjective group homomorphism from G to H. Then the following hold:*

(1) *f is a definable submersion.*
(2) *f is a continuous open map.*

In particular, if N is a normal definable subgroup of G and X is a definable group definably isomorphic to G/N (which exist by definable choice) then the t-topology of X coincides with the quotient topology inherited by X by its isomorphism with G/N.

PROOF. That f is a definable submersion follows from Theorem 3.8. The proof that this implies that f is open is well known in differential geometry:

Assume $\dim(G) = n$ and $\dim(H) = m$. To show f is open, it is enough to show that f is open around a neighborhood of any point in G. Let $g \in G$.

Let ψ and ϕ be the charts given by Theorem 1.42, so that

$$\psi \circ f \circ \phi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_m).$$

Let U be a neighborhood of $\phi(g)$.

For any $\overline{x} = (x_1, \dots, x_n) \in U$ consider the map $h : U \mapsto \mathcal{R}^m \times \mathcal{R}^{n-m}$ defined as $h(\overline{x}) = (f(\overline{x}), (x_{m+1}, \dots x_n))$. By construction the determinant of the Jacobian of $\mathrm{d}_p h$ equals the determinant of the Jacobian of $\mathrm{d}_p f$ so the Jacobian of $\mathrm{d}_p h$ is an invertible matrix. Let $(a_1 \dots, a_n)$ be the coordinates of $\phi(g)$.

By the Inverse Function Theorem, there are open subsets V of $\mathcal{R}^m$ and V' of $\mathcal{R}^{n-m}$ around $\psi(f(g))$ and $(a_{m+1}, \dots, a_n)$ and a differentiable function h^{-1} from $V \times V'$ to $\phi(U)$ such that $h \circ h^{-1}$ is the identity on $V \times V'$. This implies that $f(U) \supseteq V$, so that $f(g)$ is an interior point of $f(U)$. Since $f(g)$ was any point in $f(U)$, it follows that f is an open map.

The rest of the theorem follows from the fact that the quotient topology is the unique topology on a quotient such that the projection map is both continuous and open. □

Corollary 3.11. *If G is a definable group and N is a normal definable subgroup, then G/N is definably compact whenever G is.*

3.1.2. Definable actions of definable groups on manifolds. Let σ be a definable action from a definable group G to a space X. If X is a definable manifold, we will say that σ is smooth if the map $\sigma : G \times X$ to X is smooth when we endow $G \times X$ with the natural product manifold structure.

As usual, for $x \in X$ we define the stabilizer subgroup of G with respect to x

$$Fix_x(G) = G_x := \{g \in G \mid \sigma(g)(x) = x\}.$$

For a fixed x we define the map $\sigma_x : G \to X$ to be the map $\sigma_x(g) = \sigma(g)(x)$.

We will want to apply Theorem 1.43 to definable actions of groups on definable manifolds. The next lemma allows us to do that.

Lemma 3.12. *Let σ be a smooth definable action from a definable group G to a definable manifold X. Then the rank of σ_x is constant.*

PROOF. If for any $h \in G$ we define $l_h(x)$ to be left multiplication by h in G, then

$$\sigma_x \circ l_h = \sigma(h) \circ \sigma_x.$$

In particular, taking $g = e$ we have, by the Chain Rule,

$$\mathrm{d}_h \sigma_x \cdot \mathrm{d}_e l_h = \mathrm{d}_x \sigma(h) \cdot \mathrm{d}_e \sigma_x.$$

The maps l_h and $\sigma(h)$ are invertible so they are diffeomorphisms and the rank of σ_x is constant for all $h \in G$. □

As a corollary of Theorem 1.43 applied to the action of G on X we get the following.

Theorem 3.13. *Let G, H be definable groups. Let X be a manifold and let σ be a definable smooth action from G on X. Then the following hold:*

Let $x \in X$ and let $\sigma_x : G \to X$ be the map defined by $\sigma_x(g) = \sigma(g)(x)$. Let k be the rank σ_x. Then:

- *G_x is a Lie subgroup of G of dimension $n - k$ and such that $T_e(G_x) = \ker(\mathrm{d}_e(\sigma_x))$.*
- *The orbit $\sigma_x(G)$ of G at x is a definable submanifold of X of dimension k and $T_x(\sigma_x(G)) = (\mathrm{d}_e \sigma_x)(T_e(G))$.*

The following technical lemma will be key to understand the natural action of a definable group G to the left cosets G/H of a definable subgroup H of G.

Lemma 3.14. *Let G be a definable group of dimension n and H a definable subgroup of G of dimension k.*

Then there is a definable neighborhood U of e and a chart (ϕ, U) with $H \cap U = \phi^{-1}(\mathcal{R}^k \times \overline{0})$.

PROOF. By Theorem 1.42 (applied to the injection map), there are charts (ϕ, V) and (ψ, U) of H and G respectively, each containing the identity e and such that $\psi(e) = \phi(e) = \overline{0}$ and such that $\psi \circ \phi^{-1}(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$.

Since the manifold (and the group) topology on H is the subspace topology, we may assume $V = U \cap H$. By construction,

$$H \cap U = \psi^{-1}\left(\mathcal{R}^k \times \overline{0}\right).$$

□

We can now improve some of Subsection 3.1.1.

Theorem 3.15. *Let $(G, \odot_G)$ be a definable group.*

(1) *If H is a definable subgroup of G and G/H is the set of left cosets of H in G, then $X := G/H$ can be endowed with a definable smooth manifold structure compatible with the quotient topology on G/H, such that the action defined by left multiplication from G on X is smooth.*
(2) *Let H_1, H_2 be definable subgroups of G, and assume H_1 is definably connected. Then $H_1 \subseteq H_2$ if and only if $T_e(H_1) \subseteq T_e(H_2)$.*
(3) *Let f be a definable morphism $f : G \to H$. Let H_1 be a definable subgroup of H. Then the tangent space of $G_1 := f^{-1}(H_1)$ is $(\mathrm{d}_e(f))^{-1}(T_e(H_1))$.*

PROOF. To prove (1), Assume now that $(G, \odot)$ is a definable group, H is a definable subgroup of G and X the set of left cosets of H. Let $\pi : G \to X$ be the natural projection map $g \mapsto gH$.

We first endow X with the quotient topology. By Lemma 3.14, there is an open definable subset U of G and a chart (ϕ, U) on G with $H \cap U = \phi^{-1}(\mathcal{R}^k \times \overline{0})$ where k is the dimension of H.

Consider $S := \phi^{-1}(\overline{0} \times \mathcal{R}^{n-k})$ (so in particular $S \subset U$, and consider the map $\nu : (H \cap U) \times S \to G$ defined by $\nu(h, s) = h \odot s$. Let $\phi_S : U \cap S \to \mathcal{R}^{n-k}$ be the function sending $s \in S$ to the last $n-k$ coordinates of $\phi(s)$, and let $\phi_H : U \cap H \to \mathcal{R}^{n-k}$ be the function sending $h \in H \cap U$ to its first k coordinates of $\phi(h)$. By construction

$$\phi \circ \nu\left(\phi_H^{-1}\left(x_1, \ldots x_k\right), e\right) = \phi\left(\phi_H^{-1}\left(x_1, \ldots x_k\right)\right) = (x_1, \ldots, x_k, 0, \ldots, 0)$$

and similarly

$$\phi \circ \nu\left(e, \phi_S^{-1}\left(x_{k+1}, \ldots x_n\right)\right) = (0, \ldots, 0, x_{k+1}, \ldots x_n),$$

so $\mathrm{d}_{(e,e)}\nu(\mathrm{d}h, \mathrm{d}s) = \mathrm{d}h + \mathrm{d}s$ which implies that ν is a diffeomorphism around (e, e). This implies that there is a neighborhood of $V_1 \times V_2$ of (e, e) such that the restriction of ν to $V_1 \times V_2$ is a bijection. Taking $V = V_1 \cap V_2$ we get that the restriction of ν to $V \cap H \times V \cap S$ is a bijection.

In particular, every element in S is in a different H-coclass. If we endow G/H with the quotient topology and π is the projection map, then in the restriction of π to V we have a homeomorphism from $S \cap V$ to $\pi(V)$. Since S is diffeomorphic to $\mathcal{R}^{n-k}$ we can use this bijection as a coordinate chart $(\pi(V), \theta)$.

Finally, since for any $g \in G$ multiplication $l_g : X \to X$ on the left by g maps $lH = \pi(l)$ to glH, left translations are diffeomorphisms of G which push forward to homeomorphisms of G/H. Also, $\pi \circ l_g(V)$ is an open subset of G/H, and π maps $l_g(S)$ homeomorphically onto $l_g(V)/H$ so

$$(\pi(l_g(V)), l_g^{-1} \circ \theta) = (\pi(l_g(V)), l_{g^{-1}} \circ \theta)$$

are all coordinate maps of G/H which cover G/H. Compatibility of these charts follows from the fact that l_g is a diffeomorphism for all g, which makes

$$\{(\pi(l_g(V)), l_{g^{-1}} \circ \theta)\}_{g \in G}$$

an atlas of G/H. This completes the proof of (1).

Let H_1, H_2 be as in the statement of (2). If $H_1 \subseteq H_2$ then H_1 is a definable Lie subgroup of H_2 and by Theorem 3.9 so $T_e(H_1) \subseteq T_e(H_2)$.

For the converse, by (1) the set G/H_2 can be endowed with a manifold structure compatible with the action $\sigma(g)(aH_2) = (g \odot_G a)H_2$ of G on H/H_2. By Theorem 3.13 $T_e(H_2) = \ker(\mathrm{d}_e(\sigma_{H_2}))$ where $\sigma_{H_2}(g) = gH_2$. If we assume $T_e(H_1) \subseteq T_e(H_2)$ then for any $\eta \in T_e(H_1)$ we have $\eta \in \ker(\mathrm{d}_e(\sigma_{H_2}))$ so $\mathrm{d}_e(\sigma_{H_2})(\eta) = 0$.

Consider the restriction α of σ to $H_1 \times G/H_2$. It is the same map, so $\mathrm{d}_e(\sigma_{H_2})(\eta) = 0$ for all $\eta \in T_e(H_1)$ which implies that the rank of $\alpha_{H_2} = 0$. The second part of Theorem 3.13 implies that the set $\{gH_2\}_{g \in H_1}$ has dimension zero, and since it is definable, it must be finite. But said finite set is the image of the definably connected H_1 so it is definably connected, so it consists of the single left coclass H_2 and $H_1 \subseteq H_2$, as required.

To prove (3), let α be the action of G on $X := H/H_1$, defined by $\alpha(g)(aH_1) = (f(g) \odot_H a)H_1$ where $\odot_H$ is the group operation on H.

By definition $G_1 = Fix_{H_1}(G) = G_{H_1}$, so Theorem 3.13 implies that $T_e(G_1) = \ker(\mathrm{d}_e(\alpha_{H_1}))$. By definition

$$\alpha_{H_1}(g) = \alpha(g)(H_1) = f(g)H_1 = \sigma(f(g), H_1))$$

with σ the natural action of H on H/H_1. By the Chain Rule,

$$\mathrm{d}_e(\alpha_{H_1}) = \mathrm{d}_e(\sigma_{H_1}) \circ \mathrm{d}_e(f).$$

Since $\ker(\mathrm{d}_e(\sigma_{H_1})) = T_e(H_1)$, we have

$$T_e(G_1) = \{\eta \mid \mathrm{d}_e(f)(\eta) \in T_e(H_1)\}$$

which is precisely the statement of (3). $\square$

The statement (1) of Theorem 3.15 has the following equivalent statement. The details of this equivalence are left as an exercise:

Exercise 3.1. *Let X be a definable set and let σ be a definable transitive action of a definable Lie group G on X. Then there is a manifold structure on X such that σ is smooth.*

3.2. The Lie algebra of a definable Lie group

We will fix a definable group $(G, \odot)$ of dimension n. We will assume that it is a C^k-Lie group with $k \geq 2$. The Lie algebra of G will be a Lie algebra structure on the tangent space $T_e(G)$. From now on, we will denote $T_e(G)$ with $\mathfrak{g}$.

3.2.1. Preliminaries on Lie groups and Lie algebras over $\mathbb{R}$. To find common features between definable groups and Lie groups we will need some background on definitions and some of the main properties of Lie groups and Lie algebras over $\mathbb{R}$.

Fix a field $\mathcal{K} = (\mathbb{K}, +, \cdot)$. Given any $\mathbb{K}$-vector space V, we define a *Lie bracket* on V to be a bilinear operation $[\cdot,\cdot]\colon V \times V \to V$ such that for each $x, y, z \in V$

(i) $[x, y] = -[y, x]$,
(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$. This is the *Jacobi identity.*

A *Lie algebra* $\mathfrak{g} = (V, [\ ,\ ])$ over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space V together with a Lie bracket.

The operation $[x, y] = 0$ for each $x, y \in V$ is a Lie bracket for any vector space V. In this case we say that $(V, [,\ ,\ ])$ is *abelian.*

Other examples include the following:

Examples 3.16. The composition of maps is an associative bilinear operation on $\mathrm{End}(V)$, the set of linear endomorphisms of a vector space V. The corresponding Lie algebra with commutator bracket is denoted by $\mathfrak{gl}(V)$. The commutator

$$[\phi, \psi] := \phi \circ \psi - \psi \cdot \phi$$

defines a Lie bracket on V.

In particular, the bilinear operation in $M_n(\mathbb{K})$ defined by $[M, N] = MN - NM$ is a Lie bracket.

Such a Lie bracket can be defined given any associative bilinear operation on V:

Exercise 3.2. *Let V be a $\mathbb{K}$-vector space and $\cdot\colon V \times V \to V$ an associative bilinear operation on V. Show that the commutator*

$$[x, y] := x \cdot y - y \cdot x$$

defines a Lie bracket on V.

Exercise 3.3. *Suppose V has finite dimension n over $\mathbb{K}$, and $\mathfrak{g} = (V, [\cdot,\cdot])$ is a Lie algebra over $\mathbb{K}$. If $\{v_1, \ldots, v_n\}$ is a basis, show that $\mathfrak{g}$ is $\mathcal{K}$-definable using parameters v_i and $[v_i, v_j]$.*

Definition 3.17. Let $\mathfrak{g} = (V, [\cdot,\cdot])$ and $\mathfrak{h} = (W, [\cdot,\cdot])$ be Lie algebras over $\mathbb{K}$. A linear map $\alpha\colon V \to W$ is a *homomorphism* when for each $x, y \in V$,

$$\alpha([x, y]) = [\alpha(x), \alpha(y)].$$

We say that a homomorphism $\alpha\colon V \to W$ is an *isomorphism* when there is a homomorphism $\beta\colon W \to V$ with $\alpha \circ \beta = \mathrm{id}_W$ and $\beta \circ \alpha = \mathrm{id}_V$.

Definition 3.18. Let $\mathfrak{g} = (V, [\cdot,\cdot])$ be a Lie algebra over $\mathbb{K}$. A subset $\mathfrak{i}$ of $\mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if it is a vector subspace and $[x, y] \in \mathfrak{i}$ for each $x, y \in V$ with $y \in \mathfrak{i}$.

If $\mathfrak{i}$ is an ideal of $\mathfrak{g}$ we defined the quotient algebra $\mathfrak{g}/\mathfrak{i}$ to be the set of equivalence classes of V mod out with the equivalence relation $x \sim y$ if and only if $x - y \in \mathfrak{i}$, together with the addition and Lie bracket structure inherited from $\mathfrak{i}$.

Exercise 3.4. *Prove that a vector subspace $\mathfrak{i}$ of V is an ideal of $\mathfrak{g} = (V, [\cdot,\cdot])$ if and only if the Lie bracket is well defined on $\mathfrak{g}/\mathfrak{i}$.*

Given a Lie group $(G, \cdot)$ over $\mathbb{R}$, the Lie algebra of G may be defined in the same way as we will do for the definable case. In particular, it is a Lie algebra structure on the tangent space $T_e(G)$ of the manifold G at the identity e of G.

In this case, Lie proved that there is an *exponential map* from a neighborhood of $\bar{0}$ in $T_e(G)$ to a neighborhood of e in G as follows:

We define a one-parameter subgroup of G to be a path $\sigma(t)$ from $\mathbb{R}$ to G such that $G(t+s) = g(t) \cdot g(s)$.

Given some $\eta \in T_e(G)$ one can always find a one parameter subgroup σ_η such that the derivative of σ_η at e is η (Proposition 3.1 in Chapter II of [**44**]). The exponential map is defined as $\exp(\eta) := \sigma_\eta(1)$.

The following are Equation (34), Proposition 3.2, Proposition 3.4 in Chapter II of [**44**].

Example 3.19. If $G = GL_n(\mathbb{R})$, then its Lie algebra is $M_n(\mathbb{R})$ (see the discussion preceding Theorem 3.25 for a proof) and the exponential map is

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^n}{n!}.$$

Fact 3.20. *The exponential map maps a certain neighborhood of zero in the tangent algebra of G diffeomorphically onto a neighborhood of the identity of G.*

Fact 3.21. *If* $[\zeta, \eta] = 0$ *then* $\exp(\zeta + \eta) = \exp(\zeta) \cdot \exp(\eta)$.

3.2.2. The adjoint representation.

Given any $g \in G$ consider the conjugation map $\sigma_g : G \to G$ defined by $\sigma_g(x) = gxg^{-1}$. Taking derivations, each σ_g gives rise to a linear automorphism $Ad(g)$ of $\mathfrak{g}$. As hinted by the notation, the *adjoint map* $Ad(g)$ is the map that sends g to automorphism of $\mathfrak{g}$ induced by σ_g. Since $\sigma_g \sigma_h = \sigma_{gh}$, Ad is a representation of G in the space of linear automorphisms $GL(\mathfrak{g})$ of the vector space $\mathfrak{g}$.

In order to study this representation, we will need some notation.

Let $m(x, y)$ be the multiplication map from $G \times G$ to G. For a fixed $g \in G$, let $l_G(x) := m(g, x)$ and $r_g(x) := m(x, g)$ be the maps from G to G which result when replacing one of the variables in m by g.

Taking the derivation of m at (e, e) we have that, since $m(x, e) = m(e, x) = x$ for all x, for any $\eta \in T_e(G)$,

$$\mathrm{d}_{(e,e)}(m)(\eta, \bar{0}) = \mathrm{d}_{(e,e)}(m)(\bar{0}, \eta) = \eta$$

so that by linearity $\mathrm{d}_{(e,e)}(m)(\eta, \mu) = \eta + \mu$ for any $\eta, \mu \in T_e(G)$, which implies of course that the derivative of the inverse map is just scalar multiplication by -1.

The left multiplication l_g centered on any $h \in G$ induces the a derivation mapping $T_h(G)$ to $T_{gh}(G)$. For any h and any $\eta \in T_h(G)$, let $g\eta$ be the image under this derivation. Similarly, we will abuse notation and for any $\eta \in T_h(G)$ we will denote by ηg to be image in $T_{hg}(G)$ of the vector η under the derivation induced by the map r_g. Notice that under this notation $Ad(g)(\eta) = g\eta g^{-1}$.

As with $\mathrm{d}_{(e,e)}$, we have that $\mathrm{d}_{(e,g)}(m)(\bar{0}, \eta) = \eta$, $\mathrm{d}_{(e,g)}(m)(\eta, \bar{0}) = \eta g$, so $\mathrm{d}_{(e,g)}(m)(\eta, \mu) = \eta g + \mu$. Similar formulas are easy to derive for $\mathrm{d}_{(g,e)}(m)$ and $\mathrm{d}_{(g,h)}(m)$ and are left as exercises.

EXERCISE 3.5. *Show that for* $\eta \in T_h(G)$ *and* $\zeta \in T_g(h)$ $\mathrm{d}_{(g,h)}(m)(\zeta, \eta) = \zeta h + g\eta$.

Theorem 3.22. *For any element* $g \in G$*, the centralizer* $C_G(g)$ *of* g *in* G *is a Lie subgroup of* G *with Lie algebra* $\{\zeta \in \mathfrak{g} \mid Ad(g)(\zeta) = \zeta\}$.

Proof. Conjugation is by definition an action σ of G on itself where $\sigma(g)(h) = ghg^{-1}$. For a fixed $x \in G$, the centralizer $C_G(x)$ is precisely the subgroup consisting of the elements g such that $\sigma(g)(x) = x$, which is precisely G_x in the terminology of Theorem 3.13(1). Said theorem therefore implies that $C_G(g)$ is a Lie subgroup of G and that its Lie algebra is $\ker(\mathrm{d}_e(\sigma_g))$ where $\sigma_g(y) = ygy^{-1}$.

Claim 3.23. *For $g \in G$ and $\eta \in \mathfrak{g}$, $\eta \in \ker(\mathrm{d}_e(\sigma_g))$ if and only if $Ad(g)(\eta) = \eta$.*

Proof: $\sigma_g(y) = ygy^{-1} = m(y, l_g(y^{-1}))$ so by the chain rule,

$$\mathrm{d}_e(\sigma_g)(\eta) = \mathrm{d}_{(e,g)}(m)\left((\eta, g\,(-\eta))\right) = \eta g - g\eta$$

so that $\eta \in \ker(\mathrm{d}_e(\sigma_x))$ if and only if $\eta g = g\eta$. Again by the Chain Rule, as a map from $T_e(G)$ to $T_e(G)$ the derivative of the right multiplication r_g which maps $\eta \mapsto \eta g$ is the inverse of the derivative of the right multiplication $r_{g^{-1}}$ which we defined to be the map $\mathrm{d}_e(r_{g^{-1}}) : \eta \mapsto \eta g^{-1}$. If we compose $\eta g = g\eta$ on both sides by $\mathrm{d}_e(r_{g^{-1}})$ we get that $\eta g = g\eta$ if and only if $\eta = g\eta g^{-1} = Ad_g(\eta)$.This completes the proof of the claim. $\blacksquare_{Claim}$

So

$$C_G(g) = \{g \mid \mathrm{d}_e(\sigma_g)(\zeta) = 0\} = \{g \mid Ad(g)(\zeta) = \zeta\}$$

as required. $\square$

Corollary 3.24. *The kernel of the adjoint representation is the center $Z(G)$ of the group G.*

Proof. If $g \notin Z(G)$, then $Z(g) \neq G$ and therefore its Lie algebra is not all of $\mathfrak{g}$. By Theorem 3.22, $Ad(g)(\zeta) \neq \zeta$ for some $\zeta \in \mathfrak{g}$ which implies that $Ad(g)$ is not the identity in $GL(\mathfrak{g})$. $\square$

We will now need to identify $GL(\mathfrak{g})$ as a Lie group in order to apply Theorem 3.15.

For this we choose a basis for $\mathfrak{g}$ we can identify $\mathfrak{g}$ with $\mathcal{R}^n$ where $n = \dim(G)$ so that the adjoint representation can be studied as a morphism from G to $GL_n(\mathcal{R})$. The differential of Ad will be identified as a map from $\mathfrak{g}$ to $T_{\mathbf{I}_n}(GL_n(\mathcal{R}))$ where $\mathbf{I}_n$ is the identity matrix in $GL_n(\mathcal{R})$. Using the standard notation, we endow $GL_n(\mathcal{R})$ with a manifold structure inherited from the set of the n times n matrices $M_n(\mathcal{R})$ (naturally identified as $\mathcal{R}^{n\times n}$), either as an open subset or with a chart consisting on a single map using any invertible linear transformation.

So $T_{\mathbf{I}_n}(GL_n(\mathcal{R}))$ is naturally identified with $\mathfrak{gl}_n(\mathcal{R}) = End(\mathcal{R}^n) = M_n(\mathcal{R})$.

The following theorem can be proved in more generality (stating it for definable morphisms from G to the automorphism group of any vector space), but we will only use this version.

Theorem 3.25. *Let G be a definable group with Lia algebra $\mathfrak{g}$, and let $Ad : G \to \mathrm{Aut}(\mathfrak{g}))$ be the adjoint representation. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subspace of $\mathfrak{g}$. Then the following hold:*

(1) The tangent space of the definable subgroup $\{g \in G \mid Ad(g)(\mathfrak{h}) \subseteq \mathfrak{h}\}$ is $\{\zeta \in T_e(G) \mid (\mathrm{d}_e Ad)(\zeta)(\mathfrak{h}) \subseteq \mathfrak{h}\}$.
(2) The tangent space of the definable group $\{g \in G \mid \forall \eta \in \mathfrak{h},\ Ad(g)(\eta) = \eta\}$ is $\{\zeta \in T_e(G) \mid \forall \eta \in \mathfrak{h},\ (\mathrm{d}_e Ad)(\zeta)(\eta) = 0\}$.

Proof. We will identify $GL(\mathfrak{g})$ and its Lie algebra with $GL_n(\mathcal{R})$ and $M_n(\mathcal{R})$ as in the preceding discussion. We will use letters A, B for matrices in $GL_n(\mathcal{R})$, M, N for elements of $M_n(\mathcal{R})$ and Greek letters ζ, η for elements in $\mathfrak{g}$.

Let $L := \{A \in GL_n(\mathcal{R}) \mid \eta \in \mathfrak{h} \Rightarrow A\eta \in \mathfrak{h}\}$ and $\mathfrak{l} := \{M \in M_n(\mathcal{R}) \mid \eta \in \mathfrak{h} \Rightarrow M\eta \in \mathfrak{h}\}$. L is of course a subgroup of $GL_n(\mathcal{R})$ and by linearity $\mathfrak{l}$ is a subspace of $M_n(\mathcal{R})$ and L is an open subset of $\mathfrak{l}$, so that $\mathfrak{l}$ is the Lie algebra of L.

By Theorem 3.15,

$$Ad^{-1}(L) = \{g \in G \mid Ad(g) \in L\} = \{g \in G \mid \eta \in \mathfrak{h} \Rightarrow Ad(g)(\eta) \in \mathfrak{h}\}$$

is a definable subgroup of G with Lie algebra

$$(\mathrm{d}_e(Ad))^{-1}(\mathfrak{l}) = \{\zeta \in T_e(G) \mid \mathrm{d}_e(Ad)(\zeta) \in \mathfrak{l}\} = \{\zeta \in T_e(G) \mid \mathrm{d}_e(Ad)(\zeta)(\mathfrak{h}) \subset \mathfrak{h}\}.$$

To prove (2), we proceed in a similar manner, except we consider $GL_n(\mathfrak{g})$ as a subset of $M_n(\mathcal{R})$ with the coordinate chart where A maps to $A - \mathbf{I}_n$.

Let L be the subgroup of $GL_n(\mathfrak{g})$ defined by

$$L := \{A \in GL_n(\mathfrak{g}) \mid \eta \in \mathfrak{h} \Rightarrow A(\eta) = \eta\}.$$

L is a subgroup of $GL_n(\mathfrak{g})$ so by Theorem 3.15, the Lie algebra (tangent space at e) of $Ad^{-1}(L)$ is

$$(\mathrm{d}_e(Ad))^{-1}(T_e(L)) = \{\zeta \in \mathfrak{g} \mid Ad(\zeta) \in T_e(L)\}.$$

By definition, $Ad^{-1}(L)$ is the definable subgroup of G

$$\{g \in G \mid Ad(g) \in L\} = \{g \in G \mid \eta \in \mathfrak{h} \Rightarrow Ad(g)(\eta) = \eta\}$$

so we only need to prove that

$$\{\zeta \in \mathfrak{g} \mid Ad(\zeta) \in T_e(L)\} = \{\zeta \in T_e(G) \mid \forall \eta \in \mathfrak{h},\ (d_e Ad)(\zeta)(\eta) = 0\}.$$

As before, our coordinate system allows us to compute $T_e(L)$ quite easily. By definition,

$$L = \{A \in GL_n(\mathcal{R}) \mid \eta \in \mathfrak{h} \Rightarrow A(\eta) = \eta\} = \{A \in GL_n(\mathcal{R}) \mid \eta \in \mathfrak{h} \Rightarrow (A - \mathbf{I}_n)(\eta) = 0\}.$$

However $\mathfrak{l} := \{M \in M_n(\mathcal{R}) \mid M\eta = 0\}$ is a subspace of M_n which contains $\{A - \mathbf{I}_n\}$ for any $A \in GL_n(\mathcal{R})$. This implies that under this coordinate system $\mathfrak{l}$ is $T_e(L)$ so that

$$\{\zeta \in \mathfrak{g} \mid Ad(\zeta) \in T_e(L)\} = \{\zeta \in \mathfrak{g} \mid \eta \in \mathfrak{h} \Rightarrow Ad(\zeta)(\eta) = 0\}$$

which completes the proof of (2). □

3.2.3. Defining the Lie bracket structure on the Lie algebra of a definable group. We can now define the Lie bracket on $\mathfrak{g} = T_e(G)$ for a definable group G.

Definition 3.26. Recall that the adjoint map Ad is an action from G onto $\mathfrak{g}$ which in itself is sending g such that $Ad(g)(\eta)$, which we denoted as $g\eta g^{-1}$ understanding that for $\eta \in \mathfrak{g}$ and $g \in G$ the element $g\eta$ is the image of η under the derivative of the left multiplication by g (likewise with ηh).

The map $\mathrm{d}_e(Ad(g))$ therefore sends the tangent space $T_e(G) = \mathfrak{g}$ to the Lie algebra of $GL(\mathfrak{g})$ which by the analysis preceding Theorem 3.25, are the linear endomorphisms of $T_e(G) = \mathfrak{g}$. Define the definable Lie bracket as the bilinear operation on $T_e(G)$ defined by $[\zeta, \mu] := \mathrm{d}_e(Ad)(\zeta)(\mu)$.

Under this notation, we can restate Theorem 3.25 as follows:

Theorem 3.27. *Let G be a definable group with Lie algebra $\mathfrak{g}$, and let $Ad : G \to \mathrm{Aut}(\mathfrak{g})$ be the adjoint representation. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subspace of $\mathfrak{g}$. Then the following hold:*

(1) The tangent space of the definable subgroup $\{g \in G \mid Ad(g)(\mathfrak{h}) \subseteq \mathfrak{h}\}$ is

$$\{\zeta \in T_e(G) \mid \eta \in \mathfrak{h} \Rightarrow [\zeta, \eta] \in \mathfrak{h}\}.$$

(2) *The tangent space of the definable group* $\{g \in G \mid \forall \eta \in \mathfrak{h},\ Ad(g)(\eta) = \eta\}$ *is* $\{\zeta \in T_e(G) \mid \eta \in \mathfrak{h} \Rightarrow [\zeta, \eta] = 0\}$.

We will devote this section to proving that the definable Lie bracket is a Lie bracket and studying some of its properties.

Definition 3.28. Let G be a definable subgroup, and let H be a definable subgroup. The *normalizer* $N(H)$ of H in G is defined to be

$$\{g \in G \mid gNg^{-1} \subseteq N\}.$$

Let $\mathfrak{g}$ be a definable Lie algebra, and let $\mathfrak{h}$ be a (definable) subalgebra. The *normalizer* $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ is defined to be

$$\{\eta \in \mathfrak{g} \mid [\eta, \mathfrak{h}] \subseteq \mathfrak{h}\}.$$

Corollary 3.29. *Let G be a definably connected group with Lie algebra $\mathfrak{g}$. Let H be a definably connected definable subgroup of G with Lie algebra $\mathfrak{h}$. Then the following hold:*

(1) $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{g}$.
(2) *G is abelian if and only if $[\eta, \zeta] = [\zeta, \eta] = 0$ for any $\eta, \zeta \in G$.*
(3) *H be a subgroup of $\mathfrak{g}$ with Lie algebra $\mathfrak{h}$. Then the normalizer of H is a definable subgroup of G, the Lie algebra of which is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.*

    In particular, H is a normal subgroup if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Proof. The first item just follows from the definitions (and the fact that definable subgroups are always submanifolds).

The second item follows from Theorem 3.27 and Corollary 3.24.

The first part of (3) follows from Theorem 3.27. For the if and only if of the "in particular" statement we need Theorem 3.15 (2). □

Theorem 3.30. *Let $f : G \to H$ be a definable homomorphism of definable groups with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\mathrm{d}_e f$ be the map from $\mathfrak{g}$ to $\mathfrak{h}$ induced by f. Then $\mathrm{d}_e f$ is an homomorphism of Lie algebras. Namely,*

$$\mathrm{d}_e(f)([\eta, \zeta]_{\mathfrak{g}}) = [\mathrm{d}_e(f)(\eta), \mathrm{d}_e(f)(\zeta)]_{\mathfrak{h}}$$

for any $\eta, \zeta \in \mathfrak{g}$.

Proof. Distinguishing the group operations in G and H makes the notation quite cumbersome, so we will abuse notation and use Ad for both the adjoint action in G and H. We believe which one we are using will be clear from the context.

Because f is an homomorphism, we have that for any $g, h \in G$ we have $f(ghg^{-1}) = f(g)(f(h))(f(g))^{-1}$ so for each g, this induces a map sending h to $f(ghg^{-1}) = f(g)(f(h))(f(g))^{-1}$.

Differentiating this map at the identity, we get

$$\mathrm{d}_e(f)(Ad(g)(\zeta)) = Ad\left(f\left(g\right)\right)\left(\mathrm{d}_e f\left(\zeta\right)\right),$$

which for any fixed $\zeta \in \mathfrak{g}$ sends $g \in G$ to an element in $\mathfrak{h}$. So fixing $\zeta \in \mathfrak{g}$ we can again differentiate with respect to the identity and get

$$\mathrm{d}_e(f)(d_e(Ad)(\eta)(\zeta)) = d_e(Ad)\left(d_e f\left(\eta\right)\right)\left(\mathrm{d}_e f\left(\zeta\right)\right),$$

which by definition of the definable Lie bracket states precisely the statement of the theorem. □

3.2.4. Local Coordinates. All the Lie algebra structure depends only on the local structure around the identity. We will fix an open set U diffeomorphic with $\mathcal{R}^n$, and abuse the notation identifying the elements of U with its images in $\mathcal{R}^n$. We will also assume that e is identified with the 0-tuple $\bar{0}$. Under this identification, the operations between elements in U can come from both the $\mathcal{R}$-structure (including multiplying coordinates as in the dot product and other bilinear maps) and the group operation from G. To avoid confusion, we will be careful to use $\odot$ for multiplication in G.

We will derive many of the results in this subsection from understanding the Taylor polynomial of degree two of the group multiplication restricted to U. Given $\bar{x}$ and $\bar{y}$ in U, Let $Q(\bar{x}, \bar{y})$ be the Taylor polynomial of degree two of the group multiplication in G. Since

$$\bar{x} \odot \bar{0} = \bar{0} \odot \bar{x} = \bar{x}$$

for all $\bar{x} \in U$, we know that

$$Q(\bar{x}, \bar{y}) = \bar{x} + \bar{y} + \alpha(\bar{x}, \bar{y})$$

where α consists of degree two monomials which vanishes (as a polynomial) whenever we replace either $\bar{x}$ or $\bar{y}$ by $\bar{0}$. This implies that any monomial in α is of the form $x_i y_j$ which in turn implies that $\alpha(\bar{x}, \bar{y})$ is a bilinear quadratic form.

Example 3.31. Take $G = GL_n(\mathcal{R})$. In order to get the setup described above we need the identity matrix $\mathbf{I_n}$ to be mapped to zero, so we take the coordinate chart $\phi(A) = A - \mathbf{I}_n$ which means we are looking at the matrices as $M + \mathbf{I}_n$ where M is viewd as an element in $\mathcal{R}^{n \times n}$.

Since

$$(M + \mathbf{I}_n)(N + \mathbf{I}_n) = MN + M + N + \mathbf{I}_n$$

we have that in local coordinates the matrix group multiplication sends (M, N) to $M + N + MN$. The map $(M, N) \mapsto MN$ is a bilinear map from $\mathcal{R}^{n \times n}$ to $\mathcal{R}^{n \times n}$ so in this case the degree two Taylor polynomial Q describes the full group operation and $\alpha(M, N) = MN$.

The example above shows how the degree two Taylor polynomial of the multiplication around the origin captures the linear behavior needed to understand the Lie algebra.

Proposition 3.32. *The following hold.*

(1) The degree two Taylor polynomial of $\bar{x} \odot \bar{y} \odot \bar{x}^{-1} \odot \bar{y}^{-1}$ *is* $\alpha(\bar{x}, \bar{y}) - \alpha(\bar{y}, \bar{x})$ *and*

(2) the degree two Taylor polynomial of $\bar{x} \odot \bar{y} \odot \bar{x}^{-1}$ *is* $\bar{y} + \alpha(\bar{x}, \bar{y}) - \alpha(\bar{y}, \bar{x})$.

Proof. We will proof the second statement, and leave the first one as an exercise.

Let $c(\bar{x}, \bar{y}) := \bar{x} \odot \bar{y} \odot \bar{x}^{-1}$. Since $\bar{x} \odot \bar{y} = c(\bar{x}, \bar{y}) \odot \bar{x}$, we have that the second degree Taylor polynomials on both sides must agree, so that $\bar{x} + \bar{y} + \alpha(\bar{x}, \bar{y})$ is the same as the second degree Taylor polynomial of

$$c(\bar{x}, \bar{y}) + \bar{x} + \alpha(c(\bar{x}, \bar{y}), \bar{x}).$$

This implies of course that the second degree Taylor polynomial of $c(\bar{x}, \bar{y})$ must be of the form $\bar{y} + \mu(\bar{x}, \bar{y})$ where μ is of degree greater than or equal to two, and since α is a quadratic bilinear form, the second degree Taylor polynomial of $\alpha(c(\bar{x}, \bar{y}), \bar{x})$ is precisely $\alpha(\bar{y}, \bar{x})$. This implies that the second degree Taylor polynomial of $c(\bar{x}, \bar{y}) + \bar{x} + \alpha(\bar{y}, \bar{x})$ is the same as the second degree Taylor polynomial of $c(\bar{x}, \bar{y}) + \bar{x} + \alpha(c(\bar{x}, \bar{y}), \bar{x})$ which must be equal to $\bar{x} + \bar{y} + \alpha(\bar{x}, \bar{y})$.

Canceling $\bar{x}$ on both sides and subtracting $\alpha(\bar{y}, \bar{x})$, we get that the second degree Taylor polynomial of $c(\bar{x}, \bar{y})$ is $\bar{y} + \alpha(\bar{x}, \bar{y}) - \alpha(\bar{y}, \bar{x})$ as required. $\square$

EXERCISE 3.6. *Prove that the degree 2 Taylor polynomial of* $\bar{x} \odot \bar{y} \odot \bar{x}^{-1} \odot \bar{y}^{-1}$ *is* $\alpha(\bar{x}, \bar{y}) - \alpha(\bar{y}, \bar{x})$.

The importance of the degree two Taylor polynomial is the following.

Claim 3.33. *Let* U *be a neighborhood* $\overline{0}$ *in* $\mathcal{R}^n$ *and let* ϕ *be a* C^2*-differential map* $\phi : U \times U \to U$ *so that* $\phi(\overline{0}, \overline{0}) = \overline{0}$. *Let* $Q_\phi(x, y)$ *be the degree two Taylor polynomial of* ϕ.

For $g \in U$ *let* $\phi_g : U \mapsto U$ *be defined as* $\phi : g(y) = \phi(g, y)$ *and* $Q_g : U \mapsto U$ *be given by* $Q_g(y) = Q(g, y)$. *Let* θ *be the function mapping* g *to* $\mathrm{d}_0\phi_g$ *and* θ_Q *be the function sending* g *to* $\mathrm{d}_0 Q_g$.

Then

$$\mathrm{d}_0\theta = \mathrm{d}_0\theta_Q.$$

Proof: Let $\epsilon(x, y) = \phi(x, y) - Q_\phi(x, y)$. By definition,

$$\frac{\partial^2}{\partial x \partial y}(\epsilon(x, y))\bigg|_{(\bar{0},\bar{0})} = \frac{\partial^2}{\partial x \partial y}(\phi(x, y) - Q(x, y))\bigg|_{(\bar{0},\bar{0})} = 0.$$

Now, $\phi_g(y)$ is a C^2-function from U to U so that $\mathrm{d}_0(\phi_g) : T_e(G) \to T_{\phi(g,e)}(G)$ is a linear function between tangent spaces. We can compute this function under the local coordinate charts as follows: $h \in U$ we take the standard basis in $T_h(G)$ (taking the ith vector to be the tangency class of the curve $\sigma_i(t) = h + e_i t$ where e_i is the ith vector of the standard basis of $\mathcal{R}^n$). Then $\mathrm{d}_0(\phi_g) : T_e(G) \to T_{\phi(g,e)}(G)$ is a linear function which with the coordinates of the vectors in this basis given by the Jacobian

$$\left(\frac{\delta(\phi_g)_i}{\delta_{y_j}}\bigg|_{\bar{0}} \right)_{(i,j)} = \left(\frac{\delta(\phi)_i}{\delta_{y_j}}\bigg|_{(g,\bar{0})} \right)_{(i,j)}.$$

The map θ sends $g \in U$ to $\mathrm{d}_0(\phi_g)$ so its differential sends $T_e(G)$ to the tangent space of the linear transformations from $T_e(G)$ to $T_{\phi(g,e)}(G)$. Working with the coordinate system defined above, this will be given by a n^2 by n matrix

$$\left(\frac{\delta^2\phi_j}{\delta_{x_l y_j}}\bigg|_{(\bar{0},\bar{0})} \right)_{((l,i),j)}.$$

In a similar way, the under the same coordinate charts $\mathrm{d}_0\theta_Q$ will be given by the maps

$$\left(\frac{\delta^2 Q_j}{\delta_{x_l y_j}}\bigg|_{(\bar{0},\bar{0})} \right)_{((l,i),j)}.$$

Since

$$\frac{\partial^2}{\partial x \partial y}\phi(x, y)\bigg|_{(\bar{0},\bar{0})} = \frac{\partial^2}{\partial x \partial y}Q(x, y)\bigg|_{(\bar{0},\bar{0})},$$

the equality of both matrices follows. $\blacksquare_{Claim}$

Theorem 3.34. *Let* $\eta, \zeta \in T_e(G)$.

Identifying $U \subset G$ *with a subset of* $\mathcal{R}^n$ *with* $e = \overline{0}$, *and let* $\overline{\eta}$ *be the coordinates of* $\eta \in T_e(G)$ *with the coordinate system under the standard basis of* $T_e(G)$.

Then the following hold:

- $\mathrm{d}_e(Ad)(\overline{\zeta})(\overline{\mu}) = \alpha(\overline{\zeta}, \overline{\eta}) - \alpha(\overline{\eta}, \overline{\zeta})$, *and in particular* $[\eta, \zeta] = -[\zeta, \eta]$.
- *If* $\sigma(t)$ *and* $\mu(t)$ *are paths whose tangency equivalence class is* ζ *and* η *respectively, then* $[\zeta, \eta]$ *is the tangency equivalence class of the commutator* $\sigma(t) \odot \mu(t) \odot (\sigma(t))^{-1} \odot (\mu(t))^{-1}$.

.

PROOF. We will work in the coordinate system described in the statement of the theorem.

Claim 3.33 applied to $\phi(\bar{x}, \bar{y}) := \bar{x} \odot \bar{y} \odot \bar{x}^{-1}$ implies that if θ is the map $\overline{x} \mapsto \mathrm{d}_{\overline{0}}(Ad_{\overline{x}})$ and θ_Q is the function sending $\overline{x}$ to $\mathrm{d}_{\overline{0}}(Q_{\overline{x}})$ then $\mathrm{d}_{\overline{0}}(\theta) = \mathrm{d}_{\overline{0}}(\theta_Q)$ where Q is the degree two Taylor polynomial of ϕ.

$Q(\overline{x}, \overline{y}) = \overline{y} + \alpha(\overline{x}, \overline{y}) - \alpha(\overline{y}, \overline{x})$ by Proposition 3.32. Differentiating on both sides on $\overline{y}$ for a fixed $\overline{x}$ and using that α is bilinear we get

$$\mathrm{d}_{\overline{0}}(Q_{\overline{x}})(\overline{\eta}) = \overline{\eta} + \alpha(\overline{x}, \overline{\eta}) - \alpha(\overline{\eta}, \overline{x}).$$

But then, by definition of θ_Q and using once more that α is bilinear we have

$$\mathrm{d}_{\overline{0}}(\theta_Q)(\zeta) = \alpha(\overline{\zeta}, \overline{\eta}) - \alpha(\overline{\eta}, \overline{\zeta})$$

as required. That $[\eta, \zeta] = -[\zeta, \eta]$ follows immediately.

The last statement of the theorem is proved in exactly the same manner, using the second statement of Proposition 3.32. □

Example 3.35. By Example 3.31, with the parametrization of $GL_n(\mathcal{R})$ given by $\{M + \mathbf{I}_n\}$ we have that the map $\alpha(M, N) = MN$. So the Lie bracket of the Lie algebra $M_n(\mathcal{R})$ of $GL_n(\mathcal{R})$ is in fact the standard Lie bracket on $M_n(\mathcal{R})$ defined by

$$[M, N] = \alpha(M, N) - \alpha(N, M) = MN - NM.$$

3.2.5. The Jacobi identity. In order to have $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra, we need to prove that the definable Lie bracket satisfies the Jacobi identity. Namely, we need to prove that for any $\eta, \zeta, \nu \in \mathfrak{g}$ we have $[[\eta, \zeta], \nu] - [\eta, [\zeta, \nu]] + [\zeta, [\eta, \nu]] = 0$.

We can now do this as follows:

Consider the map $Ad : G \to GL_n(\mathfrak{g})$ and identify $GL_n(\mathfrak{g})$ with $GL_n(\mathcal{R})$ by choosing a basis of $\mathfrak{g}$. Endow $GL_n(\mathcal{R})$ with its natural Lie group structure. By Theorem 3.30, the derivative $d_e(Ad)$ is a map from $\mathfrak{g}$ to the Lie algebra $M_n(\mathcal{R})$ of $GL_n(\mathcal{R})$ preserving the Lie bracket. By Example 3.35, in $M_n(\mathcal{R})$ we have $[M, N] = MN - NM$ so that for any $\eta, \zeta \in \mathfrak{g}$ we have

$$d_e(Ad)([\eta, \zeta]) = d_e(Ad)(\eta)d_e(Ad)(\zeta) - d_e(Ad)(\zeta)d_e(Ad)(\eta).$$

Both sides of the equality are matrices representing linear transformations from $\mathfrak{g}$ to $\mathfrak{g}$ with the fixed chosen basis for $\mathfrak{g}$.

In particular, if we identify any $\nu \in \mathfrak{g}$ with its vector under said chosen basis, we have

$$d_e(Ad)\left([\eta, \zeta]\right)(\nu) = \left(d_e\left(Ad\right)(\eta)\, d_e\left(Ad\right)(\zeta)\right)(\nu) - \left(d_e\left(Ad\right)(\zeta)\, d_e\left(Ad\right)(\eta)\right)(\nu)$$

which by definition of the definable Lie bracket and associativity of composition of linear transformations implies

$$[[\eta, \zeta], \nu] = d_e(Ad)(\eta)\left([\zeta, \nu]\right) - d_e(Ad)(\zeta)\left([\eta, \nu]\right) = [\eta, [\zeta, \nu]] - [\zeta, [\eta, \nu]].$$

This implies

$$[[\eta, \zeta], \nu] - [\eta, [\zeta, \nu]] + [\zeta, [\eta, \nu]] = 0$$

and by antisymmetry of the Lie bracket (Theorem 3.34) we get

$$[[\eta, \zeta], \nu] + [[\zeta, \nu], \eta] + [[\nu, \eta], \zeta] = 0$$

This completes the proof that the Lie bracket $[\eta, \zeta] = \mathrm{d}_e(Ad)(\eta)(\zeta)$ is a Lie bracket on $\mathfrak{g}$.

The Jacobi identity, together with Theorem 3.30 is tightly related with the fact that the Lie bracket on $M_n(\mathcal{R})$ is the one given by matrix multiplication. One direction was used to prove the Jacobi identity for the definable Lie bracket. But one can also prove a converse which is left as Exercise 3.7 below.

Definition 3.36. A *representation* of a Lie algebra $\mathfrak{g}$ on the vector space V is a homomorphism $\alpha\colon \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\alpha([\eta,\zeta]_{\mathfrak{g}}) = [\alpha(\eta),\alpha(\zeta)]_{\mathfrak{gl}(V)} = \alpha(\eta)\alpha(\zeta) - \alpha(\zeta)\alpha(\eta)$.

EXERCISE 3.7. *Prove that the map* $\mathrm{d}_e(Ad) : \mathfrak{g} \to M_n(\mathfrak{g})$ *induced by the adjoint map from* G *to* $GL(\mathfrak{g})$ *is precisely the map*

$$\eta \mapsto (\zeta \mapsto [\eta,\zeta]).$$

We generalize this to all Lie algebras.

Definition 3.37. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. The *adjoint representation* of $\mathfrak{g}$ is the map

$$\begin{aligned} \mathrm{ad}\colon \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ X &\mapsto (Y \mapsto [X,Y]). \end{aligned}$$

EXERCISE 3.8. *Assuming one has a bilinear antisymmetric operation on a vector space* $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$*, prove that the Jacobi identity of a Lie bracket* $[\ ,\ ]$ *is equivalent to* $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ *preserving the Lie bracket.*

3.2.6. Jacobi Identity through vector fields. Probably the most direct way to prove the Jacobi identity in standard Lie algebras is via the exponential map, which we don't have access to in the o-minimal setting. Another nice way to prove the Jacobi identity in the o-minimal setting is to develop the theory of vector fields up until the definition of the Lie bracket as is done, for example, in Chapter 5 of [**68**]. Vector fields have had some very important applications in o-minimal theories, but developing the theory was beyond what we thought should be included in this book.

That being said, we do take the opportunity to sketch an introduction to this fascinating subject and how a vector field proof of the Jacobi identity would look like:

A vector field is by definition an assignment which for each $g \in G$ we give a vector in $T_g(G)$. The two most natural families of vector fields in a Lie group are the right invariant vector fields $g \mapsto \eta g$ for a fixed $\eta \in \mathfrak{g}$ and the left invariant vector fields $g \mapsto g\eta$ (they are called left and right invariant because they are invariant under left and right multiplication, respectively).

The advantage of working with vector fields is that the (vector field) Lie bracket is defined in a way which is very amenable to proving the Jacobi identity.

Any vector space $X(g)$ can be seen as a map sending $f : G \to \mathcal{R}$ to the function $X(f) : G \to \mathcal{R}$ where $X(f)(g)$ is the directional derivative of f at g in the direction of $X(g)$. This interpretation is reversible so that one can define multiplication of two vector fields where XY is the vector field such that $XY(f) = X(Y(f))$ for every differentiable $f : G \to \mathcal{R}$. The vector field Lie bracket is then going to be defined by $[X,Y]_{vf} = XY - YX$.

The Jacobi identity follows easily from this, exactly as it does for matrices:

$$[[X,Y]_{vf},Z]_{vf}(f) = [XY - YX, Z]_{vf}(f) = (XYZ - YXZ - ZXY + ZYX)(f)$$

So

$$[[X,Y]_{vf},Z]_{vf}(f) = XYZ(f) - YXZ(f) - ZXY(f) + ZYX(f).$$

Expanding $[[Y,Z]_{vf},X]_{vf}$ and $[[Z,X]_{vf},Y]_{vf}$ one easily verifies that $([[X,Y]_{vf},Z]_{vf} + [[Y,Z]_{vf},X]_{vf} + [[Z,X]_{vf},Y]_{vf})(f) = 0$ for all f, which is precisely what the Jacobi identity states.

Back to proving (sketching to proof) the Jacobi identity holds for the derivative of the adjoint representation, if we use the coordinate system we used in Subsection 3.2.4 and denote by $\overline{\eta}$ the coordinates of $\eta \in T_e(G)$ under this coordinate system and $\overline{g}$ the coordinates under the coordinate system for $g \in U$, one can compute the coordinates $\overline{g}\eta$ and $\overline{\eta}g$ of vectors in $T_g(G)$ for any $g \in G$ and $\eta \in T_h(G)$ which will give:

$$\overline{\eta g} = \eta + \alpha(\overline{\eta}, \overline{g}) + \epsilon_1(\overline{g}, \overline{\eta})$$

and

$$\overline{g\eta} = \eta + \alpha(\overline{g}, \overline{\eta}) + \epsilon_2(\overline{g}, \overline{\eta})$$

where ϵ_1 and ϵ_2 are functions for which all the second derivatives cancel at the origin.

If we define, for any $\eta \in T_e(G)$ one defines η^* to be the vector field $\eta^*(g) = \eta g$, and one computes the equations for the coordinates of the Lie bracket for the right invariant vector fields $\{\eta^*\}_{\eta \in T_e(G)}$ in terms of this coordinate system, one can verify that the coordinates of the vector field Lie bracket $[\ ,\ ]_{vf}$ yield

$$\overline{[\eta^*, \zeta^*]_{vf}} = \alpha\left(\overline{\eta^*}, \overline{\eta^*}\right) - \alpha\left(\overline{\eta^*}, \overline{\eta^*}\right).$$

This implies that $(T_e(G), [\ ,\ ])$ with the definable Lie bracket is Lie isomorphic to the Lie algebra $(T_e(G)^*, [\ ,\ ]_{vf})$ where the isomorphism sends η to η^*. Since the Jacobi identity holds for the latter, it must hold for $(T_e(G), [\ ,\ ])$.

Historical notes and comments

Most of the results in this Chapter were proved for o-minimal structures in the first sections of [**51**] and some of the proofs (Proposition 3.6, Theorem 3.25 and Corollary 3.29, for example).

The development of the theory we present here come from a variety of sources.

Section 3.1 is based on an adaptation of results in [**71**] to o-minimal theories. The proofs in Section 3.1.1 as Theorem 3.13 follow ideas for analogous results in [**44**], as do the computations behind Subsection 3.2.2 and Subsection 3.2.4.

Corollary 3.24 was proved in [**49**] using uniqueness of solutions of differential equations in o-minimal theories.

CHAPTER 4

Coverings, Homotopy and central extensions

The main purpose of this chapter is to study definable coverings and definable central extensions. This will entail studying definable versions of the fundamental group and the universal cover, which, as in the classical case, will be defined in terms of paths and homotopy.

Many of the results and proofs in this chapter will follow the classical methods, so we will sometimes be brief in the arguments. We will need, though, to provide constructions which make sense in the definable setting, so we will do this with greater detail.

4.1. Coverings

Definition 4.1. As in the classical setting, we will say that a definable continuous map $p : Y \to X$ between definable sets Y and X is a *definable covering map* if there is a finite cover $\{U_l\}_l \in L$ of open definable subsets of X such that for each $l \in L$ and each definably connected component V of $p^{-1}(U_l)$ the restriction of p to each V is an isomorphism onto U_l.

The definition for a finite cover $p : G \to H$ of definable groups is exactly the same, except that the U_l's are required to be definable t-open sets and p t-continuous.

The definable universal covering will be defined, as in the classical case, using (definable) paths. In general, there is always a choice to make on whether or not having paths be maps with domains in any closed interval, or restrict the definitions to maps with $[0, 1]$ in their domain. We find that the latter, where the start and endpoint of the path is always set, allows for better notation.

4.1.1. Paths and liftings.

Definition 4.2. Let X be a definable space.

A *definable path* in X is a definable continuous function from $[0, 1]$ to X.

For $a \in X$, we will call i_a the constant path at a. Namely, $i_a(t) = a$ for all $t \in [0, 1]$.

Given $p, q \in X$, if X is convex we will refer to l_{pq} as the path that runs through the line segment joining p and q at constant speed, this is, $l_{pq}(t); = (1-t)p + tq$ for $t \in [0, 1]$.

If γ is a path on X, the inverse γ^{-1} is the map $\gamma^{-1}(t) := \gamma(1-t)$.

Given a definable covering map $p : Y \to X$ and a definable map $f : Z \to X$, we will say that $\widetilde{f} : Z \to Y$ is *a lifting of* f if $p(\widetilde{f}(z)) = f(z)$ for all $z \in Z$.

Proposition 4.3. *Let $p : Y \to X$ be a definable covering map, and let $\gamma : [0, 1] \to X$ be a definable path. Then for any $y \in Y$ with $p(y) = \gamma(0)$ there is a unique lifting $\widetilde{\gamma} : [0, 1] \to Y$ of γ with $\widetilde{\gamma}(0) = y$.*

PROOF. We will first prove uniqueness. Let $U_i \subseteq X$ be as in the definition of covering and V_i^j be the connected components of $p^{-1}(U_i)$. The sets V_i^j are connected components of an open subset $p^{-1}(U_i)$ of Y, so they are open in Y.

Claim 4.4. $\{t \mid \widetilde{\gamma}_1(t) = \widetilde{\gamma}_2(t)\}$ *is an open subset of* $[0,1]$.

Proof: Assume that for some a we have $\widetilde{\gamma}_1(a) = \widetilde{\gamma}_2(a) \in V_i^j$. By continuity there is some neighborhood $(a-\epsilon, a+\epsilon)$ of a such that both $\widetilde{\gamma}_1((a-\epsilon, a+\epsilon))$ and $\widetilde{\gamma}_2((a-\epsilon, a+\epsilon))$ are contained in V_i^j. Since the restriction of p to V_i^j is an isomorphism and

$$p \circ \widetilde{\gamma}_1 = p \circ \widetilde{\gamma}_2 = \gamma,$$

we have that $\widetilde{\gamma}_1(t) = \widetilde{\gamma}_2(t)$ for all $t \in (a-\epsilon, a+\epsilon)$. $\blacksquare_{Claim}$

By continuity $\{t \mid \widetilde{\gamma}_1(t) \neq \widetilde{\gamma}_2(t)\}$ is open, so by definable connectedness, one of $\{t \mid \widetilde{\gamma}_1(t) \neq \widetilde{\gamma}_2(t)\}$ or $\{t \mid \widetilde{\gamma}_1(t) = \widetilde{\gamma}_2(t)\}$ is empty. By hypothesis the latter is not empty, so $\widetilde{\gamma}_1(t) = \widetilde{\gamma}_2(t)$ for all $t \in [0,1]$ which completes the proof of uniqueness.

Now, let $\gamma : [0,1] \to X$ be any path. We have $X \subset \bigcup_i U_i$. By Fact 1.19 there are $W_i \subset U_i$ with $\overline{W_i} \subset U_i$ and $X \subset \bigcup_i W_i$. By cell decomposition there are $0 = x_0 < x_1 < \cdots < x_l = 1 \in [0,1]$ such that for all j the set $\gamma((x_j, x_{j+1}))$ is contained in one of W_i for some i, which implies $\gamma([x_j, x_{j+1}])$ is entirely contained in the corresponding U_i.

Since $x_0 = 0$. Assume $\gamma(0) \in U_i$. Since $p(y) = \gamma(0)$, let V_i' be the connected component of $p^{-1}(U_i)$ to which y belongs to. For $t \in [x_0, x_1]$ let $\widetilde{\gamma}(t)$ be he unique preimage of $\gamma(t)$ in V_i', so that $\widetilde{\gamma}|_{[x_0,x_1]}$ is continuous and $\widetilde{\gamma}_1(0) = y$.

We can repeat the process inductively for every interval $[x_j, x_{j+1}]$ with $\widetilde{\gamma}(x_j)$ (which will be defined in the previous step) instead of y. This results in a continuous $\widetilde{\gamma} : [0,1] \to Y$ with $\widetilde{\gamma}(0) = y$ and $p \circ \widetilde{\gamma} = \gamma$, as required. $\square$

Definition 4.5. If $\gamma : [0,1] \to X$ is any path in X and $0 \leq r \leq s \leq 1$, we define *the restriction of* γ *to* $[r,s]$ to be the path

$$\gamma|_{[r,s]}(t) := \gamma\left(\frac{t-r}{s-r}\right).$$

If $\gamma : [0,1] \to X$ and $\delta : [0,1] \to X$ are two paths in X, we defined the *concatenation of* γ *and* δ to be the map

$$(\gamma * \delta)(t) := \begin{cases} \gamma(2t) \text{ if } 0 \leq t \leq 1/2 \\ \delta(2t-1) \text{ if } 1/2 \leq t \leq 1. \end{cases}$$

4.1.2. Definable path homotopy. The constructions of the fundamental group and the universal will be defined as sets of paths quotiented by "definable homotopy".

Definition 4.6. Let X, Y be definable sets.

- An *definable homotopy between continuous definable functions* $f, g : X \to Y$ is a continuous definable function
$$H : [0,1] \times X \to Y$$
such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for any $x \in X$.
- A *definable homotopy between paths* $\gamma_1, \gamma_2 : [0,1] \to Y$ on Y is a continuous definable function
$$H : [0,1] \times [0,1] \to Y$$
such that $H(0,x) = \gamma_1(x)$ and $H(1,x) = \gamma_2(x)$ for any $x \in [0,1]$, and such that the endpoints are fixed: $H(s,0) = \gamma_1(0) = \gamma_2(0)$ and $H(s,1) = \gamma_1(1) = \gamma_2(1)$ is fixed for all $s \in [0,1]$.
- Two definable sets X, Y are *definably homotopically equivalent* if there are definable continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is definably homotopically equivalent to the identity map on X.

- A definable set X is *contractible* if it is definably homotopically equivalent to a point. In this case, for any $a \in X$ there is a *contraction from X to a*: A definable function $H : [0,1] \times X \to X$ where $H(0,x) = x$ and $H(1,x) = a$ for all $x \in X$.

EXERCISE 4.1. *Prove that given any path γ, the path $\gamma * \gamma^{-1}$ is definably homotopic to the constant path $i_{\gamma(0)}(s) = \gamma(0)$.*

EXERCISE 4.2. *Prove that any cell is contractible.*

EXERCISE 4.3. *Prove that if X is contractible, any path γ on X with $\gamma(0) = \gamma(1) = a$ is homotopically equivalent to the constant path $i_a(s) = a$.*

EXERCISE 4.4. *Prove that if γ_1, γ_2 and γ_1, γ_2 are pairs of definably homotopic paths on X with $\gamma_1(1) = \gamma_2(1) = \delta_1(0) = \delta_2(0)$, then $\gamma_1 * \delta_1$ is definably homotopic to $\gamma_2 * \delta_2$.*

Proposition 4.7. *Let X be a definably contractible set. Then any two paths on X with the same endpoints are definably homotopic.*

PROOF. Let γ, σ be two paths with the same endpoints a and b. Recall that $\gamma^{-1} * \gamma$ is definably homotopic to $i_{\gamma(1)} = i_b$ and $\sigma * \gamma^{-1}$ is definably homotopic to $i_{\sigma(0)} = a$. Then

$$\sigma \sim \sigma * i_b \sim \sigma * (\gamma^{-1} * \gamma) \sim (\sigma * \gamma^{-1}) * \gamma \sim i_a * \gamma \sim \gamma$$

where $\sim$ stands for definably homotopic. □

Proposition 4.8. *Let $p : X \to Y$ be a definable covering map. Let H be a definable homotopy between paths f_1 and f_2 on on X. Suppose that $\widetilde{f_1}$ is a lifting of f_1. Then there is a unique lifting $\widetilde{H}$ of H such that $\widetilde{H}(t,0) = \widetilde{f_1}$.*

PROOF. Let $\{U_i\}$ be a covering of X by open sets given by the definition of a cover.

By Fact 1.19, there is a covering $\{V_j\}$ of Y such that for any j there is some i with $\overline{V}_j \subset U_i$. In particular, the restriction of p to any connected component of $p^{-1}(\overline{V}_j)$ is a definable homeomorphism to $\overline{V}_j$ for all j.

Let $\mathcal{C}$ be a cell decomposition of $[0,1] \times [0,1]$ into cells such that for any $C \in \mathcal{C}$ we have $H(C) \subseteq V_j$ for some j.

The set projections of cells in $\mathcal{C}$ is a cell decomposition of $[0,1]$ so we have $0 = t_1 < t_2 < \cdots < t_n = 1$ and the projection of every cell in $\mathcal{C}$ is either a point t_l or an interval (t_l, t_{l+1}). We enumerate the cells in $\mathcal{C} = \{C_{l,k}\}$ with l, k integers such that $l_1 \leq l_2$ implies $\pi_1(C_{l_1,j}) \leq \pi_1(C_{l_2,i})$ for all i, j and for any $a \in \pi(C_{l,i}) = \pi(C_{l,j})$ we have $\pi_1^{-1}(C_{l,i}) < \pi_1^{-1}(C_{l,j})$ whenever $i < j$.

The rest of the proof is essentially as in Proposition 4.3. For any C_i the $H(C_i)$ lies completely inside a U_j we have a unique lifting of H to any connected component of $p^{-1}(U_j)$. We require $\widetilde{H}(0,t) = \widetilde{f}_1$ so we have a unique lifting to any cells projecting onto $\{0\}$. We prove existence and uniqueness inductively on the indices (l,k) (ordered lexicographically) as follows. By construction, given any cell $C_{i,j}$ there is $(s,t) <_{lex} (i,j)$ such that $\overline{C_{i,j}} \cap \overline{C_{s,t}} \neq \emptyset$. Choose any such (s,t). By hypothesis the lifting of H the restriction of H to $\overline{C_{s,t}}$ has been defined in a unique manner. This implies that there is a unique connected component of $p^{-1}(H(\overline{C_{i,j}}))$ intersecting $\widetilde{H}(\overline{C_{s,t}})$ and by the definition of a covering there is a unique lifting of the restriction of H to $\overline{C_{i,j}}$ which agrees with $\widetilde{H}(\overline{C_{s,t}})$ on $\overline{C_{i,j}} \cap \overline{C_{s,t}}$.

One needs to check that this unique extension is compatible between the multiple intersections, an exercise which is left to the reader. □

EXERCISE 4.5. *Check that in the proof of Proposition 4.8, the lifting of the restriction of H to $C_{i,j}$ is independent of the choice of s, t.*

4.1.3. The case $\mathcal{R} = \mathbb{R}$. When $\mathcal{R} = \mathbb{R}$ we can study the homotopy equivalence of continuous paths and functions in definable sets. The resulting equivalence relations will coincide with the ones one gets in the definable case.

We will need the Lebesgue Number Theorem, Lemma A.19 in [**28**]:

Fact 4.9. *Given any covering of a compact metric space K by open sets, there is an $\epsilon > 0$ such that any subset of K of diameter less than ϵ is contained in some open set in the covering.*

Proposition 4.10. *If $\mathcal{R} = \mathbb{R}$ and G is a definable group, then any (possibly non definable) continuous path in X is homotopically equivalent to a definable path.*

If two definable paths in X are homotopically equivalent, then they are definably homotopically equivalent.

PROOF. Recall that Theorem 2.1 implies the existence of an open definable V where the t-topology and the order topology coincide. Take the open cells of a cell decomposition of V, so the union of such cells has small codimension and by Exercise 2.10 finitely many translates of such cells cover G. By Exercise 4.2 G can be covered with finitely many contractible sets $\{U_i\}_i$ each definably isomorphic to an open convex set $B_i \subset \mathbb{R}^n$.

Take a continuous path $\sigma : [0, 1] \to G$.

By the Lebesgue's Number Theorem, there is some $\delta > 0$ such that any subinterval of $[0, 1]$ of size less than δ is contained in some $\sigma^{-1}(U_i)$. Let m be such that $\frac{1}{m} < \delta$ and so that if $I_j := \left[\frac{j}{m}, \frac{(j+1)}{m}\right]$, then the image of I_j under σ is fully contained in $\mathcal{U}_{i_j}$. We find a definable σ_{def} as follows. Let μ_j be the pushforward of σ_j in B_j. By convexity, the straight line $l_{\mu_j(\frac{j}{m}),\mu_j(\frac{(j+1)}{m})}$ is fully contained in B_{i_j}, and because B_{i_j} is simply connected, μ_j is homotopic to l_j. We then pull back l_j to U_{i_j} and the resulting path is σ_j^{def}. By construction, the starting and endpoints of σ_j^{def} are the precisely the same as those of σ_j which by definition define the connected path σ. The gluing of the σ_j^{def} defines a definable connected path homotopic to σ, as required.

The proof of the fact that two definable maps which are homotopically equivalent are definably homotopically equivalent can be done in a very similar manner (using Lebesgue Number Theorem to build a linear homotopy out of a –non definable– homotopy between two definable curves). We will leave it as an exercise. $\square$

EXERCISE 4.6. *If $\mathcal{R} = \mathbb{R}$ and G is a definable group, then if two definable paths in X are homotopically equivalent, they are definably homotopically equivalent.*

4.2. The definable fundamental group

We will now define the definable fundamental group. Proposition 4.10 already implies that these cannot always be definable objects: If we take $G := SL_2(\mathbb{R})$ for example, the fundamental group, which is also the center of the universal cover, is $\mathbb{Z}$. We will show they are "close" to definable but we will need to expand the category of definable sets.

Definition 4.11. A group $(G, \odot)$ is *locally definable* if there is a countable collection $\{X_i\}_{i\in I}$ of definable subsets of $\mathcal{R}^n$ such that $G = \bigcup_{i\in I} Z_i$, for any $i, j \in I$ there is some k such that $Z_i, Z_j \subseteq Z_k$ and the group multiplication restricted to $Z_i \times Z_j$ is a definable map from $Z_i \times Z_j$ to Z_k.

Both our definition of the definable fundamental group and of the definable universal covering (which we will define in the next section) will be locally definable groups, which by Proposition 4.10 will coincide with (will be isomorphic to) the standard constructions in the case $\mathcal{R} = \mathbb{R}$.

The purpose of this section is to construct the fundamental group as a locally definable group.

Let X be a connected definable set. Let $a \in X$ be a fixed point. Let Γ_{X,x_0} be the set of all definable continuous paths on X starting and ending on x_0. Let $\pi_1(X, x_0)$ be the set of elements in Γ_{X,x_0} modulo being definably definably homotopic (Definition 4.6). Notice that concatenation defines a group operation on $\pi(X, x_0)$.

Definition 4.12. The set $\pi_1(X, x_0)$ with concatenation as the group operation is the *fundamental group of* X (it is, modulo an isomorphism, independent of x_0).

EXERCISE 4.7. *Prove that if X is connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Notice that $\pi_1(X, x_0)$ is not definable, not it is immediately clear how it can be represented by a definable object. Uniformly finding an element in each homotopy class is the next goal in this section.

We now work in the setting of Section 1.1.3 of Chapter 1. By Fact 1.23 and Theorem 2.30, we may assume that X is a definable simplicial complex K as in Definition 1.20. Even more, we may assume that x_0 (which whenever X is a definable group will be chosen to be the group identity) is the barycenter of an n-simplex for $n = \dim(X)$.

For each simplex C_i in K, let v_i be the barycenter, and let $\mathcal{B}$ be the set of all such barycenters. For any $v_i, v_j \in \mathcal{B}$, recall that $l_{v_i v_j}$ represents the (closed) line segment between v_i and v_j.

Let Γ_1 be the union of a all $l_{v_i v_j}$ such that C_i is a face of C_j, or viceversa. Alternatively, Γ is the 1-skeleton (union of all the 0- and 1-simplices) of the barycentric subdivision of K (see Figure 1).

Lemma 4.13. *Any path connecting elements in $\mathcal{B}$ is definably homotopic to a path in Γ_1.*

PROOF. Let γ be a path in X connecting v_0 and v_1, so that $\gamma(0) = v_0$ and $\gamma(1) = v_1$. Let r be the largest element in $\mathcal{R}$ such that $\gamma(r) \in \mathcal{B}$ and such that the restriction of $\gamma([0, r])$ is entirely contained in Γ_1.

Let $L_1, \ldots, L_k$ be a decomposition of $[r, 1]$ into connected sets such that $\gamma(L_i) \in C_j$ for some $C_j \in K$. We will do an induction on k.

If $k = 0$ then $\gamma([0, 1])$ is contained in Γ_1 and we are done.

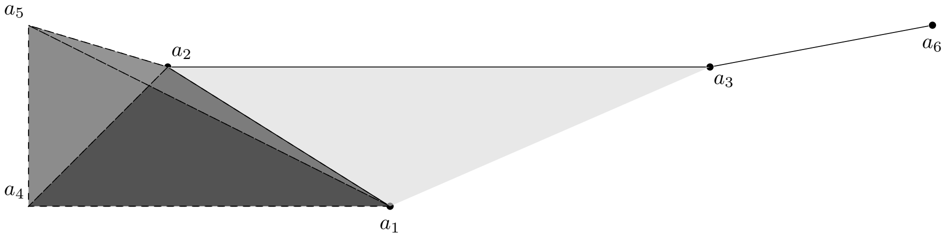
By construction and hypothesis, $\gamma(r)$ is the barycenter or C_1. Let $s \in C_2$.

The restriction of γ to $[r, s]$ is contained in $C_1 \cup C_2$ so either C_1 intersects $\overline{C_2}$, or vice versa. Since they are both simplices, one is a face of the other. Either way, $C_1 \cup C_2$ is definably simply connected convex set.

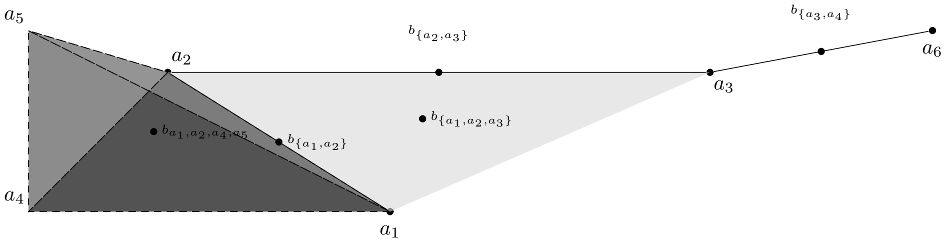
Let w be the barycenter of C_2. Consider the path δ which concatenates the line segments $l_{\phi(r)w}$ and $l_{w\phi(s)}$ connecting $\phi(r)$ with w and w with $\phi(s)$, respectively. Since $C_1 \cup C_2$ is convex, δ is contained in $C_1 \cup C_2$ and since $C_1 \cup C_2$ is definably simply connected, δ is definably homotopic to the restriction of γ to $[r, s]$.

Let

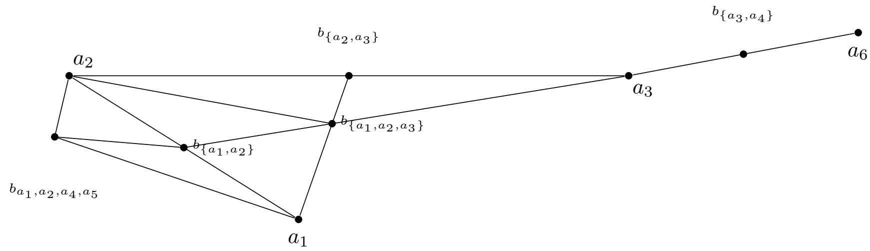
$$\gamma'(t) := \begin{cases} \gamma(t) \text{ if } 0 \leq t \leq r \\ \delta\left(\frac{t-r}{s-r}\right) \text{ if } r \leq t \leq s \\ \gamma(t) \text{ if } s \leq t \leq 1. \end{cases}$$



(A) Complex



(B) Barycenters



(C) 1-Skeleton

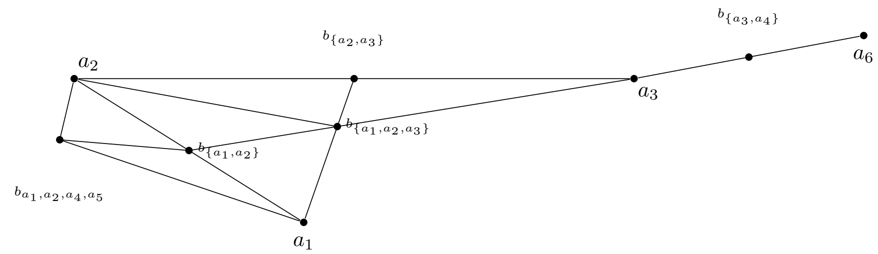
FIGURE 1. 1-Skeleton of the barycentric subdivision.

By construction γ' is a path definably homotopic to $\gamma|_{[0,r]} * \delta|_{[r,s]} * \gamma|_{[s,1]}$, and by Exercise 4.4, γ' is definably homotopic to γ. Furthermore, $\gamma'((r+s)/2) = \delta(1/2) = w$, and $\gamma'|_{[r,\frac{r+s}{2}]}$ is the line segment joining the elements $\phi(r), w \in \mathcal{B}$, so $\gamma'([0, \frac{r+s}{2}]) \subset \Gamma_1$. By induction hypothesis, γ' is definably homotopic to a path entirely contained in Γ_1 which completes the proof of the lemma. $\square$

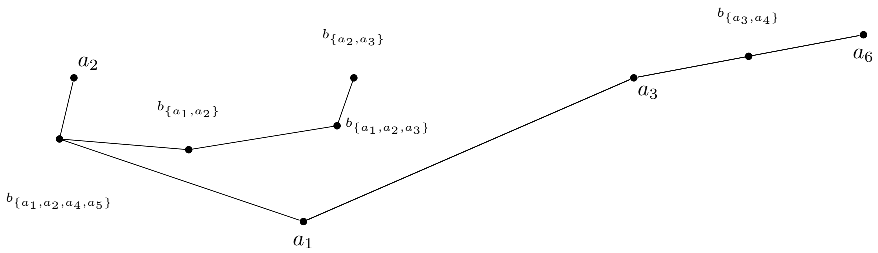
Γ_1 is by definition the realization of a connected simplicial complex K_1 of dimension 1 (i.e., consisting only on lines and points). So any path γ contained in a single simplex of Γ_1 is definably homotopic to the straight line connecting its endpoints, so that we can find a definable path definably homotopic to γ which is piecewise linear. In particular we have the following.

Corollary 4.14. *Let $\mathcal{R}$ be an o-minimal expansion of a real closed field let X be an $\mathcal{R}$-definable set, let $x_0 \in X$ be any point, and let $\mathcal{R}'$ be an expansion of $\mathcal{R}$. Then $\pi_1(X, x_0)$ is the same calculated in $\mathcal{R}$ and $\mathcal{R}'$.*

PROOF. Let K be a simplicial complex $\mathcal{R}$-isomorphic to X. By Lemma 4.13 and the comment after, any $\mathcal{R}'$-definable path is $\mathcal{R}'$-definably homotopic to a semilinear path in going through the barycenters of K. Since we always assume we are expanding a field, any semilinear path is $\mathcal{R}$-definable. $\square$



(A) 1-Skeleton



(B) Maximal tree

FIGURE 2. 1-Skeleton and its maximal tree.

The other advantage is that we can view Γ_1 as a graph (whose vertices are the 0-simplices and the edges are the 1-simplices), which will allow us to easily obtain generators of the fundamental group and use them to describe it as a locally definable group.

Remark 4.15. It is *not* true in general that $\pi_1(X, x_0) = \pi_1(\Gamma_1, x_0)$, since paths that are definably homotopic in X might not be in Γ_1. In fact, Γ_1 will be in some ways a "free group" version of $\pi_1(X, x_0)$. One can take the 2-skeleton of the barycentric subdivision (as mentioned above, Γ_1 is the 1-skeleton) and one would obtain a combinatorial object definably homotopic to X, which is a way to compute $\pi_1(X, x_0)$ when X is triangulizable. In our case (definable groups) things are much simpler, so, although fascinating, we will not pursue this line further than this remark.

4.2.1. $\pi_1(X, x_0)$ is a (i.e. can be represented by a) locally definable group. We keep the notation and definitions of $X, K, \mathcal{B}$ and Γ_0 as above. In particular, we may assume that $\pi_1(X, x_0)$ are paths starting and ending in x_0 and consisting of concatenation of line segments $l_{v_i v_j}$ where $v_i, v_j \in \mathcal{B}$ are barycenters of adjacent simplices in K. In particular, we are choosing representatives of each definable homotopy class to be paths in Γ_1.

Viewing Γ_1 as a graph, let Γ_0 be a maximal tree. This is, a maximal set of edges with no cycles (see Figure 2). In particular given any two vertices v, w in $\mathcal{B}$, there are $v = v_0, v_1, \ldots, v_{n-1}, v_n = w$ such that there is an edge in Γ_0 between v_i and v_{i+1} and this is the unique way to go from v to w over edges in Γ_0 without repeating any edge (see Exercise 4.8).

EXERCISE 4.8. *Prove that if Γ_0 is a tree then there is a unique path between any two elements (without repeating vertices). Prove that if Γ_0 is a maximal tree of a connected graph Γ_1, then Γ_0 is connected.*

For $v, w \in \mathcal{B}$, let $\gamma_{v,w}$ be the (topological) definable path in Γ_0 which goes from v to w by successively concatenating $l_{v_i v_{i+1}}$ for $i = 0$ until $n - 1$.

Going back to the viewing Γ_0 as a subspace of Γ_1 and, as such, as a definable set consisting of the points in $\mathcal{B}$ and line segments joining (some of) them, we have:

Claim 4.16. *Any finite tree is simply connected.*

Proof: This is proved by induction on the number of vertices in the tree.

Any finite tree has leaves, this is, elements which are either isolated or connected to exactly one element in the graph: Starting at any vertex, one moves, if possible, from vertex to vertex through edges in the graph with the only condition of not going back the same edge one just came from. In this process one can never go back to a vertex one already went through, since this would witness a cycle. So the process must terminate and the only way it can terminate is if one is standing in a leaf.

In the definable representation of the graph, if v is a leaf connected to w then one can shrink (in a not endpoint preserving definable homotopy) the segment from v to w while leaving the rest of the tree constant. This is a definable homotopy between the tree and another tree with one fewer vertex. $\blacksquare_{Claim}$

So in particular, Γ_0 is definably connected and all paths in Γ_0 starting and ending in the same vertex are definably homotopic.

If e is an edge between vertices v and w with $e \in \Gamma_1 \setminus \Gamma_0$ let δ_e be the path consisting of the concatenation of $\gamma_{x_0,v}$, e and γ_{w,x_0} (recall we are identifying e with the path l_{vw}).

Claim 4.17. *The set*

$$\{\delta_e \mid e \in \Gamma_1 \setminus \Gamma_0\}$$

generates $\pi_1(X, x_0)$ as a group. Namely, every path in $\pi_1(X, x_0)$ is definably homotopic to a concatenation of elements in $\{\delta_e \mid e \in \Gamma_1 \setminus \Gamma_0\}$.

Proof: Any path in Γ_1 with endpoints in $\mathcal{B}$ is definably homotopic to the concatenation of line segments of the form $l_{v_iv_j}$ where v_i, v_j are vertices in $\mathcal{B}$ with $l_{v_iv_j} \in \Gamma_1$. So we will identify a path starting and ending at x_0 with the edges it goes through. Let γ be such a path. Let $e_1, \dots, e_k$ be the edges in $\Gamma_1 \setminus \Gamma_0$ that γ goes through, so that γ is identified with the concatenation

$$\lambda_0 * e_1 * \lambda_1 * e_2 * \dots \lambda_{k-1} * e_k * \lambda_k$$

where all the λ_i-s are inside Γ_0. Let v_i, w_i be the vertices in e_i written in the order that γ runs through. The concatenation of γ_{v,x_0} and $\gamma_{x_0,v}$ is contractible for any v, so γ is definably homotopic to

$$\lambda_0*(\gamma_{v_1,x_0}*\gamma_{x_0,v_1})*e_1*(\gamma_{w_1,x_0}*\gamma_{x_0,w_1})*\lambda_1*\cdots*(\gamma_{v_k,x_0}*\gamma_{x_0,v_k})*e_k*(\gamma_{w_k,x_0}*\gamma_{x_0,w_k})*\lambda_k$$

which by definition is equal to

$$\lambda_0 * \gamma_{v_1,x_0} * \delta_{e_1} * \gamma_{x_0,w_1} * \lambda_1 * \gamma_{v_2,x_0} * \delta_{e_2} * \gamma_{x_0,w_2} * \lambda_2 * \cdots * \gamma_{v_k,x_0} * \delta_{e_k} * \gamma_{x_0,w_k} * \lambda_k.$$

Each of $\lambda_0 * \gamma_{v_1,x_0}, \gamma_{x_0,w_k} * \lambda_k$ and for $1 < i < k$ each $\gamma_{x_0,w_i} * \lambda_i * \gamma_{v_{i+1},x_0}$ is a path in the contractible space Γ_0 starting and ending in x_0, which implies they are all definably homotopic to the identity in $\pi_1(X, x_0)$. Therefore γ is definably homotopic to $\delta_{e_1} * \delta_{e_2} * \cdots * \delta_{e_k}$, as required. $\blacksquare_{Claim}$

One can now give a local definition of $\pi_1(X, x_0)$. Let F_0 be the set $\{\delta_{e_1}, \delta_{e_2}, \dots, \delta_{e_k}, i\}$ where i is the constant path at x_0, taking its quotient by identifying paths with definable homotopic equivalence (in X) between them. We define F_n inductively as follows: Assuming F_k is defined, then F_{k+1} is the set of all possible concatenations of pairs of elements in F_k again modding out by the definable homotopy.

For every n there are finitely many objects in F_n (by induction the size of F_n is always less than $(k+1)^{(2^n)}$) so F_n is definable for all n. We proved that

$\pi_1(X, x_0) = \bigcup_{i\in\mathbb{N}} F_i$, and by definition $F_n * F_n = F_{n+1}$, which completes the proof of local definability.

Proposition 4.18. *Let $p : Y \to X$ be a covering map, let $y \in Y$ and $x = p(y)$. Then p induces a natural well defined injection $p^\pi : \pi(Y, y) \to \pi(X, x)$ which is a locally definable map.*

Proof. The map $\gamma \mapsto p \circ \gamma$ is a natural map between paths on Y starting and ending in y and paths in X starting and ending in x. For this to induce a map between $\pi(Y, y)$ and $\pi(X, x)$, we need to show that the images of two definably homotopic definable maps on Y are definably homotopic in X (so that p^π is well defined). This follows from the fact that if $H : [0, 1] \times [0, 1] \to Y$ is a definable homotopy between definable paths in Y, then $p \circ H$ will be a definable homotopy between their image paths on X.

For injectivity, we need to show that given any path γ in Y such that $p\circ\gamma$ and a definable homotopy H between $p\circ\gamma$ and the constant map at x, there is a definable homotopy $\widetilde{H}$ between γ and the constant map at y. But this is a particular case of Proposition 4.8. □

4.2.2. Fundamental group of a definable group. Assume that X is a definable group $(G, \cdot)$ and that $x_0 = e$, the identity element of G. Given paths γ_1, γ_2 in G, it makes sense to define the path

$$(\delta_1 \cdot \delta_2)(t) := \delta_1(t) \cdot \delta_2(t).$$

This will, by continuity of the group operation, be a definable path in G. Notice that if both paths begin (or end) in e then so will the $(\cdot)-$product.

The following exercise proves that

$$[\delta_1] \cdot [\delta_2] = [\delta_1 \cdot \delta_2]$$

is a well defined operation on $\pi_1(G, e)$.

Exercise 4.9. *Prove that if δ_1, δ_1' and δ_2, δ_2' are pairwise definably homotopic, then*

$\delta_1 \cdot \delta_2$ and $\delta_1' \cdot \delta_2'$ are definably homotopic.

This implies that in this case we have two group operations in $\pi_1(G, e)$: Concatenation and $\cdot$. The following lemma proves that they coincide.

Lemma 4.19. *Let δ_1, δ_2 be paths such that $\delta_1(0) = \delta_1(1) = \delta_2(0) = e$, so that $\delta_1 * \delta_2$ is defined. Then*

$$[\delta_1 \cdot \delta_2] = [\delta_1 * \delta_2].$$

In particular, $[\delta_1 \cdot \delta_2] = [\delta_1 * \delta_2]$ whenever $[\delta_1], [\delta_2] \in \pi_1(G, e)$.

Proof. Consider the function

$$H(s,t) := \begin{cases} \delta_1 (2t - st) \cdot \delta_2 (st) & \text{if } 0 \le t \le 1/2 \\ \delta_1 (1 + s(t-1)) \cdot \delta_2 ((2t-1) + s(1-t)) & \text{if } 1/2 \le t \le 1. \end{cases}$$

For any s, at $t = 1/2$ both top and bottom give $\delta_1\left(1 - \frac{s}{2}\right)\delta_2\left(\frac{s}{2}\right)$ so H is continuous..

$$H(0,t) := \begin{cases} \delta_1 (2t) \cdot \delta_2 (0) = \delta_1 (2t) & \text{if } \ 0 \le t \le 1/2 \\ \delta_1 (1) \cdot \delta_2 ((2t-1)) = \delta_2(2t-1) & \text{if } \ 1/2 \le t \le 1. \end{cases}$$

So that by definition $H(0, t)$ is the concatenation $\delta_1 * \delta_2$ of δ_1 and δ_2.

On the other hand,

$$H(1,t) := \begin{cases} \delta_1 (2t - t) \cdot \delta_2 (t) = \delta_1 (t) \cdot \delta_2 (t) & \text{if } \ 0 \le t \le 1/2 \\ \delta_1 (1 + (t-1)) \cdot \delta_2 ((2t-1) + (1-t)) = \delta_1(t) \cdot \delta_2(t) & \text{if } \ 1/2 \le t \le 1. \end{cases}$$

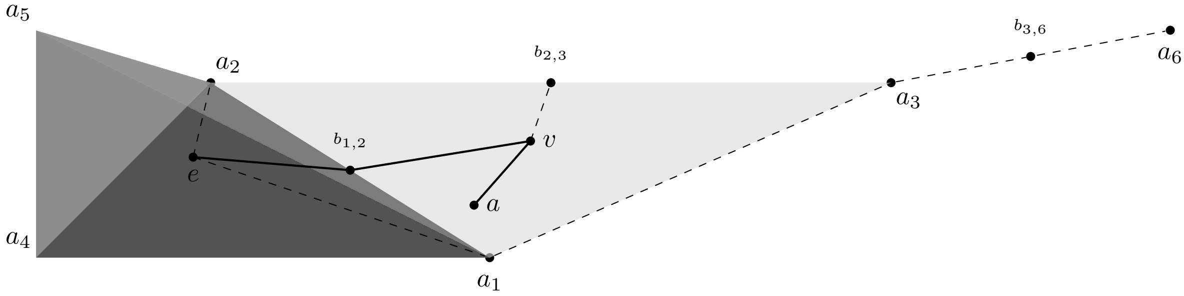


FIGURE 3. $\gamma_{e,a}$ together with Γ_0 (dashed)

So $H(1,t) = \delta_1 \cdot \delta_2(t)$.

Finally, $H(s,0) = \delta_1(0)\cdot\delta_2(0) = e$ and $H(s,1) = \delta_1(1)\cdot\delta_2(1) = e\cdot\delta_2(1) = \delta_2(1)$, so we have a definable homotopy between $\delta_1 * \delta_2$ and $\delta_1 \cdot \delta_2$, which completes the proof of the lemma. □

4.3. The definable universal cover

The definable universal cover is constructed in analogy with the construction of the standard universal cover of a manifold. Namely, for any definable definably connected X, the *definable universal cover* $\widetilde{X}$ *of* X is the set of paths that start in a fixed point $x_0 \in X$, quotiented by definable homotopy. As with the fundamental group, it is quite easy to prove that modulo definable isomorphism the definition is independent of the choice of x_0. We will use the fact that $\widetilde{X}$ is isomorphic to $\pi_1(X, x_0) \times X$ to give a locally definable version of $\widetilde{X}$.

Whenever X is a definable group $(G, \cdot)$ and the starting points of the paths defining $\widetilde{G}$ is the identity e of G we get that the group operation $\cdot$ on $\widetilde{G}$ makes $(\widetilde{G}, \cdot)$ into a locally definable group.

EXERCISE 4.10. *Prove that* $(\widetilde{G}, \cdot)$ *is a group operation, whose identity is the constant map at* e *and such that the inverse of a path* λ *is the map defined by*

$$\lambda^{-1}(t) := (\lambda(t))^{-1}$$

for any $t \in [0,1]$ *and where the right hand inverse is the group inverse in* G.

As with the fundamental group, we will need locally definable versions of this process. We keep the assumptions and definitions of the previous section, in particular of $K, \Gamma_0, \Gamma_1, \mathcal{B}, \gamma_{v,w}$ whenever $v, w \in \mathcal{B}$ are barycenters of simplices in K, and F_n.

The group $\widetilde{G}$ will project naturally onto G (sending a path to its endpoint) with kernel $\pi_1(G, g)$. The first step in the construction is to choose a section of this projection. This is, for every $a \in G$, a path which starts in e and ends in a. We will do this as follows: For any $a \in G$, let v be the barycenter of the simplex C that a belongs to. Let $\gamma_{e,a}$ be the concatenation of $\gamma_{e,v}$ and l_{va}. We are assuming that $e \in \mathcal{B}$ so that $\gamma_{e,v}$ is a path (in fact the unique path between e and v) in Γ_0 and $a, v \in C$ is convex so $\gamma_{e,a}$ is a path in G. Similarly, we define $\gamma_{a,e}$ as the concatenation of l_{av} and $\gamma_{v,e}$ (see Figure 3).

There are finitely many simplices in K and they are all definable, which implies that the set $\{\gamma_{e,a} \mid a \in G\}$ is a definable collection of paths in G.

Notice that any path λ in G starting at e will be definably homotopic to the concatenation of $\lambda * \gamma_{\lambda(1),e} * \gamma_{e,\lambda(1)}$. Since $[\lambda \cdot \gamma_{\lambda(1),e}] \in \pi_1(G, e)$ and for any two

$a \neq b \in G$ the paths $\gamma_{e,a}$ and $\gamma_{e,b}$ are not definably homotopic (they don't have the same endpoints), we have the following:

Remark 4.20. Any path in G is definably homotopic to the concatenation of δ and $\gamma_{e,a}$ for some $[\delta] \in \pi_1(G,e)$ and some $a \in G$, and any element in $\widetilde{G}$ is uniquely determined by $[\delta]$ and a.

In particular, if for any $[\delta] \in \pi_1(G,e)$ and $a \in G$ we send $([\delta], a)$ to the concatenation of δ and $\gamma_{e,a}$, we have a bijection between $\widetilde{G}$ and $\pi_1(G,e) \times G$.

By Lemma 4.19, the concatenation $\delta * \gamma_{e,a}$ is homotopic to the path $\delta \cdot \gamma_{e,a}$.

Recall the local definability $\bigcup_{n\in\mathbb{N}} F_n$ of $\pi_1(G,e)$. Under the above isomorphism we have a group operation on $\bigcup_{n\in\mathbb{N}} F_n \times G$ which sends $(\delta_1, a_1) \in F_n \times G$ and $(\delta_2, a_2) \in F_m \times G$ to $(\delta_1 * \gamma_{e,a_1}) \cdot (\delta_2 * \gamma_{e,a_2})$. By Lemma 4.19,

$$(\delta_1 * \gamma_{e,a_1}) \cdot (\delta_2 * \gamma_{e,a_2}) = \delta_1 \cdot \gamma_{e,a_1} \cdot \delta_2 \cdot \gamma_{e,a_2},$$

the following will be key to prove definability:

Proposition 4.21. *$\pi_1(G,e)$ is a central subgroup of $(\widetilde{G}, \cdot)$.*

Proof. Let γ be a path in G starting at e and let δ be a path starting and ending in e.

Consider the map from $[0,1] \times [0,1]$ to G defined by

$$H(s,t) := \mu(st) \cdot \delta(t) \cdot \mu^{-1}(st).$$

Then $H(0,t) = \delta(t), H(1,t) = \mu(t) \cdot \delta(t) \cdot \mu^{-1}(t)$ and $H(s,0) = H(s,1) = e$ for all s so that H is a definable homotopy with fixed endpoints between δ and $\gamma \cdot \delta \cdot \gamma^{-1}$. So

$$[\delta] = [\gamma \cdot \delta \cdot \gamma^{-1}] = [\gamma] \cdot [\delta] \cdot [\gamma]^{-1}$$

for any $[\delta] \in \pi_1(G,e)$ and any $[\gamma] \in \widetilde{G}$, as required. $\square$

4.3.1. $(\widetilde{G}, \cdot)$ is a locally definable group. Recall the definition of the path $\gamma_{e,a}$ in G from e to a defined at the beginning of this section.

Theorem 4.22. *Identify $\widetilde{G}$ with $(\bigcup_n F_n) \times G$ by the isomorphism that sends any path μ in G to the unique pair $(\delta, a) \in (\bigcup_n F_n) \times G$ such that $\mu \sim \delta \cdot \gamma_{e,a}$.*

Then $\widetilde{G}$ is a locally definable group.

Proof. That the universe is locally definable is given by the representation we are choosing. So we need to prove that the multiplication is locally definable.

As such, we need to prove that the map that sends $(\delta_1, a_1) \in F_n \times G$ and $(\delta_2, a_2) \in F_m \times G$ to (δ, a_1a_2) where

$$(\delta_1 * \gamma_{e,a_1}) \cdot (\delta_2 * \gamma_{e,a_2}) \sim \delta * \gamma_{e,a_1a_2}$$

is definable.

By the discussion in the previous section, this is equivalent to finding δ such that

$$\delta_1 \cdot \gamma_{e,a_1} \cdot \delta_2 \cdot \gamma_{e,a_2} \sim \delta \cdot \gamma_{e,a_1a_2}.$$

By Proposition 4.21,

$$\delta_1 \cdot \gamma_{e,a_1} \cdot \delta_2 \cdot \gamma_{e,a_2} \sim \delta_1 \cdot \delta_2 \cdot \gamma_{e,a_1} \cdot \gamma_{e,a_2}$$

and since $\pi_1(G,e) := \bigcup_n F_n$ is a locally definable group, it is enough to prove that the map h from $G \times G$ to $\bigcup_n F_n$ sending (a_1, a_2) to the unique $\delta \in \bigcup_n F_n$ such that $\gamma_{e,a_1} \cdot \gamma_{e,a_2} \cdot (\gamma_{e,(a_1 \cdot a_2)})^{-1} \sim \delta$ is definable.

Let $C_1, \ldots, C_k$ be the simplices that make up G. Let $X_{ijl} := \{(a,b) \in G \times G \mid a \in C_i, b \in C_j, ab \in C_l\}$, which is a partition of $G \times G$. We will prove that for any fixed i, j, l the restriction of h to X_{ijk} sending (a_1, a_2) to $[\gamma_{e,a_1} \cdot \gamma_{e,a_2} \cdot \gamma^{-1}_{e,(a_1 \cdot a_2)}]$ is definable. We will in fact prove that the map is constant for all elements in X_{ijl} which is stronger. Since there are finitely many X_{ijl} this will complete the proof of the theorem.

Fix some i, j, l with $X_{ijl} \neq \emptyset$ and let (a, b) and (a', b') be pairs of points in X_{ijl}. Let $a_s := (1-s)a + sa'$, $b_s := (1-s)b + sb'$ and $(ab)_s := (1-s)(ab) + s(a'b')$. all simplices are convex, we have $a_s \in C_i, b_s \in C_j$ and $(ab)_s \in C_j$.

Consider the map

$$H(s,t) := (\gamma_{e,a} * l_{aa_s}) \cdot (\gamma_{e,b} * l_{bb_s}) \cdot (\gamma_{e,(ab)} * l_{(ab)(ab)_s})^{-1}.$$

By construction $H(s, 0) = H(s, 1) = \{e\}$ for all e so $H(s, t)$ is definable homotopy between $H(0, s) = \gamma_{e,a} \cdot \gamma_{e,b} \cdot (\gamma_{e,(ab)})^{-1}$ and $H(1, s) := \gamma_{e,a} * l_{aa'}) \cdot (\gamma_{e,b} * l_{bb'}) \cdot (\gamma_{e,(ab)} * l_{(ab)(a'b')})^{-1}$ which proves that

$$\gamma_{e,a} \cdot \gamma_{e,b} \cdot \gamma^{-1}_{e,(ab)} \sim (\gamma_{e,a} * l_{aa'}) \cdot (\gamma_{e,b} * l_{bb'}) \cdot (\gamma_{e,(ab)} * l_{(ab)(a'b')})^{-1}.$$

Now, by definition $\gamma_{e,a} = \gamma_{e,v_i} * l_{v_i a}$ and $\gamma_{e,a'} = \gamma_{e,v_i} * l_{v_i a'}$. Since $v_i, a, a' \in C_i$ and C_i is contractible, we have $\gamma_{e,v_i} * l_{v_i a} * l_{aa'} \sim \gamma_{e,v_i} * l_{v_i a'}$, so that $\gamma_{e,a} * l_{aa'} \sim \gamma_{e,a'}$. Similarly, $\gamma_{e,b} * l_{bb'} \sim \gamma_{e,b'}$ and $\gamma_{e,ab} * l_{(ab)(a'b')} \sim \gamma_{e,(a'b')}$

Multiplication of paths is well defined in the homotopy classes, so

$$\gamma_{e,a} \cdot \gamma_{e,b} \cdot \gamma^{-1}_{e,(ab)} \sim \gamma_{e,a'} \cdot \gamma_{e,b'} \cdot (\gamma_{e,(a'b')})^{-1}$$

and $[\gamma_{e,a} \cdot \gamma_{e,b} \cdot \gamma^{-1}_{e,(ab)}] = [\gamma_{e,a'} \cdot \gamma_{e,b'} \cdot (\gamma_{e,(a'b')})^{-1}]$, as required. □

Remark 4.23. We have an exact sequence of groups

$$\{e\} \to \pi_1(G, e) \to \widetilde{G} \to G \to \{e\}$$

with $\pi_1(G, e)$ central in $\widetilde{G}$.

The map $h : G \times G \to \pi_1(G, e)$ defined above as $h(a, b) = \gamma_{e,a} \cdot \gamma_{e,b} \cdot \gamma^{-1}_{e,(ab)}$ is precisely the *2-cocycle* of the exact sequence associated with the embedding from G into $\widetilde{G}$ given by $a \mapsto \gamma_{e,a}$. So the argument at the end of the proof of Theorem 4.22 states precisely that the 2-cocycle h is definable and has finite image. This will be important for the applications in the next section.

Corollary 4.24. *Let p be a covering map between definable connected groups H and G. Then $\widetilde{H}$ is locally definably isomorphic to $\widetilde{G}$ and the kernel of p is isomorphic to $\pi_1(G, e)/(p^\pi(\pi_1(H, e))$ where p^π is the injection from $\pi_1(H, e)$ to $\pi_1(G, e)$ given by proposition 4.18.*

Proof. By Proposition 4.3, the projection by p of any path in H starting at the identity onto a path in G is a bijection, so $\widetilde{H}$ is isomorphic to $\widetilde{G}$. □

4.3.2. Topology on $\widetilde{G}$. Recall we are assuming that e is inside a n-dimensional simplex C, where $n = \dim(G)$ and that the endowed topology on $G \subset \mathcal{R}^k$ is the t-topology. Let $\mathcal{N}_e$ be the set of $B_{<\epsilon(e)} \cap C$ (where $B_{<\epsilon(e)}$ is defined to be the set of elements in $\mathcal{R}^k$ whose distance to e is less than ϵ). Let $\mathcal{N} := \{gN \mid g \in G, N \in \mathcal{N}_e\}$. So $\mathcal{N}$ is a basis for the topology on G consisting on definably contractible open sets ($B_{<\epsilon(e)} \cap C$ is convex).

We will not use this subsection much, except in the case where $\mathcal{R} = \mathbb{R}$ for which our construction is precisely the same as the classic construction in [**28**]. We we will not give many details and leave many proofs as exercises.

Definition 4.25. For any $N \in \mathcal{N}$ let $\widetilde{N}_z$ be the set of paths in N starting with z.

For any $N \in \mathcal{N}$ and any $\gamma \in \widetilde{G}$ with $\gamma(1) \in N$, let

$$N_{[\gamma]} := \{[\gamma * \delta] \mid \delta \in \widetilde{N}_{\gamma(1)}\}.$$

Given a subset $\mathcal{O}$ of $(\widetilde{G}, \cdot)$, we say that $\mathcal{O}$ *is open* if for any $\gamma \in \widetilde{G}$ with $[\gamma] \in \mathcal{O}$ there is some $N \in \mathcal{N}$ with $N_{[\gamma]} \subseteq \mathcal{O}$.

For any $a \in G$ and any path γ in G, let $a\gamma$ be the path defined by $a\gamma(t) := a \cdot \gamma(t)$, so that paths in $\widetilde{(gN)}_g = \{g\gamma \mid \gamma \in \widetilde{N}_e\}$.

Lemma 4.26. *The following hold:*

1. *If σ is a path in $N \in \mathcal{N}$ starting in $\gamma(1)$ then $N_{[\gamma]} = N_{[\gamma * \sigma]}$.*
2. *$N_{[\gamma]} \subset N'_{[\gamma]}$ whenever $N \subset N'$.*
3. *The projection from $N_{[\gamma]}$ to N sending $[\gamma]$ to $\gamma(1)$ is an homeomorphism.*
4. *If $\gamma, \gamma' \in \widetilde{G}$ are such that $\gamma(1) = \gamma'(1)$, then $N_{[\gamma]} = N_{[\gamma']}$ if γ and γ' are definably homotopic. $N_{[\gamma]} \cap N_{[\gamma']} = \emptyset$, otherwise.*

Proof. The first two items follow from the definition and definable contractibility of N, and the third item follows from the first two. The details are left as exercises.

As for (4), Let γ, γ' be as in the statement. If $N_{[\gamma]} \cap N_{[\gamma']} \neq \emptyset$ then by definition we can find paths β, β' in N such that $[\gamma * \beta] = [\gamma' * \beta']$ which implies in particular that $\beta(0) = \gamma(1) = \gamma'(1) = \beta'(0)$ and $\beta(1) = \beta'(1)$. Since β, β' are paths in the definably contractible N we know that $\beta \sim \beta'$ where $\sim$ stands for definably homotopic.

So

$$\gamma \sim (\gamma * \beta) * \beta^{-1} \sim (\gamma' * \beta') * \beta'^{-1} \sim \gamma'$$

and γ and γ' are definably homotopic. $\square$

Exercise 4.11. *Prove that $(\widetilde{G}, \cdot)$ is a topological group, that $\pi_1(G, e)$ is a discrete subgroup, and that the projection from $\widetilde{G}$ to G is a covering.*

Exercise 4.12. *Prove that $\widetilde{G}$ is definably connected. This is, there are no locally definable subgroups H_1, H_2 of $\widetilde{G}$ which are open and disjoint and such that $H_1 \cup H_2 = \widetilde{G}$.*

4.4. Central extensions

Definition 4.27. Let G, H, A be groups. We say that G is a *central extension of H by A* if there is a short exact sequence

$$1 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 1$$

such that $i(A) \subset Z(G)$.

When we say that G be a central extension of H by A, we will implicitely identify A with its image $i(A) = \ker \pi$.

Definition 4.28. Let $\pi \colon G \to H$ be a surjective homomorphism of groups. A map $s \colon H \to G$ is called a *section of* π if $\pi(s(x)) = x$ for each $x \in H$.

Definition 4.29. Let G be a central extension of H by A. Suppose $s \colon H \to G$ is a section of $\pi \colon G \to H$. Let $f \colon H \times H \to A$ be the function defined by

$$f(x, y) = s(x)s(y)s(xy)^{-1}.$$

We call f *the 2-cocycle associated to s.*

Proposition 4.30. *Let G be a central extension of H by A. Suppose $s \colon H \to G$ is a section of $\pi \colon G \to H$ such that $s(1) = 1$, and $f \colon H \times H \to A$ is the 2-cocycle associated to s. Set $G' = H \times A$ and define on G' the operation*

$$(x, a) * (y, b) = (xy, \; f(x, y) + a + b).$$

Then $(G', *)$ is a group isomorphic to G through the map $(x, a) \mapsto s(x)a$.

Proof. To see that $*$ is associative, we need to check that for each $x, y, z \in H$

$$f(x, y) + f(xy, z) = f(x, yz) + f(y, z),$$

which can be verified by writing $s(x)s(y)s(z)$ in two different ways, using associativity of the group operation in G.

For every $x \in H$, $f(x, 1) = s(x)s(x)^{-1} = f(1, x) = 1_G$. Remembering that we write A in additive notation, so $1_G = 0_A$, we get that for each $(x, a) \in G'$,

$$(x, a) * (1, 0) = (x, f(x, 1) + a) = (x, a) = (1, 0) * (x, a),$$

so $(1, 0)$ is an identity for $*$ and

$$(x, a)^{-1} = (x^{-1},\ -f(x, x^{-1}) - a).$$

Therefore $(G', *)$ is a group. Define $\Phi\colon G' \to G$, $\Phi(x, a) = s(x)a$. Hence

$$\Phi((x, a)*(y, b)) = \Phi(xy,\ f(x, y)+a+b) = s(xy)f(x, y)ab = s(x)s(y)ab = \Phi(x, a)\Phi(y, b)$$

where the last equality follows from the fact that A is central in G. So Φ is a group homomophism. Note that for any $g \in G$ there are unique $a \in A$ and $x \in H$ such that $g = s(x)a$. That is, $x = \pi(g)$ and $a = s(\pi(g))^{-1}g$. Thus Φ is an isomorphism, as required. □

Example 4.31. The group $G = \mathrm{SL}_n(\mathbb{R})$ is centerless when n is odd and has finite center $Z(G) = \{\pm I\}$ when n is even. The quotient $\mathrm{SL}_n(\mathbb{R})/Z(G)$ is denoted by $\mathrm{PSL}_n(\mathbb{R})$. Suppose n is even. For any section $s\colon \mathrm{PSL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$ of the canonical homomorphism $\pi\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$ the associate 2-cocycle is given by

$$f(x, y) = \begin{cases} I & \text{if } s(xy) = s(x)s(y) \\ -I & \text{otherwise.} \end{cases}$$

After identifying $Z(G)$ with $\mathbb{Z}_2 = \{0, 1\}$, $\mathrm{SL}_n(\mathbb{R})$ is isomorphic to $\mathrm{PSL}_n(\mathbb{R}) \times \mathbb{Z}_2$ with the group operation given in Proposition 4.30.

For the rest of this section, we work in the setting where we have a central extension H of some definable group G. Namely, we have an exact sequence

$$0 \to A \to H \xrightarrow{\pi} G \to 0$$

with A central in H.

We will use Remark 4.23 repeatedly. So throughout this section, let $\widetilde{G}$ be the definable universal cover of G, let $\pi_1(G, e) := \bigcup_n F_n$ be the fundamental group of G and let $h : G \times G \to \pi_1(G, e)$ be the definable 2-cocycle in Remark 4.23.

4.4.1. Finite Extensions. Here we will study finite coverings, so $H \xrightarrow{p} G$ with $\ker(p)$ finite. There are two contexts we will deal with, one assuming H is definable, and the other one assuming that $\mathcal{R} = \mathbb{R}$ and H is a Lie group. In both cases we will prove that H is isomorphic (in the correct category) to a group which is defined using only the structure of G and $\ker(p)$ and the definable 2-cocycle h from Remark 4.23. Both cases will be quite similar, but the definable one has more details to cover, so we will prove the definable case and make comments about how the proof changes in the the real Lie case.

So assume that $H \xrightarrow{p} G$ be a definable covering map from a connected definable group H to a connected definable group G with $\ker(p)$ finite. By Corollary 4.24 $\ker(p)$ is isomorphic to $\pi_1(G, e)/p^{\pi}(\pi_1(H, e))$.

By Corollary 4.24 (or the analogue result for Lie groups) the pushforward (composition by p) sending maps in H starting at the identity to maps in G starting at the identity induces an isomorphism from $\widetilde{H}$ to $\widetilde{G}$. The restriction of this map to

$\pi_1(H,e)$ is by definition the map p^π which was proved to be an embedding from $\pi_1(H,e)$ to $\pi_1(G,e)$ in Proposition 4.18. So we have a diagram

$$\begin{tikzcd} \widetilde{G} \arrow[d, two heads, "\pi_H"'] \arrow[dr, two heads, "\pi_G"] & \\ H \arrow[r, two heads, "p"] & G. \end{tikzcd}$$

We can definably embed G into H by taking $a \mapsto \pi_H([\gamma_{e,a}])$ where $\gamma_{e,a}$ is the map defined in the previous section. The 2-cocycle associated with this embedding is the definable map

$$h_H(a,b) := (\pi_H \circ h)(a,b) = \pi_H\left([\gamma_{e,a} \cdot \gamma_{e,b} \cdot \gamma_{e,(ab)}^{-1}]\right).$$

This map goes from $G \times G$ to $\pi_H(\pi_1(G,e)) = \pi_1(H,e_H)$.

We define a group H' with universe $G \times \ker(p)$ and the group operation is given by

$$(a,t) \odot_{H'} (b,s) = \left((a \cdot_G b), (t \cdot_H s \cdot_H h^p(a,b))\right).$$

It is standard (and easy to check using that π_H is a group homomorphism) that the map $(a,s) \mapsto \pi_H([\gamma_{e,a}]) \cdot_H s$ is a group isomomorphism.

Now, since h is definable and has finite image, the restriction of the locally definable map π_H to the image of h is definable, so h_H is definable.

These construction implies the following.

Theorem 4.32. *Let $\mathcal{N}$ be an o-minimal expansion of $\mathcal{R}$, and let H be an $\mathcal{N}$-definable finite cover of an $\mathcal{R}$-definable group G. Then there is a $\mathcal{R}$-definable H' which is $\mathcal{N}$-definably isomorphic to G*

Proof. In the construction above, $\ker(p)$ is finite (so we can find an $\mathcal{R}$-definable copy) and h_H is the composition of the $\mathcal{R}$-definable map h and a map from the image of h which is a finite subset of an $\mathcal{R}$-locally definable set to the $\mathcal{R}$-definable copy of $\ker(p)$, so it is also definable. So the construction will prove that H is $\mathcal{N}$-definably isomorphic to the definable $\mathcal{R}$, as required. □

Theorem 4.33. *Assume $\mathcal{R} = \mathbb{R}$ and let H be a a Lie group which is a (Lie)-finite cover of an $\mathcal{R}$-definable group G. Then there is a definable H' which is Lie isomorphic to G.*

Proof. For this we need the fact that $\widetilde{G}$ is a Lie group and π_H and π_G are Lie homomorphisms[1]. The definition of the topology in $\widetilde{G}$ implies that for in the simplex C containing the identity the map $a \mapsto [\gamma_{e,a}]$ is a diffeomorphism.

This implies that restricting to $C \times \ker(p)$, the isomorphism between H' and H defined by $(a,s) \mapsto \pi_H([\gamma_{e,a}]) \cdot_H s$ is continuous. This implies that the isomorphism between H' and H is continuous, so it is a Lie isomorphism. □

4.4.2. Definable groups containing $\widetilde{G}$. We assume in this subsection that $\mathcal{R} = \mathbb{R}$. Let G be a definable group and $\widetilde{G}$ be its definable/topological universal cover.

Suppose that we have an homomorphism j from $\pi_1(G,e)$ to a definable group A, and let $\Gamma := \{l, j(l) \mid l \in \pi_1(G,e)\}$. Let $\widetilde{G}_A := A \times_\Gamma \widetilde{G}$ be the direct product

[1]This is a well known fact in differential geometry, but can be reproved using the same methods we have done here. In particular, we define the topology in $\widetilde{G}$ as in Subsection 4.3.2 and the proof that this gives $\widetilde{G}$ a Lie group structure is done analogously to the definable case in Chapter 3

of $A \times \widetilde{G}$ quotiented by the subgroup Γ. Taking the composition of the projection from $\widetilde{G}_A$ to $\widetilde{G}$ and the natural projection from $\widetilde{G}$ to G we obtain an exact sequence

$$0 \to A \to \widetilde{G}_A \to G \to 0$$

with A central in $\widetilde{G}_A$.

The group $\widetilde{G}_A$ is therefore Lie isomorphic to a group whose underlying set is $A \times G$ and whose group operation is determined by a 2-cocycle

$$h_A(a,b) := (j \circ h)(a,b) = j\left(\left[\gamma_{e,a} \cdot \gamma_{e,b} \cdot \gamma_{e,(ab)}^{-1}\right]\right).$$

Now, h has finite image in Γ and the preimage of every point is a definable subset of $G \times G$, so h_A is definable.

This proves the following:

Theorem 4.34. *Let j be an embedding from $\pi_1(G,e)$ to a definable group A, let*

$$\Gamma := \{(l, j(l)) \mid l \in \pi_1(G,e)\},$$

and let $\widetilde{G}_A := A \times_\Gamma \widetilde{G}$ be the direct product of $A \times \widetilde{G}$ quotiented by the subgroup Γ.

Then there is a definable group Lie isomorphic to $\widetilde{G}_A$.

Example 4.35. The group $\widetilde{SL_2(\mathcal{R})}$ is not definable. However, $\pi_1(SL_2(\mathcal{R}))$ is $\mathbb{Z}$, and we can take j to be the natural embedding from $\mathbb{Z}$ to the additive group $\mathcal{R}$. If we then define

$$\mathcal{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathcal{R})}$$

we get a definable group, containing $\pi_1(SL_2(\mathcal{R}))$. If $\mathcal{R} = \mathbb{R}$ this is a definable Lie group whose Levi subgroup is the non definable $\widetilde{SL_2(\mathcal{R})}$, which has been an inspiring example in the development of definable groups in $\mathbb{R}$.

Historical notes and comments

The definition of definable paths and homotopy already appeared in [**72**], but has since been developed by several authors. Uniqueness of liftings of both paths and homotopy to covers were studied in [**26**].

The definition of fundamental group for definable groups appeared in [**3**] assuming (as we do) that the o-minimal structure expands a field. The definable universal cover appeared in [**24**] although our definition is that of [**25**]. The barycentric subdivision and the 1-skeleton of a graph for definable sets comes from [**72**]. Our computations of these constructions (i.e. th proofs of local definability) as well as several of the proofs are definable versions of the classical case which appear for example in [**28**] and in [**23**].

Definable central extensions and most of the results in Section 4.4 appear in [**39**]. The example in Remark 4.35 appeared in [**20**].

CHAPTER 5

Semisimple groups and the Levi decomposition

A Lie group G is defined to be *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple. That is, $\{0\}$ is the only abelian ideal of $\mathfrak{g}$. Equivalently, the trivial subgroup is the only abelian connected normal subgroup of G. For definable groups, the corresponding definition is the following:

Definition 5.1. Let G be a definable group. We say that G is *semisimple* if G contains no infinite abelian definably connected normal definable subgroup.

Lemma 2.26 implies that the definable hull of any abelian normal subgroup is still normal and abelian. Therefore, the following holds:

Remark 5.2. A definable group G is semisimple if and only if G contains no infinite abelian normal subgroup (definable or not). In particular, for definable Lie groups the two notions coincide.

Moreover, note that if G is a definable group or a Lie group, then G is semisimple if and only if G^0 is semisimple.

Exercise 5.1. *Show that a definable group G is semisimple if and only if G contains no infinite normal solvable definable subgroup.*

5.1. The solvable radical

Definition 5.3. Let $\mathfrak{g}$ be a Lie algebra.

(1) $\mathfrak{g}$ is called *simple* if $\mathfrak{g}$ is nonabelian and $\mathfrak{g}$ has no ideals except itself and $\{0\}$.
(2) $\mathfrak{g}$ is called *semisimple* if its unique abelian ideal is $\{0\}$.
(3) Define recursively

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j].$$

The decreasing sequence

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots$$

is called the *commutator series* for $\mathfrak{g}$. We say that $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^j = \{0\}$ for some j.

For the following see, for instance, Proposition 1.12 and Proposition 1.14 in [**42**].

Fact 5.4. *In every finite-dimensional Lie algebra $\mathfrak{g}$ there is a unique solvable ideal $\mathfrak{r}$ of $\mathfrak{g}$ containing all solvable ideals in $\mathfrak{g}$. We say that $\mathfrak{r}$ is **the radical** of $\mathfrak{g}$ and denote it by* $\operatorname{rad}\mathfrak{g}$. *The quotient algebra $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is semisimple.*

Exercise 5.2. *Show that $\mathfrak{g}$ is semisimple if and only if* $\operatorname{rad}\mathfrak{g} = \{0\}$.

Given a Lie group G with Lie algebra $\mathfrak{g}$, the (Lie) subgroup of G corresponding to the Lie subalgebra $\operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ given by Theorem 3.25 is the unique maximal normal closed connected solvable subgroup $R(G)$ of G. The subgroup $R(G)$ is called the *solvable radical* of G and as the following proposition states, analogous to the Lie case, the definable radical of a definable group, namely a maximal *definably connected* normal solvable *definable* subgroup can always be found.

Remark 5.5. Even in the Lie context where the exponential function is always defined, it is not true in general that $R(G) = \exp(\operatorname{rad} \mathfrak{g})$.

The problem is that the exponential map is not surjective on solvable subgroups. In fact, determining surjectivity of the Lie exponential for connected groups was achieved by Dixmier in [**22**]: For solvable groups the exponential is surjective on compact groups but for simply connected groups it is equivalent to the group being *completely solvable*, a concept which will play a very important role in Chapter 7.

Proposition 5.6. *Let G be a definably connected group. Then G has a unique normal solvable definably connected subgroup R such that G/R is trivial or semisimple. The subgroup R is the maximal normal solvable definably connected subgroup of G.*

Proof. By induction on $n = \dim G$. If G is semisimple, then $R = \{e\}$ by Exercise 5.1. If G is not semisimple, let $A < G$ be an infinite normal definable subgroup. If G/A is semisimple, take $R = A^0$. Otherwise, by induction there is a normal solvable definably connected subgroup $S < G/A$ such that $(G/A)/S$ is semisimple. Then take R to be the definably connected component of the identity of the pre-image of S in G.

If S is another normal solvable definably connected subgroup of G, then RS/R is a normal solvable definably connected subgroup of the semisimple group G/R. Therefore $RS = R$ and $S \subset R$. If $S \neq R$, then $\dim S < \dim R$ and R/S is an infinite solvable definable subgroup of G/S. Therefore G/S is not semisimple and R is unique. □

Definition 5.7. The unique maximal normal solvable definably connected subgroup of G from Proposition 5.6 is called the *solvable radical* of G.

Although the maximal solvable ideal of the Lie algebra $\mathfrak{g}$ of G is called the radical of $\mathfrak{g}$, we specify that the subgroup R is the *solvable* radical because there are other important radicals in G, like the nilpotent radical or the amenable radical.

Example 5.8. Let $G = \mathrm{GL}_n^+(\mathbb{R})$ be the semialgebraic group of real matrices with positive determinant and D the central subgroup of diagonal matrices in G. The quotient G/D is samialgebraically isomorphic to the semisimple $\mathrm{SL}_2(\mathbb{R})$, so D is the solvable radical of G both as a Lie group and as a semialgebraic group. Indeed, this is always the case:

Remark 5.9. If G is a definable Lie group, then the solvable radical of G as a Lie group coincides with the solvable radical of G as a definable group.

Proof. Let R be the solvable radical of G as a definable group. By Proposition 5.6, G/R is a semisimple definable group. Hence G/R is a semisimple Lie group (see Remark 5.2), and R is the solvable radical of G as a Lie group too. □

5.2. Semisimple groups

In this section (comprising several subsections) definable or Lie semisimple groups are treated simultaneously. This can be done because the two classes of groups share several key properties. However, not every semisimple Lie group can be found definably in an o-minimal structure, and we will characterize the semisimple Lie groups that are Lie isomorphic to a group definable in an o-minimal expansion of the real field. For simplicity, we give a shorter name to this property.

Definition 5.10. Let $\mathcal{R} = (\mathbb{R}, <, +, \cdot)$ and G be a Lie group. We say that G *has an o-minimal copy* when G is Lie isomorphic to a group definable in an o-minimal expansion of $\mathcal{R}$. We say that G *has a semialgebraic copy* when G is Lie isomorphic to a group definable in $\mathcal{R}$.

Semisimple groups are very strongly related to semialgebraic groups in several ways. For instance, we will show that whenever a semisimple Lie group G has an o-minimal copy, then G has a semialgebraic copy (Theorem 5.26). More generally, in any o-minimal real closed field any semisimple definable group is definably isomorphic to a semialgebraic group (Corollary 5.22).

In the linear case, remarkably, every connected semisimple Lie group is semialgebraic (Corollary 5.31), and so is every homomorphism (Proposition 5.33).

5.2.1. The Lie algebra of semisimple groups. Definition 5.1 of a semisimple group was introduced for groups definable in an arbitrary o-minimal structure, where the notion of Lie algebra of a definable group is not always available. However, in our context of o-minimal expansion of fields, we can see that the definition given in terms of the Lie algebra gives rise to the same class of groups.

Since it is central to this chapter, we recall some definitions from Chapter 3.

For a definable group G the Lie algebra $L(G) = \mathfrak{g}$ is defined as the tangent space at the identity T_eG equipped with the Lie bracket operation, and the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is defined to be

$$\mathrm{Ad}(g) = \mathrm{d}_e(\Psi_g),$$

where $\Psi_g : G \to G$ is the conjugation map $\Psi_g(h) = ghg^{-1}$, and d_e denotes the differential at the identity.

We defined the Lie bracket $[\xi, \eta] := \mathrm{d}_e(Ad)(\xi, \eta]$. So if we define the *adjoint representation of the Lie algebra* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is

$$\mathrm{ad}(\xi)(\eta) = [\xi, \eta]$$

for all $\xi, \eta \in \mathfrak{g}$, we have

$$\mathrm{ad} = \mathrm{d}_e(\mathrm{Ad}),$$

meaning ad is the derivative at the identity of the adjoint representation of G.

Fix a definably connected group G with Lie algebra $\mathfrak{g}$.

Theorem 3.27(2) states that if V is a subspace of $\mathfrak{g}$ and

$$H_V := \{g \in G : \mathrm{Ad}(g)(v) = v \text{ for all } v \in V\},$$

then H_V is a definable subgroup of G and its Lie algebra is

$$\{\xi \in \mathfrak{g} : [\xi, v] = 0 \text{ for all } v \in V\}.$$

Using the group centralizer instead of the algebra centralizer we get

Lemma 5.11. *Let H be a definable subgroup of G and $\mathfrak{h}$ its Lie algebra. Then the Lie algebra of $C_G(H)$, the centralizer of H in G, is*

$$\{\xi \in \mathfrak{g} : [\xi, \mathfrak{h}] = 0\}.$$

Proof. Since $\mathrm{Ad}(g)$ fixes every element of $\mathfrak{h}$ precisely when $g \in C_G(H)$, we have $C_G(H) = H_V$ for $V = \mathfrak{h}$ and Theorem 3.27 applies. □

Proposition 5.12. *Let G be a definably connected group. Then G is semisimple if and only if its Lie algebra $\mathfrak{g}$ is semisimple.*

Proof. Suppose first $\mathfrak{g}$ is semisimple. For any $A \le G$ definably connected abelian normal subgroup, the corresponding Lie algebra $\mathfrak{a}$ is a nonzero abelian ideal of $\mathfrak{g}$. It follows that G has no non trivial definably connected abelian normal subgroup, so it is semisimple.

Conversely, suppose G is semisimple and, by way of contradiction, suppose $\mathfrak{g}$ has a nonzero abelian ideal $\mathfrak{j}$. Set

$$N = \{g \in G : \mathrm{Ad}(g) \restriction_{\mathfrak{j}} = \mathrm{id}_{\mathfrak{j}}\}.$$

That is, $N = H_V$ above, where $V = \mathfrak{j}$. Moreover, we will now prove that N is normal. Indeed, since $\mathfrak{j}$ is an ideal of $\mathfrak{g}$, we have $[\xi, \mathfrak{j}] \subseteq \mathfrak{j}$ for every $\xi \in \mathfrak{g}$. Consider the definable subgroup

$$K = \{g \in G : \mathrm{Ad}(g)(\mathfrak{j}) \subseteq \mathfrak{j}\}.$$

Its Lie algebra is $\{\xi \in \mathfrak{g} : [\xi, \mathfrak{j}] \subseteq \mathfrak{j}\} = \mathfrak{g}$, since $\mathfrak{j}$ is an ideal. As G is definably connected and $\dim K = \dim G$, $K = G$. In particular $\mathrm{Ad}(g)(\mathfrak{j}) = \mathfrak{j}$ for all $g \in G$, so the map

$$\varphi\colon G \longrightarrow \mathrm{GL}(\mathfrak{j}), \qquad g \longmapsto \mathrm{Ad}(g)\restriction_{\mathfrak{j}},$$

is a well-defined definable group homomorphism with $N = \ker \varphi$. As the kernel of a homomorphism, N is normal in G.

As recalled above, the Lie algebra of N is

$$\mathfrak{n} = \{\xi \in \mathfrak{g} : [\xi, \mathfrak{j}] = 0\}.$$

Since $\mathfrak{j}$ is abelian, $\mathfrak{j} \subseteq \mathfrak{n}$, so N is infinite. Consider $C_N(N) = Z(N)$. By Lemma 5.11, its Lie algebra is

$$\{\xi \in \mathfrak{n} : [\xi, \mathfrak{n}] = 0\}$$

and contains the non-trivial $\mathfrak{j}$ again. Hence $Z(N)$ is an infinite. Since $Z(N)$ is characteristic in the normal N, it is normal as well. That is, $Z(N)$ is an infinite abelian normal (definable) subgroup of G, contradiction. It follows that $\mathfrak{g}$ cannot have a nonzero abelian ideal, and $\mathfrak{g}$ is semisimple, as claimed. $\square$

Semisimple Lie algebras over any field of characteristic 0 are characterized by the following (unique) decomposition into simple ideals. See, for instance, Theorem 5.2 in [**40**].

Fact 5.13. *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic* 0*. Then $\mathfrak{g}$ is semisimple if and only if*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$$

with $\mathfrak{g}_j$ ideals that are each simple Lie algebras. In this case the decomposition is unique, and the only ideals of $\mathfrak{g}$ are the sums of various $\mathfrak{g}_j$.

5.2.2. Centerless semisimple groups. In Chapter 3 we noticed that the kernel of the adjoint representation is the center of the group. It follows that centerless (definable/Lie) groups are precisely the groups (definable/Lie) isomorphic to the image of the adjoint representation. In this subsection we will see that for semisimple groups this is particularly significant, because it will give us an isomorphism with a semialgebraic group.

Recall that for any Lie algebra $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$ denotes the group of invertible linear maps on $\mathfrak{g}$ that preserve the Lie bracket. Notice that if $\mathfrak{g}$ is a finite-dimensional Lie algebra over a real closed field, then both $\mathfrak{g}$ and $\mathrm{Aut}(\mathfrak{g})$ are semialgebraic.

Lemma 5.14. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a real closed field $\mathcal{R}$. Then*

$$\dim \mathfrak{g} = \dim \mathrm{Aut}(\mathfrak{g}).$$

Proof. If $\mathcal{R} = \mathbb{R}$, this is a classical result following from the facts that for any Lie algebra $\mathfrak{g}$, the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is the space of derivation of $\mathfrak{g}$ and, when $\mathfrak{g}$ is semisimple, every derivation of $\mathfrak{g}$ is inner. See, for instance, Corollary 5.3 in [**40**].

For an arbitrary real closed field, fix $n \in \mathbb{N}$. One can express with a first order sentence that if $\mathfrak{g}$ is a semisimple n-dimensional Lie algebra, then $\dim \mathrm{Aut}(\mathfrak{g}) = n$. This sentence is true in the real field and therefore it is true in $\mathcal{R}$ as well. $\square$

EXERCISE 5.3. *Use the fact that the Lie bracket can be checked at the level of a basis to prove that* $\mathrm{Aut}(\mathfrak{g})$ *(the group of Lie-algebra automorphisms) is a semialgebraic subgroup of* $GL(\mathfrak{g})$.

Proposition 5.15. *Let* G *be a definable group or a Lie group with Lie algebra* $\mathfrak{g}$. *If* G *is (definably) connected, semisimple and centerless, then the adjoint representation* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ *is a (definable/Lie) isomorphism between* G *and the semialgebraic linear group* $(\mathrm{Aut}(\mathfrak{g}))^0$.

PROOF. If G is a definable group, let $\mathcal{R}$ be the universe of the o-minimal expansion of a real closed field G is defined in. If G is a Lie group, let $\mathcal{R} = \mathbb{R}$.

If $\dim \mathfrak{g} = n$, after fixing a basis for $\mathfrak{g}$, we can identify $\mathfrak{g}$ with $\mathcal{R}^n$ and $\mathrm{Aut}(\mathfrak{g})$ with a semialgebraic subgroup A of $\mathrm{GL}_n(\mathcal{R})$.

Recall that for both a definable group or a Lie group, the adjoint representation is a continuous homomorphism $\mathrm{Ad}\colon G \to A$ such that $\ker \mathrm{Ad} = Z(G)$. Moreover, when G is definable, Ad is definable too.

Since our group G is centerless, G is (definably/Lie) isomorphic to $\mathrm{Ad}(G)$. Set $H = \mathrm{Ad}(G)$. Then

$$\dim H = \dim G = \dim \mathfrak{g} = \dim A,$$

where the last equality follows from Lemma 5.14. Since G is (definably) connected, H is (definably) connected too and it must be $H = A^0$, as required. □

Proposition 5.16. *Let* G *be a definably connected centerless semisimple definable group. Then* $(G, \cdot)$ *is elementarily equivalent to a connected semisimple semialgebraic linear Lie group* $(G^s, \odot^s)$, *meaning that every first order sentence which is true in* $(G, \cdot)$ *is true in* $(G^s, \odot^s)$.

PROOF. Let $\mathfrak{g}$ be the Lie algebra of G. By Proposition 5.15 we can assume $\mathfrak{g} \subset \mathfrak{gl}_n(\mathcal{R})$, for some $n \in \mathbb{N}$.

By the classification of simple Lie algebras over $\mathbb{R}$ and the decomposition from Fact 5.13, there are only finitely many semisimple subalgebras of $\mathfrak{gl}_n(\mathbb{R})$, up to isomorphism. So there is some natural number s and a sentence σ in the language of fields, which is true in $(\mathbb{R}, +, \cdot)$, expressing that there are semisimple Lie subalgebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ of $\mathfrak{gl}_n(\mathbb{R})$ such that any semisimple subalgebras of $\mathfrak{gl}_n(\mathbb{R})$ is isomorphic to one of the $\mathfrak{g}_i$. The sentence σ is true in $(\mathbb{R}_{alg}, +, \cdot)$ the field of real algebraic numbers, which is the prime model of the theory of real closed fields. Therefore $\mathfrak{g}$ is isomorphic to some Lie subalgebra L of $\mathfrak{gl}_n(\mathcal{R})$ that is defined over $\mathbb{R}_{alg}$. In other words, there is a basis of L with real algebraic entries. Moreover, by Proposition 5.15 G is definably isomorphic to $\mathrm{Aut}(L)^0$, which is a semialgebraic linear group defined over $\mathbb{R}_{alg}$ by a formula φ. Let $H = \varphi(\mathbb{R})$. Then $(H, \cdot)$ is a semisimple Lie group that is elementarily equivalent to the connected semisimple linear semialgebraic $(\mathrm{Aut}(L)^0, \cdot)$, and to $(G, \cdot)$, as required. □

We can use Proposition 5.16 to transfer first order group-theoretic properties of centerless semisimple Lie groups to definable groups. An example is the following that was proved by Borel (see [**7**]).

Fact 5.17. *Every connected semisimple Lie group is perfect. Moreover, if* G *is a connected semisimple Lie group, then* $G = [G, G]_s$ *for some finite* $s \in \mathbb{N}$ *if and only if* $Z(G)$ *is finite.*

The following is Grün's Lemma (see [**32**]):

Fact 5.18. *Let* G *be a group such that* $G = [G, G]$. *Then* $G/Z(G)$ *is centerless.*

Corollary 5.19. *Let* G *be a connected semisimple Lie group. Then* $G/Z(G)$ *is centerless.*

For definable groups, we will first show that $G/Z(G)$ is centerless, and then deduce from it that G is perfect.

Lemma 5.20. *Let G be a definably connected group. If $Z(G)$ is finite, then $G/Z(G)$ is centerless.*

Proof. Set $\bar{G} = G/Z(G)$, let $\pi : G \to \bar{G}$ be the canonical projection map and $H = \pi^{-1}(Z(\bar{G}))$ be the preimage of the center of $\bar{G}$. We need to show that $H = Z(G)$. Let $h \in H$. For all $g \in G$:

$$\pi(h)\pi(g) = \pi(g)\pi(h),$$

so

$$\pi(hgh^{-1}g^{-1}) = e_{\bar{G}}.$$

In other words, the commutator $[h, g] = hgh^{-1}g^{-1}$ lies in the kernel of π, which is $Z(G)$. Consider the commutator map defined by h:

$$\phi_h : G \to G, \quad \phi_h(g) = [h, g] = hgh^{-1}g^{-1}.$$

Since G is definably connected and the map ϕ_h is definable and continuous, the image $\phi_h(G)$ must be a definably connected subset of the finite $Z(G)$, hence a point. Since $\phi_h(e) = e$, it must be $\phi_h(G) = \{e\}$. That is, $h \in Z(G)$, as we wanted. □

Proposition 5.21. *Every definably connected semisimple group G is perfect and has finite commutator width. That is, $G = [G, G]_s$ for some $s \in \mathbb{N}$.*

Proof. Let G be a definably connected semisimple group group. If G is centerless, then Proposition 5.16 implies that G is elementarily equivalent to a semialgebraic (centerless) semisimple Lie group which is perfect with finite commutator width by Fact 5.17. These are elementary properties, so G would be a perfect group with finite commutator width.

Suppose now that $Z(G)$ is not trivial. The quotient $G/Z(G)$ is centerless by Lemma 5.20, so again $G/Z(G)$ is perfect. Since $Z(G)$ is finite, $[G, G]$ has finite index in G. This will be the key property to prove that G is perfect:

By definition

$$[G, G] = \bigcup_{i \in \mathbb{N}} [G, G]_i,$$

$[G, G]_i$ being the product of i commutators from G so that $[G, G]$ is $\bigvee$-definable.

On the other hand, $[G, G]$ has finite index so that the complement of $[G, G]$ in G is $\bigvee$-definable too:

If

$$G = \bigcup_{k=0}^{n} g_k[G, G]$$

where $g_0 = e$, then

$$G \setminus [G, G] = \bigcup_{k=1}^{n} g_k[G, G],$$

$$[G, G] = \bigcap_{k=1}^{n} (G \setminus g_k\,[G, G]) = \bigcap_{k=1}^{n} \left(\bigcap_{i \in \mathbb{N}} G \setminus g_k\,[G, G]_i \right)$$

and

$$\bigcup_{i \in \mathbb{N}} [G, G]_i = \bigcap_{k=1}^{n} \left(\bigcap_{i \in \mathbb{N}} G \setminus g_k\,[G, G]_i \right)$$

Finite width now follows either by compactness of first order logic, or using saturated models:

Enumerating the sets $\{G \setminus g_k [G,G]_i\}_{i,k}$ as $\{B_j\}_{j\in\mathbb{N}}$ so that $[G,G] = \bigcap_{j\in\mathbb{N}} B_j$, then $[G,G] \neq [G,G]_n$ for each $n \in \mathbb{N}$ implies that $\{x \in B_j \setminus [G,G]_i : i,j \in \mathbb{N}\}$ is a consistent type that is therefore realized in a saturated elementary extension by an element

$$x \in \bigcap_{j\in\mathbb{N}} B_j = [G,G] \qquad x \notin \bigcup_{i\in\mathbb{N}} [G,G]_i$$

contradicting $\bigcup_{i\in\mathbb{N}} [G,G]_i = \bigcap_{j\in\mathbb{N}} B_j$.

So $[G,G] = [G,G]_n$ for some $n \in \mathbb{N}$, which implies that $[G,G]$ is definable. By connectedness, $G = [G,G]$. □

Corollary 5.22. *Every definably connected semisimple definable group is definably isomorphic to a semialgebraic group.*

Proof. If G is definably connected semisimple definable group, $G/Z(G)$ is centerless (Corollary 5.19), so definably isomorphic to $\mathrm{Aut}(\mathfrak{g})^0$ by Proposition 5.15. Namely, G is a central extension of a linear semialgebraic group by a finite group so in particular a finite cover of a semialgebraic group. Theorem 4.32 states that in this case G is definably isomorphic to a semialgebraic group. □

Remark 5.23. While semialgebraicity is preserved for central extensions by finite groups, linearity is not. An example is the double cover of $\mathrm{SL}_2(\mathbb{R})$, that is semialgebraic but not linear. More generally, no proper central extension of $\mathrm{SL}_2(\mathbb{R})$ is linear. Linear groups will be treated in Section 5.2.4.

Connected semisimple Lie group can have an infinite discrete center, so an isomorphism with a semialgebraic group like in Corollary 5.22 is in general out of question. The best we can get is the following.

Corollary 5.24. *Every connected semisimple Lie group is a central extension of a semialgebraic linear group.*

Proof. By Corollary 5.19 and Proposition 5.15. □

5.2.3. Semisimple Lie groups with an o-minimal copy.

In this subsection we characterize the semisimple Lie groups that are Lie isomorphic to a group definable in an o-minimal expansion of the real field (see Definition 5.10).

Theorem 5.25. *Let H be a semialgebraic semisimple Lie group and $p\colon G \to H$ be a cover of Lie groups. There is a section $s\colon H \to G$ of p such that $s(1) = 1$, the 2-cocycle f associated to s has finite image F and the map $f\colon H \times H \to F$ is semialgebraic.*

Proof. Suppose first $G = \widetilde{H}$ and $p\colon G \to H$ is the universal covering map. Then $\pi_1(H) = \ker p \subseteq Z(G)$ is a finitely generated group, so countable. The group G can be realized as a countable directed union of semialgebraic sets

$$G = \bigcup_{n\in\mathbb{N}} X_n$$

and for each i,j there is some k such that the image of the restriction of the multiplication map on $X_i \times X_j$ is contained in X_k and it is a semialgebraic map.

By compactness, there is some $i \in \mathbb{N}$ such that the restriction of p to X_i maps onto H. By definable choice, there is a semialgebraic map $s\colon H \to X_i$ that is a section for p. We can choose $s(1) = 1$ because 1 is contained in every X_n. Set $Y = s(H)$ and let $j \in \mathbb{N}$ such that X_j contains $Y^{-1} \cdot Y \cdot Y$. Hence the 2-cocycle

$f\colon H \times H \to X_j$ associated to s is a semialgebraic map. Moreover, because the restriction of p to X_j is semialgebraic, its kernel must be finite, since $\ker p$ is a discrete group. It follows that the image of f is a finite set, and we are done.

Suppose now $p\colon G \to H$ is an arbitrary cover of H. Then G and H have same universal cover $\widetilde{H}$ and if $\pi\colon \widetilde{H} \to G$ is the universal covering map, then the composition $\pi \circ p$ is the universal covering map of H.

Let $f\colon H \times H \to F \subseteq \ker(\pi \circ p)$ be the 2-cocycle associated to the section $s\colon H \to \widetilde{H}$ defined above. Proposition 4.30 states that $\widetilde{H}$ is isomorphic to $H \times \ker \pi$ with the operation given by

$$(x, a) * (y, b) = (xy, f(x, y) + a + b),$$

and the universal covering map is given by $(x, a) \mapsto x$. Let $\pi\colon \widetilde{H} \to G$ be the universal covering map of G. Since $G = \pi(\widetilde{H})$, G is isomorphic to $H \times (\ker(\pi \circ p)/\ker \pi)$ with operation given by

$$(x, a) * (y, b) = (xy, \pi(f(x, y)) + a + b).$$

In this notation, the map $p\colon G \to H$ is $(x, a) \mapsto x$. Therefore, if $s\colon H \to \widetilde{H}$ is the section we found above, define $s'\colon H \to G$ as $s'(x) = \pi(s(x))$. Then $s'(1) = \pi(1) = 1$, because π is a homomorphism. Moreover, the 2-cocycle associated to s' coincides with the composition $\pi(f(x, y))$ for $(x, y) \in H \times H$, it has finite image $\pi(F)$ and it is semialgebraic, as required. $\square$

Theorem 5.26. *Let G be a connected semisimple Lie group. The following are equivalent:*

(1) G has an o-minimal copy.
(2) G has a semialgebraic copy.
(3) G has finite center.

Proof. (3) $\Rightarrow$ (2) : Set $H = G/Z(G)$. By Corollary 5.19 and Proposition 5.15, we can assume H is semialgebraic and centerless. Let $f\colon H \times H \to Z(G)$ be a semialgebraic 2-cocycle from Theorem 5.25. Set $G' = H \times Z(G)$ and let $(G', *)$ be the semialgebraic group isomorphic to G from Proposition 4.30. Because G' is semialgebraic, G' is a Lie group. On the other hand, the Lie structure on a group is unique, so the topology induced by the isomorphism $\Phi^{-1} : G \to G'$ from Proposition 4.30 agrees with the semialgebraic topology. That is, G' as a semialgebraic group is Lie isomorphic to G.

Since the real field in the ring language is o-minimal, (2) $\Rightarrow$ (1). Finally, since the center of a semisimple group is always discrete, (1) $\Rightarrow$ (3) follows from o-minimality. $\square$

5.2.4. Linear semisimple groups. The following fact was proved by Chevalley in [**11**].

Fact 5.27. *Let K be a field of characteristic 0. Every semisimple Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{gl}_n(K)$ is an algebraic Lie algebra, which by definition is the Lie algebra of an algebraic (i.e. Zariski closed) group. In other words, there exists a unique Zariski-connected algebraic subgroup $H \subseteq GL_n(K)$ such that $L(H) = \mathfrak{s}$. The group H is the Zariski closure of the group generated by all one-parameter subgroups $\exp(t\xi)$ for $\xi \in \mathfrak{s}$.*

Proposition 5.28. *Let $G < \mathrm{GL}_n(\mathcal{R})$ be a definable subgroup, and let $\mathfrak{g} = L(G)$ be its Lie algebra. If $\mathfrak{s} \subseteq \mathfrak{g}$ is a semisimple Lie subalgebra, there exists a* unique *definably connected definable subgroup $S < G$ such that $L(S) = \mathfrak{s}$. Moreover, S is semialgebraic.*

Proof. By Fact 5.27, there exists a unique Zariski-connected algebraic subgroup $H \subseteq \mathrm{GL}_n(\mathcal{R})$, defined by polynomial equations over $\mathcal{R}$, such that $L(H) = \mathfrak{s}$. In particular, H is semialgebraic. Set $S = H^0$. Clearly S is semialgebraic, definably connected, and $L(S) = \mathfrak{s}$.

Uniqueness follows from the fact that two definably connected subgroups with same corresponding Lie subalgebra must coincide.

Finally, we need to check that $S \subseteq G$. Consider $K := S \cap G$. For any $\xi \in \mathfrak{s} \subseteq \mathfrak{g}$, the one-parameter subgroup $\gamma(t) = \exp(t\xi)$ satisfies $\gamma'(0) = \xi$ and lies locally in both S and G. Hence $L(K) = \mathfrak{s} = L(S)$. Since S is definably connected, it follows that $K = S$ and $S \subseteq G$. □

The following is Corollary 7.9 in [**6**] (and an easy consequence).

Fact 5.29. *Let S be a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$, and let $\bar{S}$ be the Zariski closure in $\mathrm{GL}_n(\mathbb{R})$ of S. Then the Lie algebra of $[S, S]$ is the same as the Lie algebra of $[\bar{S}, \bar{S}]$, which is the Lie algebra of an algebraic subgroup.*

In particular, proposition 5.21 implies that if $S < \mathrm{GL}_n(\mathbb{R})$ is a closed connected semisimple subgroup, then S is the connected component of its Zariski closure in $\mathrm{GL}_n(\mathbb{R})$.

The following is the main result by Goto in [**31**]

Fact 5.30. *Any faithfully representable Lie group is isomorphic to a closed subgroup of a general linear group.*

Corollary 5.31. *Any connected semisimple linear Lie group is semialgebraic.*

Remark 5.32. The linearity assumption in Corollary 5.31 is important. There are connected semisimple Lie groups with infinite center, hence not definable in any o-minimal structure, such as the universal cover $\widetilde{\mathrm{SL}_2}(\mathbb{R})$, which has center isomorphic to $\mathbb{Z}$. However, $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ is not linear, and neither is any non trivial cover of $\mathrm{SL}_2(\mathbb{R})$.

Proposition 5.33. *Every homomorphism between linear connected semisimple Lie groups is semialgebraic.*

Proof. Let $\varphi\colon G \to H$ be a homomorphism between connected semisimple linear Lie groups G and H. By Corollary 5.31, G and H are semialgebraic. Cartan-van der Waerden-Freudenthal automatic continuity theorem (see [**9**], [**74**] or [**27**]) states that every abstract homomorphism between semisimple Lie groups is continuous, so that φ is continuous. Said continuity implies that the graph

$$\Gamma_\varphi = \{(g, \varphi(g)) : g \in G\}$$

is a closed Lie subgroup of $G \times H$ isomorphic to G. In particular, Γ_φ is a linear connected semisimple group, so it is semialgebraic. Consider the projection maps:

$$\pi_1 : G \times H \to G, \quad (g, h) \mapsto g,$$

$$\pi_2 : G \times H \to H, \quad (g, h) \mapsto h.$$

Both π_1 and π_2 are semialgebraic, and the restriction $\pi_1|_{\Gamma_\varphi} : \Gamma_\varphi \to G$ is a semialgebraic isomorphism. Therefore its inverse $\iota : G \to \Gamma_\varphi$ given by $\iota(g) = (g, \varphi(g))$ is semialgebraic too.

Finally, the homomorphism $\varphi : G \to H$ is the composition of semialgebraic maps

$$\varphi = \pi_2 \circ \iota : G \xrightarrow{\iota} \Gamma_\varphi \xrightarrow{\pi_2|_{\Gamma_\varphi}} H$$

from which we conclude that φ is semialgebraic, as claimed. □

5.2.5. Simple groups and a decomposition theorem. The literature does not agree on the definition of simple Lie group. For some, a Lie group G is simple when its Lie algebra is simple. This definition allows G to have an infinite discrete center, as for the universal cover of $\mathrm{SL}_2(\mathbb{R})$, and to be disconnected. For others, a simple Lie group is a connected Lie group that contains no normal connected closed subgroups, and this definition includes $\mathbb{R}$. Yet another option is the following, that we adopt:

Definition 5.34. We say that a Lie group G is *simple* when G is infinite, non-abelian and simple as an abstract group.

The natural corresponding definable notion is the following:

Definition 5.35. We say that a definable group G is *definably simple* when G is infinite, non-abelian and its only normal definable subgroups are $\{e\}$ and G.

Examples 5.36. $\mathrm{SO}_3(\mathbb{R})$ is a compact simple Lie group and $\mathrm{PSL}_2(\mathbb{R})$ is a non-compact one. If we replace $\mathbb{R}$ with any real closed field $\mathcal{R}$, we get the definably compact definably simple $\mathrm{SO}_3(\mathcal{R})$ and the non-definably compact definably simple $\mathrm{PSL}_2(\mathcal{R})$.

The definability of the adjoint maps will now allow us to prove that centerless definable semisimple groups can be decomposed into a direct product of finitely many definably simple groups. Note that the same proof applies to Lie groups as well, since every connected semisimple centerless group has a semialgebraic copy by Proposition 5.15.

Theorem 5.37. *Let G be a definably connected semisimple centerless definable group. Then*
$$G = G_1 \times \cdots \times G_m$$
where each G_i is definably simple.

Proof. Let $\mathfrak{g} = Lie(G)$. Since G is semisimple, $\mathfrak{g}$ is semisimple (Proposition 5.12), so by Fact 5.13 we have
$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m,$$
where each $\mathfrak{g}_j$ is a simple ideal of $\mathfrak{g}$, and this decomposition is unique. For each index i, set
$$\mathfrak{s}_i \;=\; \bigoplus_{j\neq i} \mathfrak{g}_j,$$
which is an ideal of $\mathfrak{g}$ (being a sum of ideals), and $\mathfrak{g} = \mathfrak{g}_i \oplus \mathfrak{s}_i$.

For each i, define
$$N_i \;=\; \{g \in G : \mathrm{Ad}(g)\,\xi = \xi \;\text{ for all }\; \xi \in \mathfrak{s}_i\},$$
so that $N_i = H_{\mathfrak{s}_i}$ in the notation of Theorem 3.27, applied with $V = \mathfrak{s}_i \subseteq \mathfrak{g}$. With this notation Theorem 3.27 states that N_i is a definable subgroup of G with
$$Lie(N_i) \;=\; C_{\mathfrak{g}}(\mathfrak{s}_i) \;=\; \{\xi \in \mathfrak{g} : [\xi,\, \mathfrak{s}_i] = 0\}.$$
We claim $C_{\mathfrak{g}}(\mathfrak{s}_i) = \mathfrak{g}_i$. Write any $\xi \in \mathfrak{g}$ as $\xi = \sum_k \xi_k$ with $\xi_k \in \mathfrak{g}_k$. If $[\xi, \eta] = 0$ for every $\eta \in \mathfrak{s}_i = \bigoplus_{j\neq i} \mathfrak{g}_j$, then in particular for each $k \neq i$ we have $[\xi_k, \eta] = 0$ for all $\eta \in \mathfrak{g}_k$ (taking $j = k$), so $\xi_k \in Z(\mathfrak{g}_k) = 0$ (since $\mathfrak{g}_k$ is simple, hence has trivial center). Thus $\xi = \xi_i \in \mathfrak{g}_i$, giving $C_{\mathfrak{g}}(\mathfrak{s}_i) \subseteq \mathfrak{g}_i$. The reverse inclusion $\mathfrak{g}_i \subseteq C_{\mathfrak{g}}(\mathfrak{s}_i)$ holds because distinct simple ideals commute: $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $i \neq j$. Hence
$$Lie(N_i) = \mathfrak{g}_i.$$
Set $G_i = N_i^0$, the definably connected component of the identity in N_i. Then G_i is a definably connected definable subgroup of G with $Lie(G_i) = \mathfrak{g}_i$.

$\mathfrak{s}_i$ is an ideal of $\mathfrak{g}$ so $[\xi, \mathfrak{s}_i] \subseteq \mathfrak{s}_i$ for every $\xi \in \mathfrak{g}$. Define

$$K_i = \{g \in G : \mathrm{Ad}(g)(\mathfrak{s}_i) \subseteq \mathfrak{s}_i\}.$$

This is a definable subgroup of G, and its Lie algebra is $\{\xi \in \mathfrak{g} : [\xi, \mathfrak{s}_i] \subseteq \mathfrak{s}_i\} \supseteq \mathfrak{g}$. Hence $Lie(K_i) = \mathfrak{g}$ and $\dim K_i = \dim G$. The group G is definably connected so it can't have any proper definable subgroups of the same dimension which implies $K_i = G$, and thus $\mathrm{Ad}(g)(\mathfrak{s}_i) = \mathfrak{s}_i$ for all $g \in G$.

For any $g \in G$ and $h \in N_i$, and any $\xi \in \mathfrak{s}_i$:

$$\mathrm{Ad}(ghg^{-1})\,\xi = \mathrm{Ad}(g)\big(\mathrm{Ad}(h)\big(\mathrm{Ad}(g^{-1})\,\xi\big)\big).$$

Since $\mathrm{Ad}(g^{-1})\,\xi \in \mathfrak{s}_i$ (because $\mathrm{Ad}(g^{-1})$ preserves $\mathfrak{s}_i$), and $h \in N_i$ acts as the identity on $\mathfrak{s}_i$, we get $\mathrm{Ad}(ghg^{-1})\,\xi = \mathrm{Ad}(g)\ \mathrm{Ad}(g^{-1})\,\xi = \xi$. Thus $gN_ig^{-1} \subseteq N_i$, N_i is normal in G which implies that so is its definably connected component $G_i = N_i^0$.

Claim 5.38. *The subgroups G_i centralise each other pairwise.*

Proof: For $i \neq j$ we have $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$, so by Lemma 5.11 $\mathfrak{g}_j \subseteq C_{\mathfrak{g}}(\mathfrak{g}_i) = Lie(C_G(G_i))$.
$Lie(G_j) = \mathfrak{g}_j$, so $Lie(G_j) \subseteq Lie(C_G(G_i))$ and G_j is definably connected so Theorem 3.15 implies $G_j \subseteq C_G(G_i)$. $\blacksquare_{Claim}$

G_i pairwise centralising each other implies that the map

$$\varphi \colon G_1 \times \cdots \times G_m \longrightarrow G, \qquad (g_1, \ldots, g_m) \longmapsto g_1 \cdots g_m,$$

is a definable group homomorphism and it is easy to check that is an isomorphism.

Finally, to see that each G_i is definably simple, note that its Lie algebra is simple, so G_i contains no infinite normal definable subgroup. G_i is definably connected, so every finite normal subgroup (see Exercise 5.4) is central. Any central subgroup in G_i is central in G too (the G_is commute pairwise). Since G is centerless, each G_i must be centerless too. □

Exercise 5.4. *Prove that any finite normal subgroup of a definably connected group is central.*

So any arbitrary definable semisimple group G can be decomposed by Theorem 5.37 as a product of finitely many normal definable subgroups corresponding to the ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ from Fact 5.13:

Corollary 5.39. *Every definably connected semisimple definable group is an almost direct product of almost simple definable groups. That is, for every definably connected semisimple group G there are normal definable subgroups $G_1, \ldots, G_m$ with finite pairwise intersection such that*

$$G = G_1 \cdots G_m$$

and $G_i/Z(G_i)$ is definably simple for each $i = 1, \ldots, m$.

Proof. Since $G/Z(G)$ is centerless by Lemma 5.20, Theorem 5.37 applies, and we can take as G_i the pre-images in G of the definably simple direct factors of $G/Z(G)$. Note that $G_i \cap G_j \subseteq Z(G)$ for each $i \neq j$. □

5.2.6. A definable KAN decomposition. Every 2×2 real matrix with determinant equal to 1 can be written as a product of a rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, a dilation matrix $\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$, and a shearing matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. This decomposition originated from the following Iwasawa decomposition of semisimple complex Lie algebras that is Proposition 6.43 and Lemma 6.45 in [**42**].

Fact 5.40. *For every semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ there exists a basis $\{X_i\}$ of $\mathfrak{g}$ and subspaces $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ such that $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and the matrices representing $\mathrm{ad}(\mathfrak{g})$ with respect to $\{X_i\}$ have the following properties:*

- *the matrices of $\mathrm{ad}(\mathfrak{k})$ are skew-symmetric,*
- *the matrices of $\mathrm{ad}(\mathfrak{a})$ are diagonal with real entries,*
- *the matrices of $\mathrm{ad}(\mathfrak{n})$ are upper triangular with 0's on the diagonal.*

Let G be a centerless semisimple connected Lie group Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ as above. If K, A and N are connected analytic subgroups of G with Lie algebras $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ respectively, then:

(a) *the multiplication map*

$$\begin{array}{ccc} K \times A \times N & \longrightarrow & G \\ (\,k\ ,\ a\ ,\ n\,) & \mapsto & kan \end{array}$$

is a surjective diffeomorphism;

(b) *$AN = NA$, i.e. AN is a subgroup.*

Let $m \in \mathbb{N}$. We will use the following notations for subgroups of $GL_m(\mathcal{R})$:

- $O_m(\mathcal{R}) = \{[x_{ij}] \in GL_m(\mathcal{R}) : [x_{ij}][x_{ji}] = I\}$ is the orthogonal group,
- $T_m^+(\mathcal{R}) = \{[x_{ij}] \in GL_m(\mathcal{R}) : x_{ij} = 0\ \forall\, i < j$ and $x_{ll} > 0\ \forall\, l\}$ is the group of upper triangular matrices with positive elements on the diagonal,
- $UT_m(\mathcal{R}) = \{[x_{ij}] \in GL_m(\mathcal{R}) : x_{ij} = 0\ \forall\, i < j$ and $x_{ll} = 1\ \forall\, l\}$ is the group of unipotent upper triangular matrices,
- $D_m^+(\mathcal{R}) = \{[x_{ij}] \in GL_m(\mathcal{R}) : x_{ij} = 0\ \forall\, i \neq j,\ x_{ll} > 0\ \forall\, l\}$ is the group of diagonal matrices with positive elements on the diagonal.

Theorem 5.41. *Let G be a definably connected centerless semisimple group. There is some $m \in \mathbb{N}$ such that G is definably isomorphic to a semialgebraic linear group $G_1 < \mathrm{GL}_m(\mathcal{R})$ with the following properties:*

- *$G_1 = KH$, with $K = G_1 \cap O_m(\mathcal{R})$ and $H = G_1 \cap T_m^+(\mathcal{R})$,*
- *$H = AN$, with $A = G_1 \cap D_m^+(\mathcal{R})$ and $N = G_1 \cap UT_m(\mathcal{R})$.*

Proof. As noted in the proof of Proposition 5.16, G is definably isomorphic to the semialgebraic linear group $G_1 = \mathrm{Aut}(L)^0$, where L is a semisimple Lie subalgebra of $\mathfrak{gl}_n(\mathcal{R})$ with real algebraic entries. Moreover, with a first order formula we can require that L is a direct sum of subspaces $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ such that the matrices representing $\mathrm{ad}(L)$ have the following first order properties:

- the matrices of $\mathrm{ad}(\mathfrak{k})$ are skew-symmetric,
- the matrices of $\mathrm{ad}(\mathfrak{a})$ are diagonal,
- the matrices of $\mathrm{ad}(\mathfrak{n})$ are upper triangular with 0's on the diagonal.

Let $G_1(\mathbb{R})$ be the Lie group defined in $\mathbb{R}$ by the same formula over $\mathbb{R}_{alg}$ defining G_1 in $\mathcal{R}$. It is a centerless semisimple Lie group equal to $\mathrm{Aut}_{\mathbb{R}}(L)^0$. If K, A, N are connected subgroups of $G_1(\mathbb{R})$ corresponding to the Lie subalgebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$, then by Fact 5.40 $AN = NA$ and $G_1(\mathbb{R}) = KAN$. Moreover,

- the matrices of $\mathrm{ad}(\mathfrak{k})$ are skew-symmetric $\Rightarrow\ K \subseteq O_m(\mathbb{R})$,
- the matrices of $\mathrm{ad}(\mathfrak{a})$ are diagonal $\Rightarrow\ A \subseteq D_m^+(\mathbb{R})$,
- the matrices of $\mathrm{ad}(\mathfrak{n})$ are upper triangular with 0's on the diagonal $\Rightarrow$ $N \subseteq UT_m(\mathbb{R})$.

Since $D_m^+(\mathbb{R}) \cap UT_m(\mathbb{R}) = \{I\} = O_m(\mathbb{R}) \cap T_m^+(\mathbb{R})$, we get that

- $K = G_1(\mathbb{R}) \cap O_m(\mathbb{R})$,
- $A = G_1(\mathbb{R}) \cap D^+_m(\mathbb{R})$,
- $N = G_1(\mathbb{R}) \cap UT_m(\mathbb{R})$.

Thus the first order formula in the language of ordered fields which says that every element $g \in G_1$ can be written in a unique way as a product $g = kan$, with $k \in G_1 \cap O_m$, $a \in G_1 \cap D^+_m$, $n \in G_1 \cap UT_m$ and $an = na\ \forall\, a \in G_1 \cap D^+_m, \forall\, n \in G_1 \cap UT_m$ is true in $\mathbb{R}$ and therefore by transfer it is true in $\mathcal{R}$ as well, providing the claimed decomposition. □

We will see in Chapter 7 how this definable KAN decomposition can be generalized to any definably connected group with a definably compact solvable radical (Proposition 7.38 and Corollary 7.39).

5.3. Reductive groups

Definition 5.42. A Lie algebra $\mathfrak{g}$ is called *reductive* when $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

Definition 5.43. A connected Lie group G is called *reductive* when its Lie algebra is reductive. Equivalently, the solvable radical of G coincide with the connected component of the center of G. That is, *G is a central extension of a semisimple group.*

Examples 5.44. The following are examples of reductive groups.

- $\mathrm{GL}^+_n(\mathbb{R}) = \{A \in \mathrm{GL}^+_n(\mathbb{R}) : \det A > 0\}$ is a central extension of the semisimple group $\mathrm{SL}_n(\mathbb{R})$, so it is reductive for each $n > 1$. More precisely, $\mathrm{GL}^+_n(\mathbb{R}) = D \times \mathrm{SL}_n(\mathbb{R})$, where D is the central subgroup of diagonal matrices with positive determinant.
- Every connected compact Lie group and every definably connected definably compact group is reductive.

The following is a decomposition of abstract groups that, since semisimple groups are perfect, can be applied to (definable or not) reductive groups.

Proposition 5.45. *If G is a central extension of a perfect group, then*

$$G = Z(G)[G, G].$$

Moreover, $[G, G]$ is perfect and it is the only perfect subgroup P of G such that $G = ZP$, where Z is any central subgroup of G such that G/Z is perfect.

Proof. Suppose G is a group such that $G/Z(G)$ is perfect. Fix Z a subgroup contained in $Z(G)$ such that G/Z is perfect and set $\bar{G} = G/Z$, $S = [G, G]$, $N = ZS$ and $\bar{N}$ the image of N in $\bar{G}$ under the natural homomorphism $G \to \bar{G}$.

Because Z and S are normal subgroups of G, so is their product N. The quotient G/N is abelian, as it is a quotient of $G/[G, G]$, and we have

$$G/N = (G/Z)/(N/Z) = \bar{G}/\bar{N}.$$

However, $\bar{G}$ is perfect, so it must be $\bar{G} = \bar{N}$ and $G = ZS$. It follows that $[G, G] = [S, S] = S$, and S is perfect. By taking $Z = Z(G)$ we obtained the claimed decomposition.

Suppose P is *any* subgroup such that $G = ZP$. Let $Y \subset P$ be the image of a section $\bar{G} \to G$ of the canonical projection $G \to \bar{G}$. Then for any $a, b \in G$ there are $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ such that $a = z_1 y_1$ and $b = z_2 y_2$. Hence $[a, b] = [y_1, y_2] \in P$ and P contains the set of commutators of G. Since S is the subgroup generated by the set of commutators of G, it follows that S is the *unique* smallest subgroup of G on which the canonical homomorphism $G \to G/Z$ is surjective.

Suppose now P is any perfect subgroup of G. It is easy to see that $P \subseteq S$. Indeed, the group $(PS)/S$ is abelian, as it is a subgroup of $G/[G,G]$. Since $(PS)/S = P/(P \cap S)$, it follows that $P \cap S = P$, because the quotient of any perfect group is still perfect. Therefore, $S = [G,G]$ is the only perfect subgroup P of G such that $G = ZP$, as claimed. $\square$

Corollary 5.46. *Let G be a definably connected group. If G is reductive, then its commutator subgroup $[G,G]$ is perfect and it is the only perfect subgroup completing its solvable radical $Z(G)^0$.*

The decomposition of Proposition 5.45 is a particular case of the Levi decomposition of Lie groups and definable groups we will discuss in the next section.

5.3.1. Reductive Lie groups with an o-minimal copy. The following is a well known theorem (Theorem 2.12 in Chapter I of [**29**].

Fact 5.47. *Every abelian Lie group G is Lie isomorphic to*

$$D \times \mathbb{R}^n \times \mathrm{SO}_2(\mathbb{R})^k$$

for some discrete (countable) abelian group D and $n, k \in \mathbb{N}$. In particular, G^0 is linear and a direct product of 1-dimensional connected closed subgroups.

Because every connected abelian Lie group has a semialgebraic copy by Fact 5.47, the description of the semisimple Lie groups with an o-minimal copy from Theorem 5.26 can be generalized to all reductive Lie groups as follows:

Theorem 5.48. *Let G be a connected reductive Lie group. The following are equivalent:*

1. *G has an o-minimal copy.*
2. *G has a semialgebraic copy.*
3. *The center of $G/Z(G)^0$ is finite.*
4. *$Z(G)$ has finitely many connected components.*

Proof. (3) $\Rightarrow$ (2) : For ease of notation, set $Z = Z(G)^0$ and $S = [G,G]$, the commutator subgroup of G. Because G is reductive, then S is semisimple and $G = ZS$ (Definition 5.42). Therefore the semisimple Lie group G/Z can be identified with $S/(Z \cap S)$. Note that $Z \cap S \subseteq Z(S)$, so the natural homomorphism $\pi \colon S \to S/(Z \cap S)$ is a cover of semisimple Lie groups.

By assumption, G/Z has finite center, so $S/(Z \cap S)$ has a semialgebraic copy by Theorem 5.26. Let f be the semialgebraic cocycle associated to a section $s \colon S/(Z \cap S) \to S$ of π and set $H = G/Z = S/(Z \cap S)$. Note that the finite image of f is contained in $Z \cap S \subseteq Z$.

After identifying Z with its semialgebraic copy from Fact 5.47, by Proposition 4.30 G is isomorphic to the semialgebraic group $H \times Z$ with group operation given by

$$(x, a) * (y, b) = (xy,\ f(x, y)ab).$$

Implications (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (4) follow from the fact that real closed fields are o-minimal, and that in an o-minimal theory any definable set has finitely many connected components.

(4) $\Rightarrow$ (3) : Assume $Z(G)/Z(G)^0$ is finite. Suppose, by way of contradiction, $G/Z(G)^0$ has an infinite center Z. Therefore, Z maps to an infinite central subgroup of $G/Z(G)$. However, because $G/Z(G)$ is a linear semisimple connected Lie group, it is semialgebraic by Corollary 5.31, so it must have finite center. Contradiction. $\square$

Corollary 5.49 (Chevalley's Theorem). *Every compact Lie group has a semialgebraic copy.*

Proof. As noted in Examples 5.44, every connected compact Lie group is reductive. Every compact Lie group has finitely many connected components, so semialgebraicity follows. □

Example 5.50. Let $a \in \mathrm{SO}_2(\mathbb{R})$ be a rotation through an angle incommensurable with π, and let $b \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ be a generator of the center of the universal cover of $\mathrm{SL}_2(\mathbb{R})$, which is an infinite cyclic group. Set $H = \mathrm{SO}_2(\mathbb{R}) \times \widetilde{\mathrm{SL}}_2(\mathbb{R})$ and Γ to be the central subgroup generated by (a, b) in H. Define $G = H/\Gamma$. As G is a connected central extension of the semisimple $\mathrm{SL}_2(\mathbb{R})$, G is a reductive group. Neither H nor Γ are definable in any o-minimal expansion of the real field, as both define infinite cyclic groups. However, the center of G has two connected components, so we know by Theorem 5.48 that G has a semialgebraic copy.

Indeed, given $\pi\colon \widetilde{\mathrm{SL}}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ the universal covering map, let

$$f\colon\ \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to F$$

be a semialgebraic 2-cocycle from Theorem 5.25, where F is a finite subset of $\mathbb{Z} \cong \langle b \rangle = Z(\widetilde{\mathrm{SL}}_2(\mathbb{R}))$. Set $G' = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R})$ with group operation given by

$$(x, u) * (y, v) = (xy, f(x, y)a + u + v)$$

where we use additive notation $+$ for the group operation in $\mathrm{SO}_2(\mathbb{R})$. Then $(G', *)$ is a semialgebraic copy of G.

Note that the parameter a used in the definition of the group operation can be eliminated by choosing a to be real algebraic.

5.4. Levi subgroups and the Levi decomposition

When studying Lie groups one is often able to reduce to the solvable and semisimple cases thanks to the Levi decomposition (see Fact 5.53). In this section we will first recall the classical results for Lie algebras and Lie groups, and then we will show corresponding results for definable groups.

5.4.1. Levi-Malcev Theorem and classical case. The following appears in Section 5.6 in [**33**].

Fact 5.51 (Levi-Malcev Theorem)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic zero. Then there exists a semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$ such that*

$$\mathfrak{g} = \mathfrak{s} \oplus \mathrm{rad}(\mathfrak{g}).$$

Equivalently, the canonical projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ admits a Lie algebra homomorphism section $\sigma : \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \to \mathfrak{g}$ such that $\pi \circ \sigma = \mathrm{id}$.

The semisimple subalgebra $\mathfrak{s}$ is called a Levi factor *or* Levi subalgebra *of $\mathfrak{g}$.*

Any two Levi factors of $\mathfrak{g}$ are conjugate by an automorphism of $\mathfrak{g}$, i.e., if $\mathfrak{s}$ and $\mathfrak{s}'$ are two Levi factors, then there exists $\phi \in Aut(\mathfrak{g})$ such that $\phi(\mathfrak{s}) = \mathfrak{s}'$.

Remark 5.52. When $\mathfrak{g}$ is reductive, $[\mathfrak{g}, \mathfrak{g}]$ is its unique Levi factor.

The following appears in Chapter II, Section 4 in [**29**].

Fact 5.53 (Levi Decomposition)**.** *Let G be a connected Lie group. Then:*

(i) There exists a maximal connected semisimple Lie subgroup $S \subseteq G$ such that

$$G = RS,$$

where R is the solvable radical of G. Moreover, $R \cap S$ is discrete. The subgroup S is called a Levi subgroup *of G.*

(ii) Any two Levi subgroups of G are conjugate in G, i.e., if S and S' are Levi subgroups, then there exists $g \in G$ such that $S' = gSg^{-1}$.

5.4.2. Levi decomposition of definable groups. Levi subgroups of connected Lie groups are not always closed. This can happen also for semialgebraic Lie groups like the one in Example 5.50. In particular, it is not always possible to find definable Levi subgroups in definable groups. However, it is always possible to find Levi subgroups that are a directed countable union of definable sets and central extensions of a definable semisimple group, generalizing Example 5.50 where the Levi subgroup is $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. In several particular cases it is however possible to find definable Levi subgroups. For instance, this is always the case for linear groups.

Definition 5.54. We say that a definably connected group G has a *definable Levi decomposition* if there is some semisimple definable subgroup S such that $G = RS$, where R is the solvable radical of G. When this is the case, we say that S is a *definable Levi subgroup* of G.

Remark 5.55. Semisimple definable groups cannot have infinite normal solvable definable subgroups, so if S is a definable Levi subgroup then $R \cap S$ is finite and $\dim S = \dim(G/R)$.

Remark 5.56. Let G be a definably connected group. If G is an extension of a definable group with a definable Levi decomposition by a finite group, then G has a definable Levi decomposition.

Proof. Suppose N is a finite normal subgroup of G and $G/N = \bar{R}\bar{S}$ is a definable Levi decomposition of G/N. Then a definable Levi decomposition of G is $G = RS$, where the solvable radical R is the definably connected component of the pre-image of $\bar{R}$ in G and S is the definably connected component of the pre-image of $\bar{S}$ in G. □

Proposition 5.57. *Every linear definably connected group has a definable Levi decomposition. Moreover, every definable Levi subgroup is semialgebraic.*

In particular, for any definably connected group G, the quotients $G/Z(G)$ and $G/Z(G)^0$ have definable Levi decompositions.

Proof. Let $G < \mathrm{GL}_n(\mathcal{R})$ be a definably connected subgroup with solvable radical R. Fix $\mathfrak{s} \subseteq \mathfrak{gl}_n(\mathcal{R})$ a Levi factor of its Lie algebra. By Proposition 5.28, there is a unique definably connected semialgebraic subgroup $S < G$ such that $L(S) = \mathfrak{s}$. By Fact 5.51 and dimension reasons, it must be $G = RS$, and S is a definable Levi subgroup. Again by Fact 5.51 and Proposition 5.28, any other definable Levi subgroup must be semialgebraic too.

The definably connected linear group $G/Z(G)$ has a definable Levi decomposition by the previous result. And since the definably connected $G/Z(G)^0$ is an extension of the linear $G/Z(G)$ by the finite $Z(G)/Z(G)^0$ Remark 5.56 implies $G/Z(G)^0$ has a definable Levi decomposition. □

For the definable groups that do not have a definable Levi decomposition (like Example 5.50), we will need ind-definable ($\vee$-definable, locally definable) groups, for which the notion of connectedness is encompassed by the following:

Definition 5.58. Let G be an ind-definable group. We say that G is *locally definably connected* if G has no proper subgroup H such that for each definable subset Y of G, $Y \cap H$ is definable and Y meets only finitely many cosets of H in G.

Definition 5.59. Let G be a an ind-definable group. We say that G is *semisimple* if

(i) G is locally definably connected;
(ii) G is perfect;

(iii) for any definable subset X of G, $X \cap Z(G)$ is finite (that is, $Z(G)$ is *discrete*);
(iv) $G/Z(G)$ is a centerless semisimple definable group.

Remark 5.60. Notice the following.

(1) Any definably connected semisimple definable group is ind-definable semisimple by Proposition 5.21 and Corollary 5.19.
(2) Any connected semisimple Lie group G is ind-definable semisimple (with the respect to the real field). This follows from the fact that $Z(G)$ is discrete and, as we saw in Corollary 5.31, $G/Z(G)$ is semialgebraic. Moreover, G is perfect (Fact 5.17) and $G/Z(G)$ is centerless (Corollary 5.19)
(3) If G is a reductive definably connected group, then $[G, G]$ is ind-definable semisimple.

Proposition 5.61. *Let G be a definably connected group and $\mathfrak{s}$ a semisimple Lie subalgebra of $L(G)$. Then there is a unique ind-definable semisimple subgroup S of G such that $\mathfrak{s} = L(S)$.*

Proof. By induction on $n = \dim G$. If $n = 1$, G is abelian and so is its Lie algebra, so there is nothing to prove. Assume $n > 1$.

Set $\overline{G} = G/Z(G)$. Then $\mathfrak{s}$ is a semisimple Lie subalgebra of $L(\overline{G})$. By Fact 5.27, there is a unique definably connected subgroup $\overline{S}$ of $\overline{G}$ such that $L(\overline{S}) = \mathfrak{s}$. By Proposition 5.12, $\overline{S}$ is semisimple. Let H be the definably connected component of the pre-image of $\overline{S}$ in G.

If $\dim H = \dim G$, by connectedness $G = H$, $\mathfrak{s} = L(\overline{G})$ and G is a central extension of the semisimple $\overline{S}$. By Proposition 5.45, $[G, G]$ is the unique perfect subgroup completing $Z(G)$, hence it is the unique perfect subgroup P of G such that $L(P) = \mathfrak{s}$. It is easy to check that $[G, G]$ is ind-definable semisimple.

If $\dim H < \dim G$, by induction hypothesis there is a unique ind-definable semisimple subgroup S of H such that $\mathfrak{s} = L(S)$. Since every subgroup of G whose Lie algebra is $\mathfrak{s}$ maps onto $\overline{S}$, we are done. □

The Levi decomposition of connected Lie groups Fact 5.53 for definable groups becomes the following.

Theorem 5.62. *Let G be a definably connected group. Then:*

(i) There exists a maximal ind-definable semisimple subgroup $S \subseteq G$ such that

$$G = RS,$$

where R is the solvable radical of G. The subgroup S is called an ind-definable Levi subgroup *of G.*

(ii) Any two ind-definable Levi subgroups of G are conjugate in G, i.e., if S and S' are Levi subgroups, then there exists $g \in G$ such that $S' = gSg^{-1}$.

Assuming the definition of Levi subgroup from Theorem 5.62, we will prove the Levi decomposition for definable groups using the following equivalent characterization of Levi subgroups. The proof will go through the following characterization of Levi subgroups.

Proposition 5.63. *Let G be a definably connected group and S an ind-definable subgroup of G. Then S is a ind-definable Levi subgroup if and only if S is a smallest perfect subgroup that maps surjectively onto G/R through the canonical homomorphism $G \to G/R$, where R is the solvable radical of G.*

Proof. Let $\mathfrak{g} = L(G)$ be the Lie algebra of G. Suppose S is an ind-definable Levi subgroup. To see that S is a minimal perfect complement of the solvable radical, suppose $P \subseteq S$ is a subgroup such that the canonical homomorphism $P \to$

G/R is surjective. Since S is a central extension of G/R and P maps surjectively onto G/R, it follows that P is a normal subgroup of S and the quotient S/P is abelian, as it is isomorphic to a quotient of $Z(S)$. However, S is perfect, so $S = P$, as claimed.

Conversely, suppose S is an ind-definable Levi subgroup of G and let $\mathfrak{s}$ be its Lie algebra. By the Levi decomposition of Lie algebras Fact 5.51, $\mathfrak{s}$ contains a maximal semisimple Lie subalgebra $\mathfrak{s}_1$, that is a maximal semisimple Lie subalgebra of $\mathfrak{g}$ too, because $G = RS$. Let S_1 be the unique ind-definable semisimple subgroup S_1 of G whose Lie algebra is $\mathfrak{s}_1$ from Proposition 5.61. Then S_1 is an ind-definable semisimple Levi subgroup of G contained in S. As showed above, S_1 is a minimal complement of the solvable radical, and it must be $S = S_1$ by minimality of S. □

Proof of Theorem 5.62. Let G be a definably connected group. Referring to Proposition 5.63, we need to show that G contains minimal perfect ind-definable complements of the solvable radical, all conjugate to each other.

If $Z(G)$ is finite, then G has a definable Levi decomposition by Proposition 5.57 and Remark 5.56, as noted in Proposition 5.57. Minimality follows from the fact that semisimple definable groups are perfect (Proposition 5.21), similarly to the first part of Proposition 5.63. All definable Levi subgroups are conjugate to each other because they are semialgebraic (Proposition 5.15), and one can reduce to the real case, where we have uniqueness by conjugation (Fact 5.53).

Assume $Z(G)^0$ is infinite. If G is reductive, then $Z(G)^0 = R$ is the solvable radical of G and $S = [G, G]$ is the unique ind-definable Levi subgroup of G by Proposition 5.21 and Proposition 5.45.

Suppose $Z(G)^0 \subsetneq R$ and set $\overline{G} = G/Z(G)^0$. Let $\overline{S}$ be a definable Levi subgroup of $\overline{G}$ (that we know exist by Proposition 5.57) and let H be the definably connected component of the pre-image of $\overline{S}$ in G. Because H is reductive, we know from the case above that $S = [H, H]$ is the unique ind-definable Levi subgroup of H. Note that because $\overline{S}$ is a complement of the solvable radical of $\overline{G}$, then H is a complement of the solvable radical of G. That is, $G = RH$ and S is an ind-definable Levi subgroup of G too. Finally, it is easy to check that since the definable Levi subgroups of $\overline{G}$ are conjugate to each other, this is also the case for the ind-definable Levi subgroups in G. □

Remark 5.64. If G is a connected definable Lie group, then its Levi subgroups (as a Lie group) coincide with its ind-definable Levi subgroups (as a definable group).

Corollary 5.65. *Let G be a definably connected definable group with center Z and S a in-definable Levi subgroup of G. Then ZS and Z^0S are definable (reductive) subgroups of G (and Z^0S is definably connected).*

Let G be a connected real Lie group with center Z and S a Levi subgroup. If G is definable then ZS and Z^0S are definable subgroups of G.

In particular, in both cases, $\langle S\rangle_{\mathrm{def}} \subseteq Z^0S$.

Proof. In both statements the hypothesis imply the groups ZS and Z^0S are pre-images in G of definable Levi subgroups in G/Z and G/Z^0, respectively. The corollary follows. □

5.5. Definability of continuous maps between definable reductive groups

In this section we will prove that definability of continuous homomorphisms between reductive definably connected groups is reduced to the restriction of the homomorphism to the center being definable.

Theorem 5.66. *Let H, G be definable groups with H definably connected and reductive. Suppose $\alpha\colon H \to G$ is a continuous homomorphism. If $\alpha(Z(H))$ is definable and the restriction of α to $Z(H)$ is a definable map, then $\alpha(H)$ is definable and α is a definable map.*

To show that the image is definable, we do not even need continuity.

Lemma 5.67. *Let H, G be definable groups and $\alpha\colon H \to G$ be a homomorphism. Suppose H is definably connected and reductive with Levi decomposition $H = Z(H)^0 S$. If $\alpha(Z(H))$ is a definable subgroup of G and the restriction of α to $Z(H)$ is definable, then the subgroup $\alpha(H)$ is definable.*

Proof. We need to show that the map α from H into $\langle\alpha(H)\rangle_{\mathrm{def}}$ is surjective. By way of contradiction, let G be a definable group of smallest possible dimension for which there is a non-surjective homomorphism $\alpha\colon H \to G$ such that $G = \langle\alpha(H)\rangle_{\mathrm{def}}$ and H is definably connected and reductive.

Suppose $A = \alpha(Z(H))$ is infinite and note that $A \subseteq Z(\alpha(H))$ because α is a homomorphism. In particular, A is a normal subgroup of $\alpha(H)$. By Lemma 2.26, A is normal in $G = \langle\alpha(H)\rangle_{\mathrm{def}}$. So G/A has smaller dimension than G, and by minimality of G, the composition of α with the canonical homomorphism onto G/A is surjective. However, this implies that α is surjective, contradiction (this is an instance of the *Surjective Four Lemma* for groups which is left as a (fun) exercise). This means that $\alpha(Z(H))$ must be finite.

Since the restriction of α to $Z(H)$ is a definable homomorphism, $Z(H)^0 \subseteq \ker\alpha$. It follows that $\alpha(H)$ is isomorphic to a quotient of the definable semisimple group $H/Z(H)^0$ that is perfect with finite commutator width by Proposition 5.21. Since every (abstract) quotient of a perfect group with finite commutator width $n \in \mathbb{N}$ is a perfect group with finite commutator width $m \leq n$, we know that $\alpha(H) = \alpha(S)$ is a perfect group with finite commutator width $m \in \mathbb{N}$. Set $K = \alpha(H) = \alpha(S)$.

Let $Ad\colon G \to \mathrm{Ad}(G)$ be the adjoint representation and note that $Ad(K)$ is a semisimple linear group, hence semialgebraic by Proposition 5.28. Because $\ker Ad = Z(G)$, it follows that $Z(G)K$ is a definable subgroup of G containing K, so $G = Z(G)K$. Therefore, $K = [K,K]_m = [G,G]_m$ is definable, as required. $\square$

Exercise 5.5. *Let $\phi : H \to G$ be a group homomorphism, let B be a normal subgroup of H, let $\phi(B) = A$ be a normal subgroup of G and assume that the induced map from H/B to G/A is surjective. Prove that ϕ is surjective.*
Hint: This is a diagram chasing in the commuting diagram

$$\begin{tikzcd}
e_H \arrow[r] & B \arrow[r] \arrow[d, "\phi"] & H \arrow[r] \arrow[d, "\phi"] & H/B \arrow[r] \arrow[d, "\bar{\phi}"] & e_G \\
1 \arrow[r] & A \arrow[r] & G \arrow[r] & G/A \arrow[r] & 1
\end{tikzcd}$$

Remark 5.68. The requirement that α maps into a *definable* group is necessary. If we had that H and G were connected Lie groups with H definable and reductive, then definability of the restriction to the center is not enough to prove that the image is definable, even if α is a smooth isomorphism.

As a counterexample, take G to be the quotient of $\mathrm{SO}_2(\mathbb{R}) \times \widetilde{\mathrm{SL}}_2(\mathbb{R})$ described in Example 5.50 and H its o-minimal copy. The natural Lie isomorphism $\alpha\colon H \to G$ is the identity on the center $\mathrm{SO}_2(\mathbb{R})$, but $G = \alpha(H)$ is not definable.

Lemma 5.69. *Let α be a continuous surjection from a definable group H onto a definable group G such that the restriction of α to Z_H is definable. Assume that H is definably connected and reductive, an extension of a semisimple group S_H by the center Z_H. Then α is definable.*

PROOF. By construction $\alpha(Z_H)$ is a subset of $Z(G)$, so α induces a well defined map σ from H/Z_H onto $G/Z(G)$ so $G/Z(G)$ it is semisimple, so G is a central extension of a semisimple group.

Let Ad_H and Ad_G be the adjoint representation maps on H and G respectively, and let $\overline{S_H}$ and $\overline{S_G}$ be the (semisimple) images of H and G under said representations.

Both $\overline{S_H}$ and $\overline{S_G}$ are semisimple and linear (they are the images of the adjoint representations of H and G respectively), so they have finite center, and Fact 5.17 implies $\overline{S_H} := [\overline{S_H}, \overline{S_H}]_k$ and $\overline{S_G} := [\overline{S_G}, \overline{S_G}]_l$ for finite k and l, so that $Ad_H([H,H]_k)$ and $Ad_G([G,G]_l)$ are surjective. Let s_G and s_H be definable sections from $\overline{S_G}$ and $\overline{S_H}$ into the definable subsets $[G,G]_l$ and $[H,H]_k$ of G and H, respectively. We get the following diagram:

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d, shift left, "Ad_H"] & G \arrow[d, shift left, "Ad_G"] \\ \overline{S_H} \arrow[r, "\sigma"] \arrow[u, shift left, bend left, "s_H"] & \overline{S_G}. \arrow[u, shift left, bend left, "s_G"] \end{tikzcd}$$

with $s_H(\overline{S_H}) \subseteq [H,H]$ and $s_G(\overline{S_G}) \subseteq [G,G]$: Since G and H are reductive, $[H,H] := L$ and $[G,G] := S$ are the unique Levi subgroups of H and G, respectively.

Linearity of the groups $\overline{S_H}$ and $\overline{S_G}$ implies (by Proposition 5.33) that the map σ is a definable continuous quotient map. So except for α, all maps in the diagram above other are definable. We will show that α can be constructed from the other maps, thus proving definability.

By cell decomposition, there are open dense $V \subseteq \overline{S_H}$ and $W \subseteq \overline{S_G}$ in which s_H and s_G are continuous. σ is a covering map (open and continuous) so $\sigma^{-1}(W)$ and $\sigma(V)$ are open dense subsets of $\overline{S_H}$ and $\overline{S_G}$, respectively. Replacing V with $V \cap \sigma^{-1}(W)$ (intersection of open dense is open dense) and W with $W \cap \sigma(V)$ we can assume that σ maps V onto W.

Consider the map ϕ from V to G given by $c \mapsto s_G(\sigma(c)) \odot_G (\alpha(s_H(c)))^{-1}$. All maps in the construction of ϕ are continuous, so ϕ is continuous.

We have the equality $Ad_G(s_G(\sigma(c))) = \sigma(c) = Ad_G(\alpha(s_H(c)))$ which implies

$$(s_G(\sigma(c))) \odot_G (\alpha(s_H(c)))^{-1} \in Ker(Ad_G) = Z(G).$$

The images of both $s_G(\sigma(c))$ and $\alpha(s_H(c))$ are contained in $\alpha(L) = S$ so the image of ϕ is contained in the set $\alpha(L) \cap Z(G)$.

Claim 5.70. *$S \cap Z(G)$ is a discrete subset of G.*

Proof: S is the V-definable Levi subgroup G so it is semisimple. In particular, since $[G,G]_n$ is a definable subgroup of S we know that $Z_G \cap [G,G]_n$ is finite. Since S is equal to the union $\bigcup_{n\in\mathbb{N}}[G,G]_n$ of sets contained in one another, this implies the claim. $\blacksquare_{Claim}$

The claim implies that $\phi(x)$ is open for every $x \in S \cap Z(G)$, so the set

$$\{\phi^{-1}(x)\}_{x \in S \cap Z(G)}$$

is an open cover of the definable V by disjoint open sets so ϕ must be constant in each of the finitely many (definable) connected components of V implying that ϕ is a definable map with finite image.

Restricted to V we have then $\alpha \circ s_H = (s_G \circ \sigma) \odot_G \phi$ so that $\alpha \circ s_H$ is a definable map from V to G and the restriction of α to $s_H(V)$ is definable. For any $w \in L_0 := S_H(V) \odot_G S_H(V)$ we have $w = x \odot_G y$ with $x, y \in s_H(V)$ and $\alpha(w) = \alpha(x) \odot_G \alpha(y)$ so that the restriction of α to L_0 is definable

By construction $Ad_H(L_0)$ contains $V \odot_G V$ which, since V is open dense, is equal to $\overline{S_H}$. It follows that $L_0 \cdot Z_H = H$ and since α is definable in L_0 and in Z_H (by hypothesis), α is definable in its domain H, as required. $\square$

Proof of Theorem 5.66. By Lemma 5.67 we can reduce to surjective maps and Lemma 5.69 applies. $\square$

Historical notes and comments

The theory of definable semisimple groups has been developed by Peterzil, Pillay and Starchenko in [**51, 52, 53**]. It has to be noted that in this chapter we omit several significant results of their work, particularly on definably simple groups and their connection with simple algebraic groups. See [**52**] and [**53**] for more details. The definable KAN decomposition is from [**15**].

Proposition 5.6 appeared in [**13**] and [**16**], although the solvable radical had been known and widely used for many years before. Theorem 5.25 and Proposition 5.21 are from [**39**].

The characterizations of semisimple (Theorem 5.26) and reductive (Theorem 5.48) Lie groups with an o-minimal copy were obtained respectively in [**47**] and [**18**].

The Levi decomposition Theorem 5.62 of definable groups was proved in [**21**] for expansions of fields, and the equivalent formulation Proposition 5.63 in [**17**] for arbitrary o-minimal structures.

CHAPTER 6

Definably compact definable subgroups

As with their Lie analogues, definably compact definable subgroups and quotients of definable groups play a very important role. This chapter is devoted to their study.

Although in many ways inspired by the theory of compact Lie groups, there is significant difference when developing the theory of definable groups stemming from the lack of analogues of closed one-parameter subgroups which we reviewed in Chapter 3. Any compact Lie group has closed one-parameter subgroups, and any closed one-parameter subgroup is a copy of $SO_2(\mathbb{R})$. This doesn't happen in a general o-minimal setting, even assuming even when $\mathbb{R} = \mathcal{R}$, since one never has access to a global Lie exponential definable map with compact image (the kernel would violate o-minimality) and even Lie exponential maps with restricted domain are not available in some o-minimal settings. Two examples to keep in mind are the following:

- In the real field without any added structure, there are definable compact abelian groups which do not split as the product of one-dimensional tori (Abelian varieties).
- Also in the real field, there are examples of definable abelian groups with compact quotients but without infinite definable compact infinite subgroups. This was exhibited by Strzebonski and the precise definition is below (see Example 6.14).

Probably that arises from the lack of the exponential function comes when proving the existence of torsion points in definably compact abelian groups. In the real Lie group setting this can be proved by proving that the exponential map onto a compact Lie groups is surjective and that the kernel is a lattice, which immediately implies the existence of torsion points (which by the way, this also implies that any compact abelian real Lie group is isomorphic to a direct product of the one dimensional tori $SO_2(\mathbb{R})$.

The structure of torsion points in a definably compact abelian group is the same as with real compact Lie groups: It was proved in [**26**] (although announced earlier by Edmundo) that the k-torsion subgroup of a n-dimensional definably compact abelian definable groups coincides with k-torsion subgroup of a n-dimensional real torus (so the k-torsion is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$). A weaker version of this will be needed throughout this section.

Fact 6.1 (Torsion Theorem). *Let K be a definably compact group. Then K has points of order n for any $n \in \mathbb{N}$.*

We will need and use this result in this chapter, but we will not include a proof. As mentioned in the introduction, our focus is on definable groups over $\mathbb{R}$ for which the Torsion Theorem holds by classic results since we already know that any definably compact group over $\mathbb{R}$ is compact. This focus has not hindered us from developing the theory in the more general context before, but, as we will explain in Section 6.5, to explain one of the known proofs of the Torsion Theorem we would need to develop o-minimality in one of several possible areas which, although we find fascinating, it is beyond the scope of this book.

The Torsion Theorem will imply that the structure of definably compact groups is very much implied by the structure of its torsion points. So a big part of this chapter will be devoted to understanding the torsion points of definable groups and the geometric implications the existence of said torsion points imply. We follow Strzebonski's approach by showing that the existence of torsion elements in an arbitrary definable group can be detected by the Euler characteristic and study the properties that this implies in definable groups. This, in turn, leads to an important analogue to maximal compact abelian groups called 0-groups, which are definable groups (such as the one described in Example 6.14) that "torsion-wise" behave like compact abelian groups do. These generalize definably compact abelian groups (this follows from the Torsion Theorem), while preserving many of the properties: 0-groups are abelian and definably connected and in fact, if the universe is $\mathbb{R}$ the 0-groups are precisely the definable hulls of compact connected abelian subgroups.

In Section 6.1, we will understand the connection between torsion and Euler characteristic. Section 6.2 studies properties of 0-groups, which are in many ways, and particularly in the sense of the structure of the torsion points, the correct generalization of definably compact abelian definable groups. Section 6.4 deals with definably compact linear groups. The adjoint representation witnesses that any definably compact definable group is a central extension of a linear definably compact group. In $\mathbb{R}$ these are definably compact and are closed and bounded (so compact) and we hall prove (Theorem 6.28) that they can be embedded into the orthogonal group $O_n(\mathbb{R})$. These results transfer to elementary extensions of o-minimal expansions of $\mathbb{R}$, but they actually hold in arbitrary o-minimal fields, and the proofs in the general case entail structure theorems over Cartesian products of $S_1(\mathcal{R})$ on $\mathcal{R}$.

Finally, we devote Section 6.5 to give an overview of some of the known proofs of the Torsion Theorem and the different paths that an interested reader would need to undertake to understand them.

6.1. Euler characteristic and torsion points: Strzebonski's approach

In this section we will provide definable analogues for Cauchy's and Lagrange's Theorems about the order of elements in finite groups. The proofs are quite analogous to the finite case, except that we use the Euler characteristic instead of the cardinality of the group. Recall from Chapter 1 that the Euler characteristic of a definable set X is the number of even-dimensional cells minus the number of odd-dimensional cells in a cell decomposition of X and the following properties hold:

Fact 6.2. *Let X, Y, Z be definable sets.*

- *If there is a definable bijection from X to Y, then $E(X) = E(Y)$.*
- *If X, Y are disjoint, then $E(X \cup Y) = E(X) + E(Y)$.*
- *If $f : Z \to Y$ is a definable surjection and $E(f^{-1}(y)) = k$ for all $y \in Y$, then $E(Z) = kE(Y)$. In particular, $E(X \times Y) = E(X)E(Y)$.*

Definition 6.3. For any group H we define $Tor_k(H) := \{h \in H \mid h^k = e\}$ for any $k \in \mathbb{N}$ and $Tor(H) := \bigcup_{k\in\mathbb{N}} Tor_k(H)$.

If G is a definable group and H is a definable subgroup, we denote by $[G : H]$ the set of left cosets of H in G. By definable choice this is isomorphic to a definable set, and we denote by $E([G : H])$ the Euler characteristic of the isomorphic definable set.

EXERCISE 6.1. *Prove that if G is any group, H, K are subgroups of G with $H \lhd G$, then $KH = HK$ is a subgroup of G, and $E(G : H) = E([G : KH])E(H : H \cap K)$.*

Remark 6.4. Given any definable subgroup H of a definable group G, by Fact 6.2 $E(G)$ is equal to the product of $E(H)$ and $E([G : H])$. Therefore, if $x \in Tor_k(G)$ and k is the smallest positive integer such that $x^k = e$, then $|\langle x \rangle| = k$ and k divides $E(G)$.

Definition 6.5. A definable group G *is a p-group* if p divides $E([G : H])$ for any proper definable subgroup H of G. By "*0 divides $E([G : H])$*" we mean "$E([G : H]) = 0$" which may sound strange but will allow us to use a single statement for results which hold with both $p > 0$ and $p = 0$. When we say that H is a definable *p-subgroup* of a definable group G, we mean that H is a definable subgroup that is a p-group.

EXERCISE 6.2. *Show that if G is a p-group, then G^0 is a p-group.*

EXERCISE 6.3. *Show that if p divides $E(G)$, then a minimal subgroup H of G such that p divides $E(H)$ is a p-group.*

For any group H with an action σ of a set S, we will define

$$Fix(H) = \{x \in S \mid \forall h \in H \ \sigma_h(x) = x\}.$$

Lemma 6.6. *Let H be a p-group, S be a definable set, and let σ be an action on S. Then p divides $E(S) - E(Fix(H))$. If $p = 0$, then $E(S) = E(Fix(H))$.*

PROOF. Let $Hx := \{\sigma_h(x) | h \in H\}$ be the orbit of x under the action of H. Then $S = \bigcup_{\{Hx|x \in G\}} Hx$. Now, Hx is definably isomorphic to $H/(\{h \in H \mid \sigma_h(x) = x\})$ and, by definition of p-group, p divides $E(Hx)$ whenever $\{h \in H \mid \sigma_h(x) = x\} \neq H$. So p divides $E(Hx)$ whenever $x \notin Fix(H)$.

By Fact Fact 6.2,

$$\begin{aligned} E(S) &= \left(\sum_{\{Hx|x \notin Fix(H)\}} E(Hx) \right) + E\left(\bigcup_{x \in Fix(H)} Hx \right) \\ &= \left(\sum_{\{Hx|x \notin Fix(H)\}} E(Hx) \right) + E\left(Fix\left(H\right)\right) \end{aligned}$$

so

$$E(S) - E\left(Fix\left(H\right)\right) = \sum_{\{Hx|x \notin Fix(H)\}} E(Hx),$$

and since p divides the right hand side, it must divide $E(S) - E\left(Fix\left(H\right)\right)$. ☐

A first corollary of Lemma 6.6 (there will be several) is the following proposition which relates Euler characteristic to the existence of torsion points.

Proposition 6.7. *If p is a prime number and there is some $a \in G$ of order p, then p divides $E(G)$. Conversely, if $(G, \cdot)$ is a definable group with $E(G) = n$ and p is a prime number dividing n, then there is some $a \in G$ of order p.*

In particular, G is torsion-free if and only if $E(G) \in \{1, -1\}$, so that no 0-group can have a torsion free quotient.

Proof. As mentioned in Remark 6.4, the first statement is a corollary of Fact 6.2. To prove the second statement, let

$$S := \{(g_1, \dots, g_p) \mid g_i \in G \wedge g_1 \cdot g_2 \cdot \cdots \cdot g_p = e\},$$

and let σ be the action of $\mathbb{Z}/p\mathbb{Z}$ on S by (cyclic) translations, so $\sigma_i((g_1, \dots, g_p)) = (g_{i+1}, g_{i+2}, \dots, g_n, g_1, \dots, g_i)$.

$\mathbb{Z}/p\mathbb{Z}$ is of course a p-group, so Lemma 6.6 implies that p divides $E(S) - E(Fix(\mathbb{Z}/p\mathbb{Z}))$. By Fact Fact 6.2 and definition of S we have $E(S) = pE(G)$, so p must divide $E(Fix(\mathbb{Z}/p\mathbb{Z}))$.

Since $(e, e, \dots, e) \in Fix(\mathbb{Z}/p\mathbb{Z})$, $Fix(\mathbb{Z}/p\mathbb{Z}) \neq \emptyset$ and since $E(Fix(\mathbb{Z}/p\mathbb{Z})) \neq 1$ there must be another element, say $(a, a, \cdots, a)$, in $Fix(\mathbb{Z}/p\mathbb{Z})$ with $a \neq e$. By definition of S, $a^p = e$, as required. $\square$

We will only use the full strength of the Torsion Theorem in order to prove the existence of a characteristic solvable torsion free subgroup in Chapter 7. Specifically, we will use the following corollary.

Corollary 6.8. *Every definably compact definably connected abelian group is a 0-group.*

Proof. The Torsion Theorem (together with Proposition 6.7) implies that the Euler characteristic of any definably compact infinite definable group is 0. By Proposition 2.17 if G is definably compact and abelian definable group and H is a subgroup, G/H is definably compact. If G is definably connected then G/H is either trivial or infinite. The result follows. $\square$

Proposition 6.7 and Fact 6.2 imply that torsion free groups are closed under quotients. We will prove that $Tor_p(G)$ is finite for any abelian group G and any prime p.

When $H \leq G$ is not a normal subgroup, $[G : H]$ will not be a subgroup. Several inductive arguments (mostly induction on the dimension) we will need the co-classes to be groups. The next lemma will allow us to work, in some cases, with the group $N_G(H)/H$ instead of the set of co-classes $[G : H]$ where $N_G(H) := \{g \in G \mid gHg^{-1} = H\}$ denotes the normalizer of H in G.

Lemma 6.9. *Let H be a definable p-subgroup of a definable group G. Then p divides*

$$E(N_G(H)/H) - E([G : H]).$$

In particular, if H is a 0-group then $E(N_G(H)/H) = E([G : H])$.

Proof. Consider the action of H on $[G : H]$ by left multiplication. By Lemma 6.6 p divides $E([G : H]) - E(Fix(H))$. Since $hgH = gH$ if and only if $g^{-1}hg \in H$ if and only if $g \in N_G(H)$, $g \in Fix(H)$ if and only if $g \in N_G(H)$, and

$$Fix(H) = \{gH \mid \forall h \in H,\ hgH = gH\} = \{gH \mid gHg^{-1} = H\} = N_G(H)/H$$

and p divides $E(N_G(H)/H) - E([G : H])$ as required. $\square$

Proposition 6.10. *There are no infinite definable groups G with $G = Tor_p(G)$ for $p > 0$. In particular, if G is abelian then $Tor_n(G)$ is a finite subgroup for any $n \in \mathbb{N}$ with $n \neq 0$.*

Proof. We will prove the first item, leaving the second one as an exercise. Suppose G is an infinite group with $G = Tor_p(G)$ and of minimal dimension among all such groups. By Exercise 6.2 we may assume G is definably connected. $E(H)$ is bounded by $E(G)$ for any subgroup H of G by Fact 6.2, so we can find a finite subgroup H of G of maximal Euler characteristic. By definition of Euler characteristic and Cauchy's Theorem $E(H) = |H| = \mathbb{Z}/(p^k)\mathbb{Z}$ for some k.

G only has elements of order p so by Proposition 6.7 the only prime which divides $E([G : H])$ is p. Lemma 6.9 implies p divides $E(N_G(H)/H)$ so there is some $a \in N_G(H) \setminus H$ of order p (every element of G has order p), and its image $[a]_H$ in $N_G(H)/H$ generates a subgroup A of order p. The pullback K in $N_G(H)$ of A under the natural projection from $N_G(H)$ to H is a subgroup K of G contained in $N_G(H)$ with $|K/H| = p$. K is therefore a finite subgroup of G and $E(K) = pE(H)$ contradicting the maximality of $E(H)$. $\square$

EXERCISE 6.4. *Complete the proof of Proposition 6.10: Show that if G is abelian then $Tor_n(G)$ is a finite definable subgroup of G for any $n \in \mathbb{N}^{>0}$.*

EXERCISE 6.5. *Let A be a definably connected abelian group. Then A is divisible.*

Remark 6.11. The interplay between torsion and torsion free will be key in this chapter. An important tool we will use is that when proving the truth of first order statements in o-minimal theories we can always assume that any definable abelian group contains a torsion free subgroup: If we are only interested in the case $\mathcal{R} = \mathbb{R}$ this follows from Exercise 6.4. For an arbitrary $\mathcal{R}$, Proposition 6.10 implies that given any definable group G, in a saturated elementary extension $\mathcal{R}'$ of $\mathcal{R}$ the group $G(\mathcal{R}')$ contains a torsion free element.

We will end this section with the following theorem, which collects several consequences of having a Euler characteristic in the definable context.

Theorem 6.12. *Let G be a group definable in an o-minimal structure. Then the following hold:*

- *If G is torsion free and H is a definable normal subgroup, then G/H is torsion free.*
- *If G is definably connected, abelian and divisible, then $E(G) \in \{0, -1, 1\}$.*
- *If G^0 is abelian and not torsion-free, then $E(G) = 0$ and G^0 has elements of order p for every prime p.*
- *If G is semisimple, then $E(G) = 0$.*

PROOF. If G is torsion free, then $E(G) = 1$ or $E(G) = -1$ by Proposition 6.7. If $H \lhd G$ is a definable subgroup then $E(G) = E(H)E(G/H)$ by Fact 6.2 so $E(H), E(G/H) \in \{-1, 1\}$ and G/H is torsion free by Proposition 6.7.

If G is definably connected, abelian and divisible, there are two cases. If it is torsion free, then $E(G) \in \{-1, 1\}$. If G is not torsion free then for some $n \in \mathbb{N}$ the homomorphism $\phi : x \mapsto x^n$ has non trivial kernel. Since G is divisible, ϕ is a surjective map, so that $E(G) = E(\ker(\phi))E(G)$. But since $ker(\phi)$ is finite by Proposition 6.10, $E(ker(\phi)) = |ker(\phi)|$ and since it is non trivial, $|ker(\phi)| \geq 2$. It follows that $E(G) = 0$, as required.

By Exercise 6.5 G^0 is divisible, so $E(G^0) = 0$ if G^0 is not torsion free.

Finally, if G is semisimple and centerless, Corollary 3.24 implies that G is definably isomorphic via the adjoint representation to the semialgebraic linear group $\mathrm{Aut}(\mathfrak{g})$ and by Theorem 5.37 it is a direct product of definably simple groups, each elementarily equivalent to a simple Lie group. Therefore for each $n \in \mathbb{N}$ there is a first order formula in the language of rings saying that G has n-torsion elements, which is true in the real field, and therefore it is true in $\mathcal{R}$ too. Hence $E(G) = 0$ by Proposition 6.7. If G is not centerless, then the quotient of G by its center is centerless and the claim follows from Fact 6.2. $\square$

We end this section with another useful consequence of Lemma 6.6.

Lemma 6.13. *If G is a 0-group, then $E(Z(G)) = 0$. In particular, $Z(G)$ is infinite and no semisimple definable group is a 0-group.*

PROOF. Consider the action of G on itself by conjugation.

$$Fix(G) = \{g \in \mathbb{G} \mid \forall x \in G \; xgx{-}1 = g = Z(G)\}.$$

So $0 = E(G) = E(Z(G))$ by hypothesis and Lemma 6.6. □

6.2. 0-groups.

Maximal compact abelian subgroups play an important role in the study (and classification) of Lie groups. As mentioned in the introduction, the corresponding subgroups in the definable category are not always available. The following example of a 0-group with no infinite definable compact abelian subgroups, due to A. Strzebonski, is a good one to keep in mind as the reason we must work with the more general concept of 0-groups.

Example 6.14. ([**69**, 5.3]). Let $G = \mathbb{R} \times [1, e)$ be the group defined in $\langle \mathbb{R}, <, +, \cdot \rangle$ with universe the direct product of the additive group $\mathbb{R}$ and the interval $[1, e)$ where we define the group operation by

$$(r_1, t_1) * (r_2, t_2) = \begin{cases} (\; r_1 + r_2 \;, \quad t_1 \cdot t_2 \;) & \text{if } t_1 \cdot t_2 < e \\ (\; r_1 + r_2 + 1, \; t_1 \cdot t_2/e \;) & \text{otherwise.} \end{cases}$$

The proper definable subgroups of G are $\{(0, 1)\}$, $\mathbb{R} \times \{1\}$, $\mathbb{R} \times F$, $\{(-\log t, t) : t \in F\}$, F ranging between the finite subgroups of $([1, e), \otimes_e)$, where $\otimes_e$ denotes the product modulo e. Therefore, G is a 0-group. The definable subgroup $\mathbb{R} \times \{1\}$ is definably connected and has o-minimal Euler characteristic equal to -1, so it is is torsion free.

This example is a 0-group, as are all compact abelian groups in any o-minimal expansion of $\mathbb{R}$. This section studies the properties of all such groups.

Lemma 6.15. *Let G be a definable abelian group and Φ be a definable (abelian) group of automorphisms of G without infinite definable subgroups. For any $g \in G$ we define $\Phi_g := \{\phi \in \Phi \mid \phi(g) = g\}$ and $Fix(\Phi) := \{g \in G \mid \forall \phi \in \Phi, \; \phi(g) = g\}$.*

Then the following hold.

(1) If $\phi \in \Phi_g$ and $\phi \notin Tor(\Phi)$, then $\Phi_g = \Phi$.
(2) If $g \in Tor(G)$ and $\phi \notin Tor(\Phi)$ then $\phi(g) = g$.
(3) If Φ is torsion free then $Tor(G) \subseteq Fix(\Phi)$.

PROOF. By hypothesis on Φ, either $\Phi_g = \Phi$ of Φ_g is finite. In particular, if $\phi \in \Phi_g$ and $\Phi_g \neq \Phi$ then $\phi \in Tor(\Phi)$ which proves (1).

First, notice that since $(\theta(x))^n = \theta(x^n)$ for any $x \in G$ and $\theta \in \Phi$, we have $\theta(Tor_n(G)) \subseteq Tor_n(G)$ for any $\theta \in \Phi$.

Let $g \in Tor(G)$, $\phi \in \Phi \setminus Tor(\Phi)$ and k be the order of g. Then $f(n) := (\phi^n)(g)$ is an homomorphism from $\mathbb{Z}$ to the finite set $Aut(Tor_k(G))$. This implies that ϕ^l is the identity in $Tor_k(G)$ for some l so in particular $\phi^l(g) = g$. Since $\phi^l \notin Tor(\Phi)$, $\Phi_g = \Phi$ by (1) and $\phi(g) = g$ implying (2).

(3) follows immediately from (2). □

Theorem 6.16. *Let $(G, \cdot)$ be a definable group. Then the following hold.*

(1) If K is a normal definable subgroup of G and $[g]_K \in Tor(G/K)$ for some $g \in G$, then there is $h \in [g]_K$ with $h \in Tor(G)$.

(2) *If $K \leq G$ is a normal definable subgroup with $Tor(G) \subseteq K$ then G/K is either trivial or torsion free. In particular, $G/\langle Tor(G)\rangle_{\text{def}}$ is wither trivial or torsion free.*
(3) *If G is an abelian 0-group, H is an abelian group without infinite definable subgroups acting on G and $h \in H$ is torsion free, then h fixes G.*
(4) *If G is an abelian 0-group, then there is no infinite definable family of automorphisms of G.*
(5) *If G is is an abelian 0-group and A is definably connected, there is no non trivial definable action from A to G.*

Proof. Let $K \triangleleft G$, and let $g \in G$ be such that $g^l \in K$ for some l. If $g^l \in Tor(G)$ then $g \in Tor(G)$ and the conclusion of (1) will hold. So assume that $g^l \notin Tor(G)$ so that $A := \langle\{g^l\}\rangle_{\text{def}}$ is an infinite definable subgroup of K. The groups $C_A(C_A(\{g^l\}))$ and $C_G(\{g\})$ are both definable groups containing g^l, so A is an abelian group and every element of A commutes with g.

A is the definable hull of a single element, so it is definably connected. By Exercise 6.5 A is divisible, so let $\gamma \in A$ be such that $\gamma^l = g^l$.

Then $g\gamma^{-1} \in gA \subseteq gK$ and, since g commutes with γ,

$$(g\gamma^{-1})^l = (g^l)(\gamma^{-l}) = e.$$

This completes the proof of (1).

Let $K \leq G$ be a normal definable subgroup and $Tor(G) \subset K$ then (1) implies that $g^l \notin K$ for $g \notin K$, so that G/K is either trivial or torsion free and (2) holds.

Let $h \in H$ be torsion free. By Lemma 6.15(2), $\gamma_h(b) = b$ for any $b \in Tor(b)$. So $Tor(b) \subseteq Fix(\{h\})$ and by (2) the quotient $G/Fix(h)$ is torsion, so it must be trivial by definition of 0-groups and $G = Fix(h)$. This proves (3).

(4) follows from (3): Let Φ be an infinite definable family of automorphisms of G. By DCC Φ contains an infinite definable (abelian) subgroup Φ' without infinite definable subgroups. If Φ' is uncountable then Φ' must contain a torsion free element Proposition 6.10 contradicting (3). The general result (without assuming Φ' uncountable) follows by looking at realizations of ϕ' in an elementary extension of $\mathcal{R}$ where Φ' is uncountable.

Given definable 0-groups A and G and a group action $\gamma : A \times G \to G$, if we define an equivalence relation a_1Ea_2 on A by $a_1Ea_2 \iff \forall x \in G\gamma_{a_1}(x) = \gamma_{a_2}(x)$, the quotient A/E is finite by (3) which implies that $E(A/E) = |A/E|$ so it must be trivial by definition of 0-group. This proves (5).

□

Theorem 6.17. *Any 0-group G is abelian, divisible and $G = \langle Tor(G)\rangle_{\text{def}}$.*

Proof. Let G be a 0-group. We will prove first by induction on $\dim(G)$ that G is abelian. If $\dim(G) = 1$ then G is abelian by Lemma 6.13.

Assume towards a contradiction that $G \neq Z(G)$.

By Lemma 6.13 again, $Z(G)$ is a 0-group and by Exercise 6.3 there is a non trivial 0-group K which is a subgroup of $Z(G)$, so a normal subgroup of G. By induction hypothesis both $G/Z(G)$ and G/K are abelian.

Claim 6.18. *For any $a \in G$ the maps $x \mapsto xax^{-1}a^{-1}$ and $x \mapsto axa^{-1}x^{-1}$ are homomorphisms from G to K.*

Proof: Since G/K is abelian, $[G,G] \subset K$ and $x, y \mapsto xyx^1y^{-1}$ is a map from $G \times G$ to K. Let $a \in G$.

If we define $a^{-1}xax^{-1} = z_x$ for $x \in G$ then $ax^{-1} = a^{-1}hz_x$ and an easy computation (using that z_x is central for all x) shows that $(xy)a(xy)^{-1}a^{-1} = z_xz_y$

so that for any a the map $\phi : x \mapsto xax^{-1}a^{-1}$ is an homomorphism from G/K to K.

The other direction is quite similar. $\blacksquare_{Claim}$

Let $\phi_a : x \mapsto xax^{-1}a^{-1}$. By the claim, ϕ_a is an homomorphism from G to K and $\phi_a \circ \phi_b = \phi_{ab}$, so $\{\phi_a\}_{a\in G}$ is a definable family of homomorphisms from G to K. The kernel of ϕ_a is by definition $C_G(a)$.

By Remark 6.11 we may assume that there is $g \in G \setminus Z(G)$ such that the projection $\overline{g}$ of g in $G/Z(G)$ is torsion free. Since $C_G(g) \subset C_G(g^n)$ for any $n \in \mathbb{N}$, DCC implies that for some power h of g we have $C_g(h) = C_g(h^n)$ for all $n \in \mathbb{N}$. By hypothesis $\overline{g}$ is torsion free in $G/Z(G)$, so $h^n \notin Z(G)$ for all $n \in \mathbb{N}$.

By the claim, $\phi_{h^n}(x) = (z_x)^n = (\phi_h(x))^n$. The image $Im(\phi_h)$ is abelian (contained in K), definably connected (it is the image of the definably connected G under a continuous map) so it is divisible and $Im(\phi_h) = Im(\phi_{h^n})$.

So our hypothesis on h yield

$$\ker(\phi_h) = C_G(h) = C_G(h^n) = \ker(\phi_{h^n})$$

and $Im(\phi_h) = Im(\phi_{h^n})$. These are definable conditions, so if we let Φ be the smallest definable subgroup of $\{\phi_a\}_{a\in G}$ containing ϕ_h, then Φ is abelian, and every map in Φ has kernel $C_G(h)$ and image $Im(\phi_h)$. This implies that $\{\psi \circ (\phi_h)^{-1}\}_{\psi\in\Phi}$ is a definable family of automorphisms of the 0-group $G/C_G(h)$ which by Theorem 6.16(4) must be finite. This implies that for some $k \in \mathbb{N}$ the map $(\phi_h)^{k+1}(x) = \phi_h(x)$ for all $x \in G$, so $(z_x)^k = e$ and $h^k \in Z(G)$ contradiction our hypothesis in h.

For divisibility, note that for any $n \in \mathbb{N}$, the group $G^n := \{x^n \mid x \in G\}$ is normal and G/G^n is a definable quotient with every element of order n, so by definable choice and Proposition 6.10, must be finite. It follows that $E(G/G^n) \neq 0$ so if G is a 0-group then $G = G^n$.

By item (2) in Theorem 6.16, $G/\langle Tor(G)\rangle_{\mathrm{def}}$ is torsion free. By Theorem 6.12 and the definition of 0-group, $G/Tor(G)\rangle_{\mathrm{def}}$ is trivial, and $G = \langle Tor(G)\rangle_{\mathrm{def}}$. $\square$

By Corollary 6.8, any definably connected abelian definably compact definable group is a 0-group. Since definable subgroups of definably compact groups are definably compact, these are 0-groups for which every definable subgroup is also a 0-group. This allows us to prove something stronger.

Theorem 6.19. *Let G be an abelian definably compact definable group.*

(1) G has no infinite family of definably connected definable subgroups.
(2) If A is an abelian definably compact definably connected group, then there is no infinite family of definable homomorphisms from A to G.

Proof. Let $(G, \cdot)$ be a definably compact abelian definably connected definable group. Since $(G : G^0)$ is finite, we may assume that G is definably connected. We will prove that there is no definable infinite family of definable subgroups by induction on $\dim(G)$.

If $\dim(G) = 1$ there is nothing to prove. Assume now that the statement holds for any 0-group of dimension less than $\dim(G)$. Let $\{A_\alpha\}_{\alpha\in X}$ be an infinite definable family of subgroups of G. Fix some $A \neq G$ in the family.

$\{A \cdot A_\alpha/A\}_{\alpha\in X}$ is family of subgroups of G/A, so it must be finite by induction hypothesis. It follows that $A \cdot A_\alpha$ is constant for α in a definable subset X_1 of X for which $\{A_\alpha\}_{\alpha\in X_1}$ is infinite. Let $G' = A \cdot A_\alpha$ for $\alpha \in X_1$. Now, $\dim(A) < \dim(G)$ by definable connectedness and $\{A \cap A_\alpha\}_{\alpha\in X_1}$ is a definable family of disjoint subgroups of A so by induction hypothesis it must be finite. So there is a definable subset X_2 of X_1 such that $\{A_\alpha\}_{\alpha\in X_1}$ is infinite and $A \cap A_\alpha = A \cap A_\beta = K$ for any $\alpha, \beta \in X_2$.

We will work with the images of the A_α's in the abelian definably compact group G'/K. In order to make the notation lighter, we will assume $K = \{e\}$ and $A \cap A_\alpha = \{e\}$. Given any $\alpha, \beta \in X_2$, any $a \in A_\alpha$ belongs to $A \cdot A_\beta$ so there is a unique $b \in A_\beta$ such that $a \in b \cdot A$. This defines a homomorphism $\phi_{\alpha,\beta}$ from A_α to A_β for any α, β. We can extend $\phi_{\alpha,\beta}$ to G' by $\phi(ka) = k\phi_{\alpha,\beta}(a)$. We will abuse the notation and refer to this automorphism of G' as $\phi_{\alpha,\beta}$.

Finally, notice that $\phi_{\alpha,\beta}(A_\alpha) = A_\beta$ so that $\phi_{\alpha,\beta} \neq \phi_{\alpha,\gamma}$ whenever $A_\beta \neq A_\gamma$. So $\{\phi_{\alpha,\beta}\}$ is an infinite family of automorphisms of G', contradicting Theorem 6.16(3). This proves (1).

The proof of (2) is basically a reduction to the previous cases. Any morphism ϕ from A to G is an isomorphism from $A/\ker(\phi)$ to $im(\phi)$ which is a subgroup of G. By (1) the sets $\ker(\phi)$ and $im(\phi)$ achieved as ϕ varies in a definable set of homomorphisms can only achieve finitely many values. So an infinite family of homomorphisms between A and G necessarily translates into infinitely many isomorphisms between a quotient of A and a subgroup of G. Fixing one of these isomorphisms and composing with the inverse of all the rest, we would have an infinite set of automorphisms of a (abelian definably compact connected) quotient of A, contradicting condition (4) in Theorem 6.16. □

6.2.1. 0-Sylow subgroups.

Definition 6.20. A 0-*Sylow subgroup* of a definable group G is a maximal definable 0-subgroup of G.

EXERCISE 6.6. *Let G be a definable group, H be a 0-Sylow subgroup of G and N a definable normal subgroup of G. Prove that HN/N is a 0-group. Prove that if N is torsion free, then HN/N is a 0-Sylow subgroup of G/N.*

Lemma 6.21. *Let G be a definable group. Then the following hold:*

- *Every 0-group is definably connected and every 0-subgroup of G is contained in a 0-Sylow subgroup.*
- *If $E(G) = 0$, then there is a non trivial definable 0-group of G.*

PROOF. The identity is always a 0-subgroup of G. Any definable 0-group must be definably connected, since $E(H/(H)^0$ is equal to $|[H : H^0]|$. This implies that if $H_1 \leq H_2$ are definable 0-groups $\dim(H_1)$ is strictly smaller than $\dim(H_2)$. Thus, any ascending chain of definable 0-subgroups must stabilize to a 0-Sylow subgroup. This proves the first item.

For the second item, if $E(G) = 0$ then by DCC there is a minimal subgroup H of G with $E(H) = 0$. By definition, it must be a 0-group. □

Proposition 6.22. *Let H be a 0-subgroup of a definable group G.*

(1) H is 0-Sylow if and only if $E(G : H) \neq 0$.
(2) Let S be a 0-Sylow of a definable group G. Then there is $g \in G$ such that $H \subset gSg^{-1}$. Therefore if K is a definable subgroup of G then K is a 0-Sylow if and only if $K = gSg^{-1}$ for some $g \in G$.

PROOF. Assume that H is a 0-group and $E(G : H) \neq 0$. Let $H \leq K$ be any definable subgroup. There is a definable isomorphism between $[G : H]$ and $[G : K] \times [K : H]$ so by Fact 6.2 $E([G : H]) = E([G : K])E([K : H])$. So $E([K : H]) \neq 0$ and K is not a 0-group, and H is a maximal 0-group and a 0-Syllow.

We will prove the contrapositive of the converse, so assume that $E(G : H) = 0$. Then $E(N_G(H)/H) = E(G : H)$ by Lemma 6.9 and by Exercise 6.3 there is some 0-group $K_H \leq N_G(H)/H$. Let $K = K_H H$.

H is normal in K so for any $L \leq K$

$$E([K : L]) = E([K : HL])E([H : H \cap L])$$

by Exercise 6.1. Since H was a 0-group, $E(H : H \cap L) = 0$ so $E([K : L]) = 0$ for any $L \leq K$ which implies that K is a 0-group, so H is not maximal.

To prove (2), let S be a 0-Sylow subgroup of G and consider the left multiplication action σ from H on $[G : S]$. Since $hgS = gS$ if and only if $g^{-1}hg \in S$ if and only if $h \in gSg^{-1}$, the fixed points $Fix(H)$ of σ are precisely the classes gS in $[G : S]$ for which $H \subset gSg^{-1}$.

By Lemma 6.6,

$$E(\{g \mid H \subset gSg^{-1}\}) = E(Fix(H)) = E([G : S]).$$

By the previous item $E([G : S]) \neq 0$ so $\{g \mid H \subset gSg^{-1}\} \neq \emptyset$ and $H \subset gSg^{-1}$ for some $g \in G$, as required. □

Remark 6.23. Let G be a definable group. If there is a 0-Sylow which is definably compact then all 0-subgroups of G are definably compact.

Proof. Since every 0-subgroup of G is contained in a 0-Sylow and they are all conjugate to each other, if there is a definably compact 0-Sylow every 0-subgroup is definably compact. □

We can see that 0-Sylow are characterized by the fact to be definably connected of minimum dimension such that $E(G/P) \neq 0$:

Proposition 6.24. *Let H be a definably connected definable subgroup of a definable group G. Then H is a 0-Sylow if and only if H is minimal among the definable subgroups P such that $E(G/P) \neq 0$.*

Proof. Suppose P a definable subgroup of G such that $E([G : P]) \neq 0$ and $\dim P < \dim H$. By Proposition 6.22, $P \subset gHg^{-1} = H^g$, for some $g \in G$. Since H^g is a 0-group, $E([H^g : P]) = 0$, in contradiction with $E([G : H^g])E([H^g : P]) = E([G : P]) \neq 0$.

For the other direction, let H be minimal among the definable subgroups P such that $E(G/P) \neq 0$.

If $E(G) \neq 0$ then $H = \{e\}$. By Exercise 6.1 $E(H) \neq 0$ for any $H \leq G$ so the only 0-group of G is the identity and the implication holds.

If $E(G) = 0$ then $E(H) = 0$. Let A be a 0-Sylow subgroup of H. By Proposition 6.22 $E(H/A) \neq 0$, and $E(G : A) = E(E : H)E(H : A) \neq 0$ by Exercise 6.1, contradicting minimality of H.

□

Proposition 6.25. *Let G be a solvable definably connected definably compact group. Then G is abelian.*

Proof. We will prove it by induction on $\dim(G)$, so we will assume that any definably connected solvable definably compact group of dimension less than $\dim(G)$ is abelian.

If $\dim(G) = 1$, then G is abelian by Theorem 2.32. Suppose $\dim G > 1$. By Lemma 2.26, if H is an abelian normal subgroup of G, then $\langle H \rangle_{\text{def}}$ is also an abelian normal subgroup of G. So assume A is a definable abelian normal subgroup of G. By Proposition 2.13, any definable subgroup of G is closed, so A and G/A are definably compact by Proposition 2.18 and 2.17.

If $A \neq G$, then G/A is abelian by induction hypothesis. So A and G/A are 0-groups by Corollary 6.8. In particular, $E(G/A) = 0$. By Proposition 6.22, A is not a 0-Sylow subgroup of G. Let S be a 0-Sylow subgroup of G containing A.

Then $E(G/S) \neq 0$, again by Proposition 6.22. But G/S is definably isomorphic to $(G/A)/(S/A)$ and G/A is 0-group, so it must be $G = S$ and G is abelian by Theorem 6.17. $\square$

6.3. The maximal normal definably compact definably connected subgroup

Definable groups may not have a maximal definably compact subgroup: Example 6.14 is a group which has finite definably compact subgroups of arbitrarily large order, but no infinite one. We will prove in Chapter 9 that a definable group has maximal definably compact subgroups if and only if all its 0-groups are definably compact. But if we restrict ourselves to *normal definably connected* definably compact subgroups, then there is always a unique maximal element (that in the case of Example 6.14 is trivial).

Proposition 6.26. *Every definable group G contains a unique maximal* normal *definably connected definably compact subgroup K.*

This normal subgroup K is a maximal definably compact subgroup of G if and only if G/K is torsion-free.

Proof. Let K be a normal definably compact definably connected subgroup of G of maximal dimension and H any normal definably compact definably connected subgroup of G. We want to check that $H \subseteq K$.

Consider the definably compact definably connected group $H \times K$ and the multiplication map $f : H \times K \to G$, $f(x, y) = xy$. Since f is definable and continuous, its image, coinciding with the subgroup HK, is definably compact and definably connected. Since K is of maximal dimension, $\dim HK = \dim K$. By connectedness, $HK = K$ and $H \subseteq K$, as claimed.

If G/K is not torsion-free, let F be a finite subgroup of G/K. Its pre-image in G is a definably compact subgroup containing K properly. Conversely, if K' is a definably compact subgroup of G containing K properly, then K'/K is a definably compact subgroup of G/K, and it has torsion by the Torsion Theorem. $\square$

Remark 6.27. If G/K is not torsion-free, K is not a maximal definably compact subgroup, but it might be a maximal definably connected definably compact subgroup. This may happen trivially if G is not definably connected, but it can happen even in the definably connected case: For instance, this is the case when $G = K \times H$ and H is a 0-group with no infinite definably compact subgroup (like the group described in Example 6.14).

6.4. Linear definably compact subgroups

In this section we want to study definably compact linear groups. We will generalize the following results over the real field to any o-minimal expansion $\mathcal{R}$ of a real closed field.

Theorem 6.28. *Every compact subgroup of $GL_n(\mathbb{R})$ is contained in a conjugate of the orthogonal group $O_n(\mathbb{R})$.*

Proof. Let K be a compact subgroup of $GL_n(\mathbb{R})$. Because K is a compact Lie group, it has a Haar measure μ. Consider the symmetric positive definite bilinear form b:

$$b(v, w) = \int_K \langle Xv, Xw \rangle d\mu(X)$$

where $v, w \in \mathbb{R}^n$ and $\langle , \rangle$ denotes the usual scalar product of $\mathbb{R}^n$. Note that b is invariant for K, i.e. for every $X \in K$, $b(v,w) = b(Xv, Xw)$. The matrix B of b is symmetric positive definite, thus $B = A^T A$ for some $A \in GL_n(\mathbb{R})$. Consider now K^A, the conjugate of K with respect to A. For every $X \in K$ and $v, w \in \mathbb{R}^n$,

$$\begin{array}{rcll}
\langle AXA^{-1}v,\ AXA^{-1}w\rangle & = & v^T(A^{-1})^T X^T A^T AXA^{-1}w & (B = A^TA) \\
& = & b(XA^{-1}v,\ XA^{-1}w) & \text{(invariance for K)} \\
& = & b(A^{-1}v,\ A^{-1}w) & (B = A^TA) \\
& = & v^T(A^{-1})^T A^T AA^{-1}w & \\
& = & v^T w & \\
& = & \langle v,\ w\rangle. &
\end{array}$$

Therefore $AXA^{-1} \in O_n(\mathbb{R})$ for every $X \in K$, and $K \subset O_n(\mathbb{R})^{A^{-1}}$. □

We will need to use some results on the Lie theory of compact linear groups from [**42**]. The following are Theorem 4.3 in Chapter IV of [**42**], together with its corollaries and the examples following 4.30.

Fact 6.29. *The following hold.*

- *Any two maximal compact abelian subgroups of $SO_n(\mathbb{R})$ are conjugate.*
- *$D^K_{2n}(\mathbb{R})$ is a maximal abelian subgroup of $SO_{2n}(\mathbb{R})$ and $D^K_{2n+1}(\mathbb{R})$ is a maximal abelian subgroup of $SO_{2n+1}(\mathbb{R})$.*
- *Any maximal abelian linear compact subgroup is definably isomorphic to a product of*
$$S_1(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}.$$
- *Any abelian linear compact subgroup has a conjugate which is a subgroup of $D^K_n(\mathbb{R})$ for some $n \in \mathbb{N}$.*

For any fixed particular group, Theorem 6.28 and Fact 6.29 are expressible in first order so that if we assumed we are only working in elementary extensions of o-minimal structures over $\mathbb{R}$, then the results would hold. However, we find that the results involved in the proofs are quite insightful, so we prove the general case.

We will work inside $GL_n(\mathcal{R})$. Throughout this section, we will use the multiplicative notation between elements of $GL_n(\mathcal{R})$ for matrix multiplication. The matrix $\mathbf{I}_n$ will denote the identity matrix in $GL_n(\mathcal{R})$.

Definably compact subgroups of $GL_n(\mathcal{R})$ will be closely related to algebraic subgroups of $GL_n(\mathcal{R})$, so we need the following definition.

Definition 6.30. Let G be a subgroup of $Gl_n(\mathcal{R})$. We say that G is *algebraic* if it is Zariski closed. This is, G is the zero set of polynomials with coefficients in $\mathcal{R}$.

Lemma 6.31. *Any algebraic definably compact abelian definable subgroup of $GL_n(\mathcal{R})$ is a conjugate of a subgroup of a definable product of one dimensional definably compact abelian subgroups of $GL_n(\mathcal{R})$.*

Proof. By Proposition 2.18, definable compactness for definable subsets of $GL_n(\mathcal{R})$ is equivalent to being closed and bounded.

Consider the subgroup $D^K_n(\mathcal{R})$ of $SO_n(\mathcal{R})$ defined as

$$D^K_{2n}(\mathcal{R}) := \left\{ \begin{pmatrix} \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} & 0 & \dots & 0 \\ & & \ddots & \\ 0 & \dots & 0 & \begin{pmatrix} a_n & b_n \\ -b_n & a_n \end{pmatrix} \end{pmatrix} \right\}$$

with $a_i, b_i \in \mathcal{R}$ with $a_i^2 + b_i^2 = 1$ for all i, and

$$D_{2n+1}^K(\mathbb{R}) := \left\{ \begin{pmatrix} \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} & 0 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & \dots & 0 & \begin{pmatrix} a_n & b_n \\ -b_n & a_n \end{pmatrix} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \right\}$$

with $a_i, b_i \in \mathcal{R}$ with $a_i^2 + b_i^2 = 1$ for all i.

By Fact 6.29, if A is a closed and bounded abelian subgroup of $GL_n(\mathbb{R})$ then there is an invertible $\mathbb{R}$-matrix M such that for any $X \in A$ the MXM^{-1} is in $D_n^K(\mathbb{R})$. The statement "For any sets of polynomials defining a subgroup A of $GL_n(\mathbb{R})$, if A is bounded then there is an invertible $\mathbb{R}$-matrix M such that for any $X \in A$ the MXM^{-1} is in $D_n^K(\mathbb{R})$" is expressible in first order logic by an infinite set of formulas varying over the degrees of the polynomials. By transfer theorems it is true for any $\mathcal{R}$.

Since $D_n^K(\mathbb{R})$ is definably isomorphic to a cartesian power of

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathcal{R},\ a^2 + b^2 = 1 \right\},$$

the result follows. □

From now on, we will denote by S_1 be the circle group

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathcal{R},\ a^2 + b^2 = 1 \right\}.$$

We will need to understand S_1 quite well, and in particular we will give an order to the elements.

Identifying any point $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ in S_1 with $(a, b) \in \mathcal{R}^2$, so we can identify S_1 with the unitary circle in $\mathcal{R}^2$. Under this identification, we can start at any point $p \in S_1$ and "move clockwise" which is a definable concept: The quadrants are definable and the clockwise order within each quadrant is definable. Let $<_p$ be the order this defines, so that $x \leq_p y$ if and only when starting at p, x goes before y in the clockwise order.

Given points p, q we will define $(p, q) := \{x \in S_1 \mid p <_p x <_p q\}$. Notice that for any $p \neq q$

$$S_1 = (p, q) \cup \{q\} \cup (q, p) \cup \{p\}.$$

Let $R_n := \{x \in S_1 \mid x^n = \mathbf{I}_2\}$. Either from the axioms of real closed fields or by transfer theorems with $\mathbb{R}$, $|\{x \in S_1 \mid x^n = \mathbf{I}_2\}| = n$. Even though complex exponential is never definable in an o-minimal expansion of the real field, we will still refer to the elements in R_n as $e^0, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \dots, e^{\frac{2n\pi i}{n}}$ where $e^0 = \mathbf{I}_2$ and

$$e^{\frac{2\pi i}{n}} <_{\mathbf{I}_2} e^{\frac{4\pi i}{n}} <_{\mathbf{I}_2} \dots <_{\mathbf{I}_2} e^{\frac{2n\pi i}{n}}.$$

Exercise 6.7. *Prove that any finite subgroup of S_1 is R_k for some k.*

Exercise 6.8. *Prove that for every $n \in \mathbb{N}$, S_1/R_k is isomorphic to S_1.*

Exercise 6.9. *Prove the following:*

- *R_m is a subgroup of $S_1(R)$.*
- *$R_m \subset R_{km}$ for any $k, m \in \mathbb{N}$.*
- *$e^{\frac{2l\pi i}{n}} \cdot e^{\frac{2k\pi i}{n}} = e^{\frac{2(l+k)\pi i}{m}}$ for any $k, l, m \in \mathbb{N}$ with $m + l < n$.*
- *$e^{\frac{2k\pi i}{m}} = e^{\frac{2(kl)\pi i}{ml}}$ for any $k, l, m \in \mathbb{N}$ with $m < l$.*

Hint: The statements are all equivalent to first order sentences in the language of fields.

Lemma 6.32. *The only definable automorphisms of S_1 are inversion and the identity.*

Any definable endomorphisms of S_1 is either constant or $x \mapsto x^k$ for some $k \in \mathbb{Z}$.

PROOF. Let σ be a definable automorphism of S_1 which is not the identity. $Fix(\sigma)$ is a definable subgroup of S_1 so it must be finite. σ must always fix $-\mathbf{I}_2 = e^{\frac{2\pi i}{2}}$, so $\{n \in \mathbb{N} \mid e^{\frac{2\pi i}{n}} \in Fix(\sigma)\}$ is finite and nonempty. Let k be the largest natural number such that $e^{\frac{2\pi i}{k}}$ is fixed by σ.

The automorphism σ is continuous, and $(e^0, e^{\frac{2\pi i}{k}}), (e^{\frac{2\pi i}{k}}, e^0)$ are the two connected components of $S_1 \setminus \{e^0, e^{\frac{2\pi i}{k}}\}$ so σ must either set-wise fix $(e^0, e^{\frac{2\pi i}{k}})$ or send $(e^0, e^{\frac{2\pi i}{k}})$ to $(e^{\frac{2\pi i}{k}}, e^0)$. Since σ is a group automorphism of S_1 it must permute the elements

$$R_{2k} := \{x \in S_1 \mid x^{2k} = 0 := \{e^0, e^{\frac{2\pi i}{2k}}, e^{\frac{4\pi i}{2k}}, \ldots, e^{\frac{2n\pi i}{2k}}\}.$$

By construction

$$\left| R_{2k} \cap (e^0, e^{\frac{2\pi i}{k}}) \right| = \left| \left\{ e^{\frac{2\pi i}{2k}} \right\} \right| = 1,$$

so σ can't send $(e^0, e^{\frac{2\pi i}{k}})$ to $(e^{\frac{2\pi i}{k}}, e^0)$ if $1 \neq 2k-3$. However, if σ fixes $(e^0, e^{\frac{2\pi i}{k}})$ setwise, it must fix $e^{\frac{2\pi i}{2k}}$ contradicting maximality of k. So $1 = 2k-3$ and $k = 2$.

If $k = 2$ then $e^{\frac{2\pi i}{4}}$ is not fixed by σ, and since $e^0 = \mathbf{I}_2$ and $e^{\frac{4\pi i}{2}} = -\mathbf{I}_2$ are fixed by any automorphism, then σ must send $e^{\frac{2\pi i}{4}}$ to $e^{\frac{6\pi i}{4}}$. R_4 is closed under inverses, $\mathbf{I}_2^{-1} = I_2$ and $(-\mathbf{I}_2)^{-1} = -\mathbf{I}_2$ so $e^{\frac{6\pi i}{4}} = (e^{\frac{2\pi i}{4}})^{-1}$ and

$$\sigma\left(e^{\frac{2\pi i}{4}}\right) = \left(e^{\frac{2\pi i}{4}}\right)^{-1}.$$

So the composition of σ with inversion must fix $e^{\frac{2\pi i}{4}}$ so by the previous result it must be the identity and σ is inversion. This completes the proof of the first item.

Assume now we have an endomorphism σ of S_1. The subgroup $\ker(\sigma)$ of S_1 so is either finite or all of S_1. In the latter case σ is constant, so we may assume that $\ker(\sigma)$ is finite, and $x^n = \mathbf{I}_2$ for some $n \in N$ by Exercise 6.7. It follows that σ and $p_n : x \mapsto x^n$ both have kernel R_n so that they induce isomorphisms σ' and p'_n from $S_1/\ker(\sigma)$ to S_1. The composition $\sigma' \circ (p'_n)^{-1}$ is therefore an automorphism of $S_1/\ker(\sigma)$. But $S_1/\ker(\sigma)$ is isomorphic to S_1 so by the previous claim $\sigma(x) = x^n$ or $\sigma(x) = x^{-n}$. □

In S_1 the map $x \mapsto x^k$ plays a significant role: endomorphisms are defined by these maps, and subgroups of S_1 are all solution sets of $x^k = \{e\}$ for some $k \in Z$. This properties will generalize to direct products S_1^n of S_1. To prove this, we need some notation.

Definition 6.33. Let $i, j, n \in \mathbb{N}$ with $0 \leq i, j \leq n$.

Let $\pi_i : (S_1)^n \to S_1$ be the projection map to the i-th coordinate.

Let $\rho_i : S_1 \to (S_1)^n$ be the fiber map of π_i that sends x to the element of $(S_1)^n$ that has x in the i-th coordinate and is $\mathbf{I}_2$ in every other coordinate, this is, the unique homomorphism from S_1 to $(S_1)^n$ such that $\pi_i(\rho_i(x)) = x$ and $\pi_j(\rho_i(x)) = \mathbf{I}_2$ for $i \neq j$.

A *monomial* in S_1^n is a map from S_1^n to S_1 of the form

$$(x_1, x_2 \ldots, x_n) \mapsto x_1^{k_1} \cdot x_2^{k_2} \cdot \cdots \cdot x_n^{k_n}$$

for $k_1, k_2, \ldots k_n \in \mathbb{Z}$.

Given a monomial f in S_1^n, *the solution set of* f is

$$\{\overline{x} \mid f(\overline{x}) = \mathbf{I}_2\}.$$

Lemma 6.34. *Any endomorphism from S_1^n to S_1 is a monomial.*

PROOF. Let ϕ be any endomorphism from S_1^n to S_1. The map $\phi \circ \rho_i$ is an endomorphism of S_1, so $\phi \circ \rho_i(x) = x^{k_i}$ with $k_i \in \mathbb{Z}$ by Lemma 6.32.

Then

$$\phi(x_1, \ldots, x_n) = \phi\left(\prod_{i \le n} \rho_i(x_i)\right) = \prod_{i \le n} \phi\left(\rho_i(x_i)\right) = \prod_{i \le n} x_i^{k_i}.$$

□

Proposition 6.35. *Any definable subgroup of $(S_1)^n$ is the intersection of the solution sets of a (possibly trivial) finite set of monomials in $(S_1)^n$.*

PROOF. We will do an induction on n. For $n = 1$ this is Exercise 6.7.

Assume now that the statement of the theorem holds for any $n = m$ and let A be a definable subgroup of $(S_1)^{m+1}$. If $\{x \in S_1 \mid \rho_i(x) \in A\} = S_1$ for all $i \le m+1$ then $A = S_1^{m+1}$ and A is the solution set of the trivial set of monomials and the result follows. So we may assume that for some i we have $\{x \in S_1 \mid \rho_i(x) \in A\} \neq S_1$ and by changing the i and $m+1$ coordinates we may assume that $\{x \in S_1 \mid \rho_{m+1}(x) \in A\} \neq S_1$ and in particular $\{x \in S_1 \mid \rho_{m+1}(x) \in A\} = R_k$ for some k and $\{w^k \mid \rho_{m+1}(w) \in A\} = \{e\}$.

Now, take elements $\overline{x} \in (S_1)^m$. If $\alpha := (\overline{x}, w_1)$ and $\beta := (\overline{x}, w_2)$ are in A, then $\alpha\beta^{-1} = \rho_{m+1}(w_1 w_2^{-1}) \in A$ which implies by the above that $(w_1 w_2^{-1})^k = \{e\}$ and $w_1^k = w_2^k$. It follows that $\{(\overline{x}, x_{m+1}^k) \mid (\overline{x}, x_{m+1}) \in A\}$ is the graph ϕ of a function from $(S_1)^m$ to S_1. Then $\{(\overline{x}, x_{m+1}^k) \mid (\overline{x}, x_{m+1}) \in A\}$ is an endomorphism from $(S_1)^m$ to S_1 and by Lemma 6.34 $x_{m+1}^k = f(\overline{x})$ for some monomial f with variables $x_1, \ldots, x_m$ and $f(x_1, \ldots, x_m)x_{m+1}^{-k} = e$.

Let A_m be the projection of A to the first m coordinates,

$$A = \{(\overline{x}, x_{m+1}) \mid \overline{x} \in A_m \wedge (\overline{x}, x_{m+1}) \in A\} \subseteq \left\{(\overline{x}, x_{m+1}) \mid \overline{x} \in A_m \wedge f(\overline{x})x_{m+1}^{-k} = e\right\}.$$

Fix $\overline{a} \in A_m$. By definition there is some $b \in S_1$ with $(\overline{a}, b) \in A$ so $f(\overline{a}) \cdot b^{-k} = e$. By hypothesis, $\rho_{m+1}(R_k) \subset A$. This implies that for any $r \in R_k$ the pair $(\overline{a}, b \cdot r)$ is in A and

$$\left\{(\overline{a}, x) \mid f(\overline{a}) \cdot x^{-k} = e\right\} \subseteq A.$$

It follows that

$$A = \left\{(\overline{x}, x_{m+1}) \mid \overline{x} \in A_m, f(\overline{x})x_{m+1}^{-k} = e\right\}$$

and the theorem follows by induction hypothesis. □

Proposition 6.36. *Any definable subgroup of $(S_1)^n$ is (also) definably isomorphic to a cartesian power of S_1.*

PROOF. By Theorem 6.35 A is the finite intersection of solution sets of monomials.

The proof will follow by a double induction: We will first show by induction on n that the solution set of a single monomial in $(S_1)^{n+1}$, if definably connected, is isomorphic to $(S_1)^n$. We will then show the statement by an induction on the number of monomials defining a subgroup A.

Let $f(\overline{x}) = x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n}$ be the monomial defining a subgroup A. Let $d = GCD(k_0, k_n)$ and consider the map τ from S_1 to $(S_1)^{n+1}$ defined by

$$t \mapsto (t^{\frac{k_n}{d}}, e, e, \ldots, e, t^{\frac{-k_0}{d}}).$$

Then

$$f\left(\tau\left(t\right)\right)=\left(t^{\frac{k_n}{d}}\right)^{k_0}\cdot\left(t^{\frac{-k_0}{d}}\right)^{k_n}=e$$

so τ maps S_1 to A. Also, $\tau(t)$ is the identity if and only if $t^{\frac{k_n}{d}}=t^{\frac{k_0}{d}}=e$ and $\frac{k_0}{d}$ and $\frac{k_n}{d}$ are relatively prime, this implies $t=e$. So τ is injective.

If π_1 is the projection to the first coordinate, the subgroup $\ker(\pi_1)$ of A is isomorphic a subgroup of $(S_1)^n$ defined by the monomial $x_2^{k_2}\dots x_n^{k_n}$, so by induction hypothesis it is isomorphic to $(S_1)^{n-1}$. If t is such that $\tau(t)\in\ker(\pi_1)$ then $t^{\frac{k_n}{d}}=e$ and $t\in R_{\frac{k_n}{d}}$. The intersection $\ker(\pi_1)\cap\tau(S_1)$ is finite, so by Exercise 6.7 it is equal to the subgroup R_l of $\tau(S_1)$ for some l. The following claim proves that A is isomorphic to S_n:

Claim 6.37. *Let G be an abelian group S,B be subgroups of G with $G=SB$, S isomorphic to S_1, B isomorphic to $(S_1)^{n-1}$. Let $R_k^S:=\{x\in S\mid x^k=e\}$ and assume that $S\cap B=R_k^S$. Then G is isomorphic to $(S_1)^n$.*

Proof: The map $\phi:S\times B\to G$ sending (s,b) to sb is surjective by hypothesis, and the kernel is $(x,x)\mid x\in S\cap B$ which is isomorphic to R_k^S which is isomorphic to the subgroup R_K of S_1. So it is enough to prove that the quotient of $S_1\times(S_1)^{n-1}$ by a subgroup H such that the projection to the first coordinate is an isomorphism to R_k is isomorphic to $S_1\times(S_1)^{n-1}$.

Let $H\leq S_1\times(S_1)^{n-1}$ be such a subgroup. Let ρ be a primitive k-root of unity in S_1 and consider the subgroup H_ρ of $S_1\times(S_1)^{n-1}$ generated by $(\rho;e,e,\dots,e)$ with e the identity of S_1.

The preimage of ρ in H under the projection to the first coordinate $(h;h_1,\dots h_{n-1})$ must of course have order k so $h_i\in R_k$ for all i, and in particular $h_i=\rho^{t_i}$ for some t_i. Consider the map $\mu:S_1\times(S_1)^{n-1}\to S_1\times(S_1)^{n-1}$ given by

$$(s_0;s_1,\dots,s_{n-1})\mapsto(s_0;s_1s_0^{t_1},s_2s_0^{t_2},\dots,s_{n-1}s_0^{t_{n-1}}).$$

This is a group homomorphism with $\mu(H_\rho)=H$. So

$$\left(S_1\times(S_1)^{n-1}\right)/H_\rho\cong\left(S_1\times(S_1)^{n-1}\right)/H.$$

By construction

$$\left(S_1\times(S_1)^{n-1}\right)/H_\rho\cong(S_1/\langle\rho\rangle)\times(S_1)^{n-1}\cong S_1\times(S_1)^{n-1}$$

as required. $\blacksquare_{Claim}$

Since the image of monomials under monomial maps is a monomial, the induction on the number of monomials follows. This completes the proof of the theorem. $\square$

6.4.1. Compact linear algebraic groups. We will now prove that any definably compact abelian group is semialgebraic. We will first need to give some definitions and theorems of classic algebraic groups. Let $\mathcal{R}$ be a real closed field and $\mathcal{R}(i)$ be its algebraic closure. Let $GL_n(\mathcal{R}(i))$ be the general linear group over $R(i)$. Identify $GL_n(\mathcal{R})$ with its natural image in $GL_n(\mathcal{R}(i))$.

Remark 6.38. The concept of Zariski closed is usually done in the context of algebraically closed fields. In that sense, there are two ways to interpret Definition 6.30: That G is the real points of the subset of $Gl_n(\mathbb{C})$ which is the zero set of complex polynomials, or that there are polynomials with coefficients in $\mathcal{R}$ whose zero subset of $Gl_n(\mathcal{R})$ is G.

Both definitions agree: The real polynomials imply complex ones is trivial. The other direction follows from the fact that tuples of real numbers are invariant under complex conjugation, and that the ideals of complex polynomials which are closed under complex conjugation are generated by polynomials with real coefficients.

REMARK. *In the context of linear algebraic groups in* [**30**]*, the algebraic closure is defined at the level of Lie algebras, where given a Lie subalgebra* $\mathfrak{g}$ *of* $\mathfrak{gl}_n(\mathcal{R})$ *the lie closure* $\mathfrak{g}^a$ *of* $\mathfrak{g}$ *is defined to be the smallest Lie algebra containing* $\mathfrak{g}$ *such that for some algebraic group* H *we have* $L(H) = \mathfrak{g}^a$*.*

By Corollary 3.29, if $\mathfrak{g} = L(G)$ *and* G *is a connected linear subgroup of* $GL_n(\mathcal{R})$*, then* $L((G^a)^0) = \mathfrak{g}^a$ *where* G^a *is the algebraic closure of* G *in* $GL_n(\mathcal{R})$*.*

Definition 6.39. A subset of $Gl_n(\mathcal{R})$ is *constructible* if it is the union of intersections of algebraic sets and complements of algebraic sets.

For definable subset X of $Gl_n(\mathcal{R})$, let X^a be the Zariski closure of X in $Gl_n(\mathcal{R})$.

Fact 6.40. *If* H *is a constructible subgroup of* $Gl_n(\mathcal{R})$*, then* $H^a = H$*.*

PROOF. Since H is constructible, H contains an Zariski open subset of H^a. But Zariski open subsets are dense, so $H^a \subset H \cdot H = H$. $\square$

Fact 6.41. *If* H *is a subgroup of* $Gl_n(\mathcal{R})$*, then* H^a *is a subgroup of* $Gl_n(\mathcal{R})$*.*

PROOF. Since H^a is algebraic, for any $h \in H$ the set $\{x \mid hx \in H^a\}$ is an algebraic set containing H, so $H^a \subseteq \{x \mid hx \in H^a\}$ and $hH^a \subset H^a$.

Similarly, $\{x \mid xH^a \subset H^a\}$ is now an algebraic set containing H, so it must contain H^a. $\square$

Lemma 6.42. *H is linear, algebraic and irreducible, then* $[H, H]$ *is algebraic.*

PROOF. Let $[H, H]_1 = \{xyx^{-1}y^{-1} \mid x, y \in H\}$ and we define inductively $[H, H]_{n+1} = ([H, H]_n)([H, H]_n)$.

$[H, H]_n$ is constructible and connected for all n, so $([H, H]_n)^a$ is a chain of irreducible Zariski closed subsets of $GL_n(\mathcal{R})$ which implies that it stabilizes and for some n,

$([H, H]_n)^a$ is a subgroup of $GL_n(\mathcal{R})$ containing $[H, H]_{n+1}$. But $[H, H]_n$ is constructible, so Zariski open and Zariski dense in $([H, H]_n)^a$, so $([H, H]_n)^a = ([H, H]_n)([H, H]_n) = [H, H]_{n+1}$. Therefore $[H, H]_{n+1} = [H, H]$ is a constructible subgroup, so it is algebraic. $\square$

The following appears in Sections 2.3 and 2.4 in [**6**].

Fact 6.43. *Let* H *be a closed subgroup of* $GL_n(\mathcal{R})$*. Then the following hold.*

H solvable H^a solvable

The connected components of $[H, H]$*, of* $([H, H])^a$*, and of*

$$[H^a, H^a]$$

are all equal.

Let $G \in GL_n(\mathcal{R})$. We will say that G is *diagonalizable* if it has a conjugate in $GL_n(\mathcal{R}(i))$ which is a subgroup of the diagonal subgroup D_n of $GL_n(\mathcal{R}(i))$.

Let $U_n(\mathcal{R})$ be the subgroup of $GL_n(\mathcal{R})$ consisting of upper triangular matrices with ones in the diagonal.

The following appear in [**6**], Sections 4.5, 4.6, 4.7 and 4.8.

Fact 6.44. *Let* G *be a commutative* $\mathcal{R}$*-algebraic linear group. Then there are* $\mathcal{R}$*-algebraic subgroups* G_s *and* G_u *such that:*

(1) *G_s is diagonalizable and contains all diagonalizable elements (i.e., all matrices which can be diagonalized in $\mathcal{R}(i)$).*
(2) *G_u is unipotent and is $\mathcal{R}$-conjugate to a subgroup of $U_n(\mathcal{R})$.*

(3) $G \cong G_s \times G_u$.

Definition 6.45. An $\mathcal{R}$-algebraic subgroup of $GL_n(\mathcal{R})$ is a $\mathcal{R}$-*torus* if it is diagonalizable.

Fact 6.46. *Given any $\mathcal{R}$-algebraic torus T there are algebraic subgroups $T_a(\mathcal{R})$ and $T_d(\mathcal{R})$ such that $T_d(\mathcal{R})$ has a $\mathcal{R}$-conjugate which is a subgroup to the diagonal group in $GL_n(\mathcal{R})$ and $T(\mathcal{R}) = T_a(\mathcal{R})T_d(\mathcal{R})$ and $T_a(\mathcal{R}) \cap T_d(\mathcal{R})$ is finite. If $\mathcal{R} = \mathbb{R}$, then $T_a(\mathcal{R})$ is compact.*

These are all results in Section 8 of [**6**]. The equivalence of Definition 6.45 with the one in [**6**] for linear algebraic groups is 8.5. 8.15 gives the decomposition $T(\mathcal{R}) = T_a(\mathcal{R})T_d(\mathcal{R})$ where T_a is the anisotropic tori and T_d is the $\mathcal{R}$-split tori. That being anisotropic over $\mathbb{R}$ is equivalent to being compact is 8.16.

Theorem 6.47. *Let H be a closed subgroup of $GL_n(\mathcal{R})$. Then there are algebraic groups H_1 and H_2 such that $(H_1)^0 \leq H \leq (H_2)^0$ and $(H_2)^0/(H_1)^0$ is abelian.*

Proof. Take $H_2 = H^a$ and $H_1 = [H^a, H^a] = [H, H]$. Then H_2/H_1 is abelian and $(H_1)^0 \leq H \leq (H_2)^0$ by the previous fact.

Since $H_1/(H_1)^0$ is finite, $H_2/(H_1)^0$ is a finite extension of an abelian subgroup. So $(H_2/(H_1)^0)^0 = (H_2)^0/(H_1)^0$ is abelian. □

Theorem 6.48. *The following hold.*

(1) Any abelian definably compact definable subgroup of $GL_n(\mathcal{R})$ can be conjugated into a subgroup of $D_n^K(\mathcal{R})$, and is definably isomorphic to a cartesian power of S_1.

(2) Any definably compact definable subgroup of $GL_n(\mathcal{R})$ is semialgebraic.

Proof. Let G be an abelian definably compact definable subgroup of $GL_n(\mathcal{R})$, let G^a be the algebraic closure of G.

Claim 6.49. *$G^a(\mathcal{R})$ is definably compact.*

Proof: $G^a(\mathcal{R})$ is abelian, so by Facts 6.44 and Fact 6.46, there are algebraic subgroups T_a, T_d and G_u of G such that $G(\mathcal{R}) = (T_a(\mathcal{R}))(T_d(\mathcal{R}))(G_u(\mathcal{R}))$, $G_u(\mathcal{R})$ can be $\mathcal{R}$-conjugated to a subgroup of $U_n(\mathcal{R})$ and $T_d(\mathcal{R})$ can be $\mathcal{R}$-conjugated to an $\mathcal{R}$-subgroup of the diagonal group of $GL_n(\mathcal{R})$. Let $G_s(\mathcal{R}) = (T_a(\mathcal{R}))(T_d(\mathcal{R}))$.

$G_s(\mathcal{R}) \cap G_u(\mathcal{R}) = e$ so the map sending $g \in G(\mathcal{R})$ to (g_s, g_u) with $g = g_s \cdot g_u, g_s \in G_s(\mathcal{R}), g_u \in G_u$ is definable. Now, U_n is the set of upper triangular matrices with ones in the diagonal so for $A = (a_{i,j}) \in U_n$, if $a_{i,j}$ is a non zero element with smallest value of $i + j$ among non zero non diagonal elements and $A^n = \left(a_{i,j}^{(n)}\right)$ with $a_{i,j}^{(n)}$ defined as n times $a_{i,j}$. So if the conjugate of g_u in U_n has a non zero element outside the diagonal for some g the values of the entries of conjugates of the unipotent g_u's for g in G would be unbounded contradicting definable compactness. So g_u must be the identity for any $g \in G$ and $G(\mathcal{R}) \subseteq G_s(\mathcal{R}) = T_a(\mathcal{R})T_d(\mathcal{R})$.

Similarly, if $g \in T_a(\mathcal{R})T_d(\mathcal{R})$ then the $\mathcal{R}(i)$-eigenvalues of g cannot have positive norm, which implies that if we decompose $g = g_a \cdot g_d$ with $g_a \in T_a$ and $g_d \in G_d$ then g_d can be conjugated to a diagonal matrix with only 1 and -1 in the diagonal. These are all anisotropic, so $G(\mathcal{R}) \subset T_a(\mathcal{R})$. Since $T_a(\mathbb{R})$ is compact by Fact 6.46 it is closed and bounded, so $T_a(\mathcal{R})$ is closed and bounded and $G_a(\mathcal{R})$ is definably compact. $\blacksquare_{Claim}$

$G^a(\mathcal{R})$ is an algebraic definably compact abelian subgroup of $GL_n(\mathcal{R})$, so by Lemma 6.31 it can be conjugated to a subgroup of $D_n^K(\mathcal{R})$. So G can be conjugated to a subgroup of $D_n^K(\mathcal{R})$.

Since $D_n^K(\mathcal{R})$ is definably isomorphic to a cartesian power of S_1, Proposition 6.36, implies that the conjugate of G in $D_n^K(\mathcal{R})$ (and therefore G itself) is definably isomorphic to a cartesian power of S_1's.

For the second item, let K be a definably compact subgroup of $GL_n(\mathcal{R})$. By Theorem 6.47 there are algebraic subgroups K_0 and K_1 of $GL_n(\mathcal{R})$ such that $K_0 \leq K \leq K_1$ and K_1/K_0 is abelian, and K_1 the algebraic closure of K. By (1), K_1 is definably compact. K_1/K_0 is therefore an abelian, linear (by [**6**][6.8]) definably compact group so by Lemma 6.31 it is definably isomorphic to a direct product $(S_1)^k$ for some k.

The subgroup K/K_0 of K_1/K_0 is therefore the solution set of monoids by Proposition 6.35, so it is so definable with matrix multiplication. So K is semialgebraic. □

6.5. Overview of the Torsion Theorem

The Torsion Theorem was first posed as a question in [**59**]. There are some reductions to the problem that we can achieve already. By Lemma 2.33 it is enough to prove the Torsion Theorem for definably compact *abelian* group has torsion. And Proposition 6.7 implies that it is enough to prove that any definably connected definably compact abelian group has Euler characteristic zero.

(As a side note, this already implies that the only counterexamples can only happen with even dimension: Any compact orientable real manifold of odd dimension has Euler characteristic 0. These can be expressed in first order, so it transfers to any real closed field.)

The first proof of the Torsion Theorem appeared in the paper [**26**] by Edmundo and Otero. They continued the development of o-minimal homology theory that Woerheide started in [**77**], dualizing the results to get a $\mathbb{Q}$-group cohomology. They then link the singular and o-minimal homology via triangulations to eventually prove that the Euler characteristic is zero. In this paper, they achieve the full structure of the torsion for definably compact abelian groups.

A second proof (in [**4**]) avoids the cohomology part, but to sketch the proof we should mention Lefschetz fixed point theorem. It essentially states that if K is a finite simplicial complex which is orientable as a manifold and of characteristic different from zero, then any a continuous map from K to K which is definably homotopic to the identity has fixed points. Once one has a definable version of Lefschetz, Euler characteristic zero is easily implied on any definably compact definable group, since translations are continuous maps definably homotopic to the identity with no fixed points. The proof of Lefschetz in [**4**] is done by transfer theorems: Using triangulations, they realize definable groups as simplices which, being semilinear, can be interpreted in any $\mathcal{R}$, in any o-minimal structure $\mathbb{R}$, or in the real algebraic numbers which is a common substructure of both. They then find an homological statement which is equivalent to Lefschetz which transfers from $\mathcal{R}$ to $\mathbb{R}$ where we already know is true. So one can avoid the cohomology but many results of o-minimal homology from [**77**] are still used.

The first "homology free" proof appeared in [**57**]. They introduce definable Morse functions and prove that the Euler Characteristic of a definably compact oriented manifold is the degree of a map, a degree that is then shown to be zero for any definable group using the translations. This approach uses differential geometry beyond what we mentioned in Chapters 1 and 3, as well as the theory of definable Morse functions, both of which we feel are beyond the scope of this book.

Another proof of the Torsion Theorem, as was sketched in [**38**], just uses model theory. One proves looks at realizations of a definably compact abelian G on a saturated model. One then shows that the smallest type definable group of bounded index G^{00} is not all of G, so that G/G^{00} is an infinite real Lie group, so it has torsion. Since G^{00} is divisible, said torsion must come from G.

All of these approaches develop algebraic topology, differential geometry or the model theory of definably compact groups, and having a full proof would entail developing one of this areas all of which have been developed beautifully in part by the authors mentioned above but we feel that this is beyond what we think is the right approach for this book. And somehow this complexity seems unavoidable. The classical version of torsion problem or, namely, the fact that the Euler characteristic of a compact connected real Lie group is zero, has many proofs but all entail some structure on the manifold that is implied by the Lie group operation: The existence of the exponential surjective map from a vector field, the contrapositive of the Lefchetz fixed point theorem for orientable simplices, the existence of a Morse function with no zeros or the existence of an invariant vector field (which then implies Euler characteristic zero by Poincaré-Hopf Theorem) all entail the existence of either a function with particular properties or of a vector field which are all second order statements so a "purely geometric" may be impossible.

Historical notes and comments

The use of Euler characteristic and Sylow-like theorems for groups definable in o-minimal structures was done by Strzebonski in [**69**]. All of the results in Section 6.1 appear in this paper. Although we present a different approach, many of the results of Section 6.2, appear in [**55**].

The work of 0-Sylow (Subsection 6.2.1) was started by Strzebonski in [**69**], although Proposition 6.24 appeared in [**13**]. Many of the results in Section 6.3 can be found in [**13**] and in the papers by Berarducci [**2**] and by Conversano [**14**]. The theory behind Section 6.4 was developed by Peterzil, Pillay and Starchenko in [**53**] although our approach is closed to the classical theory in [**6**].

Besides the proof of the Torsion Theorem which we spoke about in Section 6.5, another important line of research related to definably compact groups which we omit is the study of the type definable connected component and what is known as compact domination. The idea is that given any definably compact definable group one can recover a real Lie group which "dominates" by taking a quotient by a type definable subgroup. This was known as "Pillay's Conjecture" that was stated in [**62**] and proved in [**5**] and [**38**] (using in a very important way Edmundo's structure of the torsion subgroup in the abelian case). Compact domination was proved in [**39**]. Although quite important in the study of groups definable in o-minimal structures, the statements are vacuous for groups defined over the real field which is in many ways the motivating topic of this book.

CHAPTER 7

Torsion-free definable groups

In this chapter we will study definable torsion free. By the torsion theorem, and analogously with Lie groups, torsion-free definable groups are precisely the definably connected groups such that the trivial subgroup is the only definably compact subgroup.

That torsion free subgroups were a key component was probably realized in [**59**] where it was proved (in a beautiful although quite technical result included in Section 7.2) that if a group was not definably compact then it must have a definable torsion free 1-dimensional subgroup. The analogous property for Lie groups is of course true (Lemma 3.6 and Lemma 3.12 in [**41**]), and can be proved analyzing the one dimensional subgroups a tool which of course is not available in the definable category.

Another thing in common between Lie groups and definable groups is the following: Any Lie group that is not compact contains a one dimensional torsion-free closed subgroup and any definable group that is not definably compact contains a one dimensional torsion-free definable subgroup.

Despite what one might think by the previous analogies, there are important differences between torsion free definable groups and torsion free Lie groups. The definable category is much more restricted, and it is precisely this that makes torsion free definable subgroups a central tool in the analysis of definable groups.

The o-minimal Euler characteristic. An important tool in the study of definable groups (that is not available for Lie groups) is the o-minimal Euler characteristic. In Chapter 6 we discovered that definable torsion-free groups are completely characterized by the o-minimal Euler characteristic. Namely, a definable group is torsion-free if and only if $|E(G)| = 1$ (Proposition 6.7). It follows that every definable quotient of a torsion-free definable group is torsion-free too (Theorem 6.12). This is of course not true in the Lie category, where quotients by discrete subgroups may very well originate torsion elements (e.g. $\mathbb{R}/\mathbb{Z}$).

Moreover, because of this property that is preserved in definable quotients, we can think of torsion-free definable subgroups as p-subgroups with $p = 1$, and we will show the existence of 1-Sylow subgroups, in the sense that every torsion-free definable subgroup is contained in a maximal one, which is unique up to conjugation (Theorem 7.34).

There is a unique maximal normal torsion free definable subgroup. Using the above mentioned characterization of torsion-free definable groups provided by the Euler carecteristic, we will show that every definable group G contains a unique maximal torsion-free normal definable subgroup that we denote by G^{tf} (Proposition 7.22). In Lie groups such unique subgroup cannot always be found.

Example 7.1. Let G be the two dimensional torsion-free abelian group $\mathbb{R} \times \mathbb{R}$ and consider the closed subgroup $N = \{(z, z) : z \in \mathbb{Z}\}$. The one dimensional connected subgroups $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ intersect trivially N, therefore they both embed in the quotient G/N, which is the product of these two torsion-free one dimensional subgroups. However, G/N is not torsion-free, as every element in G of the form (q, q) with $q \in \mathbb{Q}$, $q \notin \mathbb{Z}$, maps to a non-trivial torsion element in G/N.

EXERCISE 7.1. *Prove that the group G/N described in Example 7.1 is isomorphic to $\mathbb{R} \times S_1(\mathbb{R})$, the direct product of the additive group and the circle group.*

The Lie group G/N above contains two different normal torsion-free closed connected subgroup of maximal dimension, while in definable groups this cannot happen and G^{tf} will be precisely the only normal torsion-free definable subgroup of maximal dimension.

G^{tf} will play a major role in the structural analysis of a definable group.

The role of torsion-free subgroups in the structure of a definable group. We will show in Theorem 7.26 that when G is solvable, then G/G^{tf} is definably compact.

REMARK. *This is another major difference with Lie groups, in two ways. There are many solvable groups without any infinite torsion-free closed normal subgroup and which are not compact. The smallest such example is a quotient of the three dimensional Heisenberg group by a central discrete subgroup.*

Moreover, Proposition 6.25 states that definably compact definably connected solvable groups are abelian, so R/G^{tf} is a definable torus where R is the solvable radical of a definably connected G. Therefore, for every definable group G there is an exact sequence of definable subgroups:

$$1 \longrightarrow G^{\mathrm{tf}} \longrightarrow R \longrightarrow R/G^{\mathrm{tf}} \longrightarrow 1$$

where G^{tf} is torsion-free, R/G^{tf} is a definable torus and G/R is semisimple, identifying three main classes of groups: torsion-free, definably compact and semisimple groups. The last two classes were studied in the previous two chapters, and in this chapter we focus on the first class.

Besides torsion free subgroups playing a significant role in the structure of any definable group, the fact that they are torsion free implies much more structure on a group in the definable case (compared with Lie and other cases).

Torsion free definable groups are solvable. There is exactly one connected torsion-free Lie group with a simple Lie algebra: the universal cover of $SL_2(\mathbb{R})$. It follows that if a connected Lie group is torsion-free, then its semisimple Levi subgroup is isomorphic to a direct product of finitely many copies of $\widetilde{SL}_2(\mathbb{R})$ ([**30**, Theorem 3.2]) (which could of course be trivial).

This may not initially seem like an obstruction, since already Example 4.35 is a definable group whose Levi subgroup is definable $\widetilde{SL}_2(\mathbb{R})$. However, the fact that the center is divisible implies that the definable example has torsion.

EXERCISE 7.2. *Let $G = \widetilde{SL}_2(\mathcal{R}) \times_{\mathbb{Z}} (\mathcal{R}, +)$ (the definable group from Example 4.35). Show that G has torsion.*

In a definable torsion-free group the semisimple Levi subgroup is always trivial:

Proposition 7.2. *Let G be a torsion-free definable group. Then G is definably connected and solvable.*

PROOF. If G is not definably connected, then G/G^0 is a finite group and $E(G) = E(G^0)E(G/G^0) = \pm|G/G^0|$ by Fact 6.2 and Proposition 6.7, since G^0

is torsion-free. However, $|E(G)| = 1$ by Proposition 6.7 again, so $G = G^0$ is definably connected.

If G is semisimple, Theorem 6.12 implies $E(G) = 0$, contradiction.

If G is not solvable and R is its solvable radical, we have seen in Chapter 5 that G/R is semisimple, and $E(G) = E(R)E(G/R) = 0$, contradiction. So G/R must be trivial and $G = R$ is solvable, as required. □

Corollary 7.3. *Let G be a torsion free definable group. Then G contains a normal definable abelian subgroup.*

PROOF. Follows from the proposition and Proposition 2.28. □

Torsion free definable groups are *completely solvable*. Consider the following example:

Example 7.4. Let γ be the action from $\mathbb{R}$ on $\mathbb{R} \times \mathbb{R}$ given by rotation, so that

$$\gamma(t)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Let D be the group $\mathbb{R} \ltimes_\gamma (\mathbb{R} \times \mathbb{R})$.

D is torsion-free but it is not definable, nor it is Lie isomorphic to any definable group:

EXERCISE 7.3. *Prove that the group D defined in Example 7.4 is torsion-free and that $Z(D)$ is a discrete infinite subgroup.*

This example of a non definable group gives the insight to the main structural theorem for torsion free definable groups: Theorem 7.32, proves that any definable torsion-free group contains a definable normal one dimensional torsion-free subgroup. This will imply that any definable torsion-free group is *completely solvable* (see Definition 7.30), a condition that will characterize which connected torsion-free (solvable simply connected) Lie groups are isomorphic to a definable group.

As mentioned above, one of the important technical tools for understanding torsion free subgroups is the fact that any non definably compact group has a definable one dimensional torsion free subgroup, which we will prove in Section 7.2. In Section 7.1 we will understand the structure of one dimensional definable groups. In Section 7.3 we will prove the existence of the maximal normal torsion free definable group G^{tf} for any definable group G. We will then prove in Section 7.4 that G^{tf} is completely solvable. Finally, Section 7.6 will be devoted to classifying all torsion free linear definable groups in o-minimal expansions of the real field.

7.1. One dimensional torsion-free groups

We already know from Proposition 7.2 that torsion-free definable groups are definably connected and solvable. If we also assume that $\dim(G) = 1$ then Theorem 2.32 states that G is abelian and divisible and it must admit a definable order by Theorem 2.31.

One dimensional torsion-free groups may occur in two kinds: those that have infinite definable families of automorphisms, and those that don't. Both kinds may or may not happen in any particular o-minimal theory: in $(\mathbb{R}.+, \cdot, <)$ the additive group has an infinite definable family of automorphisms, whereas the multiplicative group does not; in $(\mathbb{R}.+, \cdot, e^x, <)$ all one dimensional torsion-free groups are definably isomorphic to the additive group, so all admit infinite families of automorphisms.

In both examples above, the group that we exhibit as admitting an infinite definable family of automorphisms is the additive group. Turns out that this is always the case: If a torsion-free group admits an infinite definable family of automorphisms, then it is definably isomorphic to the additive group of $\mathcal{R}$ and the infinite definable family comes from the pullback of scalar multiplications under said isomorphism (Theorem 7.9). This is quite useful in understanding both torsion-free groups definable in o-minimal expansions of real closed fields and in understanding definable automorphisms of torsion-free abelian definable groups (recall that the group of linear automorphisms of $(\mathcal{R},+)^n$ is $GL_n(\mathcal{R})$, so it is always definable).

Proposition 7.5. *Let $(L,+)$ be a definable one dimensional torsion-free (abelian) subgroup and let e be the identity and $<$ be the definable order so that*

$$(L,+,e,<)$$

is an ordered group with identity e (Theorem 2.31). Assume that there is an infinite definable group Γ of automorphisms of $(L,+)$ and let Γ^+ be the set of order preserving automorphisms of L. Then Γ^+ contains the connected component of Γ, and Γ^+ is in a definable bijection with $L^{>e}$.

The bijection between Γ^+ and $L^{>e}$ endows L with a multiplication that makes L an ordered field.

Proof. Fix an element $l \in L$ with $e < l$. By Theorem 2.31 we may assume that the universe of L is an interval I. After a linear transformation we may assume that the universe is $(-1,1)$ and $e=0$. In particular, $L^{>e}=(0,1)$.

Claim 7.6. *If $\gamma,\sigma \in \Gamma$ we have that $\gamma(l)=\sigma(l)$, then $\gamma=\sigma$.*

Proof: The set $\{x \in L \mid \gamma(x)=\sigma(x)\}$ is a subgroup of L. If $\gamma(l)=\sigma(l)$ the subgroup is not trivial so it must be all of L and $\sigma(x)=\gamma(x)$ for all x, as required. $\blacksquare_{Claim}$

In particular, Γ is a one dimensional definable group, so it is commutative.

Let Γ^+ be the set of order preserving automorphisms of Γ. Since Γ^+ is a subgroup of Γ of order at most two, Γ^+ contains the connected component Γ^0 of Γ.

Furthermore, $l>0$ so $\gamma(l)>0$ for all $\gamma \in \Gamma^+$. It follows that $\gamma \mapsto \gamma(l)$ is an injection from Γ^+ to $(0,1)$.

Claim 7.7. *$\gamma(l)>l$ if and only if $\gamma(x)>x$ for all $x \in (0,1)$.*

Proof: Since $\gamma \neq Id$ implies $\gamma(x) \neq x$ for any x, $(0,1)$ is partitioned by the open sets $\{x \in L \mid \gamma(x)>x\}$ and $\{x \in L \mid \gamma(x)<x\}$. Since $(0,1)$ is definably connected, one of the two must be empty. $\blacksquare_{Claim}$

Claim 7.8. *The set $\Gamma^+(l) := \{\gamma(l) \mid \gamma \in \Gamma^+\}$ is $(0,1)$.*

Proof: By connectedness, it suffices to show that $\{\gamma(l) \mid \gamma \in \Gamma^+\}$ is unbounded above and below in $(0,1)$. We will prove it is unbounded above, the other case being analogous.

Notice that because we are in Γ^+, $\gamma(l)<l$ if and only if $\gamma^{-1}(l)>l$ so by Claim 7.7 we can fix some $\gamma \in \Gamma^+$ such that $\gamma(x)>x$ for all $x \in L$. $\Gamma^+(l)$ is infinite (it contains all elements of the form $\gamma \circ \cdots \circ \gamma(l)$). By cell decomposition, there is an interval (c,a) contained in $\Gamma^+(l)$ such that either $a=1$ or $(a,1) \cap \Gamma^+(l)$ is finite.

If $a \neq 1$ then $a \in (0,1)$ so in particular $\gamma(a)>a$ and, because γ is an order preserving homeomorphism, $\gamma((c,a))=(\gamma(c),\gamma(a))$ contradicting our choice of (c,a). So $a=1$ and $\Gamma^+(l)$ is unbounded above.

In the same way, $\Gamma^+(l)$ is unbounded below so $\Gamma^+(l)=(0,1)$. $\blacksquare_{Claim}$

For any $a \in (0,1)$, let λ_a be the unique element in Γ^+ such that $\lambda_a(l) = a$. We can now define a multiplication on L

$$a \cdot x := \begin{cases} \lambda_a(x) & a > 0 \\ 0 & a = 0 \\ \lambda_{-a}(x) & a < 0. \end{cases}$$

The $l \cdot x$ is the identity under this definition and, since λ_a is an homeomorphism,

$$a \cdot (b + c) = \lambda_a(b + c) = \lambda_a(b) + \lambda_a(c) = (a \cdot b + a \cdot c).$$

Γ^+ is commutative which implies that the operation ($\cdot$) is commutative. If $\gamma \in \Gamma^+$ we have $1/\gamma \in \Gamma^+$ and $\lambda_a((\lambda_a)^{-1}(l)) = l$ so $\lambda_a^{-1}(l)$ is the multiplicative inverse of a in L. It follows that $(L, \cdot, +)$ is a field. $\square$

Theorem 7.9. *If $(L, +_L, \cdot_L)$ is an ordered field structure on a one dimensional torsion-free group $(L, +)$, then $(L, +_L, \cdot_L)$ is definably isomorphic to the field structure on $\mathcal{R}$.*

If $(L, +_L)$ is a one dimensional ordered group and there is an infinite definable group of automorphisms of L. Then $(L, +_L)$ is definably isomorphic to the additive group in $\mathcal{R}$.

Proof. We will use $+, \cdot$ to be to operations on the field $\mathcal{R}$.

Let $(L, +_L, \cdot_L, 0_L, 1_L)$ is a ring structure with $(L, +_L)$ an ordered one dimensional torsion-free group. Once again we may assume (Theorem 2.31) that L is $(-1, 1)$ with $0_L = 0$, order preserving addition, and such that the group topology is the order topology. Let us now consider the multiplicative structure $M_L := (L^{>0}, \cdot_L)$. By Theorem 2.1 there is an open V of codimension one where the group structure on M_L and the order topology on V coincide. Since multiplication in an ordered field preserves the order, this implies that the group topology of M_L is the order topology. By the Lie structure of G proved in Chapter 4, there is a neighborhood (interval) U of 1_L where the group operations of M_L are C^1, and the C^1-group structure of M_L is comes from endowing M_L with a manifold structure given by translates of U. Multiplicative translates of intervals in an ordered field are intervals, so M_L is a Lie group with the order topology. Similarly, $(L, +_L)$ is a Lie group with the order topology, and we have that both $+_L$ and $\cdot_L$ are differentiable operations.

For any $a \in L$, consider the map $a(x) = a \cdot_L x$. It is an homoemorphism from the ordered group $(L, +_L)$ onto itself and the topology is the order topology, $a(x)$ is a continuous function and it is therefore smooth. Let $\lambda(a) \in \mathcal{R}$ be the derivative $\mathrm{d}_0(a(x))$ of the function $a(x)$ at 0. $\lambda(a \cdot_L b)$ is thus the derivative of the operation $(a \cdot_L b) \cdot_L (x)$ at 0 which by associativity of multiplication and the chain rule is equal to the $\mathcal{R}$-multiplication of the derivatives of $a(x)$ and $b(x)$ at 0 (recall that $b(0) = 0$) so that $\lambda(a \cdot_L b) = \lambda(a) \cdot \lambda(b)$.

The maps $x \mapsto x +_L e$ and $x \mapsto e +_L x$ are both the identity, so the derivative is 1 and if define $S(x, y) = x + y$ then $(\frac{\partial}{\partial x} S(x, y))(0, 0) = (\frac{\partial}{\partial y} S(x, y))(0, 0) = 1$.

If $a +_L b = c$ and $S(x, y) = x +_L y$, we know that $c(x) = a(x) +_L b(x) = S(a(x), b(x))$ and $\lambda(c) = d_0(c(x)) = d_0(a(x) +_L b(x)) = d_0(S(a(x), b(x)))$ which by the multivariable chain rule is equal to

$$\left(\left(\frac{\partial}{\partial x} S \right)(0,0), \left(\frac{\partial}{\partial y} S \right)(0,0) \right) \begin{pmatrix} d_0(a(x)) \\ d_0(b(x)) \end{pmatrix} = (1,1) \begin{pmatrix} \lambda(a) \\ \lambda(b) \end{pmatrix} = \lambda(a) + \lambda(b).$$

So the map $x \mapsto \lambda(x)$ is a definable ring homeomorphism from $(L, \cdot_L, +_L)$ to the field $\mathcal{R}$. Since there are no infinite definable subgroups in $(L, +_L)$ nor in the additive group, λ must be an isomorphism. This completes the first statement of the theorem.

The second statement follows by Proposition 7.5. $\square$

We will study actions of torsion free group on abelian torsion free groups.

Namely, we will prove that if G is a torsion-free group acting on an abelian torsion free group A then there is a subgroup V of A isomorphic to $(\mathcal{R},+)^k$ with $k \in \mathbb{N}$ closed under the action of G and such that the action of G on V is isomorphic to a definable subgroup of $GL_k(\mathcal{R})$. This will be key to prove Theorem 7.32, the main structure theorem for definable torsion-free groups.

Theorem 7.10. *Let G be a definable torsion-free group, $(A,+)$ an abelian definable torsion-free group, $L \leq A$ a one dimensional subgroup, and assume there is a non trivial definable action $(g,a) \mapsto g(a)$ of G on A.*

Then the following hold:

(1) *Let $g_1,\dots,g_n \in G$ and let $L_i := g_i(L)$. If $g_{i+1}(L) \notin g_1(L)+g_2(L)+\cdots+g_i(L)$ for any i and $v_1+\cdots+v_n = w_1+w_2+\dots w_n$ with $v_i, w_i \in L_i$ then $v_i = w_i$ for any i.*

(2) *There is a subgroup H of A containing the orbit*
$$G(L) := \{g(l) \mid g \in G, l \in L\}$$
of L under G such that H is definably isomorphic to a direct product $(L,+)^k$ for some $k \in \mathbb{N}$.

(3) *If $\{g(L) \mid g \in G\}$ is infinite, $G(L)$ is isomorphic to $(\mathcal{R},+)^k$ and the group of automorphisms of $G(L)$ induced by the action of G is isomorphic to a definable subgroup of $GL_k(\mathcal{R})$.*

Proof. L_i is a connected torsion-free one dimensional group so it has no definable subgroups. Thus, for every definable subgroup H of A, if $H \cap L_i \neq \{e\}$ then $L_i \subset H$ and therefore if $v_1+w_1 = v_2+w_2$ with $v_i \in H$ and $w_i \in L_i$ then $v_1 = v_2$ and $w_1 = w_2$. The first item follows from this observation by induction on n.

By (1), given any sequence of $g_1,\dots,g_k \in G$ with
$$g_{i+1}(L) \notin g_1(L)+g_2(L)+\cdots+g_i(L),$$
then $\dim(L_1+\cdots+L_j) = j$. Since the dimension of the group generated by $G(L)$ is bounded by $\dim(A)$, there is some k such that $g(L) \subset H := g_1(L)+g_2(L)+\cdots+g_k(L)$ for any $g \in G$, and $G(L) \subseteq H$.

The definable homomorphism $(x_1,\dots,x_k) \mapsto g_1(x_1)+g_2(x_2)+\cdots+g_k(x_k)$ from $(L,+)^k$ to H is an isomorphism by (1). This completes the proof of (2).

For (3), assume that H and g_i are as in (2). Since H is isomorphic to $(L,+)^k$, every $v \in H$ can be written uniquely as $v_1+v_2+\cdots+v_k$ with $v_i \in g_i(L)$ and the projection map π_i that sends $v \in H$ to v_i is an endomorphism from H to $g_i(L)$.

Claim 7.11. *There is an infinite definable group of automorphisms of L.*

Proof: For any $g \in G$ let $\sigma_{g,i}$ be the map from L to L given by
$$l \mapsto g_i^{-1}\left(\pi_i\left(g\left(l\right)\right)\right).$$
It is an endomorphism from L to L, so it is either an automorphism or it is the map $\mathbf{0}_e$ that sends everything to the identity $\{e\}$.

The map $\pi_i(g(v)) = g_i(\sigma_{g,i}(v))$ and $g(v) = \pi_1(g(v))+\pi_2(g(v))+\cdots+\pi_k(g(v))$ so that $g(v)$ is is determined uniquely by the tuple $(\sigma_{g,1}(v),\dots,\sigma_{g,k}(v))$. Therefore,
$$\{\sigma_{g,1}\}_{g\in G} \cup \{\sigma_{g,2}\}_{g\in G} \cdots \cup \{\sigma_{g,k}\}_{g\in G}$$

is infinite whenever $\{g(L) \mid g \in G\}$ is infinite. Since $\mathbf{0}_e$ is the only endomorphism of L which is not an automorphism,

$$\left(\{\sigma_{g,1}\}_{g\in G} \cup \{\sigma_{g,2}\}_{g\in G} \cdots \cup \{\sigma_{g,k}\}_{g\in G}\}\right) \setminus \{\mathbf{0}_e\}$$

is an infinite family of definable automorphisms of L, as required. $\blacksquare_{Claim}$

By Proposition 7.9, L is definably isomorphic to $\mathcal{R}$ which implies that there is a definable isomorphism between the group of definable automorphisms of $G(L) \cong (L,+)^k$ and the group of definable automorphisms of the k-th cartesian power of the additive group, i.e. elements of $GL_k(\mathcal{R})$. □

7.2. Existence one dimensional definable torsion-free subgroups

As mentioned in the introduction of the chapter, if a connected Lie group is not compact, then it contains a closed one dimensional torsion-free subgroup. We now show the definable analogous result:

Theorem 7.12. *Let $(G,\cdot)$ be a definable group. If G is not definably compact, then G contains a one dimensional torsion-free definable subgroup.*

Proof. We will develop a notion of "distance at 1" or "tangent at 1" between two open definable curves in G (recall that an open definable curve in G is a continuous definable map from $(0,1)$ to G).

For any open definable curve I on G given by a function f, we will denote by $I^{>t}$ the set $\{f(x) \mid x \in (t,1)\}$. Let $\mathcal{U}$ be a definable family of basis of the identity e in G closed under inverses [1]. To make notation simpler, s,t will always denote numbers between 0 and 1 and U,W will be elements of $\mathcal{U}$.

Claim 7.13. *Let I,J be open definable curves given by functions f,g respectively. Then the following are equivalent.*

(1) $\forall U \forall s \forall t,\ I^{>s} \cap (U \cdot J^{>t}) \neq \emptyset$.
(2) $\forall U \forall s \forall t,\ (U \cdot I^{>s}) \cap J^{>t} \neq \emptyset$.
(3) $\forall U \forall W \forall s \forall t,\ (W^{-1} \cdot I^{>s}) \cap (W \cdot J^{>t}) \neq \emptyset$.
(4) $\forall U \forall s \forall t,\ I^{>s} \cap (U \cdot J^{>t})$ *is an infinite set.*

If $I^{>s}$ is infinite for all s, then the following condition is also equivalent to the previous ones.

(5) $\forall U \forall t \exists s,\ I^{>s} \subset U \cdot I^{>t}$.

Proof: Let U be any element of $\mathcal{U}$. Notice that if $x \in I^{>s}$ with $x = u\cdot y$ with $u \in U$ and $y \in J^{>t}$. Then $y = u^{-1}x$ with $u^{-1} \in U^{-1}$. This proves the equivalence of the first two.

(1) is tautologically a stronger than (3). For the converse, continuity of multiplication and definition of a basis imply that for any $U \in \mathcal{U}$ there is some $W \in \mathcal{U}$ such that $W^{-1}W \subset U$. It follows that $z = w_1x_1 = w_2x_2$ for $w_1, w_2 \in W, x_1 \in I^s, x_2 \in J^t$, then $x_1 = (w_1^{-1}w_2)x_2$ with $w_1^{-1}w_2 \in U$ so (3) implies (1).

(4) $\Rightarrow$ (1) is a tautology. For the converse, if (4) fails, then for some U,s,t the set $I^{>s} \cap U \cdot J^{>t}$ is finite. So $F = \{x \in (0,1) \mid f(x) \in U \cdot J^{>t}\}$ is finite, let $S = \max(F)$. By construction $f(x) \notin U \cdot J^{>t}$ for any $x > S$ and by definition $I^{>S} \cap (U \cdot J^{>t}) = \emptyset$ contradicting (1).

If $I^{>s}$ is infinite, it is clear that (5) implies (4). The converse follows by o-minimality: If $I^{>s} \subset U \cdot J^{>t}$ is non empty for all s, then $\{x \in (0,1) \mid f(x) \in U \cdot J^{>t}\}$

[1] we will only use that for any $U \in \mathcal{U}$ there is $W \in \mathcal{U}$ with $W^{-1}W \subseteq U$.

is unbounded so it is cofinal and contains an interval $(a, 1)$ and $I^{>a} \subset U \cdot J^{>t}$. $\blacksquare_{Claim}$

It follows from the claim that if we define a binary relation $\mathbb{T}$ on definable curves where $I\mathbb{T}J$ if and only if any of the conditions in Claim 7.13 holds, then $\mathbb{T}$ is an equivalence relation.

Given any curve J defined by a function $h : (0, 1) \to G$ and any $g \in G$, we can operate J by g and define ${}_gJ$ to be the definable curve given by the function ${}_gh$ defined as ${}_gh(x) = g \cdot h(x)$.

Assume now that $(G, \cdot)$ is not definably compact. We fix a non completable definable curve I (recall Definition 2.15) defined by $f : (0, 1) \to G$. Both f and I will be fixed until the end of the proof.

The torsion-free one dimensional subgroup H of G will be the set of g for which ${}_gI$ is tangent at 1 to I. Explicitly,

$$H := \{g \in G \mid \ {}_gI\mathbb{T}I\}.$$

That H is a group follows by transitivity and symmetry of $\mathbb{T}$. For the rest of the requirements, we will first prove that it has dimension one, by showing that the dimension is at most one (using o-minimality) and showing that it is infinite, which will be done by proving that it intersects boxes of any small enough radius centered in the identity. Proving torsion-free will be more technical, and we will leave this for the end.

For $s \in I$, let $P_s := \{xy^{-1} \mid x, y \in I^{>s}\}$, let $\overline{P_s}$ and let $D := \bigcap_{s\in I} \overline{P_s}$.

Claim 7.14. *$H \subseteq D$.*

Proof: Let $l \in I$. We need to prove that $g \in \overline{P_s}$ whenever $g \in H$. Let $g \in H$.

By Claim 7.13,

$$\forall U \in \mathcal{U}, \forall s, t \in I,\ U \cdot_g I^{>s} \cap I^{>t} \neq \emptyset.$$

By definition $U \cdot_g I^{>s} = U \cdot g \cdot I^{>s}$. Notice also that if W is a neighborhood of g then $U := Wg^{-1} \in \mathcal{U}$. So for any neighborhood W of g and any $s, t \in I$ we have some $x \in I^{>s}$, $y \in I^{>s}$ and $g_1 \in W$ such that $g_1 \cdot x = y$. In particular, for any neighborhood W of g there are $x, y \in I^{>l}$ such that $yx^{-1} \in W$. This implies $W \cap P_s \neq \emptyset$ for any neighborhood W of g and $g \in \overline{P_s}$. $\blacksquare_{Claim}$

Claim 7.15. *$\dim D \leq 1$.*

Proof: Since D is contained in the (topological) closure of any $I^{>l} \cdot (I^{>l})^{-1}$ we get immediately $\dim(D) \leq 2$.

Notice that

$$D = \left(D \setminus \left(\bigcap_{l\in I} P_l\right)\right) \cup \bigcap_{l\in I} P_l.$$

We will prove that both sets in the right hand side have dimension less than one.

Let us consider first $D\setminus(\bigcap_{l\in I} P_l) = \bigcup_{l\in I} D\setminus P_l$. For each fixed l, $D\setminus P_l \subset \overline{P_l}\setminus P_l$ so by Fact 1.9 $\dim(D \setminus P_l) \leq 1$.

Let

$$S_l := (D \setminus P_l) \setminus \left(\bigcup_{s<l} D \setminus P_l\right).$$

Since $D \setminus P_s \subset D \setminus P_t$ for $s < t$, we have $D \setminus P_l = \bigcup_{s<l} S_l$ and since $\{S_l\}_{l\in I}$ is a disjoint family of sets of dimension at most one, Fact 1.9 implies that for any $l \in I$ the set $\{t < l \mid \dim(S_t) = 1\}$ is finite. This of course implies that the set

$\{l \in I \mid \dim(S_t) = 1\}$ is finite and (once more by Fact 1.9) $D = \bigcup_{l\in I} S_l$ has dimension one.

We now need to prove that $\bigcap_{l\in I} P_l$ has dimension one. Notice that if $g \in \bigcap_{l\in I} P_l$ then $X_g := \{(x,y) \in I \times I \mid xy^{-1} = g\}$ is infinite. But $\{X_g\}$ is a disjoint family of sets with $I \times I = X_g$ and $\dim(I \times I) = 2$, so by Fact 1.9

$$\dim\left(\{g \in G \mid \dim\left(X_g\right) \geq 1\}\right) \leq 1$$

so $\dim(\bigcap_{l\in I} P_l) \leq 1$ as required. $\blacksquare_{Claim}$

Claim 7.16. *Let U be a neighborhood of the identity in G which is diffeomorphic to an open set in $\mathcal{R}^n$. If we endow U with the metric in $\mathcal{R}^n$, then for any $\epsilon \in \mathcal{R}$ such that*

$$B_{\leq\epsilon}(e) := \{g \in G \mid d(g,e) \leq \epsilon\} \subset U.$$

Then there is a point a_ϵ in H such that $d(a_\epsilon, e) = \epsilon$.

In particular, H is infinite.

Proof: For simplicity we will assume that U is actually a subset of $\mathcal{R}^n$.

Fix $l \in (0,1)$, consider $(f(l))^{-1} \cdot I$. The set

$$B_{\leq\epsilon}(e) := \{g \in G \mid d(g,e) \leq \epsilon\}$$

is definably compact by Fact 2.18 so it cannot contain any non completable curve. In particular, $(f(l))^{-1} \cdot I^{\geq l} \not\subseteq B_{\leq\epsilon}(e)$. Since $e = f(l)(f(l))^{-1}$ and $f(l) \in I^{\geq l}$, if we define $b_l = \sup\{s \in (l,1) \mid f(s) \cdot (f(l))^{-1} \in B_{\leq\epsilon}(e)\}$, then $l < b_l < 1$, and by continuity

$$d\left(\left(f\left(b_l\right) \cdot f\left(l\right)^{-1}\right), e\right) = \epsilon.$$

Let $h_\epsilon(l) = f(b_l) \cdot f(l)^{-1}$, so $h_\epsilon(l) \cdot f(l) \in I^{>l}$ and the distance between $h_\epsilon(l)$ and e is ϵ. By o-minimality there is an interval $(t_1, 1)$ where h_ϵ is continuous. By definable compactness the curve $h_\epsilon(x)$ with $x \in (t_1, 1)$ is completable; let

$$g_\epsilon = \lim_{x\to 1} h_\epsilon(x).$$

By continuity $d(g_\epsilon, e) = \epsilon$. We will now show that $g_\epsilon \in H$.

Let U be any neighborhood of the identity, and $s,t \in (0,1)$. Ug_ϵ is a neighborhood of g_ϵ so by definition of the limit there is some $l > s,t$ such that $h_\epsilon(l) \in Ug_\epsilon$.

By construction, $h_\epsilon(l)f(l) \in I^{>l} \subset I^{>t}$ and $f(l) \in I^{\geq l} \subset I^{>s}$. So $h_\epsilon(l)f(l) \in (U \cdot (g_\epsilon I)^{>s}) \cap I^{>t}$ so $(U \cdot (g_\epsilon I^{>s})) \cap I^{>t} \neq \emptyset$ for all $U \in \mathcal{U}$ and all $s,t \in (0,1)$ which by Claim 7.13 and definition implies $g_\epsilon I \mathbb{T} I$ and $g_\epsilon \in H$, as required. $\blacksquare_{Claim}$

The previous two claims show that the dimension of H is one. So to complete the proof of the lemma we need to show that H^0 is torsion-free.

By Theorem 6.12, if H^0 is not torsion-free then $E((H)^0) = 0$ and contains an element α of order 2.

By Claim 7.14, α is in the closure of $I^{>s} \cdot (I^{>s})^{-1}$ which implies that α is in the closure of $I^{>s} \cdot (I^{>t})^{-1}$ for any $s,t \in I$ so that if for any $U \in \mathcal{U}$ we define

$$X_U := \{(x,y) \mid x \cdot y^{-1} \in U \cdot \alpha\}$$

then $(I^{>t} \times I^{>t}) \cap X_U$ is infinite for any $t \in (0,1)$ and any $U \in \mathcal{U}$.

Let $\mathcal{U}$ be the set of all neighborhoods of the identity such that $\alpha \notin U \cdot U^{-1}$ so that U and $U \cdot \alpha$ are disjoint. In particular, for any $U \in \mathcal{U}$, if $\Delta_{x<y}$ and $\Delta_{x>y}$ are the respective graphs of the $x < y$ and $x > y$ relations, either the set $(I^{>t} \times I^{>t}) \cap X_U \cap \Delta_{x<y}$ is infinite, or $(I^{>t} \times I^{>t}) \cap X_U \cap \Delta_{x>y}$ is infinite.

Claim 7.17. *Either $(I^{>t} \times I^{>t}) \cap X_U \cap \Delta_{x<y}$ or $(I^{>t} \times I^{>t}) \cap X_U \cap \Delta_{x>y}$ holds for all $U \in \mathcal{U}$.*

Proof: We will give a proof by contradiction. If $(I^{>t} \times I^{>t}) \cap X_{U_1} \cap \Delta_{x<y}$ and $(I^{>t} \times I^{>t}) \cap X_{U_2} \cap \Delta_{x>y}$ are both finite for some neighborhoods U_1, U_2, then $(I^{>t} \times I^{>t}) \cap X_{U_1 \cap U_2}$ would be finite with $U_1 \cap U_2 \in \mathcal{U}$, a contradiction. $\blacksquare_{Claim}$

Since both cases are analogous, we will assume that for all $U \in \mathcal{U}$ and all $t \in (0,1)$ the set $(I^{>t} \times I^{>t}) \cap X_U \cap \Delta_{x<y}$ is infinite.

By o-minimality there is a uniform bound on the number of definably connected components of the family $\{x \in (t,1) \mid f(x) \in U \cdot f(t)\}_{t \in (0,1),\ U \in \mathcal{U}}$. Let k be such that for any such U and t the number of definably connected components of

$$\{x \in (t,1) \mid f(x) \in U \cdot f(t)\}$$

is less than k.

Let $U_0 \in \mathcal{U}$.

Let U_k be such that $(U_k)^k \subset U_0$ and let $V \subset U_k$ be such that $\alpha \cdot V \subset U_k \cdot \alpha$. Let x_1 be such that $(x_1, 1) \subset \{x \in (0,1) \mid \exists y > x\ f(x) \cdot f(y)^{-1} \in V \cdot \alpha\}$. Let $x_{i+1} > x_i$ be such that $f(x_i) \cdot f(x_{i+1})^{-1} \in U_k \cdot \alpha$.

Then

$$f(x_1) \cdot f(x_2)^{-1} \in U_k \alpha$$

and

$$f(x_1) \cdot f(x_3)^{-1} = \left(f(x_1) \cdot f(x_2)^{-1}\right) \cdot \left(f(x_2) \cdot f(x_3)^{-1}\right) \in (V\alpha) \cdot (V\alpha) \subset (V \cdot U_k)\alpha^2 \subset (U_k)^2,$$

so $f(x_1) \cdot f(x_3)^{-1} \in (U_k)^2$.

In a similar way, it follows by induction that $f(x_1) \cdot f(x_{i+1})^{-1} \in (U_k)^i \alpha$ if i is odd and $f(x_1) \cdot f(x_{i+1})^{-1} \in (U_k)^i$ if i is even. By choice of U_k we have $f(x_1) \cdot f(x_{i+1})^{-1} \in U_0 \alpha$ if i is odd and $f(x_1) \cdot f(x_{i+1})^{-1} \in U_0$ if i is even. But $U_0 \in \mathcal{U}$ so $U_0 \alpha$ and U_0 are disjoint. Since $x_i < x_{i+1}$, we have that

$$\{y \in (t,1) \mid f(x_1) \cdot (f(y))^{-1} \in U_0\}$$

has at least k definably connected components, contradicting the choice of k. This completes (finally) the proof of the theorem. $\square$

Corollary 7.18. *Let G be a torsion-free definable group. Then G has a (solvable torsion-free) normal definable subgroup of codimension 1.*

PROOF. This is an induction on the dimension of G. Since G is torsion free there is a one dimensional subgroup L by theorem 7.12, and and if $H \lneq G$ is a proper subgroup of G then by Theorem 6.12 H is torsion free so connected by Proposition 7.2 and $\dim(H) < \dim(G)$. $\square$

Remark 7.19. Any definable torsion-free one dimension definable subgroup H of a definable G is the definable image of an open interval by Theorem 2.31, so it is itself an open definable curve. Since $H \subseteq \{g \mid {}_h H \mathbb{T} H\}$, Claims 7.15 and 7.16 imply the two sets are equal.

Since definably compact groups have torsion by the Torsion Theorem, the converse of Theorem 7.32 holds:

Corollary 7.20. *Let G be a definable group. Then G is definably compact if and only if G contains no torsion-free definable subgroup.*

Corollary 7.21. *Let G be a definable group and N a normal definable subgroup.*

$$G \text{ is definably compact} \iff N \ \&\ G/N \text{ are definably compact.}$$

Proof. If G is definably compact, then every definable subgroup is definably compact by Theorem 2.13 and Proposition 2.16, and every definable quotient is definably compact by Proposition 2.17.

Conversely, if G is not definably compact, then G contains a torsion-free definable subgroup H by Theorem 7.32. Because N is definably compact, $H \cap N = \{e\}$ by Corollary 7.20. Thus H embeds definably into G/N, and G/N is not definably compact again by Corollary 7.20, contradiction. Therefore, if N and G/N are definably compact, G must be definably compact. $\square$

7.3. The maximal normal torsion-free definable subgroup

In the previous chapter (Proposition 6.26) we showed that every definable group G contains a unique maximal normal definably compact subgroup. Similarly, G contains a unique maximal normal torsion-free definable subgroup G^{tf}. The latter will play a bigger role in the study of the structure of a definable group.

Proposition 7.22. *Every definable group G contains a unique maximal normal definable torsion-free subgroup. We denote it by G^{tf}.*

The subgroup G^{tf} contains all normal torsion-free definable subgroups of G. Moreover, if H is a normal definable subgroup of G, then H^{tf} is normal in G too.

Proof. Suppose N is a normal torsion-free definable subgroup of maximal dimension. The fact that N is unique and contains all other torsion-free normal definable subgroups follows from the following claim.

Claim 7.23. *Let H, L be normal torsion-free definable subgroups of G. Then $H \cdot L$ is a normal torsion-free (definable) subgroup of G.*

Proof: For any $h_1, h_2 \in H, l_1, l_2 \in L$ and $g \in G$ we have

$$h_1 l_1 h_2 l_2 = h_1 h_2 (h_2^{-1} l_1 h_2) l_2$$

and $g(h_1 l_1)g^{-1} = g(h_1)g^{-1}g(l_1)g^{-1}$ so $H \cdot L$ is a normal subgroup of G.

Now, $H \lhd H \cdot L$ and $(H \cdot L)/H$ is isomorphic to $L/(L \cap H)$. Any definable quotient of a definable torsion free is torsion free by Theorem 6.12, which implies that $E(L/L \cap H)$ is 1 or -1 by Proposition 6.7, and $E(H \cdot L) = E(H \cdot L/L)E(L)$ is 1 or -1 which implies that $H \cdot L$ is torsion-free again by Proposition 6.7. $\blacksquare_{Claim}$

So if H is another normal torsion-free definable subgroup, then HN is a torsion-free definable subgroup with same dimension of N, $HN = N$ by definable connectedness, and $H \subseteq N$.

If H is a normal subgroup of G, for every $g \in G$ the conjugation by g is a definable automorphism of H mapping H^{tf} to a torsion-free definable subgroup of H of same dimension. By uniqueness of H^{tf}, H^{tf} is therefore normal in G too. $\square$

Remark 7.24. If H is a normal torsion-free definable subgroup of a definable group G and G/H is definably compact, then $H = G^{\mathrm{tf}}$.

Proof. If H is properly contained in G^{tf}, the quotient G^{tf}/H is an infinite torsion-free definable subgroup of the definably compact G/H, in contradiction with the Torsion Theorem. $\square$

Lemma 7.25. *If R is the solvable radical of a definable group G, then $G^{tf} = R^{tf}$.*

Proof. Torsion-free definable groups are definably connected and solvable by Proposition 7.2, so $G^{\mathrm{tf}} \subseteq R$ and $G^{\mathrm{tf}} \subseteq R^{\mathrm{tf}}$. Moreover, because R is normal in G, R^{tf} is normal in G by Proposition 7.22 and $R^{\mathrm{tf}} \subseteq G^{\mathrm{tf}}$. The claimed equality follows. $\square$

The quotient R/G^{tf} is always a definable torus:

Theorem 7.26. *Let G be a solvable definable group. Then*

(i) G is definably compact if and only if G^{tf} is trivial.
(ii) The quotient G/G^{tf} is definably compact.

Proof. We will prove (i) and (ii) together by induction on $\dim G$.

If G is definably compact, then G^{tf} is trivial (Corollary 7.20) and there is nothing to prove. Assume G is not definably compact. Because $G^{\mathrm{tf}} \subseteq G^0$ and G/G^0 is definably compact, we can assume $G = G^0$ is definably connected. If $\dim G = 1$, then $G = G^{\mathrm{tf}}$. Assume $\dim G > 1$. Since G is solvable, G contains an infinite normal abelian subgroup A by Proposition 2.28. As G is definably connected, both A and G/A have smaller dimension than G.

If G/A is definably compact, then A is not. By induction hypothesis, A^{tf} is infinite and A/A^{tf} is definably compact. Therefore G/A^{tf} is definably compact by Corollary 7.21 (by setting $N = A/A^{\mathrm{tf}}$ and $G/N = G/A$), and $A^{\mathrm{tf}} = G^{\mathrm{tf}}$ by Remark 7.24.

If G/A is not definably compact, by induction hypothesis $(G/A)^{\mathrm{tf}}$ is infinite and $(G/A)/(G/A)^{\mathrm{tf}}$ is definably compact. Let H be the pre-image in G of $(G/A)^{\mathrm{tf}}$ under the projection from G to G/A. As H/A is torsion-free, H is not definably compact. By induction hypothesis, H^{tf} is infinite and H/H^{tf} is definably compact. As $G/H = (G/A)/(G/A)^{\mathrm{tf}}$ is definably compact, G/H^{tf} is definably compact by Corollary 7.21 again (where now $N = H/H^{\mathrm{tf}}$ and $G/N = G/H$). By Remark 7.24, $H^{\mathrm{tf}} = G^{\mathrm{tf}}$. □

Corollary 7.27. *Every solvable definably connected group is an extension of a definable torus by a torsion-free definable group.*

Proof. If G is a solvable definably connected group, then G/G^{tf} is definably connected and definably compact by Theorem 7.26, and abelian by Proposition 6.25. □

Proposition 7.28. *If K is a normal definably compact subgroup of a definable group G and G/K is torsion-free, then G is a definable direct product*

$$G = G^{tf} \times K.$$

Proof. We will do an induction on $\dim(G/K)$. Assume first $\dim(G/K) = 1$. Since G has a quotient that is not definably compact, G is not definably compact by Proposition 2.17. Theorem 7.12 implies that there is some definable $H \leq G$ one dimensional and torsion-free. Since definably compact groups have torsion, it must be $H \cap K = \{e\}$. Because one dimensional torsion-free definable groups have no proper non-trivial definable subgroups, H maps surjectively onto G/K, so $G = K \cdot H$.

By Theorem 6.16 any conjugation of K by an element in H is trivial so every element of H commutes with every element of K. It follows that H is normal in G and therefore $G = H \times K$. Since G/H is definably compact, $H = G^{\mathrm{tf}}$ by Remark 7.24.

For the induction step, suppose that G/K has dimension greater than one and that the statement holds for any G_1, K_1 with

$$\dim(G_1/K_1) < \dim(G/K).$$

By Theorem 7.32, there is some definable normal subgroup H_1 of G/K of codimension 1. Let $\pi : G \to G/K$ be the canonical projection, and $H = \pi^{-1}(H_1)$. By induction hypothesis,

$$H = H^{\mathrm{tf}} \times K.$$

Note that H^{tf} is normal in G by Proposition 7.22. Consider the exact sequence

$$\{e\} \to H/H^{\mathrm{tf}} \to G/H^{\mathrm{tf}} \to (G/H^{\mathrm{tf}})/(H/H^{\mathrm{tf}}) \to \{e\}.$$

The definable H/H^{tf} is definably isomorphic to K, so it is definably compact and $(G/H^{\mathrm{tf}})/(H/H^{\mathrm{tf}})$ is a definable quotient of the torsion-free G/K, so it is torsion-free as well. By induction hypothesis,

$$G/H^{\mathrm{tf}} = (G/H^{\mathrm{tf}})^{\mathrm{tf}} \times H/H^{\mathrm{tf}}.$$

The preimage N of $(G/H^{\mathrm{tf}})^{\mathrm{tf}}$ under the projection map from G to G/H^{tf} is a normal definable torsion-free subgroup by Theorem 6.12 and $N \cdot K = G$, so $G = N \times K$, as $N \cap K = \{e\}$. Since $G/N \cong H/H^{\mathrm{tf}} \cong K$ is definably compact, by Remark 7.24, $N = G^{\mathrm{tf}}$, as required. □

Remark 7.29. If we exchange the roles of K and G/K, considering K torsion-free and G/K definably compact, then wen know that $K = G^{\mathrm{tf}}$, as noticed in Remark 7.24. In this case, we cannot always find a *definable* subgroup of G isomorphic to G/K. For instance, take G to be a 0-group that is not definably compact. If there were such a definable subgroup H, then $E(G/H) = E(K) = \pm 1$, contradiction.

7.4. Torsion-free definable groups are completely solvable

Definition 7.30. A definably connected group G is *completely solvable* if there is a sequence of definably connected subgroups

$$\{e\} = G_0 < G_1 < \cdots < G_n = G$$

such that each G_i is normal in G and G_{i+1}/G_i is one-dimensional (and definably connected).

We will prove in this section that any torsion-free definable group G is completely solvable. We already know that G is solvable.

We will need to use the following technical lemma.

Lemma 7.31. *Let G be a torsion-free definable group and σ a definable and continuous action from G to a definably compact space X. Then σ has fixed points.*

Proof. Let G be a torsion-free definable group and σ a definable action from G on a definably compact space X.

We will prove the theorem by induction on the dimension of G.

If $\dim(G) = 1$, Theorem 7.12 implies that we may assume G is an ordered group structure on the interval $(0, 1)$ with the $\mathcal{R}$ order. Let $x \in X$, and consider the definable map from G to X given by $t \mapsto \sigma_t(x)$. By definable compactness $\lim_{t\to 1} \sigma_t(x) = L$ for some $l \in L$. Since it is an ordered group, translation by any element s is increasing and t-continuous so that $s \cdot t \to 1$ whenever $t \to 1$ and

$$\sigma_s(L) = \sigma_s\left(\lim_{t\to 1} \sigma_t(x)\right) = \lim_{t\to 1} (\sigma_{s\cdot t}(x)) = \lim_{(st)\to 1} (\sigma_{st}(x)) = L.$$

Assume now that any definable action σ' from a torsion-free G' with $\dim(G') < \dim(G)$ and σ' to a definably compact space X' σ has fixed points.

By Corollary 7.3 and Corollary 7.18, G contains a normal infinite definable abelian $H \lhd G$ with $\dim(H) < \dim(G)$. By induction hypothesis, the set

$$Fix_H(X) := \{x \in X \mid \forall h \in H,\ \sigma_h(x) = x\}$$

is non empty. Let $x \in Fix_H(X), g \in G$ and $h \in H$. Then H normal implies

$$\sigma_h(\sigma_g(x)) = \sigma_{hg}(x) = \sigma_{gh'}(x) = \sigma_g(\sigma_{h'}(x)) = \sigma_g(x)$$

so $\sigma_g(Fix_H(X)) \subseteq Fix_H(X)$ for any $g \in G$ so σ is an action from G on $Fix(H)$ which by definition induces an action from G/H on $Fix(H)$.

By induction hypothesis again, this action must have a fixed point x. But then $\sigma_{gh}(x) = \sigma_h(x) = x$ and x is a fixed point of σ, as required. □

Theorem 7.32. *Let G be a torsion-free (definably connected) definable group. Then G has an infinite normal subgroup of dimension 1.*

Let G be a torsion-free (definably connected) definable group. Then G is completely solvable.

Proof. Let G be a torsion-free definable group. By Corollary 7.3 there is an abelian normal definable subgroup $A \lhd G$. By Theorem 7.12, A contains a definable torsion-free subgroup L of dimension 1. Now, consider the action from G on A given by conjugation. For any $g \in G$ we denote $x^g = gxg^{-1}$ and $L^g := \{glg^{-1} \mid l \in L\}$.

If $\{L^g \mid g \in G\}$ is finite, then it must be trivial by definable connectedness of G. So $L^g = L$ for any $g \in G$ and L is a normal subgroup of G.

Otherwise, Theorem 7.10 implies that $G(L)$ is contained in a subgroup H of A such that H definably isomorphic to $(\mathcal{R}, +)^k$ and the group of automorphisms of H induced by the action of G is isomorphic to $GL_n(\mathcal{R})$. In particular, the set of conjugates of L under G are the set $Gr(A)$ of one dimensional subgroups of H.

If we identify H with $(\mathcal{R}, +)^n$, then $Gr(H)$ is isomorphic to

$$\{v \mid \lambda \in \mathcal{R} \wedge |v| = 1\}/E$$

where vEw if and only if $v = w$ or $v = -w$ for any $v, w \in A$. The set

$$\{v \mid \lambda \in \mathcal{R} \wedge |v| = 1\}$$

is a closed and bounded subset of $\mathcal{R}^n$ so it is definably compact by Proposition 2.18. Since E is a finite equivalence relation, all equivalence classes are closed so $Gr(H)$ is definably compact. G is torsion-free so Lemma 7.31 implies there is $L' \in Gr(H)$ which is fixed by the action of G. So $gL'g^{-1}$ for all $g \in G$, as required.

That every torsion-free definable group is completely solvable follows by induction on the dimension: If G is definable, then it has a normal one dimensional definable subgroup L. Now, $\dim(G/L) < \dim(G)$ so by induction hypothesis G/L is completely solvable and there are normal definable subgroups G'_i of G/L such that $\{e\}_L = G'_0 \lhd G_1 \lhd \cdots < G'_k = G/L$. If G_{i+1} is the pullback of G'_i under the natural projection $\pi : G \to G/L$, then G_{i+1} is a normal definable subgroup of G, $G_i \lhd G_{i+1}, G_{k+1} = \pi^{-1}(G/L) = G$, and G_{i+1}/G_i is a one dimensional torsion-free group for $i \geq 1$.

If we define $G_1 := L$, the sequence $G_1, \ldots, G_{k+1}$ witnesses that G is a completely solvable group, as required. □

Corollary 7.33. *Every torsion-free definable group G is a product of definable abelian subgroups.*

Proof. Let H be a normal definable subgroup of codimension 1 from Theorem 7.32, $a \notin H$ and $K = \langle a \rangle_{\text{def}}$. Note that K is abelian by Lemma 2.26. Since HK is a definable subgroup containing H properly, it must be $G = HK$ by connectedness. By induction hypothesis H is a product of definable abelian subgroups, and so is G. □

7.5. Maximal torsion-free definable subgroups and Zappa-Szep products

Although in general they are not definably characteristic subgroups of a definable group, maximal (not necessarily normal) torsion-free definable subgroups do exist and they are conjugates of each other. Namely:

Theorem 7.34. *Every torsion-free definable subgroup of a definable group is contained in a maximal torsion-free definable subgroup, that is unique up to conjugacy. Moreover, if H is a maximal torsion-free definable subgroup of a definably connected group G, the definable set G/H is definably compact and*

$$\bigcap_{g\in G} gHg^{-1} = G^{tf}$$

We will prove Theorem 7.34 by first finding a Zappa-Szep product decomposition of G/G^{tf}.

Definition 7.35. In group theory, when a group G can be decomposed as $G = KH$ for some K, H subgroups with trivial intersection, G is called a *Zappa-Szep product* of K and H.

Remark 7.36. A nice feature of the Zappa-Szep product decomposition is that, because of the trivial intersection, every element $g \in G$ can be written as $g = kh$ for *unique* $k \in K$ and *unique* $h \in H$.

Lemma 7.37. *Suppose G is a definable group containing definable subgroups K, H with K definably compact and H torsion-free such that $G = KH$. Then K is a maximal definably compact subgroup, H is a maximal torsion-free definable subgroup, $K \cap H = \{e\}$ and G is a Zappa-Szep product of K and H.*

Proof. Suppose L is a definably compact subgroup containing properly K and let $x \in L, x \notin K$. By writing $x = kh$ with $k \in K$ and $h \in H$, we see that $k^{-1}x = h \in L \cap H$. Because $x \notin K$, we have that $h \neq e$ and $L \cap H$ is a non-trivial group that is both definably compact (because a definable subgroup of L) and torsion-free (because a definable subgroup of H), in contradiction with the Torsion Theorem. Therefore K is maximal definably compact. Similarly, H is maximal torsion-free and $K \cap H = \{e\}$. □

Proposition 7.38. *Let G be a definably connected group such that $G^{tf} = \{e\}$. Then G is a Zappa-Szep product of a (maximal) definably compact subgroup K and a (maximal) torsion-free definable subgroup H.*

Proof. If G is solvable, then G is definably compact by Theorem 7.26, so we take $K = G$ and $H = \{e\}$.

Suppose G is not solvable and let R be its solvable radical. As noted above, R is definably compact because $G^{\mathrm{tf}} = R^{\mathrm{tf}} = \{e\}$ (see Lemma 7.25). The quotient of the definably connected semisimple group G/R by its finite center is centerless by Lemma 5.20, and it is a Zappa-Szep product of definably compact K' and torsion-free H' by Theorem 5.41. Set S to be the pre-image in G of the finite center of G/R. Note that S is normal, solvable and definably compact by Corollary 7.21.

Let K be the pre-image of K' in G. Then K is a definable extension of the definably compact K' by the definably compact S and it is definably compact again by Corollary 7.21.

Let L be the pre-image of H' in G. Then L is a definable extension of the torsion-free H' by the definably compact S. By Proposition 7.28, $L = L^{\mathrm{tf}} \times S$. Set $H = L^{\mathrm{tf}}$. Then $G = KH$ and G is a Zappa-Szep product of K and H by Lemma 7.37. □

Corollary 7.39. *Let G be a definably connected group such that $G^{tf} = \{e\}$. Then $G = KAN$ for definable subgroups K, A, N where K is maximal definably compact, A is torsion-free abelian, N is torsion-free nilpotent and $H = AN = NA$ is maximal torsion-free. Moreover, H is definably isomorphic to a linear (upper triangular) group.*

Proof. The maximal torsion-free definable subgroup H we found in Proposition 7.38 is definably isomorphic to the maximal definable torsion-free H' of a centerless semisimple group. Therefore H is definably isomorphic to the linear upper triangular group from Theorem 5.41. $\square$

Corollary 7.40. *Let G be a definably connected group. Then G/G^{tf} is a Zappa-Szep product of a (maximal) definably compact subgroup K and a (maximal) torsion-free definable subgroup H. Moreover, H is definably isomorphic to a linear (upper triangular) group.*

Proof. Because $(G/G^{\mathrm{tf}})^{\mathrm{tf}} = \{e\}$, Proposition 7.38 and Corollary 7.39 apply. $\square$

Proof of Theorem 7.34. Let $G/G^{\mathrm{tf}} = K'H'$ be a Zappa-Szep product decomposition from Corollary 7.40 and let H be the pre-image of H' in G.

Recall that a definable group is torsion-free if and only if its Euler characteristic is equal to 1 in absolute value (Proposition 6.7), so $|E(H)| = |E(G^{\mathrm{tf}})E(H')| = |E(G^{\mathrm{tf}})||E(H')| = 1$ and H is torsion-free as well.

Note that H is a maximal definable torsion-free subgroup, because the image in G/G^{tf} of any subgroup containing H properly intersects non-trivially the definably compact K'. Moreover, G/H is in definable bijection with K', hence it is definably compact, as claimed.

Let L be an arbitrary torsion-free definable subgroup of G. We want to show that L is contained in a conjugate of H. The group G acts on the definably compact G/H by left multiplication and by Lemma 7.31, the corresponding action of L on G/H has a fixed point. This is, for all $x \in L$, $xgH = gH$ for some $g \in G$, so $g^{-1}Lg \subseteq H$ and $L \subseteq gHg^{-1}$, as claimed.

Finally, we need to check that the intersection I of all maximal torsion-free definable subgroups of G is G^{tf}. Let H be a maximal torsion-free definable subgroup containing G^{tf}. Since G^{tf} is normal, $G^{\mathrm{tf}} \subseteq I$. Conversely, I is a normal torsion-free definable subgroup, so $I \subseteq G^{\mathrm{tf}}$, and we have the claimed equality. $\square$

Note that, as for 0-groups, the fact that definable torsion-free groups are definably connected implies that every ascending chain of torsion-free definable subgroups must stabilize by dimension reasons. It follows that every torsion-free definable group is contained in a maximal one. However, the proof we presented clarifies the general structure of a definable group through the exact sequence

$$1 \longrightarrow G^{\mathrm{tf}} \longrightarrow G \longrightarrow K'H' = G/G^{\mathrm{tf}} \longrightarrow 1$$

showing where maximal definable torsion-free subgroups fit within it.

7.6. Torsion-fee linear definable groups.

When we restrict ourselves to linear definable groups, there is a sharp contrast between the torsion-free case and the semisimple and definably compact cases we studied before. In order to give some intuition and explain why we need to make extra assumptions in this case, consider $\mathbb{R} = \mathcal{R}$. In this case, we have seen (in Section 6.4 of Chapter 6 and Section 5.2.4 of Chapter 5) that all compact and semisimple linear groups are semialgebraic. This means any linear real Lie group which is either compact or semisimple is definable in any o-minimal expansion of the real field. This is not the case for torsion-free linear groups in three different ways:

- Not every linear connected torsion-free real Lie group is isomorphic to a definable group. Furthermore, as Example 7.4 shows, there are linear Lie groups that are not definable nor are they Lie isomorphic to *any* group definable in an o-minimal expansion of the real field. This is of course, because there are torsion-free linear Lie groups which are not completely solvable, a property we already proved is implied by definability.

- Definability of a linear torsion-free group may depend on the o-minimal structure. The group
$$\left\{ \begin{pmatrix} e^x & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$
is only definable whenever the exponential function is definable.

- The above is true even if we work modulo Lie isomorphisms. The group G_3 whose universe is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and group structure
$$(x_1, y_1, z_1) \odot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + e^{x_1} y_2, z_1 + e^{x_1} z_2 + x_1 e^{x_1} y_2)$$
is a completely solvable centerless group, so in particular it is linear, and it is definable in $(\mathbb{R}, +, \cdot, e^x)$ but it is not Lie isomorphic to any semialgebraic group ([**1**]).

It will turn out that these are the only exceptions and that once we assume definability of an exponential function, all completely solvable linear groups will be definable. Before we make this assumption, we will prove some general structure theorems.

Let $T_n(\mathcal{R})$ be the group of invertible upper triangular matrices in $GL_n(\mathcal{R})$, and $T_n^+(\mathcal{R})$ be the group of invertible upper triangular matrices with positive eigenvalues.

EXERCISE 7.4. *Proof that $T_n^+(\mathcal{R})$ is torsion-free and completely solvable.*

$T_n^+(\mathcal{R})$ is an open subgroup of $T_n(\mathcal{R})$; it is in fact the connected component.

Theorem 7.41. *Let G be a definable (connected) torsion-free subgroup of $GL(V)$. Then G can be conjugated into $T_n^+(\mathcal{R})$.*

PROOF. Since G is connected, it is enough to show that it can be conjugated into a subgroup of $T_n(\mathcal{R})$.

We identify G with the action of G on V induced by matrix multiplication.

We need to show that under some basis of V, any element in G is represented by upper triangular matrices. As in the proof of Theorem 7.32 we define $Gr(V)$ to be the (definable and definably compact) space of one dimensional subspaces of V.

Let $n = \dim(V)$. We construct $v_1, v_2, \ldots, v_n$ so that G fixes
$$L_k := \operatorname{span}(\{v_1, \ldots, v_k\})$$
for every k inductively as follows. Let $v_0 = \overline{0}$, and assume that $\operatorname{span}(\{v_1, \ldots, v_k\})$ is G-invariant, so that multiplication by elements in G induces a well defined action from G on the vector space V/L_k. By Lemma 7.31 the natural action of G on $Gr(V/L_k)$ given by multiplication has a fixed point, a 1-dimensional subspace $\mathbf{L}$ of V/L_k which is fixed by G. Let v_{k+1} be any element such that the projection of v_{k+1} in V/L_k is in $\mathbf{L}$. Let $L_{k+1} = \operatorname{span}(\{v_1, \ldots, v_k, v_{k+1}\})$. Since G fixes L_{k+1}/L_k set wise, L_{k+1} is also G-invariant.

It is now an easy computation to show that under the basis $\{v_1, \ldots, v_n\}$ the representation of G is upper triangular, this is, that for any $g \in G$ the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

representing g with the basis $\{v_1, \ldots, v_n\}$ is upper triangular.

We will prove by induction on k that $a_{lk} = 0$ for any $l > k$. By construction, $g(v_1 + \cdots + v_k) \in \mathrm{span}(\{v_1, \ldots, v_k)\}$. So for any $l > k$

$$\sum_{j=1}^{k} a_{jl} = 0.$$

By induction hypothesis (or in the initial $k = 1$ case by vacuous condition) $a_{jl} = 0$ for all $j < k$. So $a_{lk} = 0$ as required. $\square$

Theorem 7.41 will be the main structure theorem for linear torsion-free definable groups. As we mentioned before, the class of definable torsion-free linear subgroups depends on the o-minimal structure we are working in, and we need to make assumptions about the field $\mathcal{R}$ beyond o-minimality to classify linear completely solvable subgroups as well as we want to. So from now on, in this section, we will assume that $\mathcal{R}$ is an o-minimal expansion (possibly with extra structure) of an elementary extension of $(\mathbb{R}, +, \cdot, e^x)$. So in particular, we assume there is an exponential map, a continuous isomorphism between $(\mathcal{R}, +)$ and $(\mathcal{R}^+, \cdot)$.

In this setting, we will prove that all linear torsion-free definable groups split as products of definable one dimensional (torsion-free, abelian) subgroups, and that every linear torsion-free subgroup is $\mathcal{R}$-definable. For this we need to give a definable version of the matrix exponential map for matrices in $T(\mathcal{R})$.

7.6.1. The matrix exponential map. We define an exponential map for matrices in $GL_n(\mathcal{R})$ with real eigenvalues as follows:

For a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda 2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix},$$

let

$$Exp(D) := \begin{pmatrix} e^{\lambda_1} & 0 & 0 & \ldots & 0 \\ 0 & e^{\lambda 2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & e^{\lambda_n} \end{pmatrix}.$$

For a nilpotent matrix N, let $Exp(N) = \sum_{i=0}^{n} \frac{N^i}{i!}$.

For A a matrix with real eigenvalues, let J be its Jordan normal form, $A = MJM^{-1}$ for some $M \in GL_n(\mathcal{R})$. Let $J = J_D + J_N$ where J_D is the matrix consisting on the diagonal entries of J, and J_N the matrix with the ones and zeros in the superdiagonal corresponding from Jordan blocks of J. We define

$$Exp(A) := M\,(Exp(J_D)Exp(J_N))\,M^{-1}.$$

Remark. *If $\mathcal{R} = \mathbb{R}$, then Exp coincide with the standard Taylor series exponential: For diagonal and nilpotent matrices it is easily verified. For general matrices with real eigenvalues, the result follows from the fact that if J is in Jordan canonical form, then J_D and J_N commute.*

As with previous cases, to characterize which solvable real Lie groups can be represented by definable groups over $\mathbb{R}$ we will need some classical results.

The following was proved by Dixmier in [**22**].

Fact 7.42. *Let G be a real Lie group such that any matrix in its image under the adjoint representation only has real eigenvalues. Let $\mathfrak{g}$ be the Lie algebra fo G. Then the exponential map is a bijection.*

Proposition 7.43. *Recall that $t_n(\mathcal{R})$ is the Lie algebra of upper triangular matrices. Then the following hold:*

- *Exp is a bijective map from $t_n(\mathcal{R})$ to $T_n^+(\mathcal{R})$.*
- *For any $v \in t_n(\mathcal{R})$ the set*

$$H_v : \{Exp\,(\lambda v) \mid \lambda \in \mathcal{R}\}$$

is a subgroup of $T_n^+(\mathcal{R})$ with

$$Exp\,(\lambda_1 v)\, Exp\,(\lambda_2 v) = Exp\,((\lambda_1 + \lambda_2)\, v)\,.$$

The Lie algebra of H_v is the subspace of $t_n(\mathcal{R})$ generated by v.

- *Let G be a closed connected subgroup $G \leq T_n^+(\mathcal{R})$ and let $\mathfrak{g}$ be the Lie subalgebra of $t_n(\mathcal{R})$ corresponding to G. Then $Exp(v) \in G$ for $v \in t_n(\mathcal{R})$ implies $v \in \mathfrak{g}$.*

Proof. All statements can be expressed in first order in any language including $\{+.\cdot, e^x\}$, so by our hypothesis on $\mathcal{R}$ it is enough to verify them for $\mathcal{R} = \mathbb{R}$.

The first item (for $\mathcal{R} = \mathbb{R}$) follows from Fact 7.42.

The statements in the second item are well known, and can easily be verified using the Taylor series characterization of matrix exponentiation.

The third item follows from the second one by corollary 3.29. □

Theorem 7.44. *Let G be a definably connected torsion-free definable linear group. Then the following holds.*

1. *For any non-trivial $g \in G$ there is a one dimensional definable subgroup of G containing g,*
2. *G is the product of one dimensional definable subgroups of G.*
3. *If $\mathfrak{g}$ is the Lie subalgebra of $gl_n(\mathcal{R})$ corresponding to G, then $G = Exp(\mathfrak{g})$.*

Proof. By Theorem 7.41, we may assume that G is a subgroup of $T_n^+(\mathcal{R})$. Let $g \in G$. By Proposition 7.43 there is some $v \in t_n(\mathcal{R})$ such that $Exp(v) = g$ and $\{Exp\,(\lambda v) \mid \lambda \in \mathcal{R}\}$ is a one dimensional definable torsion-free subgroup L of $T_n^+(\mathcal{R})$ containing g. So $L \cap G$ is a definable subgroup of L which is not trivial since it contains g, so by connectedness $L \cap G = L$ and $L \leq G$. This proves (1).

We will prove (2) by induction on $\dim(G)$. If $\dim(G) = 1$ there is nothing to prove. Otherwise, let $N \lhd G$ be a normal subgroup of dimension $n-1$. Let $g \in G \setminus N$ and let L be a connected one dimensional definable subgroup of G containing g. $L \cap N$ is a definable subgroup of L so by definable connectedness (and the fact that L is torsion-free) it must be trivial. So $N \cdot L$ is a definable subgroup of G of dimension n, so $N \cdot L = G$. By induction hypothesis N is the product of one dimensional definable subgroups, so so is G.

By Theorem 7.43, any one dimensional subgroup of G is the image of the exponential map of a one dimensional subspace of $\mathfrak{g}$. So (3) follows from (2). □

Theorem 7.45. *Let G be a closed connected subgroup of $T_n^+(\mathbb{R})$. Then G is definable in $\mathcal{R}$.*

Proof. The proof will follow from Theorem 7.44 by induction on $n = \dim(G)$. If $n = 1$, let $g \in G$ be a non-trivial element and H a one dimensional definable subgroup of $T_n^+(\mathbb{R})$ containing g from Theorem 7.44. Because H is definable torsion-free, it is connected and $G = H$. Suppose $n > 1$.

Since the Lie algebra of G is completely solvable, G contains a normal closed connected subgroup N of codimension one. By induction hypothesis, N is definable. Let $g \in G\backslash L$ and let L be a one dimensional definable subgroup of $T_n^+(\mathbb{R})$ containing g from Theorem 7.44. L is torsion-free and $L \cap N \neq L$ so by definably connected, torsion-free and dimension 1, $L \cap N = \{e\}$. It follows that $L \cdot N$ has same dimension as G, and by connectedness $L \cdot N = G$, so G is definable. □

7.6.2. Linear Unipotent subgroups. As opposed to torsion-free completely solvable groups, which linear torsion-free unipotent groups are definable does not depend on the o-minimal structure one is working with.

Recall that $U_n(\mathbb{R})$ is the subgroup of $T_n^+(\mathbb{R})$ consisting of elements with eigenvalue 1 (i.e.,1's in the diagonal)

Theorem 7.46. *Let G be a closed connected subgroup of $U_n(\mathbb{R})$. Then G is semi-algebraic.*

Proof. The proof is just a revision of Theorems 7.44 and 7.45. For any element M in $U_n(\mathbb{R})$ the matrix exponential of M is given by $\sum_{i=0}^{n} \frac{M^i}{i!}$. This is a polynomial, so when G is a closed connected subgroup of $U_n(\mathbb{R})$ all the results used in the proofs of Theorems 7.44 and 7.45 transfer from $\mathbb{R}$ to any real closed field by completeness of the theory of real closed fields. □

7.7. Torsion-free completely solvable Lie groups have o-minimal copies

In this section we will prove that every completely solvable torsion-free real Lie group has a definable Lie isomorphic copy in some o-minimal expansion of $\mathbb{R}$. Using the following two results, which are two of the main structural theorems of the Lie Group-Lie Algebra correspondence, the result will follow easily from Theorem 7.45.

The following is known both as Lie's Second Theorem or as the Homomorphism Theorem and is Theorem 2.10 in Chapter I of [**44**].

Fact 7.47. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and such that G simply connected. Then for every homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ there is a Lie group homomorphism $f : G \to H$ such that $\mathrm{d}f = \phi$.*

The following is Ado's Theorem, Theorem 5.1 in Chapter I of [**45**].

Fact 7.48. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field K of characteristic 0. Then the Lie algebra $\mathfrak{g}$ has a faithful finite-dimensional linear representation over K.*

Finally, the following is stated in Section 2 of Chapter 2 of [**45**] without proof. The proof we include is from Vinberg, provided to us in a private communication and was included in [**19**].

Fact 7.49. *Let $\mathfrak{g}$ be a solvable finite dimensional Lie algebra over $\mathbb{R}$. The following are equivalent.*

(1) $\mathfrak{g}$ is completely solvable.

(2) For any $\xi \in \mathfrak{g}$ all eigenvalues of the linear operator $ad(\xi)$ are in $\mathbb{R}$.
(3) $\mathfrak{g}$ is isomorphic to a subalgebra of the upper triangular matrices $t_n(\mathbb{R})$ for some $n \in \mathbb{N}$.

Proof. The equivalence of (1) and (2) is [**42**, Corollary 1.30].

The implication (3) $\Rightarrow$ (1) is easy: if $\mathfrak{g} \subseteq t_n(\mathbb{R})$, then the eigenvalues of $\mathrm{ad}(\xi)$ are real for every $\xi \in \mathfrak{g}$, and hence (1) follows from the equivalence of (1) and (2).

Thus we only need to see that (1) implies (3). Using Ado's theorem we can embed $\mathfrak{g}$ into $\mathfrak{gl}(V)$ for some finite dimensional $\mathbb{R}$-vector space V, and we will assume that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

As in Chapter 6, let $\mathfrak{g}^a$ be the minimal algebraic subalgebra of $\mathfrak{gl}(V)$ containing $\mathfrak{g}$, i.e. $\mathfrak{g}^a$ is the Lie algebra of an algebraic subgroup of $\mathrm{GL}(V)$, $\mathfrak{g}^a$ contains $\mathfrak{g}$ and is the minimal such. (It exists by a dimension argument.) By Fact 6.43 we have $[\mathfrak{g}^a, \mathfrak{g}^a] = [\mathfrak{g}, \mathfrak{g}]$, in particular $\mathfrak{g}$ is an ideal of $\mathfrak{g}^a$ and $\mathfrak{g}^a/\mathfrak{g}$ is abelian.

Let $G^a < \mathrm{GL}(V)$ be a connected algebraic subgroup whose Lie algebra is $\mathfrak{g}^a$. Clearly G^a is defined over $\mathbb{R}$.

Let $\mathfrak{g} = \mathfrak{g}_n > \mathfrak{g}_{n-1} > \cdots > \mathfrak{g}_0 = 0$ be a complete flag of ideals in $\mathfrak{g}$. Consider the adjoint action of G^a on $\mathfrak{g}^a$. Let

$$G' = \{g \in G^a \colon Ad(g)(\mathfrak{g}_i) \subseteq \mathfrak{g}_i \text{ for each } i = 0, \dots, n\}.$$

G' is an algebraic subgroup of G^a and, by Theorem 3.25, its Lie algebra is

$$\mathfrak{g}' = \{\xi \in \mathfrak{g}^a \colon [\xi, \mathfrak{g}_i] \subseteq \mathfrak{g}_i \text{ for each } i = 0, \dots, n\}.$$

Obviously $\mathfrak{g} \subseteq \mathfrak{g}'$, so by minimality of $\mathfrak{g}^a$ we obtain $\mathfrak{g}' = \mathfrak{g}^a$ and $G' = G^a$.

Thus each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}^a$, and since $\mathfrak{g}^a/\mathfrak{g}$ is abelian, the chain $\mathfrak{g} = \mathfrak{g}_n > \mathfrak{g}_{n-1} > \cdots > \mathfrak{g}_0 = 0$ can be extended to a complete flag of ideals of $\mathfrak{g}^a$. Hence $\mathfrak{g}^a$ is completely solvable. Replacing $\mathfrak{g}$ by $\mathfrak{g}^a$ if needed, we will assume that $\mathfrak{g}$ is the Lie algebra of a connected algebraic subgroup $G^a < \mathrm{GL}(V)$ defined over $\mathbb{R}$.

Let G^a_u be the subset of G^a consisting of unipotent elements, and $T < G^a$ a commutative algebraic group consisting of semi-simple elements defined over $\mathbb{R}$. By [**6**, Theorem III.10.6], G^a_u is a normal subgroup of G^a and $G^a = T \cdot G^a_u$ (a semi-direct product).

We now consider $\mathbb{R}$-points of groups G^a, G^a_u and T. We have $G^a(\mathbb{R}) = T(\mathbb{R}) \cdot G^a_u(\mathbb{R})$, $G^a_u(\mathbb{R})$ is a normal subgroup of $G^a(\mathbb{R})$, and $\mathfrak{g}$ is the Lie algebra of $G^a(\mathbb{R})$ viewed as a Lie group.

By Fact 6.46, we can write T as a product $T = T_d \cdot T_a$ of algebraic groups T_a and T_d defined over $\mathbb{R}$ such that $T_d(\mathbb{R})$ is isomorphic over $\mathbb{R}$ to a product of multiplicative group $G^a_m(\mathbb{R})$ and $T_a(\mathbb{R})$ is compact.

We have $G^a(\mathbb{R}) = T_a(\mathbb{R}) \cdot T_d(\mathbb{R}) \cdot G^a_u(\mathbb{R})$. Let $H = T_d \cdot G^a_u$. Then $H(\mathbb{R})$ is a normal subgroup of $G^a(\mathbb{R})$ and $G^a(\mathbb{R}) = T_a(\mathbb{R}) \cdot H(\mathbb{R})$ with finite intersection $T_a(\mathbb{R}) \cap H(\mathbb{R})$. It is not hard to see that H is an $\mathbb{R}$-split solvable group, hence, by [**6**, Theorem V.15.4], we can choose a basis in V so that every matrix in $H(\mathbb{R})$ is upper triangular.

Consider the adjoint representation $Ad \colon G^a(\mathbb{R}) \to \mathrm{GL}(\mathfrak{g})$. The differential of this representation is the adjoint representation $ad \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ of $\mathfrak{g}$. By the equivalence of (1) and (2), for every $\xi \in \mathfrak{g}$ all eigenvalues of $ad(\xi)$ are real. By [**42**, Corollary 1.30], there is a basis of $\mathfrak{g}$ such that $ad(\mathfrak{g})$ consists of upper triangular matrices.

Let $B < \mathrm{GL}(\mathfrak{g})$ be the upper triangular subgroup with respect to this basis, and let B^0 be its identity component, i.e. the subgroup of upper triangular matrices with positive diagonal entries. Since the Lie algebra of $\mathrm{Ad}^{-1}(B)$ is all of $\mathfrak{g}$, and $G^a(\mathbb{R})^0$ is connected, we have $\mathrm{Ad}(G^a(\mathbb{R})^0) \subseteq B^0$.

We note that B^0 has no non-trivial compact subgroup. Indeed, the diagonal map $B^0 \longrightarrow (\mathbb{R}_{>0})^m$ sends any compact subgroup trivially, because $(\mathbb{R}_{>0})^m$ has no non-trivial compact subgroup. The compact subgroup must therefore lie in the unipotent radical of B^0, and the real unipotent upper triangular group has no non-trivial compact subgroup, since every non-identity unipotent element has unbounded powers.

Now $T_a(\mathbb{R})^0$ is compact, and hence $\mathrm{Ad}(T_a(\mathbb{R})^0) = 1$. Let $\mathfrak{s}$ be the Lie algebra of $T_a(\mathbb{R})$ and let $\mathfrak{h}$ be the Lie algebra of $H(\mathbb{R})$. Since $\mathrm{Ad}(T_a(\mathbb{R})^0) = 1$, we have $[\mathfrak{s}, \mathfrak{g}] = 0$. Thus $\mathfrak{s}$ is central in $\mathfrak{g}$, in particular it is an ideal. Since $H(\mathbb{R})$ is normal in $G^a(\mathbb{R})$, $\mathfrak{h}$ is also an ideal of $\mathfrak{g}$. From $G^a(\mathbb{R}) = T_a(\mathbb{R}) \cdot H(\mathbb{R})$ and the finiteness of $T_a(\mathbb{R}) \cap H(\mathbb{R})$, it follows that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$ as Lie algebras.

By the choice of basis in V, we have $H(\mathbb{R}) \subseteq B_V$, where B_V is the upper triangular subgroup of $\mathrm{GL}(V)$. Hence $\mathfrak{h} \subseteq Lie(B_V)$, so $\mathfrak{h}$ has a faithful representation by upper triangular matrices. The Lie algebra $\mathfrak{s}$ is abelian, hence it also has a faithful representation by upper triangular matrices, for instance by diagonal matrices. Since $[\mathfrak{s}, \mathfrak{h}] = 0$ and $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$, taking the direct sum of these two representations gives a faithful representation of $\mathfrak{g}$ by upper triangular matrices. Therefore $\mathfrak{g}$ is isomorphic to a subalgebra of $t_n(\mathbb{R})$ for some $n \in \mathbb{N}$. □

Theorem 7.50. *Let G be a connected torsion-free solvable Lie group. Then G is Lie isomorphic to a linear upper triangular group definable in $(\mathbb{R}, +, \cdot, e^x)$.*

PROOF. Let $\mathfrak{g}$ be the Lie algebra of G. By Ado's Theorem (Fact 7.48) $\mathfrak{g}$ is isomorphic (as a Lie algebra) to a Lie linear Lie algebra $\mathfrak{g}_{lin}$, which by Fact 7.49can be assumed to be a subalgebra of the $t_n(\mathbb{R})$ for some n. By Fact 7.47, G is Lie isomorphic to the exponential $Exp(\mathfrak{g}_{lin})$ which is a linear Lie group definable in $(\mathbb{R}, +, \cdot, e^x)$ by Theorem 7.45. □

Corollary 7.51. *Let G be a simply-connected completely solvable Lie group. Then G has a representation G_{def} definable in $(\mathbb{R}, +, \cdot, e^x)$ such that any Lie automorphism of G_{def} is definable in $(\mathbb{R}, +, \cdot, e^x)$.*

PROOF. Let G_{def} be the representation of G given by Theorem 7.50. Any automorphism of G_{def} is the push forward under Exp of an automorphism of the Lie algebra $\mathfrak{g}$ which is a vector subspace of $t_n(\mathbb{R})$ for some n. But automorphisms of the linear subspace of a definable vector space are determined by a finite basis, so they are always definable. Since the restriction of matrix exponentiation Exp to $t_n(\mathbb{R})$ is definable in $(\mathbb{R}, +, \cdot, e^x)$, the corollary follows. □

Historical notes and comments

Torsion-free definable groups were first studied by Y. Peterzil and S. Starchenko in [**56**], where they showed Proposition 7.2 holds.

The existence of a one dimensional torsion-free definable subgroup in groups that are not definably compact Theorem 7.12 was proved by Y. Peterzil and C. Steinhorn in [**59**].

The results of Section 7.3 appeared in [**15**] and [**20**]. Another proof of Theorem 7.26 was given in [**16**]. Section 7.5 is from [**15**], except for uniqueness by conjugacy of the maximal torsion-free definable subgroup that was first proved by Y. Peterzil and S. Starchenko in [**58**] with types.

Theorem 7.32 and related results on torsion-free definable groups being completely solvable were showed in [**19**]. A complete classification of torsion free definable groups up to dimension four can be found in [**1**]. The results about the linear case are also based in [**19**].

CHAPTER 8

Structure of a definable group

This chapter includes important results about the structure of definable groups. In Chapter 5 we decomposed every definably connected group G into a product of the solvable radical and a definably connected central extension of a semisimple group (Theorem 5.62 and Corollary 5.65). In this chapter we will show that the solvable radical can be replaced by G^{tf}, the maximal normal definable torsion-free subgroup of G. To this end, we will first study solvable groups to then prove that any definably connected group is a product of a torsion-free (completely solvable) definable group and a definably connected reductive group. We will then use this decomposition to give necessary and sufficient conditions for the definability of continuous homomorphisms between definable groups. Finally, we will study the disconnected case.

8.1. Solvable groups

We start by showing that in a definably connected solvable definable group G, every 0-Sylow subgroup A maps surjectively onto G/G^{tf}, where G^{tf} denotes the maximal (normal) torsion-free definable subgroup, providing a definable decomposition $G = AG^{\mathrm{tf}}$.

Proposition 8.1. *Let R be a solvable definably connected definable group. Then $R = AU$, for every 0-Sylow A of R. Moreover if $A_1 < R$ is a 0-subgroup such that $R = A_1U$, then A_1 is a 0-Sylow as well (and therefore a conjugate of A).*

Proof. If G is torsion-free then $A = \{e\}$ and there is nothing to prove. So suppose $E(G) = 0$ and let A be a 0-Sylow of R.

We want to show that $R = AU$. Since $E(R/AU)E(AU/A) = E(R/A) \neq 0$, then $E(R : AU) \neq 0$. On the other side, AU/U is a definable subgroup of the abelian, definably connected, definably compact definable group R/U, which is a 0-group. Hence $E(R/AU) = E((R/U)/(AU/U))$ is 0, unless $R/U = AU/U$. Therefore it must be $R = AU$.

Let A_1 be a 0-subgroup of R such that $R = A_1U$, and let S_1 be a 0-Sylow containing A_1. Since $R = A_1H$, S_1/A_1 is definably isomorphic to a subgroup of U, so $E(S_1/A_1) = \pm 1 \neq 0$. But S_1 is a 0-group, so $S_1 = A_1$. □

Corollary 8.2. *Let G be an abelian definably connected definable group. Then there are unique definable subgroups H and A with H torsion-free and A 0-group such that $G = AH$. A is the unique 0-Sylow of G and H is the maximal torsion-free definable subgroup of G.*

We can make the decomposition in Proposition 8.1 finer, a strengthening which we will need later. For this we will need the following fact (Corollary 10.24 in [**66**]) from Group Theory.

Fact 8.3. *If a divisible group D is a subgroup of an abelian group $(G, \oplus)$ then D is a direct summand of G. Namely, there is some subgroup K of G with $G = D \oplus K$ and $D \cap K = \emptyset$.*

The proof uses Zorn's Lemma to show that the identity map on D can be extended to a group homomorphism ϕ from G to D. So we have an exact sequence

$$0 \to \ker(\phi) \to G \to D \to 0$$

and since the identity map is a section from D to G the sequence splits. We leave the details to the interested reader.

EXERCISE 8.1. *Use Zorn's Lemma to prove that if D is a divisible subgroup of an abelian group $(G, \oplus)$ then there is a group homomorphism from G to D extending the identity on D.*

Proposition 8.4. *Let G be a solvable definably connected definable group and U its maximal normal torsion-free definable subgroup. Then G is isomorphic (as an abstract group) to a semidirect product $G/U \ltimes U$.*

PROOF. Let A be a 0-Sylow of G. By the previous theorem $G = AU$. If A is definably compact, then it is definably isomorphic to G/U and the isomorphism between G and $G/U \ltimes U$ is definable.
Otherwise, $A \cap U$ is the maximal torsion-free definable subgroup of A and it is infinite. Since A and $A \cap U$ are both abelian and divisible (Proposition 6.17), Fact 8.3 implies $A \cap U$ has an abstract group complement in A isomorphic with $A/(A \cap U)$. Note that $A/(A \cap U)$ is isomorphic to the image of A by the canonical projection onto G/U, therefore $A/(A \cap U)$ and G/U are definably isomorphic.

Thus G is abstractly isomorphic to $A/(A \cap U) \ltimes U$ which is definably isomorphic to $G/U \ltimes U$. □

It is easy to verify that there is a definable isomorphism between G and $G/U \ltimes U$ if and only if every 0-Sylow ($\Leftrightarrow$ some 0-Sylow: 6.23) of G is definably compact:

Lemma 8.5. *Let A be a 0-group and let H_A be its maximal torsion-free definable subgroup. Then the definable exact sequence*

$$1 \longrightarrow H_A \xrightarrow{i} A \xrightarrow{\pi} A/H_A \longrightarrow 1$$

definably splits if and only if A is definably compact.

PROOF. If A is definably compact then $H_A = \{e\}$ and there is nothing to prove. Suppose then A not definably compact. If H_A has a definable complement K in A, then $E(A/K) = E(H_A) = \pm 1$, a contradiction with the fact that A is a 0-group. □

Corollary 8.6. *Let G be a solvable definably connected definable group and let H be its maximal normal torsion-free definable subgroup. Then either*

(i) *the 0-Sylow of G are definably compact and each of them is definably isomorphic to G/H (and therefore G is definably isomorphic to the semidirect product $G/H \ltimes H$), or*

(ii) *the 0-Sylow of G are not definably compact and for each A of them, $A/(A \cap H)$ is definably isomorphic to G/H (and G does not contain any definable subgroup definably isomorphic to G/H).*

PROOF. By Proposition 8.4, $G = AH$ for every 0-Sylow A of G. If A is definably compact, (i) holds. Otherwise, $A \cap H$ is infinite and the canonical projection $G \to G/H$ maps A onto $A/(A \cap H)$. If K is a definable subgroup of G definably isomorphic to G/H, then it is a 0-group (it is a torus) such that $E(G/K) \neq 0$ and $\dim K = \dim G/H = \dim A/(A \cap H) < \dim A$, in contradiction with 6.24. Therefore if A is not definably compact then G does not contain any definable subgroup definably isomorphic to G/H. □

8.2. A definable decomposition of definably connected groups

In this section we show that every definably connected group is a product of a torsion-free and a reductive definable subgroup. More precisely:

Theorem 8.7. *Let G be a definably connected definable group. Then*

$$G = HG^{tf}$$

where H is a definably connected reductive definable group such that $Z(H)$ contains a 0-Sylow of the solvable radical of G, $[H, H]$ is an ind-definable Levi subgroup of G and $H^{tf} = H \cap G^{tf}$.

Along the way, we will see some intermediate steps of independent interest. We start with a generalization of the Frattini's Argument, a fundamental tool in the study of finite groups.

Lemma 8.8 (Definable Frattini's Argument)**.** *Let G be a definable group, N a normal definable subgroup and S a p-Sylow subgroup of N ($p = 0$ or p prime). Then $G = N_G(S)N$.*

Proof. For $g \in G$, let S^g denote the conjugate of S by g. Then S^g is a p-Sylow of N for any $g \in G$, because the conjugation map $N \to N$, $x \mapsto gxg^{-1}$ is a definable automorphism of N, and the Euler characteristic is invariant by definable bijections. Since p-Sylow subgroups are all conjugate, there is $x \in N$ such that $S^g = S^x$ and $gx^{-1} \in N_G(S)$, the normalizer of S in G. Therefore $G = N_G(S)N$. □

Lemma 8.9. *Every normal 0-subgroup of a definably connected group is central.*

Proof. Let A be a normal 0-subgroup of a definably connected group G. For each $n \in \mathbb{N}$, the n-torsion subgroup T_n of A is definably characteristic in A. Because A is normal, T_n is a normal subgroup of G. Fix $x \in T_n$. The definable continuous function $G \to G$ given by $g \mapsto gxg^{-1}$ has image in T_n. However, G is definably connected and T_n is finite, so by taking $g = e$ we deduce that the image must be $\{x\}$, and $x \in Z(G)$. It follows that the torsion subgroup $T = \bigcup_{n\in\mathbb{N}} T_n$ of A is central in G. As $\langle T\rangle_{\mathrm{def}} = A$ by Theorem 6.17, A is central too. □

Corollary 8.10. *For any definable group G, the quotient G^0/G^{tf} is a definably connected reductive group.*

Proof. For ease of notation, suppose G is a definably connected group with solvable radical R. Then the solvable radical of G/G^{tf} is R/G^{tf}, which is a 0-group by the Torsion Theorem and Theorem 7.26. By Lemma 8.9, R/G^{tf} is central in G/G^{tf}. That is, G/G^{tf} is a central extension of the semisimple $G/R = (G/G^{\mathrm{tf}})/(R/G^{\mathrm{tf}})$ by R/G^{tf}, and it is reductive, as claimed. □

By Corollary 8.10, if G is a definably connected group with trivial G^{tf}, then G is reductive. If G^{tf} is not trivial, it suffices to show that it is central.

Proposition 8.11. *Let G be a definably connected group. Then G is reductive if and only if G^{tf} is central in G.*

Proof. We know that G is reductive if and only if its solvable radical R is central. Since $G^{\mathrm{tf}} \subseteq R$, if G is reductive G^{tf} must be central.

Conversely, suppose G^{tf} is central and let A be a 0-Sylow subgroup of R. Any other 0-Sylow subgroup of R is a conjugate of A in R. But because G^{tf} is central and $R = AG^{\mathrm{tf}}$ (Proposition 8.1), it follows that A is the only 0-Sylow subgroup of R. Hence A is definably characteristic and normal in G. By Lemma 8.9, A is central in G. Thus $R = AG^{\mathrm{tf}}$ is central too, as claimed. □

We can show this is always the case when $\dim G^{\mathrm{tf}} = 1$.

Lemma 8.12. *Let G be a definably connected group and set $G' = G/G^{tf}$. If N is a normal definable subgroup of G' such that G'/N is torsion-free, then $N = G'$.*

Proof. If G is solvable, then G' is a 0-group (Theorem 7.26) and all definable quotients of 0-groups are 0-groups (Proposition 6.7). If G is not solvable, let R be its solvable radical and $R' = R/G^{\mathrm{tf}}$ the solvable radical of G'. By the solvable case, $R' \subset N$. If G'/N is torsion-free, it is solvable (Proposition 7.2) and $G'/N = (G'/R')/(R'/N)$ is a solvable quotient of the perfect G'/R' (Proposition 5.21). It follows that $G' = N$, as claimed. □

Proposition 8.13. *If $\dim G^{tf} = 1$, then G^{tf} is central in G^0.*

Proof. For ease of notation, set $N = G^{\mathrm{tf}}$ and assume G is definably connected.

Let $s\colon G/N \to G$ be a definable section of the canonical projection $\pi\colon G \to G/N$. Then there is a unique ordered pair $(a, x) \in N \times G/N$ such that $g = s(x)a$. For every $g \in G$, the conjugation map $f_g\colon N \to N$, $a \mapsto gag^{-1}$ is a definable automorphism of N, so there is a homomorphism $\Phi\colon G \to \mathrm{Aut}(N)$, given by $g \mapsto f_g$ such that $N \subseteq \ker \Phi$, being N abelian. Thus Φ induces the definable homomorphism

$$\begin{aligned} \varphi\colon G/N &\longrightarrow \mathrm{Aut}(N) \\ x &\mapsto (a \mapsto s(x)as(x)^{-1}) \end{aligned}$$

which does not depend on the choice of the section s.

We claim that $\varphi(G/N) = \{\mathrm{id}\}$, and so $N \subseteq Z(G)$. Set $H = \varphi(G/N)$. Because G/N is definably connected, so is H and $|E(H)|$ is either 0 or 1.

Suppose $E(H) = 0$. By Proposition 6.7 there is a definable automorphism of N of order 3. However, N is an ordered group interval by Theorem 2.31, so any non-trivial automorphism is either strictly increasing or strictly decreasing. Indeed, by o-minimality any definable function is piecewise strictly increasing, strictly decreasing or constant on definable intervals, but injectivity forces the map to be monotone. Thus if σ is a strictly increasing definable automorphism, so is σ^3. Similarly, if σ is a strictly decreasing definable automorphism, so is σ^3 (same argument applies to any odd power). It follows that $E(H) \neq 0$.

If $|E(H)| = 1$, then H is torsion-free (Proposition 6.7) and $\ker \varphi$ is a normal definable subgroup such that $(G/N)/\ker \varphi$ is torsion-free. By Lemma 8.12, $\ker \varphi = G/N$, as claimed. □

Corollary 8.14. *Let G be a definably connected group. If G is not reductive, then $\dim G^{tf} \geq 2$.*

Proof. By Proposition 8.11 and Proposition 8.13. □

Example 8.15. There are many examples of non-reductive definable groups with $\dim G^{\mathrm{tf}} = 2$. For instance, the group $G = \mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$ (where the action is matrix multiplication) is centerless.

Note that by Corollary 8.10 when G is a definably connected semidirect product $G = G^{\mathrm{tf}} \rtimes K$ with K definable, then Theorem 8.7 is proved by taking $H = K$. However, G^{tf} is not always a semidirect factor (regardless of the definability of K), so a little bit more work is needed in general.

Proof of Theorem 8.7. If G is solvable, then by Proposition 8.1 $G = AG^{\mathrm{tf}}$, where A is an (abelian) 0-Sylow subgroup. Since every abelian group is reductive, we can take $H = A$. If G is semisimple, then G^{tf} is trivial, and $G = H$ is reductive.

Suppose G is neither solvable nor semisimple. Note that for any definable subgroup H, in order to get that $G = HG^{\mathrm{tf}}$, because of the Levi decomposition

Theorem 5.62 and the decomposition of solvable groups Proposition 8.1, it is enough to ensure that H contains a 0-Sylow of the solvable radical of G and an ind-definable Levi subgroup S of G. Moreover, recall that $SZ(G)^0$ is a definably connected reductive definable subgroup of G (Corollary 5.65).

Fix a 0-Sylow A of the solvable radical R of G and set $B = N_G(A)^0$, the definably connected component of the normalizer of A in G. Then $A \subseteq B$, because A is definably connected. By the definable Frattini Argument Lemma 8.8 and connectedness, $G = RB$. It follows that every ind-definable Levi subgroup S of B is an ind-definable Levi subgroup of G too. To see this, note that $G/R = (RB)/R = B/(R \cap B)$ is semisimple. Therefore, $(R \cap B)^0$ is the solvable radical of B, and we have $B = S(R \cap B)^0$. Hence $G = RB = RS$ and S is an ind-definable Levi subgroup of G, as claimed. Moreover, A is normal in B, hence central in B by Lemma 8.9.

Set $H = SZ(B)^0 A$. Then $R(H) = Z(H)^0 = Z(B)^0 A$ and H is reductive with $[H, H] = S$. Because H contains A and S, we get $G = HG^{\mathrm{tf}}$. We leave to the reader to check that $H \cap G^{\mathrm{tf}} = H^{\mathrm{tf}}$. □

8.3. Continuous homomorphisms between definable groups

Given definable groups G and H, and a continuous map ϕ between G and H, definability of ϕ is of course dependent on the o-minimal structure we are working with.

For example, in the pure real closed field $(\mathbb{R}, +, \cdot)$ an abelian variety is not definably isomorphic to a Tori of the same dimension, even though as Lie groups they are. In the same structure the additive group is not definably isomorphic to the multiplicative group. Two of the most important unsolved problems about o-minimal groups (and in o-minimality) over $\mathbb{R}$ have to do with this:

Suppose we are in the exponential case, so that we are working in an expansion of $\mathbb{R}_{exp}$ so that the additive and the multiplicative group are definably isomorphic. Let L be a definable group which is Lie isomorphic to $(\mathbb{R}, +)$. It is not known whether in this setting it is possible for L not to be definably isomorphic to $(\mathbb{R}, +)$. Surprising (to us), though, Chris Miller and Sergei Starchenko proved that the set of one dimensional torsion free definable groups has *at most two* classes modulo definable isomorphisms.

The other open problem is as follows. Suppose again $\mathcal{R} = \mathbb{R}$ and that we have a two dimensional definable abelian torsion free group A. The group A always has a definable torsion free subgroup H (Theorem 7.12), so that H and A/H are one dimensional torsion free groups. It is known that A is Lie isomorphic to $H \times A/H$. It is not known whether or not there is an o-minimal expansion of the real field with such a definable A which is not definably isomorphic to $H \times A/H$.

Both of these open problems about existence of definable maps have to do with abelian groups. We will prove that in fact definability of such functions is reduced to the abelian case. This short section is devoted to proving the following corollary of Theorem 8.7.

Theorem 8.16. *Let G, G' be definable groups and let $\phi : G \to G'$ be a continuous homomorphism such that its restriction to any definable abelian subgroup of G is definable. Then ϕ is definable.*

Proof. We can assume G is definably connected. By Theorem 8.7, G is the product of a torsion-free completely solvable definable group $U = G^{\mathrm{tf}}$, and a subgroup H which is a central extension of the semisimple group $G/R(G)$. So we need to prove that under the hypothesis of the theorem, the restrictions of ϕ to each of U and H are definable.

Since U is the product of definable abelian groups (Corollary 7.33) the restriction of ϕ to U is definable. So we only need to show that the restriction if ϕ to H is definable.

By hypothesis, the restriction of ϕ to $Z(H)$ is definable and since H is reductive, Theorem 5.66 implies that the restriction of ϕ to H is definable.

Since every element in G can be expressed as a product of elements in H and U, we get that ϕ is definable, as required. $\square$

8.4. Disconnected groups

As it happens for Lie groups with finitely many connected components, every definable group G contains a finite subgroup F such that $G = FG^0$. Moreover, in definable groups, F can be found in the normalizer of each 0-Sylow subgroup:

Theorem 8.17. *Let G be a definable group. For each 0-Sylow subgroup A of G there is a finite subgroup $F \subset N_G(A)$ such that $G = FG^0$, where $N_G(A)$ denotes the normalizer of S in G.*

Theorem 8.17 is analogous to a result in the theory of Lie groups with finitely many connected components which we will include in the next chapter. The proof of both results start with the (definably) compact case with abelian connected component (Lemma 8.18) and then the two proofs diverge. In the definable case we make strong use of the uniqueness of the maximal torsion free definable subgroup, a tool that is not available in the Lie group case. The Lie group case is proved using maximal tori and maximal compact subgroups which, as we have seen, are not always available for definable groups.

Lemma 8.18. *Let G be a definably compact definable group such that G^0 is abelian. There is a finite subgroup F of G such that $G = FG^0$.*

Proof. For any $X \in G/G^0$, choose an element $s_X \in G$ such that $|s_X|_{G^0} = X$. As is usual in this cases we will consider the "error" (cocycle) as a function of $G \times G/G^0$ to G^0,

$$\gamma(g, X) = g(s_X)(s_{(gX)})^{-1}.$$

so that

$$g(s_X) = \gamma(g, X)s_{gX}.$$

$X = aX$ for all $a \in G^0$ so $\gamma(h, X) = h$. Also,

$$\begin{aligned}
\gamma(hg, X) &= hgs_X\left(s_{(hgX)}\right)^{-1}\\
&= h\left(gs_X\right)\left(s_{(hgX)}\right)^{-1}\\
&= h\left(\gamma(g, X)s_{(gX)}\right)\left(s_{(hgX)}\right)^{-1}\\
&= h\gamma(g, X)h^{-1}\left(hs_{(gX)}\left(s_{(h(gX))}\right)^{-1}\right)\\
&= h\gamma(g, X)h^{-1}\left(\gamma\left(h, s_{(gX)}\right)\right)
\end{aligned}$$

and in particular using commutativity (and normality) of G^0,

$$(1)\qquad \gamma(hg, X) = h\gamma(g, X)h^{-1}\left(\gamma\left(h, s_{(gX)}\right)\right) = \left(\gamma\left(h, s_{(gX)}\right)\right)h\gamma(g, X)h^{-1}.$$

Let ϕ be the function from G to G^0 defined by

$$\phi(g) := \prod_{X \in G/G^0} \gamma(g, X).$$

For any $l \in G^0$ we have $\phi(l) = l^n$ where $n = |G/G^0|$, and in particular, if e_G is the identity in G and we define $E := \{x \in G \mid \phi(g) = e_G\}$, then $e_G \in E$ and $E \cap G^0 = Tor_n(G^0)$. So $E \cap G^0$ must be a finite (Proposition 6.10). We will prove that E is a finite subgroup of G which intersects all the G^0-coclasses, thus proving the theorem.

$$\begin{aligned}
\phi(hg) &= \prod_{X\in G/G^0} \gamma(hg,X) \\
&= \prod_{X\in G/G^0} h\gamma(g,X)h^{-1}\left(\gamma\left(h,s_{(gX)}\right)\right) \text{ by (1),} \\
&= \left(\prod_{X\in G/G^0} h\gamma(g,X)h^{-1}\right)\left(\prod_{X\in G/G^0}\gamma\left(h,s_{(gX)}\right)\right) \text{ by commutativity of } G^0, \\
&= \left(h\phi(g)h^{-1}\right)(\phi(h)).
\end{aligned}$$

This implies that E is closed under multiplication and replacing $h = g^{-1}$ we have $\phi(g^{-1}) = g(\phi(g))^{-1}g^{-1}$ for all g so E is closed under inverses and is therefore a subgroup of G. Finiteness follows from the fact that both $G^0 \cap E$ and G/G^0 are finite.

To prove that it meets every class in G/G^0, let $g \in G^0$. Since G^0 is definably connected and abelian, it is divisible (this is Exercise 6.5 and follows from finiteness of $Tor_m(G^0)$ for all m) so there is some $x \in G^0$ such that $x^n = \phi(g)$. Let $l_g := x^{-1}g$ so that in particular $|l_g|_{G^0} = |g|_{G^0}$. We will prove that $l_g \in E$, thus completing the proof of the theorem.

$$\phi(l_g) = \phi(x^{-1}g) = \left(x^{-1}\phi(g)x\right)\left(\phi(x^{-1})\right)$$

and since $x, \phi(g) \in G^0$ we have that everything commutes and $\phi(x^{-1}) = x^{-n}$ so

$$\phi(l_g) = \phi(g)x^{-n} = e_G$$

so $\phi(l_G) \in E$, as required. □

Even with the hypothesis of Lemma 8.18, it is not always possible to find a finite subgroup F of G isomorphic to G/G^0. After completing the proof of Theorem 8.17 we will give an example of a definable group G where any finite subgroup F intersects G^0 non-trivially (Example 8.23).

Lemma 8.19. *Let A be a 0-Sylow subgroup of a definable group G and denote by $N_G(A)$ its normalizer in G. Then $N_G(A)^0$ is solvable and $N_G(A)^0/A$ is torsion-free.*

PROOF. For ease of notation, set $H = N_G(A)$. Let R be the solvable radical of H.

As A is abelian and definably connected, $A \subseteq R$. If H^0 is not solvable, then H^0/R is an infinite semisimple definable group and $E(H^0/R) = 0$. However,

$$E(H^0/A) = E(H^0/R)E(R/A) \neq 0$$

by contradiction. It follows that H^0 is solvable.

Note that $H^{\mathrm{tf}} = (H^0)^{\mathrm{tf}}$, since torsion-free definable groups are definably connected. Moreover, we claim that $H^{\mathrm{tf}} \cap A = A^{\mathrm{tf}}$.

Since H^{tf} is torsion-free and A is abelian, clearly $A^{\mathrm{tf}} \cap A \subseteq A^{\mathrm{tf}}$. If $H^{\mathrm{tf}} \cap A \neq A^{\mathrm{tf}}$, then the quotient $A^{\mathrm{tf}}/(H^{\mathrm{tf}} \cap A)$ is an infinite torsion-free definable subgroup of $A/(H^{\mathrm{tf}} \cap A)$. However, because H^0 is solvable, $H^0 = AH^{\mathrm{tf}}$ by Proposition 8.1, so $A/(H^{\mathrm{tf}} \cap A)$ is definably isomorphic to the definably compact H^0/H^{tf}, contradiction.

Hence $(H/A)^0 = H^0/A = H^{\mathrm{tf}}/A^{\mathrm{tf}}$ is torsion-free, as required. □

We need the following facts, [**65**, 11.3.8 and 11.1.4]

Fact 8.20. *Let M be a G-module with G finite, such that M is uniquely divisible by m for all m. Then $H^n(\overline{G}, M) = 0$.*

Fact 8.21. *Let*

$$0 \to N \to G \to G/N \to 0$$

with N is abelian, G/N finite and consider the natural G/N-module structure on N given by conjugation. Then there is a bijection between equivalence classes of extensions of N by G/N inducing the G/N-module structure on N and the group $H^2(G/N, N)$.

Proof of Theorem 8.17. Assume first G has only the trivial 0-Sylow subgroup $A = \{e\}$. That is, $E(G) \neq 0$ and G^0 is torsion-free. Therefore, we want to find a finite subgroup F such that

$$G = G^0 \rtimes F.$$

Suppose first G^0 is abelian and set $\overline{G} = G/G^0$. Then G^0 is a $\overline{G}$-module. As G^0 is uniquely divisible, then Fact 8.20 implies $H^n(\overline{G}, G^0) = 0$ for all $n > 0$ and in particular Fact 8.21 implies that every extension $\overline{G}$ by G^0 splits and we can find some finite subgroup F that is a complement of G^0.

We will now prove our decomposition by induction on $d = \dim G$.

If $d = 1$, G^0 is abelian. So assume $d > 1$ and G^0 is not abelian. Set $N = \langle [G^0, G^0] \rangle_{\text{def}}$, the definable subgroup generated by the commutator subgroup of G^0. Since G^0 contains a normal definable subgroup of co-dimension 1 (Corollary 7.18), and 1-dimensional definable groups are abelian (Theorem 2.32), so N is an infinite proper definable subgroup of G^0. Any definable subgroup and definable quotient of a definable torsion-free group is torsion-free as well (Proposition 6.7), so $(G/N)^0 = G^0/N$ is torsion-free and $G/N = G^0/N \rtimes F_1$ by the abelian case above. Let G_1 be the pre-image in G of F_1. By induction hypothesis, $G_1 = N \rtimes F$, for some finite subgroup F. Since G_1 maps surjectively onto G/N, $G = NG_1$ and $G = G^0 \rtimes F$, as claimed.

Assume now G has infinite 0-Sylow subgroups and let A one of them. If A is normal in G, $G = N_G(A)$ and by Lemma 8.19 and the torsion-free case above,

$$G/A = (G/A)^0 \rtimes F'$$

for some finite $F' < G/A$. Let K' be the pre-image of F' in G. Note that A^{tf} is a definably characteristic subgroup of A, hence normal in G. The quotient $K = K'/A^{\text{tf}}$ is definably compact because K'/A and A/A^{tf} are definably compact. Set $A_1 = A/A^{\text{tf}}$. By Lemma 8.18, $K = F_1A_1$, for some finite $F_1 < K$, because $A_1 = K^0$ is abelian. Let H be the pre-image of F_1 in G. By the torsion-free case above, there is some finite subgroup F such that $H = A^{\text{tf}} \rtimes F$. Therefore, $G = FG^0$ because $G = HA$.

Suppose now $N_G(A) \neq G$. Because A is definably connected, $A \subseteq G^0$. By Lemma 8.8, $G = N_G(A)G^0$. Let $F \subseteq N_G(A)$ be a finite subgroup from above such that $N_G(A) = FN_G(A)^0$. Since $N_G(A)^0 \subseteq G^0$, it follows that $G = FG^0$, which completes the proof of the theorem. $\square$

Corollary 8.22. *Let G be a definable group. Then there are definable subgroups F of G and H, A, U of the connected component G^0 such that F is finite, H is a central extension of the semisimple group G^0/R, A is a 0-Sylow subgroup of G^0 and U is a completely solvable group.*

Moreover, the action of F on G^0 induced by conjugation must fix U.

When $E(G) = 0$, it is not always possible to find a finite subgroup that intersects G^0 trivially, unlike the case where G^0 is torsion-free:

Example 8.23. Let $\mathcal{R}$ be a real closed field and $\mathcal{K}$ its algebraic closure. Consider the definable (in $\mathcal{R}$) subgroup G of $\mathrm{SL}_2(\mathcal{K})$ of matrices of the form

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix}$$

where $z \in \mathcal{K}$, $|z| = 1$. Then G is an extension of $\mathbb{Z}/2\mathbb{Z}$ by a 1-dimensional definable torus. The unique 0-Sylow subgroup of G is G^0, corresponding to the diagonal matrices above on the left. Each anti-diagonal matrix on the right has order 4 and the subgroup generated by any of them meets both definably connected components of G. G^0 is not a semidirect factor of G, as all elements of order 2 are contained in G^0.

8.5. p-groups

In this section we describe p-groups and p-Sylow subgroups when p is a prime number. It will follow that they are not 0-groups and 0-Sylow subgroups respectively if and only if they are disconnected. To this end, the decomposition of disconnected group Theorem 8.17 will be applied.

Proposition 8.24. *Let G be a definable group and p a prime number. Then G is a p-group if and only if G^0 is a 0-group and G/G^0 is a finite p-group.*

Proof. Suppose G is a definable p-group. Then

$$E(G/G^0) = |G/G^0| \equiv 0 \mod p.$$

Suppose, by way of contradiction, G/G^0 is not a p-group. That is, $|G/G^0| = np^a$, where $a \geq 1$ and $n > 1$ is not a multiple of p. Let P be a p-Sylow subgroup of G/G^0 and H its pre-image in G. Then $E(P) = |P| = p^a$ and $E(G/H) = n$, in contradiction with the fact that G is a p-group.

Next, we want to show that G^0 is a 0-group.

If G^0 is non-trivial, then $E(G^0) = 0$, otherwise $E(G^0) = \pm 1$ and G cannot be a p-group. Let A be a 0-Sylow subgroup of G^0. Since $E(G/G^0) = |G/G^0| \neq 0$, A is also a 0-Sylow of G and $E(G/N_G(A)) = 1$. But G is a p-group, so $G = N_G(A)$ and A is normal in G. By Lemma 8.19 and Theorem 8.17, G^0 is solvable and there is a finite subgroup F such that

$$G/A = (G^{\mathrm{tf}}/A^{\mathrm{tf}}) \rtimes F.$$

Let K be the pre-image of F in G. Then $E(G/K) = E(G^{\mathrm{tf}}/A^{\mathrm{tf}}) = \pm 1$. Since G is a p-group, $G^{\mathrm{tf}} = A^{\mathrm{tf}}$ and $G^0 = G^{\mathrm{tf}}A = A$ by Proposition 8.1.

Conversely, suppose G^0 is a 0-group and G/G^0 is a finite p-group. Let H be a proper definable subgroup of G. We need to show that $E(G/H) \equiv 0 \mod p$.

If $H \cap G^0 = G^0$, then $G^0 \subseteq H$. We have $E(G/G^0) = E(G/H)E(H/G^0) = p^a$, because G/G^0 is a finite p-group. So $E(G/H) = p^{a-b}$, where $E(H/G^0) = p^b$.

If $H \cap G^0 \subsetneq G^0$, then $E(G^0/(H \cap G^0)) = 0$, because G^0 is a 0-group. Note that $E((G^0H)/H) = 0$, as $G^0/(H \cap G^0)$ is in definable bijection with $(G^0H)/H$. Moreover,

$$E(G/H) = E(G/(G^0H))E((G^0H)/H) = 0.$$

In either case, $E(G/H) \equiv 0 \mod p$, and G is a p-group, as required. □

Corollary 8.25. *If G is a definably connected group, then G is a p-group if and only if G is a 0-group.*

Theorem 8.26. *Let $H < G$ be definable groups and p a prime number. The following are equivalent:*

(i) H is a p-Sylow of G;

(ii) H^0 *is* 0*-Sylow of* G*,* $H^0 = H \cap G^0$ *and* H/H^0 *is a* p*-Sylow of* G/G^0*.*

In particular, the 0*-Sylow subgroups of* G *coincide with the* p*-Sylow subgroups of* G^0*.*

Proof. $(i) \Rightarrow (ii)$. Suppose H is a p-Sylow subgroup of G. By Proposition 8.24, H^0 is a 0-group.

If $E(G) \neq 0$, G^0 is torsion-free and $G \cong G^0 \rtimes G/G^0$ by Theorem 8.17. It follows that the trivial subgroup is the only 0-subgroup of G and all definable p-subgroups of G are finite. Therefore $H^0 = \{e\} = H \cap G^0$ is the unique 0-Sylow of G. Since $|E(G^0)| = 1$, $|E(G)| = |E(G/G^0)|$ and H is a p-Sylow of G/G^0.

Assume $E(G) = 0$. THen $E(G/H) \not\equiv 0 \mod p$. In particular, $E(G/H) \neq 0$. So

$$E(G/H^0) = E(G/H)E(H/H^0) \neq 0$$

and H^0 is a 0-Sylow subgroup of G.

Set $K = N_G(H^0)$ and note that $H \subseteq K$. By Lemma 8.19, K^0 is solvable and K^0/H^0 is torsion-free. Since $(H \cap G^0)/H^0$ is a finite subgroup of K^0/H^0, it must be trivial, and $H \cap G^0 = H^0$. It follows that H/H^0 is the image of H in G/G^0 through the canonical homomorphism $G \to G/G^0$.

Set $G_1 = G/G^0$ and $H_1 = H/H^0$. By Proposition 8.24, H_1 is a p-subgroup of G_1. Let P_1 be a p-Sylow of G_1 containing H_1 and P its pre-image in G. If $H_1 \neq P_1$, then $E(P_1/H_1) \equiv 0 \mod p$ and

$$E(G/H) = E(G/P)E(P/H) = E(G/P)E(P_1/H_1) \equiv 0 \mod p$$

in contradiction with H being a p-Sylow of G. Therefore $H_1 = P_1$ and H/H^0 is a p-Sylow of G/G^0, as we wanted.

$(ii) \Rightarrow (i)$. Suppose H is a definable subgroup of G such that H^0 is a 0-Sylow, $H^0 = H \cap G^0$ and H/H^0 is a p-Sylow of G/G^0.

By Proposition 8.24, H is a p-group. Moreover,

$$E(G/H) = E((G/G^0)/(H/H^0)) \not\equiv 0 \mod p$$

because H/H^0 is a p-Sylow of G/G^0. Therefore H is a p-Sylow of G.

We are left to show that the 0-Sylow subgroups of G coincide with the p-Sylow subgroups of G^0 for each p. Suppose H is a 0-Sylow of G. Because H is definably connected, $H \subseteq G^0$ and H is a p-Sylow of G^0 by the above.

Conversely, suppose H is a p-Sylow of G^0. By the above, $H = H^0$ and H is a 0-Sylow of G^0. As $E(G/G^0) = |G/G^0| \neq 0$, H is a 0-Sylow of G, since

$$E(G/H) = E(G/G^0)E(G^0/H) \neq 0.$$

□

Since 0-groups are abelian and finite p-groups are nilpotent, the following holds:

Corollary 8.27. *Let* p *be a prime number or* $p = 0$*. Definable* p*-groups are solvable and definably generated by torsion elements.*

Unlike finite p-groups, definable p-groups are not necessarily nilpotent, though. For instance, Example 8.23 is a semialgebraic 2-group that is not nilpotent.

Historical notes and comments

The results in Section 1 on solvable groups are from [**14**]. Proposition 8.13 appears in [**12**]. Continuous homomorphisms are studied in [**46**]. All the other results are contained in or follow from [**17**].

CHAPTER 9

Definability of Lie groups

In this chapter, we will use many of the structure results proved in the previous chapters to give necessary and sufficient conditions for when a connected Lie group has a Lie isomorphic copy definable in an o-minimal expansion of the real field. The non-connected case is more subtle, and we will need to understand both the relation between a definable group G and its connected component G^0 and the automorphisms of definable connected groups.

9.1. o-minimal copies of connected Lie groups.

In this section we will characterize which connected real Lie groups can be studied under the framework of o-minimality. Namely, we will give necessary and sufficient conditions for a Lie group to have Lie isomorphic group definable in an o-minimal extension of the real field.

Definition 9.1. Recall that a closed subgroup H of a topological group G is called *cocompact* if the homogenous space G/H is compact.

We will prove the following theorem.

THEOREM. *Let G be a connected real Lie group. The following are equivalent:*

(1) G has an o-minimal copy.
(2) G has a definable copy in $\mathbb{R}_{\exp}$.
(3) $(Z : Z^0)$ is finite and the solvable radical of G contains a normal triangular cocompact subgroup.

If the solvable radical of G has a normal nilpotent simply connected cocompact subgroup, G will have a semialgebraic copy. In particular, this is always the case when G is reductive.

9.1.1. Lie group preliminaries.

The o-minimal side of the proof will follow from the structure results from the previous chapters, but we will also need some particular representation theorems for Lie groups.

The following is also from Malcev, Theorem 5.2 of Chapter I in [**45**].

Fact 9.2. *Let G be a connected Lie group and L its Levi subgroup. Then G is linearizable (has a faithful linear representation) if and only if both L and the radical $R(G)$ of G are linearizable.*

Proposition 1.39 in [**42**] states:

Fact 9.3. *If h is solvable and linear then $[h, h]$ is nilpotent.*

Finally, the following is a particular case of Theorem 3.2 in [**35**].

Fact 9.4. *Let G be a connected Lie group, and suppose that G is a semidirect product $K \cdot U$, with U normal in K and U solvable. Let ρ be a representation of G with image in $GL_n(\mathbb{R})$ such that $\rho(yxy^{-1}x^{-l}) - \mathbb{I}_n$ is a nilpotent matrix for every $y \in G$ and every $x \in U$. Then the representation ρ can be lifted to a representation of G.*

The strategy is as follows. We will find definable copies of U and then of $R = KU$ that are linear with the copy of U consisting of upper triangular matrices. This will allow us to have all images of actions by conjugation from H to R being definable.

9.1.2. Definable copy of a solvable group. Suppose R is a connected solvable Lie group that is definable in an o-minimal structure. By Proposition 7.22 and Theorem 7.26, R contains a normal torsion-free definable cocompact subgroup U. By Theorem 7.32, U is completely solvable. We see below that these conditions are not only necessary, but also sufficient to provide a linear copy copy definable in the real exponential field.

Proposition 9.5. *Let R be a connected solvable Lie group. The following are equivalent:*

(1) R has an o-minimal copy.
(2) R contains a torsion-free normal completely solvable cocompact subgroup.
(3) R has a linear definable copy in $\mathbb{R}_{\exp}$, $R = K \ltimes U$ with K connected compact abelian and U connected upper triangular.

PROOF. Let R be a connected solvable real Lie group. Assume that R has a definable copy R_{def}, and let R^{tf} be the subgroup of R corresponding to $R^{\text{tf}}_{\text{def}}$. By Theorem 7.32 R^{tf} is completely solvable, and by Theorem 7.26 R/R^{tf} is definably compact, so compact by 2.16. Since $R^{\text{tf}}_{\text{def}}$ is torsion-free, R^{tf} is simply-connected. So (1) implies (2),

Assume (2), and let U be the normal torsion-free completely solvable cocompact subgroup. Theorem 7.50 gives a faithful representation ρ' of U in $t_n(\mathbb{R})$ with $\rho'(U)$ a subgroup of $t_n(\mathbb{R})$ definable in $\mathbb{R}_{\exp}$. By Fact 9.3, $[R,U] \subseteq [R,R]$ is nilpotent, so that $\rho'(xyx^{-1}y^{-1})$ belongs to the nilradical of $t_n(\mathbb{R})$ for all $x \in R, y \in U$. But the nilradical of $t_n(\mathbb{R})$ is the subgroup with ones in the diagonal, which implies that $\rho'(xyx^{-1}y^{-1}) - \mathbb{I}_n$ is a strictly upper triangular matrix, so it is nilpotent. By Fact 9.4 ρ' can be lifted to a representation ρ of R.

On the other hand, R/U is compact, so it has a faithful representation τ' with algebraic linear image. If τ is the representation of R induced by τ', then the product $\tau \times \rho$ of τ and ρ mapping $g \in R$ to the pair $(\tau(g), \rho(g))$ gives a faithful representation of R as a semidirect product of the algebraic $K = \tau(R)$ and the upper triangular $U = \rho(R)$ definable in $\mathbb{R}_{\exp}$. Since K is a definable solvable (definably) compact (definably connected) group, it is a torus by Proposition 6.25. This proves (3).

Since $\mathbb{R}_{\exp}$ is o-minimal, (1) follows immediately from (3). □

9.1.3. Definable copy of a linear group. Let G be a connected Lie group such that $(Z : Z^0)$ is finite and the solvable radical of G contains a normal triangular cocompact subgroup. Let L be a Levi subgroup and R the solvable radical. Let $L_{\text{lin}} = L/(Z(G) \cap L)$.

L acts on R by conjugation, an action that induces a well defined action from L_{lin} on R. Let $\overline{\phi}$ be the action of L_{lin} on R and set $G_{\text{lin}} := L_{\text{lin}} \ltimes_{\overline{\phi}} R$. To prove that G has a definable copy we will first need to prove that G_{lin} has a linear definable copy in $\mathbb{R}_{\exp}$.

Lemma 9.6. *Let L_{lin} be a semisimple connected linear Lie group and let R be a connected solvable Lie group with an o-minimal copy R_{def}. Let ϕ be a continuous action of L_{lin} on R. Then the connected Lie group $G_{\text{lin}} := L_{\text{lin}} \ltimes_{\phi} R$ has a linear copy definable in $\mathbb{R}_{\exp}$ whose solvable radical R_{def} splits definably as $R_{def} = K_{\text{def}} \ltimes U_{\text{def}}$*

with K_{def} a linear algebraic compact connected group and U_{def} a closed connected subgroup of upper triangular matrices.

PROOF. Let $R_{\mathrm{def}} = K_{\mathrm{def}} \ltimes U_{\mathrm{def}}$ be a linear definable copy of R in $\mathbb{R}_{\exp}$ given by Proposition 9.5. Because L_{lin} is a (linear) Levi subgroup of G_{lin}, by Fact 9.2 we may assume that G_{lin} is a closed subgroup of $GL_n(\mathbb{R})$ with solvable radical R_{def}. The Levi subgroup of G_{lin} is semialgebraic by Corollary 5.31, and so G_{lin} is definable in $\mathbb{R}_{\exp}$ with the claimed properties. □

9.1.4. The main theorem: characterization of connected Lie groups with definable copies. We will start with the following lemma.

Lemma 9.7. *Let $G = RL$ be a Levi decomposition of a connected real Lie group G. Set $H = ZL$, where Z is the center of G. Then $H \cap R = D \times Z^0$ for some discrete group D central in H. In particular, $H \cap R$ is abelian.*

PROOF. Note that $Z(H) = Z \cdot Z(L)$, so $H/Z(H) \cong L/Z(L)$ is a centerless semisimple group, since L is a connected semisimple group. Because $H \cap R$ is a solvable normal subgroup of H, then $H \cap R/(H \cap R) \cap Z(H)$ is a normal solvable subgroup of the centerless semisimple group $H/Z(H)$. The following claim proves that $H \cap R/(H \cap R) \cap Z(H)$ is the identity, so that $H \cap R \subseteq Z(H)$ thus completing the proof of the lemma.

Claim 9.8. *Let G be a connected centerless semisimple group. Then the only normal solvable subgroup of G is the trivial one.*

Proof: Suppose N is a solvable normal subgroup of G. We can prove that N is trivial by induction on $n = \dim N$. If $n = 0$, then N is discrete, therefore it is central, and it must be trivial. If $n > 0$, then N^0 is also solvable and normal in G. We know that N^0 cannot be abelian, because G is semisimple. So the derived subgroup of N^0 is a non-trivial connected solvable normal subgroup of G with dimension less than n, and it must be trivial by induction hypothesis. But then N^0 is abelian, contradiction. $\blacksquare_{Claim}$ □

Theorem 9.9. *Let G be a connected real Lie group. Assume that its center $Z(G)$ has finitely many connected components and that $R(G)$ has a cocompact completely solvable torsion-free closed subgroup. Then G is Lie isomorphic to a group G_{def} definable in $\mathbb{R}_{\exp}$.*

Furthermore, G_{def} is such that for any Levi subgroup L of G if $H = Z(G)L$ and $R = R(G)$ and we let $H_{\mathrm{def}}, R_{\mathrm{def}}$ be the corresponding subgroups of G_{def}, then

(1)
$$G_{\mathrm{def}} := (H_{\mathrm{def}} \ltimes_\psi R_{\mathrm{def}})/\{(x, \alpha(x)^{-1}) : x \in X\}$$
where α is a definable embedding from a subgroup of H_{def} isomorphic to $H \cap R$ into R_{def},

(2) H_{def} is a semialgebraic group which is a central extension of $\overline{S} := L/L\cap Z$ by $Z(G_{\mathrm{def}})$. We will identify $L/L\cap Z$ with the image of L under the adjoint representation of G.

(3) R_{def} is linear, and $R_{def} = K_{\mathrm{def}} \ltimes U_{\mathrm{def}}$ with K_{def} is a linear algebraic compact group and U_{def} is a closed subgroup of upper triangular matrices.

PROOF. Let G be a connected Lie group with center $Z := Z(G)$ and solvable radical $R := R(G)$ satisfying the hypothesis. Let $G = RL$ be a Levi decomposition of G. We will build a definable copy of G as a quotient of a semidirect product of definable copies of $H := ZL$ and R. This construction will be done in $\mathbb{R}_{\exp}$, so for the rest of the proof we will use *definable* to mean *definable in* $\mathbb{R}_{\exp}$.

Let $\overline{L}$ be the image of L under the adjoint representation of G. So $\overline{L}$ is linear and semisimple, so semialgebraic by Proposition 5.31 and $\overline{L} \cong H/(H \cap Z) = L/(L \cap Z)$.

Because R is a normal subgroup in G, the subgroup H acts on R by conjugation. Let $\gamma\colon H \to \mathrm{Aut}(R)$ be defined by $\gamma(x)(r) = x^{-1}rx$ for each $x \in H$ and each $r \in R$. Since $Z \subseteq \ker(\gamma)$, γ induces an homomorphism $\overline{\gamma}\colon \overline{L} \to \mathrm{Aut}(R)$ so that given $\overline{x} \in \overline{L}$ and $r \in R$, $\overline{\gamma}(\overline{x})(r) = x^{-1}rx$, for any $x \in H$ with $x \in \overline{x}$. We therefore get the following diagram.

$$\begin{tikzcd} H \arrow[r, "\gamma"] \arrow[d, "\pi"'] & \mathrm{Aut}(R) \\ \overline{L} \arrow[ur, "\overline{\gamma}"] & \end{tikzcd}$$

By Lemma 9.6 the group $\overline{L} \rtimes_{\overline{\gamma}} R$ has a linear copy $\overline{L}_{\mathrm{def}} \rtimes_{\overline{\gamma}_{\mathrm{def}}} R_{\mathrm{def}}$ definable in $\mathbb{R}_{\exp}$. The groups $\overline{L}_{\mathrm{def}}$ and $\overline{L}$ are linear and semisimple, so the isomorphism between them is semialgebraic and we may assume $\overline{L}_{\mathrm{def}} = \overline{L}$.

$Z^0 \leq R$ and the image Z^0_{def} of Z^0 in R_{def} splits as $R_{\mathrm{def}} = K_{\mathrm{def}} \ltimes U_{\mathrm{def}}$ with K_{def} is a linear algebraic compact group and U_{def} is a closed subgroup of upper triangular matrices and by Theorem 6.48 we may assume that K_{def} is a subgroup of D_n^K. In particular, Z^0_{def} is definable. Let F be the finite group with $Z = Z^0 \times F$ and let $Z_{\mathrm{def}} = Z^0_{\mathrm{def}} \times F$.

Our next step is to build a definable copy H_{def} of H as a definable central extension of $\overline{L}$ having $Z^0(R_{\mathrm{def}})$ as its solvable radical and a definable action $H_{\mathrm{def}} \times R_{\mathrm{def}} \to R_{\mathrm{def}}$ mimicking the conjugation map of H on R:

Claim 9.10. *There is a definable copy H_{def} with $R(H_{def}) = Z^0_{\mathrm{def}}$, a Lie isomorphism $h\colon H_{def} \to H$ and a definable action $\psi\colon H_{def} \times R_{def} \to R_{def}$ such that*

$$\psi(x, r_1) = r_2 \quad \text{if and only if} \quad h(x)^{-1} g(r_1) h(x) = g(r_2)$$

where $g\colon R_{def} \to R$ is a fixed Lie isomorphism.

Proof: H is a central extension of the semisimple group linear group $\overline{L}$ by Z, and we proved in Proposition 4.30 that any such H can be assumed to be a group structure on the set $\overline{L} \times Z$ with product

$$(x, a) * (y, b) = (xy,\ f(x, y) + a + b).$$

for a 2-cocycle f.

Let H' be the structure on $\overline{L}_{\mathrm{def}} \times Z_{\mathrm{def}}$ induced by the group structure on H and the respective isomorphisms from Z and $\overline{L}$ to Z_{def} and $\overline{L}_{\mathrm{def}}$. H' is therefore a cover of $\overline{L}_{\mathrm{def}}$ and $\overline{L}_{\mathrm{def}}$ is semialgebraic, so Theorem 5.25 states that there is a section s from $\overline{L}_{\mathrm{def}}$ to H' such that the 2-cocycle f' induced by s is semialgebraic. We reparametrize using this cocycle, and define a group H_{def} on $\overline{L}_{\mathrm{def}} \times Z_{\mathrm{def}}$ with product

$$(x, a) \cdot (y, b) = (xy,\ f'(x, y) + a + b).$$

H_{def} is thus a semialgebraic group which by construction is isomorphic to H', so it is isomorphic to H, as required.

Let $\pi_{\mathrm{def}}\colon H_{\mathrm{def}} \to \overline{L}$ be the projection onto the first coordinate. For notation purposes, we will identify the subgroup defined by $\pi_{\mathrm{def}}(x) = e_{\overline{L}}$ with Z_{def} and assume that Z_{def} is a subgroup of H_{def}.

We define $\gamma_{\mathrm{def}}\colon H_{\mathrm{def}} \times R_{\mathrm{def}} \to R_{\mathrm{def}}$ by

$$\gamma_{\mathrm{def}}(x)(r) = \overline{\gamma}_{\mathrm{def}}(\pi_{\mathrm{def}}(x))$$

so that we get the following commuting diagram:

$$\begin{tikzcd} H_{\mathrm{def}} \arrow[r, "\gamma_{\mathrm{def}}"] \arrow[d, "\pi_{\mathrm{def}}"] & \mathrm{Aut}(R_{\mathrm{def}}) \\ \overline{L} \arrow[ur, "\overline{\gamma}_{\mathrm{def}}"'] & \end{tikzcd}$$

so that $\psi(x,r) = \gamma_{\mathrm{def}}(x)(r)$ is the required definable action. $\blacksquare_{Claim}$

By construction, $R \rtimes_\gamma H$ is a Lie group isomorphic to $R_{\mathrm{def}} \rtimes_{\gamma_{\mathrm{def}}} H_{\mathrm{def}}$. Since G is isomorphic to $R \rtimes_\gamma H/(R \cap H)$, in order to have a definable G_{def} copy of G we need to find a definable isomorphism α between the corresponding copies of $R \cap H$ in R_{def} and H_{def}. For notation purposes, let $X := R \cap H$, $X_H := h^{-1}(X)$ and $X_R := g^{-1}(X)$ where $h : H_{\mathrm{def}} \to G$ and $g : R_{\mathrm{def}} \to G$ are the Lie homomorphisms that send H_{def} to H and R_{def} to R, respectively.

By Lemma 9.7, $X = H \cap R$ is abelian and $X = A \times Z^0$ for some finite $A < X$. This implies that $X_H := h^{-1}(A) \times h^{-1}(Z^0)$ and, since by construction H_{def} is defined on the set $\overline{L}_{\mathrm{def}} \times Z_{\mathrm{def}}$ with $h^{-1}(Z^0) = \{e\} \times Z_{\mathrm{def}}$, $h^{-1}(Z^0) = \{e\} \times Z^0_{\mathrm{def}}$. Similarly, $X_R = g^{-1}(F) \times Z^0_{\mathrm{def}}$ and the map $g^{-1} \circ h$ restricted to $h^{-1}(Z^0)$ is the projection on the second coordinate.

The complement of $h^{-1}(Z^0)$ in X_H is a finite group, so the isomorphism $\alpha := g^{-1} \circ h$ from X_H to X_G is definable.

We will now show that the definable group

$$R_{\mathrm{def}} \rtimes_{\gamma_{\mathrm{def}}} H_{\mathrm{def}}/\{(\alpha(x), x) \mid x \in X_H\}$$

is isomorphic to G, as follows:

The definable semi-direct product $H_{\mathrm{def}} \ltimes_\psi R_{\mathrm{def}}$ and the map

$$\Phi\colon H_{\mathrm{def}} \ltimes_\psi R_{\mathrm{def}} \to G$$

given by $\Phi(x,r) = h(x)g(r)$ where $h\colon H_{\mathrm{def}} \to H$ and $g\colon R_{\mathrm{def}} \to R$ are Lie isomorphisms. Φ is a surjective smooth homomorphism and its kernel is the set of $(x,r) \in H_{\mathrm{def}} \times R_{\mathrm{def}}$ such that $h(x)g(r) = e$, which of course implies that $x \in X_H$, $r \in X_R$ and $g^{-1} \circ h(x) = r^{-1}$. So

$$\ker \Phi = \{(x, \alpha(x)^{-1}) : x \in X\}$$

and $(H_{\mathrm{def}} \ltimes_\psi R_{\mathrm{def}})/\ker \Phi$ is Lie isomorphic to G, as required.
This ends the proof of Theorem 9.9. $\square$

9.1.5. The disconnected case. A Lie group that has a definable copy in an o-minimal expansion of the real field, it must of course have a connected component of finite index. The conditions beyond this deal with the inner automorphisms of the connected component:

If G is a definable Lie group, then G^{tf} is a definably characteristic subgroup. In particular, for a Lie automorphism f of G to be definable, it is necessary that $f(G^{\mathrm{tf}}) = G^{\mathrm{tf}}$. This prompts the following definition:

Definition 9.11. For definable groups G_1, G_2 in an o-minimal expansion of $\mathbb{R}$, we define a Lie homomorphism $\sigma : G_1 \to G_2$ to be *good* if $\sigma(U_1) = U_2$, where U_1 and U_2 are the maximal normal definable torsion-free subgroups of G_1 and G_2 respectively.

So being good is a necessary condition for a Lie automorphism of a definable group to be definable. We see below that the condition is also sufficient, if we choose the definable copy from Theorem 9.9.

Lemma 9.12. *Let G be a definable group satisfying the conditions* $(1), (2)$ *and* (3) *stated in Theorem 9.9 for the definable group G_{def}. Then every good Lie automorphism of G is definable in $\mathbb{R}_{exp}$.*

In particular, any connected Lie group that admits an o-minimal copy, admits a definable copy in $\mathbb{R}_{exp}$ in which every good Lie automorphism is definable.

Proof. Let σ be a good (Lie) automorphism of G_{def}.

By hypothesis, G_{def} is a product of groups $H_{\mathrm{def}}, U_{\mathrm{def}}$ and K_{def} where K_{def} is algebraic compact linear, U_{def} is a closed subgroup of the upper triangular matrices and H_{def} is a central extension of a linear semisimple group.

Since σ is good the restriction of σ to U_{def} is an automorphism of the upper triangular linear group U_{def} which by Corollary 7.51 is definable. Theorem 5.66 implies that the restriction of σ to H_{def} is definable. Finally, $\sigma(K_{\mathrm{def}})$ is a compact linear group, so the graph of the restriction of σ to K_{def} is a compact linear group, so algebraic. □

We can now prove the following. It is mentioned in several texts and apparently it is well known, but we could not find a complete proof, so we include one.

Lemma 9.13. *Let G be a Lie group with finitely many connected components. Then there is a finite subgroup F of G such that $G = FG^0$.*

The lemma will follow as a consequence of known facts.

The first one appears in [**36**]. The proof is identical to the proof of Lemma 8.18.

Fact 9.14. *Let G be a compact Lie group such that the connected component G^0 is abelian. There is a finite subgroup F of G such that $G = FG^0$.*

The following follows from Theorem 3.1 in Chapter XV of [**34**].

Fact 9.15. *Let G be a Lie group with finitely many connected components. Let K be a maximal compact subgroup of G. Then the following hold:*

- *All maximal compact subgroups of G are conjugates to K.*
- *K meets every connected component of G.*
- *$G^0 \cap K$ is connected and itself a maximal compact subgroup of G^0.*

We will need the following facts about maximal tori of compact groups. The following is Corollary 4.35 in [**42**].

Fact 9.16. *For a compact connected Lie group, any two maximal tori are conjugate.*

Fact 9.17. *Let K be a compact group and T ma maximal tori of K. Then the normalizer $N_K(T)$ of T in K has T as a connected component and $N_K(T)/T$ is finite.*

This is well known, since $N_K(T)/T$ is the Weyl group of K (see Sections 6 of Chapter II and Section 6 of Chapter 4 in [**42**]). One way to prove it is as follows: maximality of T implies that the centralizer $Z_K(T)$ of T in K is T itself, and $N_K(T)/Z_K(T)$ is a subgroup of the endomorphisms of T, which as we saw in Section 6.4 is a subgroup of $Gl_n(\mathbb{Z})$ for some n, so a discrete group. Since $N_K(T)$ is compact, finiteness follows.

Proof of Lemma 9.13. Let K be a maximal compact subgroup of G, and let T be a maximal tori in K. By Fact 9.17 the normalizer $N_K(T)$ satisfies the conditions of Fact 9.14, so there is a subgroup F of $N_K(T)$ with $N_K(T) = FT$. T is of course connected, so a subgroup of G^0. We will prove that F meets all the connected components of G.

By Fact 9.15 it is enough to show that F meets every connected component of K. And since F meets every connected component of $N_K(T)$, it is enough to show that $N_K(T)$ meets every connected component of K. This is the following claim:

Claim 9.18. *Let K be a compact group, let T be a maximal tori of K and let $N_K(T)$ be the normalizer of T in K. Then $N_K(T)$ meets every connected component of K*

Proof: Let $g \in K$ and consider the group gTg^{-1}. It is of connected, so a subgroup (and a maximal tori) of K^0. By Fact 9.16 gTg^{-1} and T are conjugates inside K^0. So $gTg^{-1} = kTk^{-1}$ for some $k \in K^0$, which implies that $k^{-1}g$ is an element in the same K^0-class as g which by definition is in $N_K(T)$. $\blacksquare_{Claim}$ $\square$

Theorem 9.19. *Let G be a real Lie group with finitely many connected components. Assume that $Z(G^0)$ has finitely many connected components and that the solvable radical $R(G)$ contains a torsion-free completely solvable cocompact subgroup U that is normal in G.*

Then G is Lie isomorphic to a group G_{def} definable in $\mathbb{R}_{\mathrm{exp}}$.

PROOF. By Lemma 9.13 we have that $G = FG^0$ so that G is isomorphic to a quotient of the semidirect product $F \ltimes_\gamma G^0$ by a subgroup isomorphic to $G^0 \cap F$ where γ is conjugation in G. Since F is finite, it is enough to find a definable copy of $F \ltimes_\gamma G^0$.

Let G^0_{def} be a definable copy of G^0 in $\mathbb{R}_{exp}$ with the properties from Theorem 9.9 and let $f\colon G^0 \to G^0_{\mathrm{def}}$ be a Lie isomorphism. Note that by construction, $f(U)$ is the maximal normal definable torsion-free subgroup of G^0_{def}.

For each $g \in G$ the inner automorphism of G associated to g induces an automorphism of G^0 that fixes U by hypothesis. This automorphism is therefore mapped through f to a good automorphism σ_g of G^0_{def}. By Lemma 9.12, σ_g is definable in $\mathbb{R}_{exp}$. Let $\mathrm{Aut}(G^0_{\mathrm{def}})$ be the group of *definable* automorphisms in $\mathbb{R}_{exp}$ of G^0_{def} and let $\alpha\colon F \to \mathrm{Aut}(G^0_{\mathrm{def}})$ be the *definable* homomorphism given by $\alpha(g) = \sigma_g$. Then $F \rtimes_\alpha G^0_{\mathrm{def}}$ is a definable Lie group in $\mathbb{R}_{exp}$ that is Lie isomorphic to $F \ltimes_\gamma G^0$. $\square$

Example 9.20. The group $G^0 := S_1(\mathbb{R}) \times \mathbb{R}$ used in Example 7.1 as an example of a Lie group with more than one maximal torsion free subgroups can be used to build an example of a Lie group which is a finite extension of a definable group but such that the group itself cannot be definable in any o-minimal expansion of the real field.

G^0 is definable in the real field, and we will treat it as such. Only for notation purposes we will use complex exponential notation to describe $S_1(\mathbb{R})$.

Consider the automorphism $\phi : G^0 \to G^0$ given by

$$\phi(e^{i\theta}, s) = (e^{-i\theta}e^{is}, s).$$

Then

$$\phi(\phi(e^{i\theta}, s)) = \phi(e^{-i\theta}e^{is}, s) = \phi(e^{-i(\theta+s)}, s) = (e^{-(-i\theta+s)}e^{is}, s) = (e^{(i\theta-s+s)}, s)$$

so $\phi \circ \phi$ is the identity.

It is also clear that ϕ maps the definable torsion free subgroup $\{(1, s) \mid s \in \mathbb{R}\}$ to the group $\{(e^{is}, s) \mid s \in \mathbb{R}\}$ which cannot be definable in any o-minimal expansion (among many reasons because this would imply two maximal torsion free normal subgroups of the definable group G^0). But if we take the action γ of the finite group $\mathbb{Z}/2\mathbb{Z}$ on G^0 with $\gamma(0)(x) = x$ and $\gamma(1)(x) = \phi(x)$ then the group $\mathbb{Z}/2\mathbb{Z} \ltimes_\gamma G^0$ is a Lie group with finitely many definable connected components which is not definable in any o-minimal expansion of $(\mathbb{R}, +, \cdot)$.

This example can be generalized:

EXERCISE 9.1. *Show that for any finite cyclic group F there is an action γ of F on $S_1(\mathbb{R}) \times \mathbb{R}$ such that $F \ltimes_\gamma S_1(\mathbb{R}) \times \mathbb{R}$ is not definable in any o-minimal expansion of the real field.*

9.2. Lie morphisms and isomorphisms between definable groups.

Theorem 9.21. *Let G, H be two definable Lie groups and let $\phi : G \to H$ be a Lie morphism. Then ϕ is definable in an o-minimal expansion of the real field if and only if $\phi(G)$ is definable and ϕ sends the maximal normal torsion-free definable subgroup U_G of G to the maximal normal torsion-free definable subgroup $U_{\phi(G)}$ of $\phi(G)$.*

PROOF. If ϕ is definable $\phi(G)$ and $\phi(U_G)$ are both definable. $\ker(\phi) \cap U_G$ is a definable subgroup of U_G so by multiplicativity of the Euler characteristic the Euler characteristic of of $U_G/\ker(\phi) \cap U_G$ must be 1 or -1 so that it is torsion free by 6.7. In particular, $\phi(U_G)$ is a torsion free normal subgroup of the definable $\phi(G)$ so it must be contained in $U_{\phi(G)}$ by Proposition 7.22.

For the other direction, assume that $\phi(G)$ is definable and ϕ sends the maximal torsion free connected subgroup U_G of G to the maximal torsion free connected subgroup $U_{\phi(G)}$ of $\phi(G)$. We will prove that ϕ is definable in $\mathcal{R}_{\mathrm{Pfaff}}$

Let R be the solvable radical of G, Z^0 the connected component of the center of G and L be a Levi subgroup of G. Since $G = (R) \cdot (L \cdot Z^0)$ and both R and $L \cdot Z^0$ are definable, it is enough to prove that the restriction of α to R and to $L \cdot Z^0$ are definable in $\mathcal{R}_{\mathrm{Pfaff}}$.

We begin by proving that the restriction of α to R is definable in $\mathcal{R}_{\mathrm{Pfaff}}$.

Claim 9.22. *The restriction of α to R is definable in $\mathcal{R}_{Pfaff}$.*

Proof: Let K be a maximal compact Lie subgroup of R. By Corollary A.14 K, $\alpha(K)$ and the graph of the restriction of α to K are definable in $\mathcal{R}_{\mathrm{Pfaff}}$. Since $R = K \cdot U_1$, we need to prove that the restriction of α to U_1 is definable in $\mathcal{R}_{\mathrm{Pfaff}}$. Corollary 7.33 states that U_1 is a product of definable abelian subgroups, so it is enough to prove that the restriction of α any abelian subgroup of U_1 is definable in $\mathcal{R}_{\mathrm{Pfaff}}$. Let A be an abelian subgroup of U_1. Let $a \in A$ be such that $C_{G_1}(C_{G_1}(a)) \supseteq A$. Since α is an isomorphism, then $C_{G_2}(C_{G_2}(\alpha(a))) \supseteq \alpha(A)$. This, together with the assumption that $\alpha(U_1) = U_2$, imply that both A and $\alpha(A)$ are subgroups of torsion free abelian definable subgroups of G_1 and G_2 (namely $C_{G_1}(C_{G_1}(a)) \cap U_1$ and $C_{G_2}(C_{G_2}(\alpha(a))) \cap \alpha(U_1)$) and by Corollary A.9 both A and $\alpha(A)$ are definable and definably isomorphic (in $\mathcal{R}_{\mathrm{Pfaff}}$) to $(\mathbb{R}^k, +)$ for some k. This implies that α is interdefinable in $\mathcal{R}_{\mathrm{Pfaff}}$ with a linear automorphism of $(\mathbb{R}^k, +)$, which is of course definable. $\blacksquare_{Claim}$

Since Z^0 is a subgroup of R, the above proof shows that the restriction of α to Z^0 is definable in $\mathcal{R}_{\mathrm{Pfaff}}$, and Theorem 5.66 implies that the restriction of α to the reductive definable group $H := L \cdot Z^0$ is definable in $\mathcal{R}_{\mathrm{Pfaff}}$. This completes the proof of the theorem. $\square$

9.3. A Lie approximation to the category of definable groups.

Let $\mathcal{R}$ be an o-minimal expansion of the real field. Consider the category of definably connected definable groups $Def_{\mathcal{R}}$ whose objects are groups definable in $\mathcal{R}$ and whose morphisms are $\mathcal{R}$-definable group homomorphisms.

The results in Chapter 3 imply that there the identity is a functor from $Def_{\mathcal{R}}$ into the category of connected Lie groups. This is of course faithful but not full:

The results in this book imply that the definable groups, seen as Lie groups, one must have that:

- The center of G has finitely may connected components.
- $R(G)$ has a cocompact completely solvable torsion-free subgroup.

Restricting the category of connected Lie groups in this manner, we would get a functor which is surjective on objects for any $\mathcal{R}$ expanding the exponential field. However, Example 7.1 shows that there are Lie group embeddings between definable groups that cannot be definable in *any* o-minimal expansion of the real field. Of course the problem is, as mentioned in Chapter 7, that in any definable group there is a unique maximal cocompact definable subgroup of the radical, and by uniqueness it is categorical in the category of definable groups. So if we have any hope of having an equivalence of categories we would need Lie automorphisms to preserve some a maximal cocompact definable subgroup of the radical.

We can therefore try the following:
Consider the category Lie_{def} defined as follows:
Let the objects of Lie_{def} be pairs (G, U) such that:

- G is a connected Lie group such that $Z(G)$ has finitely many connected components and such that $R(G)$ has a cocompact completely solvable subgroup.
- U is a completely solvable subgroup of $R(G)$.

A morphism ϕ in Lie_{def} between objects (G_1, U_1) and (G_2, U_2) is a Lie homomorphism from G_1 to G_2 such that $\phi(U_1) \subseteq U_2$.

If $\mathcal{R}$ expands the Pfaffian closure of the real field $\mathbb{R}_{\text{Pfaff}}$ described in Appendix A then we have a functor from Lie_{def} to $Def_{\mathcal{R}}$: Theorem 9.9 implies that any object in Lie_{def} has an image in $Def_{\mathcal{R}}$ and Theorem 9.21 states that any morphism in Lie_{def} gives rise to a morphism between the corresponding objects in $Def_{\mathcal{R}}$.

This choice of a cocompact subgroup of the radical and the requirement of preserving it comes, however, at a price. Consider $Gl_2(\mathbb{R})$. It's solvable radical is the center consisiting of

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a, \lambda \in \mathbb{R} \right\}.$$

so that we would need to choose $U = Z(Sl_2(\mathbb{R}))$. Similarly, if we choose $\mathbb{R} \times \mathbb{R}$ the only choice of the cocompact subgroups of the readical is the group itself. However, this would imply that the definable map from $\mathbb{R} \times \mathbb{R}$ to $Sl_2(\mathbb{R})$ sending (a, b) to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is not in the category Lie_{def}.

Essentially, the problem is that the conditions of Theorem 9.21 gives sufficient conditions for Lie morphisms between definable groups (in $\mathcal{R}_{\text{Pfaff}}$) to be definable, but as the example above shows, these conditions are not necessary. So we are still missing a characterization of when a Lie morphism between groups definable in an o-minimal expansion of a real closed fields are definable in some o-minimal theory. Any such definability would most likely happen in $\mathcal{R}_{\text{Pfaff}}$, but we don't know this for sure.

Historical notes and comments

The characterization of Lie groups that admit definable copies was carried out in a series of papers by the authors of this book, together with Sergei Starchenko and Sacha Post. Lemma 9.5 appears in [**19**], Lemma 9.6 is the main theorem in [**47**] and Theorem 9.9 is the main theorem in [**18**].

Lemma 9.13 is, as mentioned there, a compilation of facts which lead to the result which is claimed in several books as part of exercises but for which we couldn't find a complete proof. Theorem 9.19 is the correct version of a result in [**18**]: In said paper the definability was claimed without assuming that the action of the finite group on the connected component fixed a cocompact solvable group (although this was implicitly assumed in the proof). This hypothesis is of course needed as Example 9.20 shows.

Finally, early versions of the results in Section 9.2 were proved by Onshuus in [**46**].

APPENDIX A

The expansion $\mathcal{R}_{\mathbf{Pfaff}}$ whenever $\mathcal{R} = \mathbb{R}$:

In order to define the structure $\mathcal{R}_{\text{Pfaff}}$, we need a couple of definitions. We will always assume that we are working with an o-minimal expansion $\mathcal{R}$ of the real closed field (so $\mathcal{R} = \mathbb{R}$).

A.1. Definition of $\mathcal{R}_{\mathbf{Pfaff}}$

We will first formalize the introduction to vector fields which we hinted at the end of Chapter 3.

Definition A.1. Working in $\mathbb{R}^n$, we will assume that the tangent space $T_p(\mathbb{R}^n)$ over a point p is given with the canonical basis.

Given an open subset U or $\mathbb{R}^n$, we define a *vector field on* $U \subset \mathbb{R}^n$ to be a function v defined over U where $v(p)$ is a vector in $T_p(\mathbb{R}^n)$. We will say that the function is *definable* or C^l if the functions from U to $\mathbb{R}$ giving each of the coefficients of the vector $v(p)$ are definable or C^l, respectively.

Given an n-dimensional manifold M, a *vector field on* M is a function v such that $v(p) \in T_p(M)$ for every $p \in M$ and such that for each coordinate chart (U, ϕ) in an atlas of M after identifying U with the corresponding open subset of $\mathbb{R}^n$ the resulting vector field on $\pi(U)$ is a vector field over $\phi(U)$. We will refer to such vector space as *definable* or C^l if this resulting vector space over $\phi(U)$ is definable or, respectively, C^l for every chart (U, ϕ) in the atlas of M.

We now move to the context of [**43**].

By definition, given a n-dimensional submanifold N of a manifold M, for every point $p \in N$ we have by definition that $T_p(N)$ is a subspace of $T_p(M)$. So any manifold N of M induces a choice where for each $p \in N$ we choose a subspace of $T_p(M)$ of dimension n. This process can sometimes be inverted: Say we have a vector field v over M. Then an integral curve of v will be a 1-dimensional manifold N such that $T_p(N)$ is generated by $v(p)$ for every $p \in N$. This idea gives rise to the following definition.

Definition A.2. Let $M \subseteq \mathbb{R}^n$ be an m-dimensional submanifold and fix $k \in \mathbb{N}$ with $k \leq n$. For every $p \in M$ let Δ_p be an k-dimensional subspace of $T_p(M)$. Assume furthermore that there are vector fields $X_1, \ldots, X_k$ on M such that for every $p \in M$ the subspace Δ_p is generated by $X_1(p), \ldots, X_k(p)$. Then the function $p \mapsto \Delta_p$ is a *k-distribution on M*. Such a distribution will be *definable* of C^l if each of the vector fields $X_1, \ldots, X_k$ defining Δ are.

Given a k-distribution on M, an immersed manifold N of M is an *integral manifold of* N if $T_p(N) = \Delta_p$ for all $p \in N$.

A *leaf of a k-distribution* Δ *on* M is a maximal connected integral manifold of Δ. we will say that N is *an integral manifold* of Δ.

The definition of a Rolle leaf is key for the results we will need.

Definition A.3. If $\dim(M) = m$ and Δ is a $(m-1)$-distribution on M, a *closed* submanifold $N \subset M$ of N is said to be a *Rolle leaf of* Δ if N is a leaf of Δ and such that for every C^1-curve $\gamma : [0,1] \to M$ with $\gamma(0), \gamma(1) \in N$ there is some $p \in [0,1]$ such that $\gamma'(p) \in \Delta_p$.

The following is the main theorem proved by Speissegger in [**67**], and is the key result in this section.

Fact A.4. *Let $\mathcal{R}$ be an o-minimal expansion of the real field, let M be a definable manifold with* $\dim(M) = n$ *and let N be a Rolle leaf of a definable $(n-1)$-distribution Δ on M. Then we can add a predicate defining the points in N to the structure $\mathcal{R}$ preserving o-minimality.*

In particular, given any o-minimal expansion of the real field $\mathcal{R}$, we can close under adding Rolle leafs of definable distributions and obtain an o-minimal structure $\mathcal{R}_{\mathrm{Pfaff}}$ which includes $\mathcal{R}$ and is closed under adding predicates of Rolle leafs of definable distributions.

Definition A.5. Let $d = (d_0, \dots, d_k)$ be a tuple of distributions on M. We call d *nested* if each d_j is an $(m-j)$-distribution on M and $d_{i+1}(p) \subseteq d_i(p)$ and $d_0(p)$ is the distribution $d_0(p) = T_p(M)$ for all $p \in M$. We call $V := (V_0, \dots, V_k)$ a *nested integral manifold of d* if each V_i is an integral manifold of d_i and $V_{i+1} \subseteq V_i$. Moreover, V is a *nested (Rolle) leaf* if $V_0 = M$ and V_{j+1} is a nested (Rolle) leaf of the restriction $d_{j+1}|_{V_j}$ of d_{j+1} to V_j.

It follows by induction that if a set V appears in the sequence of a nested Rolle lead of a nested definable distribution of a definable submanifold M of $\mathbb{R}^n$, then V is definable in $\mathbb{R}_{\mathrm{Pfaff}}$. The importance of nested Rolle leafs is that if $\mathcal{R}$ admits an analytic cell decomposition, then $\mathbb{R}_{\mathrm{Pfaff}}$ is precisely the structure resulting on adding all such nested Rolle leafs as predicates to the structure.

The reason we are using nested Rolle leafs is only to avoid induction argument. The reader can easily adapt all of our proofs to the original context of [**67**].

The reason for calling it *Pfaffian closure* is the following:

Definition A.6. Let $V \subset \mathbb{R}^n$ be an open definable non empty subset, and let $f : V \to \mathbb{R}$ be a C^1 function. We will say that f satisfies a definable Pfaffian system if there are definable C^1-functions $F_i : V \times \mathbb{R} \Rightarrow \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_i} = F_i(x, f(x)).$$

If f satisfies a definable Pfaffian system, then for any $p \in V$ the tangent plane of the graph Γ of f at a point $(p, f(p))$ will be given by the equations

$$x_{n+1} = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i)$$

so that $T_{(p,f(p))}(\Gamma)$ is the tangent space $\Delta_{(p,f(p)}$ where Δ is the distribution generated by vector the vector fields

$$X_i((p, f(p)) = e_i + \frac{\partial f}{\partial x_i}(p)e_{n+1} = e_i + F_i(p, f(p))e_{n+1}.$$

In particular, the graph of f is a leaf of the definable distribution

$$X_i((p,t)) = e_i + F_i(p,t)e_{n+1}.$$

Speissegger proved in [**67**] that the graph of any such f is a Rolle leaf, so that we have the following:

Fact A.7. *Let f satisfy a definable Pfaffian system. Then the graph of f is definable in $\mathcal{R}_{Pfaff}$.*

We can now deduce several facts.

Fact A.8. *Let $\mathcal{R}$ be an o-minimal expansion of the field of real numbers. By a* group interval *we mean a subset B with an identity element e, an open subset V of B^2 containing (e,e) and a binary operation defined on V which satisfies the group laws whenever all elements in said group laws are defined.*

We abuse notation and by $(\mathbb{R}/\mathbb{Z},+)$ we denote the definable group with universe $(-1/2,1/2]$ and group operation is

$$x \oplus y = \begin{cases} x+y-1 & x+y>1/2 \\ x+y & -1/2<x+y\leq 1/2 \\ x+y+1 & x\leq -1/2. \end{cases}$$

Then the following hold.

(1) *Let $(A,<,U,*)$ be a group interval of class C^1 definable in $\mathcal{R}$. Assume that $(B,<,V,+)$ is a sub-group interval of $(\mathbb{R},<,\mathbb{R}^2,+)$ and that $f:(B,<,V,+)\to(A,<,U,*)$ is a C^1-isomorphism. Then the graph of f is in $\mathcal{R}_{Pfaff}$.*
(2) *Let $(G,*)$ be a torsion free one dimensional definable group. Then $(G,*)$ is $\mathcal{R}_{Pfaff}$-definably isomorphic to $(\mathbb{R},+)$.*
(3) *Let $(G,*)$ be a definably compact one dimensional definable group. Then $(G,*)$ is $\mathcal{R}$-definably isomorphic to $(\mathbb{R}/\mathbb{Z},+)$.*

Proof. The first item is proved in [**54**], where they show that any such f satisfies the Pfaffian system on V, given by $f'=h(f(x))$ where $h(a):=\frac{\partial(a*b)}{\partial b}\mid_{(a,1)}\cdot f'(0)$. The reader can verify that this in fact holds.

The second item is an immediate consequence of (1).

To prove (3), by Theorem 3.1 any group G definable in $\mathcal{R}$ has a C^1-structure that makes G a Lie group and by Proposition 2.18 compactness is equivalent to definable compactness. In particular, since G is a definably compact abelian one dimensional Lie group and the Lie exponential map Exp from the Lie algebra $(\mathbb{R},+)$ to G is a C^1 surjective group morphism ([**8**, Theorem 3.6], the kernel must be discrete and generated by its first positive element. After a linear transformation, we may assume that the kernel is $\mathbb{Z}$. The map Exp restricted to the interval $(-1/2,1/2)$ (with the standard order inherited from $\mathbb{R}$) will be a C^1 group interval isomorphism, which by Fact A.8 is definable. This map can be extended definably to a group isomorphism from $\mathbb{R}/\mathbb{Z}$ to G. □

A.2. Lie homomorphisms between torsion free abelian definable groups

In this section we will prove the following.

Theorem A.9. *Let $(H,\odot_H)$ be a torsion free abelian group definable in an o-minimal expansion $\mathcal{R}$ of the real closed field. Then H definably splits in $\mathcal{R}_{Pfaff}$.*

In particular, H is $\mathcal{R}_{Pfaff}$-definably isomorphic to $(\mathbb{R}^n,+)$ for some n and every Lie subgroup of H is definable.

Proof. Theorem 7.12 states that any any non definably compact definable group (so in particular any torsion free definable group) contains a one dimensional definable subgroup. We will prove the theorem by induction on $\dim(H)$. The most important case is $\dim(H)=2$.

Claim A.10. *Let $(H,\odot_H)$ be a two dimensional abelian torsion free group definable in an o-minimal expansion $\mathcal{R}$ of the real closed field, and let U be a definable one dimensional subgroup of H. Then the sequence*

$$0\to U\to H\to H/U\to 0$$

definably splits in $\mathcal{R}_{Pfaff}$. This is, there is a definable subgroup H' of H which projects bijectively onto H/U (and such that $H = U \odot H'$).

Proof: By Fact A.7 in $\mathcal{R}_{\text{Pfaff}}$ both H and H/U are $\mathcal{R}_{\text{Pfaff}}$-definably isomorphic to $(\mathbb{R}, +)$. By definable choice, we may assume that the universe of H is $\mathbb{R} \times \mathbb{R}$ and that the identity of H is $(0, 0)$. The group structure will be given by

$$(a, b) \odot (c, d) = ((a + c), (b + d + F(a, c)))$$

for any $a, b, c, d \in \mathbb{R}$, where $F(x, y)$ is a 2-cocycle. By definition F can be computed directly from the group operation, so it is definable in $\mathcal{R}_{\text{Pfaff}}$.

In the category of Lie groups all abelian groups split, so there is a Lie group section, i.e. a C^1-function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $(x, \gamma(x))$ is a subgroup of H. So $(x, \gamma(x)) \odot (y, \gamma(y)) = (x + y, \gamma(x + y))$, so

$$F(x, y) = \gamma(x + y) - \gamma(x) - \gamma(y).$$

Then

$$\gamma'(x) = \lim_{h\to 0} \frac{\gamma(x + h) - \gamma(x)}{h} = \lim_{h\to 0} \frac{F(x, h)}{h} + \frac{\gamma(h)}{h}.$$

Since $\lim_{x\to 0} \frac{\gamma(h)}{h} = \gamma'(0)$ is a constant, $\lim_{x\to 0} \frac{F(x,h)}{h}$ exists and $\gamma'(x)$ is definable. So $\gamma(x)$ is definable in $\mathcal{R}_{\text{Pfaff}}$, which implies that H definably splits witnessed by $H' := \{(x, \gamma(x) \mid x \in \mathbb{R})\}$. $\blacksquare_{Claim}$

From here the inductive argument is quite straightforward and in fact, the only place where we need to go to $\mathcal{R}_{\text{Pfaff}}$ is for the dimension 2 case.

Assume we have the result for $n \geq i \geq 2$ and let H have dimension $n + 1$. By Theorem 7.12 there is a definable one dimensional subgroup U of H. Now, H/U splits into products of one dimensional groups $R_1 \times \cdots \times R_n$. The preimage H_i of R_i in H has dimension 2, so by Lemma A.10 it splits so that there is a definable isomorphism ϕ_i from $R_i \times U$ to H_i. The map $\phi : R_1 \times \cdots \times R_n \times U$ defined by

$$\phi(x_1, \ldots, x_n, g) \mapsto \phi_1(x_1, g) \odot_H \cdots \odot_H \phi_1(x_n, g)$$

is therefore a group isomorphism.

Since H is a direct product of definable torsion free one dimensional subgroups and by Corollary A.8 each one is $\mathcal{R}_{\text{Pfaff}}$-definably isomorphic to $(\mathbb{R}, +)$, it follows that H is $\mathcal{R}_{\text{Pfaff}}$-definably isomorphic to $(\mathbb{R}^{n+1}, +)$. It is therefore a vector space and any Lie subgroup is a vector subspace, so it is definable. This completes the proof of the corollary. $\square$

Theorem A.11. *Let H_1 and H_2 be torsion free abelian definable groups and let ϕ be a surjective Lie homomorphism from H_1 to H_2. Then ϕ is definable in $\mathcal{R}_{Pfaff}$.*

Proof. By Theorem A.9 the subgroup $\ker(\phi)$ of H_1 is definable in $\mathcal{R}_{\text{Pfaff}}$ and by 6.12 $H_1/\ker(\phi)$ is torsion free. So we can replace ϕ by the isomorphism between definable $H_1/\ker(\phi)$ and H_2 and it is enough to prove the theorem assuming that ϕ is an isomorphism, and Theorem A.9 implies that ϕ is $\mathcal{R}_{\text{Pfaff}}$-definably isomorphic to a group automorphism of $(\mathbb{R}, +)^n$. But these are simply linear transformations so they are all definable. $\square$

A.3. Compact Abelian Groups in $\mathcal{R}_{\text{Pfaff}}$

We will fix $(A, \odot_A)$ be a to be a $\mathcal{R}$-definable abelian group. Let $\mathfrak{a}$ be its Lie algebra. The results in this subsection will be corollaries to the following proposition, which was pointed to us by Gal Binyamini.

Lemma A.12. *Let H be a connected (closed) Lie subgroup of A. Then there is a neighborhood W of the identity e_A of A such that $H \cap W$ is definable in $\mathcal{R}_{Pfaff}$.*

PROOF. Since H is connected it lives in the connected component, so we may assume A is connected. Let n be the dimension of A. There is an open subset U of the identity with $A \cap U$ an open subset of $\mathbb{R}^n$. We will assume $A \cap U \subseteq \mathbb{R}^n$ in order to apply the Rolle leaf technology.

Let $\mathfrak{a}$ and $\mathfrak{h}$ be the Lie algebras of A and H, respectively. Both A/H and H are abelian Lie groups, so by Fact 5.47 they split as a product of a cartesian power of $\mathbb{R}$ with a cartesian power of the circle group $S_1(\mathbb{R})$. We can therefore find a sequence of (closed) Lie subgroups

$$A_n = \{e\} \leq A_{n-1} \leq \cdots \leq A_l \leq A_{l+1} \leq \cdots \leq A_0 = A$$

of A where $A_l = H$ and A_i/A_{i+1} is a one dimensional group isomorphic to either $(\mathbb{R}, +)$ or to $S_1(\mathbb{R})$.

Let $\mathfrak{a}_l$ be the Lie algebra of A_l, and let $v_1, \ldots, v_n \in \mathfrak{a}$ be elements in $\mathfrak{a}_0$ be such that $\mathfrak{a}_j = Span\left(\{v_1, \ldots, v_{n-j}\}\right)$, so that in particular $\mathfrak{h} := \mathfrak{a}_k = Span\left(\{v_1, \ldots, v_{n-k}\}\right)$ is the Lie algebra of K.

In the context of [**68**, Chapter 10], let $T(A_l)$ be the tangent bundle of A_l, and let $d_l : A \to T(A)$ be distribution associated with the Lie algebra $\mathfrak{a}_l$, so that $d_l(x) = L_x(\mathfrak{a}_l)$ where L_x is the derivative of the left translation (in A_l) by the element x.

Intersecting everything with U so $A \cap U$ is a submanifold (open subset in fact) of $\mathbb{R}^n$, the tuple $\mathcal{A}_U := (A_0 \cap U, A_1 \cap U, \ldots, A_n \cap U)$ is a nested integral manifold of the restriction of the nested distribution $d := (d_0, d_1, \ldots, d_n)$.

Claim A.13. *For some neighborhood $W \subseteq U$ of e_A in A, $\mathcal{A} \cap W = (A_0 \cap W, A_1 \cap W, \ldots, A_n \cap W)$ is a nested Rolle leaf of the restriction of d to W.*

Proof: Recall that the nested leaf $(A_0 \cap W, A_1 \cap W, \ldots, A_n \cap W)$ of the nested distribution $d|_W$ is Rolle if $A_{i+1} \cap W$ is a closed (embedded) submanifold of $\subseteq A_i \cap W$ (which we have by the definition of Lie subgroups) and it satisfies the separating condition: For each l we have that $A_{l+1} \cap W$ is a closed submanifold of $A_l \cap W$ and for any C^1-curve $\gamma : [0,1] \mapsto A_l \cap W$ with $\gamma(0), \gamma(1) \in A_{l+1}$ there exists a $t \in [0,1]$ such that $\gamma'(t) \in d_l(\gamma(t))$.

Now, by hypothesis $\mathfrak{a}$ is abelian, so it is isomorphic to $\mathbb{R}^n$, which implies ([**68**] Theorem 5) there is a local C^2-diffeomorphism between A and $\mathfrak{a}$. This is, there is a neighborhood W of A and a diffeomorphism sending $A_l \cap W$ into $\mathfrak{a}_l$. If we send the basis $v_1, \ldots v_n$ to the canonical basis $e_1 \ldots e_n$ of $\mathbb{R}^n$, the diffeomorphisms sends d to the nested trivial distribution over $\mathbb{R}^n$ whose nested integral manifold $\langle \mathbb{R}^{n-0}, \ldots \mathbb{R}^{n-i}, \ldots \mathbb{R}^{n-n} \rangle$ (and the restriction to the image of W) is trivially a nested Rolle leaf. The separating condition is, by definition, preserved under diffeomorphisms, so $(A_0 \cap W, A_1 \cap W, \ldots, A_n \cap W)$ is a nested Rolle leaf of the restriction of d to W. $\blacksquare_{Claim}$

By [**53**] the Lie algebra $\mathfrak{a}$ is $\mathcal{R}$-definable, and subspaces of vector spaces are always definable, which implies that $\mathfrak{a}_l$ is $\mathcal{R}$-definable for all l. Right translation by an element $x \in A$ is of course $\mathcal{R}$-definable, and therefore so is its differential L_x, which implies that the distribution d is $\mathcal{R}$-definable. By Fact A.4 and definition of $\mathcal{R}_{\text{Pfaff}}$ the definability of the Rolle leaf $A_l \cap W$ in $\mathcal{R}_{\text{Pfaff}}$ implies that $A_{l+1} \cap W$ is definable in $\mathcal{R}_{\text{Pfaff}}$. By induction $A_l \cap W$ is definable in $\mathcal{R}_{\text{Pfaff}}$ for all l, so in particular $H \cap W = A_k \cap W$ is $\mathcal{R}_{\text{Pfaff}}$-definable, as required. □

Theorem A.14. *The following holds.*

1. *Any compact connected subgroup of A is $\mathcal{R}_{Pfaff}$-definable.*
2. *If B is any $\mathcal{R}$-definable compact abelian group and $\phi : B \to A$ is a Lie isomorphism, then the restriction of ϕ to some open neighborhood of the identity is definable in $\mathcal{R}_{Pfaff}$.*

PROOF. Let K be any compact subgroup of A. By compactness the connected component of K has finite index and the definability of K is equivalent to the definability of its connected component, so we may assume that K is connected. Lemma A.12 implies that for some open W the set $K \cap W$ is definable in $\mathcal{R}_{\mathrm{Pfaff}}$. By compactness and connectedness,

$$K = \underbrace{(K \cap W) \odot_A \cdots \odot_A (K \cap W)}_{m-\text{times}}$$

for some $m \in \mathbb{N}$. Since multiplication in A is definable, (1) follows.

For the second item, notice that the graph of ϕ is a compact Lie subgroup of the $\mathcal{R}$-definable abelian $B \times A$. $\square$

Bibliography

[1] E. Barriga and A. Onshuus. Classifying torsion free groups in o-minimal expansions of real closed fields. *Ann. Pure Appl. Logic*, 167(12):1267–1297, 2016.

[2] A. Berarducci. Zero-groups and maximal tori. In *Logic Colloquium 2004*, volume 29 of *Lect. Notes Log.*, pages 33–46. Assoc. Symbol. Logic, Chicago, IL, 2008.

[3] A. Berarducci and M. Otero. o-minimal fundamental group, homology and manifolds. *J. London Math. Soc. (2)*, 65(2):257–270, 2002.

[4] A. Berarducci and M. Otero. Transfer methods for o-minimal topology. *J. Symbolic Logic*, 68(3):785–794, 2003.

[5] A. Berarducci, M. Otero, Y. Peterzil, and A. Pillay. A descending chain condition for groups definable in o-minimal structures. *Ann. Pure Appl. Logic*, 134(2-3):303–313, 2005.

[6] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[7] A. Borel. Class functions, conjugacy classes and commutators in semisimple lie groups. In *Algebraic Groups and Lie Groups: Australian Mathematical Society Lecture Series 9*, pages 1–19. Cambridge University Press, Cambridge, 1997.

[8] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[9] É. Cartan. Sur les représentations linéaires des groupes clos. *Commentarii Mathematici Helvetici*, 2(1):269–283, 1930.

[10] G. Cherlin. Groups of small Morley rank. *Ann. Math. Logic*, 17(1-2):1–28, 1979.

[11] C. Chevalley. *Classification des groupes de Lie algébriques*. Secrétariat mathématique, Paris, 1958. Séminaire C. Chevalley, École Normale Supérieure, Paris, 1956–1958, notes multigraphiées; reprinted 2005 by Springer-Verlag with a postface by P. Cartier.

[12] A. Conversano. Definable rank, o-minimal groups and wiegold's problem. to appear.

[13] A. Conversano. *On the connections between definable groups in o-minimal structures and real Lie groups: the non-compact case*. PhD thesis, University of Siena, 2009.

[14] A. Conversano. Maximal compact subgroups in the o-minimal setting. *J. Math. Log.*, 13(1):1–15, 2013.

[15] A. Conversano. A reduction to the compact case for groups definable in o-minimal structures. *J. Symb. Log.*, 79(1):45–53, 2014.

[16] A. Conversano. Groups definable in o-minimal structures:various properties and a diagram. *J. Appl. Log.*, 8(1):2235–2255, 2021.

[17] A. Conversano. A Jordan–Chevalley decomposition beyond algebraic groups. *Journal of the London Mathematical Society*, 111(6), 2025.

[18] A. Conversano, A. Onshuus, and S. Post. Real Lie groups and o-minimality. *Proceedings of the American Mathematical Society*, 150(6):2701–2714, 2022.

[19] A. Conversano, A. Onshuus, and S. Starchenko. Solvable Lie groups definable in o-minimal theories. *J. Inst. Math. Jussieu*, 17:1–12, 2018.

[20] A. Conversano and A. Pillay. Connected components of definable groups and *o*-minimality I. *Adv. Math.*, 231(2):605–623, 2012.

[21] A. Conversano and A. Pillay. On Levi subgroups and the Levi decomposition for groups definable in o-minimal structures. *Fundamenta Mathematicae*, 222(1):1–45, 2013.

[22] J. Dixmier. L'application exponentielle dans les groupes de Lie résolubles. *Bull. Soc. Math. France*, 85:113–121, 1957.

[23] J. J. Duistermaat and J. A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.

[24] M. J. Edmundo. Covers of groups definable in o-minimal structures. *Illinois J. Math.*, 49(1):99–120, 2005.

[25] M. J. Edmundo and P. E. Eleftheriou. The universal covering homomorphism in o-minimal expansions of groups. *MLQ Math. Log. Q.*, 53(6):571–582, 2007.

[26] M. J. Edmundo and M. Otero. Definably compact abelian groups. *J. Math. Log.*, 4(2):163–180, 2004.

[27] H. Freudenthal. Die Topologie der Lieschen Gruppen als algebraisches Phänomen. I. *Annals of Mathematics*, 42:1051–1074, 1941.
[28] W. Fulton. *Algebraic topology*, volume 153 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course.
[29] V. Gorbatsevich, A. Onishchik, and E. Vinberg. *Lie Groups and Lie Algebras I: Foundations of Lie Theory Lie Transformation Groups*, volume 20 of *Encyclopaedia of Mathematical Sciences*. Springer, 1993.
[30] V. Gorbatsevich, A. L. Onishchik, and È. B. Vinberg. Structure of lie groups and lie algebras. In *Lie groups and Lie algebras, III*, volume 41 of *Encyclopaedia Math. Sci.*, pages 1–246. Springer, Berlin, 1994.
[31] M. Goto. Faithful representations of lie groups ii. *Nagoya Mathematical Journal*, 1:91–100, 1950.
[32] O. Grün. Beiträge zur Gruppentheorie. II. *J. Reine Angew. Math.*, 188:165–169, 1945.
[33] J. Hilgert and K.-H. Neeb. *Structure and Geometry of Lie Groups*. Springer Monographs in Mathematics. Springer, 2012.
[34] G. Hochschild. *The structure of Lie groups*. Holden-Day, Inc., San Francisco-London-Amsterdam, 1965.
[35] G. Hochschild and G. D. Mostow. Extensions of representations of Lie groups and Lie algebras. I. *Amer. J. Math.*, 79:924–942, 1957.
[36] K. H. Hofmann and S. A. Morris. *The structure of compact groups*, volume 25 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1998. A primer for the student—a handbook for the expert.
[37] E. Hrushovski. *CONTRIBUTIONS TO STABLE MODEL THEORY*. ProQuest LLC, Ann Arbor, MI, 1986. Thesis (Ph.D.)–University of California, Berkeley.
[38] E. Hrushovski, Y. Peterzil, and A. Pillay. Groups, measures, and the NIP. *J. Amer. Math. Soc.*, 21(2):563–596, 2008.
[39] E. Hrushovski, Y. Peterzil, and A. Pillay. On central extensions and definably compact groups in o-minimal structures. *J. Algebra*, 327:71–106, 2011.
[40] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.
[41] K. Iwasawa. On some types of topological groups. *Ann. of Math.*, 50(3):507–558, 1949.
[42] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.
[43] J.-M. Lion and P. Speissegger. The theorem of the complement for nested sub-Pfaffian sets. *Duke Math. J.*, 155(1):35–90, 2010.
[44] A. L. Onishchik and È. B. Vinberg. Foundations of Lie theory. In *Lie groups and Lie algebras, I*, volume 20 of *Encyclopaedia Math. Sci.*, pages 1–94. Springer, Berlin, 1993.
[45] A. L. Onishchik and E. B. Vinberg, editors. *Lie groups and Lie algebras, III*, volume 41 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1994. Structure of Lie groups and Lie algebras, A translation of *Current problems in mathematics. Fundamental directions. Vol. 41* (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990 [MR1056485 (91b:22001)], Translation by V. Minachin [V. V. Minakhin].
[46] A. Onshuus. Continuous isomorphisms between groups definable in o-minimal expansions of the real field. to appear.
[47] A. Onshuus and S. Post. A definability criterion for connected Lie groups. *Bulletin of the London Mathematical Society*, 48(5):1232–1249, 2022.
[48] M. Otero. A survey on groups definable in o-minimal structures. In Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, editors, *Model Theory with Applications to Algebra and Analysis*, volume 2 of *London Mathematical Society Lecture Note Series*, pages 177–206. Cambridge University Press, Cambridge, 2008.
[49] M. Otero, Y. Peterzil, and A. Pillay. On groups and rings definable in o-minimal expansions of real closed fields. *Bull. London Math. Soc.*, 28(1):7–14, 1996.
[50] Y. Peterzil. Pillay's conjecture and its solution—a survey. In F. Delon, U. Kohlenbach, P. Maddy, and F. Stephan, editors, *Logic Colloquium 2007*, volume 35 of *Lecture Notes in Logic*, pages 177–203. Cambridge University Press, Cambridge, 2010.
[51] Y. Peterzil, A. Pillay, and S. Starchenko. Definably simple groups in o-minimal structures. *Trans. Amer. Math. Soc.*, 352(10):4397–4419, 2000.
[52] Y. Peterzil, A. Pillay, and S. Starchenko. Simple algebraic and semialgebraic groups over real closed fields. *Trans. Amer. Math. Soc.*, 352(10):4421–4450, 2000.
[53] Y. Peterzil, A. Pillay, and S. Starchenko. Linear groups definable in o-minimal structures. *J. Algebra*, 247(1):1–23, 2002.

[54] Y. Peterzil, P. Speissegger, and S. Starchenko. Adding multiplication to an o-minimal expansion of the additive group of real numbers. In *Logic Colloquium '98 (Prague)*, volume 13 of *Lect. Notes Log.*, pages 357–362. Assoc. Symbol. Logic, Urbana, IL, 2000.

[55] Y. Peterzil and S. Starchenko. Definable homomorphisms of abelian groups in o-minimal structures. *Ann. Pure Appl. Logic*, 101(1):1–27, 2000.

[56] Y. Peterzil and S. Starchenko. On torsion-free groups in o-minimal structures. *Illinois J. Math.*, 49(4):1299–1321 (electronic), 2005.

[57] Y. Peterzil and S. Starchenko. Computing o-minimal topological invariants using differential topology. *Trans. Amer. Math. Soc.*, 359(3):1375–1401, 2007.

[58] Y. Peterzil and S. Starchenko. Topological groups, μ-types and their stabilizers. *Journal of the European Mathematical Society*, 19(10):2965–2995, 2017.

[59] Y. Peterzil and C. Steinhorn. Definable compactness and definable subgroups of o-minimal groups. *J. London Math. Soc. (2)*, 59(3):769–786, 1999.

[60] A. Pillay. First order topological structures and theories. *J. Symbolic Logic*, 52(3):763–778, 1987.

[61] A. Pillay. On groups and fields definable in o-minimal structures. *J. Pure Appl. Algebra*, 53(3):239–255, 1988.

[62] A. Pillay. Type-definability, compact Lie groups, and o-minimality. *J. Math. Log.*, 4(2):147–162, 2004.

[63] G. Pinilla. La exponencial de lie en expansiones o-minimales de un campo real cerrado. Master's thesis, Universidad de los Andes, 2023.

[64] V. Razenj. One-dimensional groups over an o-minimal structure. *Ann. Pure Appl. Logic*, 53(3):269–277, 1991.

[65] D. J. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, second edition, 1982.

[66] J. J. Rotman. *An introduction to the theory of groups*, volume 148 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, fourth edition, 1995.

[67] P. Speissegger. The Pfaffian closure of an o-minimal structure. *J. Reine Angew. Math.*, 508:189–211, 1999.

[68] M. Spivak. A comprehensive introduction to differential geometry. Vol. I. 2nd ed. Berkeley, California: Publish or Perish, Inc. XIII, 668 p. (1979)., 1979.

[69] A. W. Strzebonski. Euler characteristic in semialgebraic and other o-minimal groups. *J. Pure Appl. Algebra*, 96(2):173–201, 1994.

[70] A. W. Strzebonski. One-dimensional groups definable in o-minimal structures. *J. Pure Appl. Algebra*, 96(2):203–214, 1994.

[71] T. Tao. *Hilbert's fifth problem and related topics*, volume 153 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2014.

[72] L. van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[73] L. van den Dries, A. Macintyre, and D. Marker. The elementary theory of restricted analytic fields with exponentiation. *Ann. of Math. (2)*, 140(1):183–205, 1994.

[74] B. L. van der Waerden. Stetigkeitssätze für halbeinfache Liesche Gruppen. *Mathematische Zeitschrift*, 36(1):780–786, 1933.

[75] A. Weil. On algebraic groups of transformations. *Amer. J. Math.*, 77:355–391, 1955.

[76] A. J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Amer. Math. Soc.*, 9(4):1051–1094, 1996.

[77] A. A. Woerheide. *O-minimal homology*. ProQuest LLC, Ann Arbor, MI, 1996. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign.