DIVERGENCE AND SQUARE OF NORM

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Abstract. It is known that there are no nontrivial continuously differentiable solutions $\mathbb{R}^2 \to \mathbb{R}^2$ to the vector-field equation $\operatorname{div} Y = |Y|^2$. Shown here is that for all $n \geq 3$ there are nontrivial analytic square-integrable solutions $\mathbb{R}^n \to \mathbb{R}^n$. However, the exhibited solutions are obtained by a fixed-point argument, and there are no nontrivial C^1 solutions that are polynomial, radial, conservative, subharmonic or $o(|x|^{1-n})$ as $|x| \to \infty$.

This note addresses a question that I once thought might be an exercise in advanced multivariable calculus, albeit possibly labelled as difficult. But I did not find it in any textbooks that I checked; now I know better why not. Neither could I find any treatment in the advanced literature, and occasional consultations with colleagues yielded only partial answers. I attempt to make the exposition accessible (at least, with some guidance) to advanced undergraduate mathematics students, in particular, I tend to use notation and terminology from undergraduate mathematics where possible and reasonable. But there will be some more advanced material toward the end that I do not know how to avoid.

The basic question: If $n \geq 2$, are there nontrivial (real-)analytic solutions $\mathbb{R}^n \to \mathbb{R}^n$ to the vector-field equation $\operatorname{div} Y = |Y|^2$? (Throughout, Y denotes a vector-field variable. Specific vector fields are denoted by F, G and so on. The euclidean norm is indicated by $|\ |$.) The solution set of $\operatorname{div} Y = |Y|^2$ is closed under translation and $Y \mapsto cY \circ (cx)$ for each fixed $c \in \mathbb{R}$, hence the question is equivalent to whether there exist nonzero $c \in \mathbb{R}$ and analytic solutions to $\operatorname{div} Y = c\,|Y|^2$, $Y(0) \neq 0$.

A basic answer is as follows. No, if $n = 2$ (indeed, then there are no nontrivial C^1 solutions to $\operatorname{div} Y \geq |Y|^2$). Yes, if $n > 2$, but one might question the naturality of the solutions exhibited here, and there are no nontrivial C^1 solutions that are polynomial, radial, conservative or of decay $o(|x|^{1-n})$ as $|x| \to \infty$, conditions that would make the question more reasonable for advanced calculus students. Also, if n is even, then there are no nontrivial solutions that are complex-differentiable as maps $\mathbb{C}^{n/2} \to \mathbb{C}^{n/2}$. Evidently, there are no nontrivial divergence-free solutions.

We now proceed to more precise statements. Vector fields are defined on all of the ambient $\mathbb{R}^n$ unless explicitly indicated otherwise. Integrals without specified domains are taken over the ambient $\mathbb{R}^n$. Integrands will often be nonnegative and continuous, hence integrals can usually be taken in the sense of Riemann integration (allowing for improper), but Lebesgue integration will be used later for some of the more sophisticated material.

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Proposition A. *There exist analytic* $\rho\colon (2,+\infty) \to (0,1)$ *such that for all* $n > 2$ *and nontrivial analytic Lipschitz* $H\colon \mathbb{R}^n \to \mathbb{R}^n$, *if* $|H| + \int |H| \le \rho(n)$, *then there exist analytic* $h\colon \mathbb{R}^n \to \mathbb{R}$ *such that* $\Delta h + 2H\cdot\nabla h + |H|^2 h = 0$, $\int |\nabla h|^2 \in \mathbb{R}$ *and* $\inf h > 0$.

While one can think of H as small in various ways, the zero set of H is closed and has Hausdorff dimension at most $n-1$, so in this sense, H is far from 0. (The result is also true for $H = 0$, but then solutions are harmonic and positive, hence constant; this is not useful for current purposes.)

I postpone a proof of Proposition A in order to smooth the exposition because all arguments of which I am aware are well beyond the scope of standard undergraduate mathematics. However, the following consequence requires only calculus.

Theorem A. *If* $n > 2$ *and* $c \in \mathbb{R}$ *is nonzero, then there exist analytic* $F\colon \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\operatorname{div} F = c|F|^2$, $\int |F|^2 \in \mathbb{R}$ *and* $F(0) \neq 0$.

(Observe that without the condition $\int |F|^2 \in \mathbb{R}$, the case $n > 3$ follows easily from the case $n = 3$: consider the vector field $(F, 0)$ where F is a solution for $n = 3$ and 0 indicates the trivial vector field on $\mathbb{R}^{n-3}$.)

Proof. As indicated earlier, it suffices to consider the case $c = 1$. Let $n > 2$ and H as in Proposition A be divergence free and not a gradient, say,

$$H = \frac{1}{N(1+|x|^{2n})}(x_2, -x_1, 0, \ldots, 0)$$

for some sufficiently large $N \in \mathbb{N}$. With h as in Proposition A, put $F = -H - \nabla \log h$; then F is analytic and nontrivial (because $H \neq \nabla(-\log h)$), and $\operatorname{div} F = -\operatorname{div}\nabla\log h = -\Delta \log h = |F|^2$. By translation, we may take $F(0) \neq 0$. Because $|H| \le \rho(n) \le 1$, we have $\int |H|^2 \le \int |H|$. Hence, $\int |F|^2 \le 2\int |H| + 2(\int |\nabla h|^2)/(\inf h)^2 < +\infty$. □

Except for being analytic and square integrable (that is, $\int |\quad|^2 \in \mathbb{R}$), F appears not to belong to any class of examples of tractable (or "analyzable") vector fields usually encountered in advanced calculus. Indeed, we now proceed to several nonexistence results, many of which may be known, either in the literature (possibly as exercises in textbooks) or as folklore. I contend that most of them can be done as calculus exercises (allowing for some guidance). In order to expose a variety of methods, efficiency is not a high priority.

Fact 1. *There are no nontrivial polynomial solutions to* $\operatorname{div} Y \ge |Y|^2$.

Hint of proof. If $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is polynomial, then the degree of $|F|^2$ is twice the maximum of the degrees of the components of F. □

Fact 2. *There are no nontrivial entire (analytic and infinite radius of convergence) solutions to* $\operatorname{div} Y \ge |Y|^2$ *such that the maximum modulus function of its complexification is polynomially bounded.*

Hint of proof. Combine Fact 1 with Liouville's Theorem in several complex variables. □

Fact 3. *There are no nontrivial conservative* C^1 *solutions to* $\operatorname{div} Y = |Y|^2$.

Hint of proof. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is twice differentiable and $\operatorname{div}\nabla f = |\nabla f|^2$, then e^{-f} is harmonic and positive, hence constant (an exercise via the harmonic mean value property). □

We now turn to the Divergence Theorem as a tool. For $r > 0$ and $p \in \mathbb{R}^n$, let $B(p, r)$ be the open ball of radius r centered at p and $S(p, r)$ be its boundary. Integration over $S(p, r)$ is always with respect to $(n-1)$-dimensional surface measure. For $p = 0$, we write just $B(r)$ and $S(r)$. Let α_n be the volume of $B(1) \subseteq \mathbb{R}^n$. Our first application is straightforward.

Fact 4. *If F is a C^1 solution to $\operatorname{div} Y \geq |Y|^2$ and $r > 0$, then* $\int\limits_{B(r)} |F|^2 \leq n\alpha_n \max\limits_{S(r)} |F| \, r^{n-1}$.

Proof. For $r > 0$, $\int\limits_{B(r)} |F|^2 \leq \int\limits_{B(r)} \operatorname{div} F = \int\limits_{S(r)} F \cdot \frac{x}{|x|} \leq n\alpha_n \max_{S(r)} |F(x)| \, r^{n-1}$. □

As an immediate corollary,

Fact 5. *There are no nontrivial C^1 solutions F to $\operatorname{div} Y \geq |Y|^2$ such that* $\lim\limits_{r\to\infty} r^{n-1} \max\limits_{S(r)} |F| = 0$.

Next is another direct consequence of the Divergence Theorem.

Fact 6. *Let F be a C^1 solution to $\operatorname{div} Y \geq |Y|^2$. If $U \subseteq \mathbb{R}^n$ is an open ball and $F = 0$ on the boundary of U, then $F = 0$ on U.*

(Of course, this holds for many open sets other than balls, but in calculus textbooks such sets are often described only by "having boundary to which the Divergence Theorem applies".)

Fact 7. *There are no nontrivial C^1 solutions F to $\operatorname{div} Y = |Y|^2$ for which there exist $f \colon [0, +\infty) \to \mathbb{R}$ such that $|x| \, F(x) = f(|x|)x$ for all $x \in \mathbb{R}^n$.*

(I avoid the use of "radial" here for F because of inconsistencies or vagueness in the definitions given in calculus textbooks. But the notion of "radial symmetry" is unproblematic.)

Proof. Let F be a C^1 solution to $\operatorname{div} Y = |Y|^2$. Let $f \colon [0, +\infty) \to \mathbb{R}$ be such that $|x| \, F(x) = f(|x|)x$ for all $x \in \mathbb{R}^n$. We show that $F = 0$. By radial symmetry and Fact 6, exactly one of the following holds:

— F has no zeros except possibly at the origin;
— there exists $R \in (0, +\infty)$ such that $\{ x \in \mathbb{R}^n : |x| \leq R \}$ is the zero set of F;
— $F = 0$.

We rule out the first two possibilities.

Assume that F has no zeros except possibly at the origin. We derive a contradiction. Observe that $\lim\limits_{r\downarrow 0} |f(r)| = |F(0)|$ and f^2 is C^1 (because $f^2(|x|) = |F(x)|^2$ for all $x \neq 0$); then f is C^1 on $(0, +\infty)$. For $x \neq 0$, we have $f^2(|x|) = |F(x)|^2 = \operatorname{div} F(x)$, and so

$$f^2(|x|) = \operatorname{div}\left(f(|x|) \frac{x}{|x|} \right) = f'(|x|) + \frac{n-1}{|x|} f(|x|).$$

Thus, for $r > 0$, $f'(r) = f^2(r) - (n-1)f(r)/r$. Then $(r^{n-1}f)' = (r^{n-1}f)f$, hence $r^{n-1}f(r) = f(1)\bigl(\exp \int_0^1 (-f)\bigr)\bigl(\exp \int_0^r f\bigr)$. Letting $r \downarrow 0$ contradicts $f(1) \neq 0$. (*Aside.* Some readers will know that $ry' - ry^2 + (n-1)y = 0$ can be solved more explicitly, but we do not need this.)

The case that the zero set of F is a closed ball of radius $R \in (0, +\infty)$ is similar: f is C^1 on $(R, +\infty)$ and $f(R + 1) \neq 0$. □

We require a more sophisticated application of the Divergence Theorem, beginning with an estimate relying on the integral Cauchy-Schwarz inequality

Fact 8. *Let $p \in \mathbb{R}^n$, $0 < R \leq +\infty$ and $F \in C^1(B(p,R),\mathbb{R}^n)$. For $0 < r < R$, put $f(r) = \int_{B(p,r)} |F|^2$. Then f is differentiable, $f'(r) = \int_{S(p,r)} |F|^2$ and*

$$\int\limits_{B(p,r)} \operatorname{div} F \leq \sqrt{n\alpha_n r^{n-1} f'(r)}.$$

Proof. It is a standard calculus fact that $f'(r) = \int_{S(p,r)} |F|^2$. We have:

$$\int\limits_{B(p,r)} \operatorname{div} F \leq \int\limits_{S(p,r)} |F| \leq \left[\int\limits_{S(p,r)} 1^2 \cdot \int\limits_{S(p,r)} |F|^2\right]^{1/2} = \sqrt{n\alpha_n r^{n-1}\, f'(r)}. \qquad \square$$

Proposition B. *With data as in Fact 8, assume moreover that $R = +\infty$ and $\operatorname{div} F \geq |F|^2$.*

(1) *For all $0 < a < b \leq +\infty$, if $f(a) \neq 0$, then*

$$\frac{1}{f(a)} - \frac{1}{f(b)} \geq \begin{cases} \dfrac{1}{n(n-2)\alpha_n}\left(\dfrac{1}{a^{n-2}} - \dfrac{1}{b^{n-2}}\right), & n \neq 2 \\[2ex] \dfrac{\log b - \log a}{2\pi}, & n = 2. \end{cases}$$

(2) *If $n = 2$, then $F = 0$.*

(3) *For all $r > 0$, $\int_{B(p,r)} |F|^2 \leq n(n-2)\alpha_n r^{n-2}$.*

(4) *If $|F|^2$ is subharmonic, then $F = 0$.*

Proof. (1). By Fact 8, we now have $f^2 \leq n\alpha_n r^{n-1} f'$, hence also $r^{1-n}/(n\alpha_n) \leq (-1/f)'$.

(2). Assume that $n = 2$ and $F(p) \neq 0$. Then for all $b > 1$,

$$\frac{1}{f(1)} \geq \frac{1}{f(1)} - \frac{1}{f(b)} \geq \frac{\log b}{2\pi},$$

contradicting that log is unbounded above.

(3). By (2), we may take $n \geq 3$. The result is trivial for r such that $f(r) = 0$, so assume that $f(r) \neq 0$. By (1), setting $a = r$ and $b = +\infty$ yields $\int_{B(p,r)} |F|^2 \leq n(n-2)\alpha_n r^{n-2}$.

(4). By (3) and subharmonicity, we have $|F(p)|^2 \leq n(n-2)/\epsilon^2$ for all $\epsilon > 0$. $\square$

It is worth noting that result (3) applies to F as in Theorem A, and thus the uniform bound of $n(n-2)/r^2$ on the average of $|F|^2$ over balls of positive radius r does not entail that F is trivial (unless $n = 2$). And for emphasis, we restate item (2).

Theorem B. *If $n = 2$, then there are no nontrivial C^1 solutions to $\operatorname{div} Y \geq |Y|^2$.*

Fact 9. *There are no nontrivial C^2 solutions to $\operatorname{div} Y \geq |Y|^2$, $\Delta(|Y|^2) \geq 0$.*

(If f is C^2, then $\Delta f \geq 0$ if and only if f is subharmonic.)

Fact 10. *If n is even, then there are no nontrivial solutions to $\operatorname{div} Y \geq |Y|^2$ that are holomorphic as maps $\mathbb{C}^{n/2} \to \mathbb{C}^{n/2}$.*

(If $F\colon \mathbb{C}^m \to \mathbb{C}^m$ is holomorphic, then $\Delta(|F|^2) = 4 \sum_{1\le j,k\le m} |\partial F_j/\partial z_k|^2$.)

This ends the statements of results.

Proof of Proposition A. For $t > 2$, put

$$\begin{aligned}\alpha(t) &= \frac{\pi^{t/2}}{\Gamma(1+t/2)}\\ \beta(t) &= 1 + \frac{1}{2(t-2)} + \frac{1}{t\alpha(t)} + \frac{1}{t(t-2)\alpha(t)}\\ \rho(t) &= \frac{1}{6\beta(t)}.\end{aligned}$$

Observe that ρ is analytic, $0 < \rho < 1/6$, and for $n \ge 3$ we have $\alpha(n) = \alpha_n$. Fix $n \ge 3$ and let $H\colon \mathbb{R}^n \to \mathbb{R}^n$ be analytic and Lipschitz such that $|H| + \int |H| \le \rho(n)$. We find analytic $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $-\Delta f = 2H\cdot\nabla f + |H|^2(1+f)$ and $|f| \le 1/3$; then $h := 1+f$ satisfies the conclusion of Proposition A.

Put $X = \{\, u \in C^1(\mathbb{R}^n,\mathbb{R}) : \sup|u| + \sup|\nabla u| \in \mathbb{R}\,\}$, which is a Banach space when equipped with the norm $\|u\|_X := \sup|u| + \sup|\nabla u|$.

Define $K\colon \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ by $K(x) = |x|^{2-n}/\big(n(n-2)\alpha_n\big)$. Note that $\nabla K = -x/\big(n\alpha_n|x|^n\big)$, $\int_{B(1)} |\nabla K| = 1$ and $\int_{\mathbb{R}^n\backslash B(1)} |\nabla K|^2 = K(1)$. By the choice of $\rho(n)$, the map $\Phi\colon X \to X$ given by

$$\Phi(u)(x) = \int K(x-\zeta)[2H\cdot\nabla u + |H|^2(1+u)](\zeta)d\zeta, \quad x \in \mathbb{R}^n$$

is Lipschitz with constant at most 1/2. (Standard estimates for the Newtonian kernel, obtained by splitting the integrals over $B(x,1)$ and its complement, show that the choice of $\rho(n)$ makes Φ a contraction of X into itself, with Lipschitz constant at most 1/2. In more detail, one uses Lebesgue integration, $\int = \int_{B(x,1)} + \int_{\mathbb{R}^n\backslash B(x,1)}$ for each fixed $x \in \mathbb{R}^n$, and that here the gradient passes through the integrals. The case $x = 0$ actually illustrates well the essential estimates.) Hence, by the contraction mapping principle, there exists $f \in C^1(\mathbb{R}^n,\mathbb{R})$ such that $\Phi(f) = f$. We now have

$$\|f\|_X = \|\Phi(f)\|_X \le \|\Phi(f) - \Phi(0)\|_X + \|\Phi(0)\|_X \le \frac{1}{2}\|f\|_X + \frac{1}{6}.$$

Thus, $\|f\|_X \le 1/3$; in particular, $|f| \le 1/3$ and $|\nabla f| \le 1/3$. Moreover, $\int |\nabla f|^2 \in \mathbb{R}$, which follows from $\Phi(f) = f$ and that there exists $C \in \mathbb{R}$ such that for all $u \in X$, the L^2 norm of $|\nabla(\Phi(u))|$ is bounded by C times the sum of the L^2 and L^1 norms of $2H\cdot\nabla u + |H|^2(1+u)$. (We have $|\nabla(\Phi(u))(x)|$ bounded by the sums of

$$\int_{B(x,1)} |\nabla K(x-\zeta)|\left|2H\cdot\nabla u + |H|^2(1+u)\right|(\zeta)\,d\zeta$$
$$\int_{\mathbb{R}^n\backslash B(x,1)} |\nabla K(x-\zeta)|\left|2H\cdot\nabla u + |H|^2(1+u)\right|(\zeta)\,d\zeta$$

though now it is probably best to regard each of these integrals as defined over $\mathbb{R}^n$ in the usual manner. The integral Cauchy-Schwarz inequality and Tonelli's Theorem suffice for the result, but the assumption on $|H| + \int |H|$ should be kept in mind. In particular, $\int |H|^2 \leq \int |H|$ because $|H| \leq 1$.)

Finally, we show that f is an analytic solution of $-\Delta y = 2H \cdot \nabla y + |H|^2 (1+y)$; for this, we use some advanced PDE theory. Because H is analytic and $\Delta u + 2H \cdot \nabla u = \Delta(1+u) + 2H \cdot \nabla(1+u)$, it suffices to show that f is a classical solution (by the Morrey-Nirenberg Theorem on analyticity of solutions of linear elliptic systems with analytic coefficients, which can be seen as a generalization of the result that harmonic functions are analytic). Now, f is a weak (but C^1) solution and $2H \cdot \nabla f + |H|^2 (1+f)$ is locally bounded, so $W^{2,2n}$-regularity applies to yield $f \in W^{2,2n}_{\text{loc}}(\mathbb{R}^n)$. Thus, $f \in C^{1,1/2}_{\text{loc}}(\mathbb{R}^n)$ by Morrey's Theorem. Since H is Lipschitz, we have $2H \cdot \nabla f + |H|^2(1+f) \in C^{0,1/2}_{\text{loc}}$. Schauder theory yields $f \in C^{2,1/2}_{\text{loc}}$. Hence, Δf exists, and f is an actual solution. □

Some remarks on the proof: (a) Experts will know that the Lipschitz condition on H can be weakened. But Theorem A is more about existence than classification, and recall the goal of keeping the exposition reasonably accessible. (b) Any attempts toward optimization of $\rho(n)$ are left to the interested reader. (c) For $n = 2$, our proof of the existence of a solution of $\Delta y + 2H \cdot \nabla y + |H|^2 (1+y)$ with $\sup |y| < 1$ does not make sense, indeed, the kernel K (as given) does not make sense. Of course, this does not rule out such a solution (indeed, there is still the trivial solution), but the conjunction of Theorem B and the proof of Theorem A prohibit any nontrivial solutions under the extra conditions that H be divergence free and not a gradient.

This ends our results. To summarize: If $n > 2$, then there are nontrivial square-integrable analytic solutions $\mathbb{R}^n \to \mathbb{R}^n$ to $\operatorname{div} Y = |Y|^2$, even though many of the standard model examples from calculus (including $n = 2$) rule out nontrivial C^1 solutions.

We now consider some further questions.

In Proposition A, how "tame" is h relative to H? To illustrate, if H is semialgebraic (hence also Nash), can h be semialgebraic? One way to make the question more precise is via the notion of "structure on the real field" (see van den Dries and Miller [1] for definitions and basic facts): What can be said about $(\mathbb{R}, +, \cdot, H, h)$ relative to $(\mathbb{R}, +, \cdot, H)$? (And note that $-H - (\nabla h)/h$ is definable in $(\mathbb{R}, +, \cdot, H, h)$.) If $(\mathbb{R}, +, \cdot, H)$ does not define $\mathbb{N}$, can the same be true of $(\mathbb{R}, +, \cdot, H, h)$? (Results from Hieronymi and Miller [3] might be helpful.) There are many H such that $(\mathbb{R}, +, \cdot, H)$ is o-minimal (that is, every definable subset of $\mathbb{R}$ has only finitely many connected components); can then $(\mathbb{R}, +, \cdot, H, h)$ also be o-minimal? If so, and H is also divergence free and not a gradient field, then with F as in the proof of Theorem A, $(\mathbb{R}, +, \cdot, F)$ is o-minimal. (*Aside:* By van den Dries and Speissegger [2], $(\mathbb{R}, +, \cdot, \rho)$ is o-minimal. I concede that I do not see any immediate use for this observation, but it is intriguing.) More generally, are there nontrivial analytic solutions F to $\operatorname{div} Y = |Y|^2$ such that $(\mathbb{R}, +, \cdot, F)$ is o-minimal? Or even just that $(\mathbb{R}, +, \cdot, F)$ does not define $\mathbb{N}$? Because F is specified as analytic, we do know that if $E \subseteq \mathbb{R}^n$ is compact and subanalytic, then the restriction of F to E is definable in the o-minimal structure $\mathbb{R}_{\text{an}}$. Hence, the restriction of F to any bounded ball is definable in $\mathbb{R}_{\text{an}}$.

One could ask about solutions to $\operatorname{div} Y = |Y|^2$ over other real-closed fields, though for $n > 2$, this is perhaps either too trivial (by transfer) or too esoteric (requiring complicated extra assumptions). Perhaps more interesting would be whether there can be nontrivial solutions for $n = 2$ of $\operatorname{div} Y = |Y|^2$ that yield a definably complete structure, thus indicating that there should be no *differential* calculus proof of Theorem B—that something like the Divergence Theorem is necessary, not merely convenient.

I close with some history and acknowledgements. I had thought about the basic question intermittently over several years, but had collected only nonexistence results (with help from other researchers such as Thierry De Pauw, Mateusz Kwaśnicki and Daniel Panazzolo). An answer to the basic question for $n > 2$ eluded me until I became interested in the extent to which "generative AI" might be useful as a tool in mathematical research. I applied ChatGPT (various 5.x versions) to the basic question. Results were mixed, including some overly complicated potential proofs, conflicting answers on different attempts, and even some outright invalid arguments. Nevertheless, it did suggest attacking the basic question via the "Helmholtz approach" (summing a divergence-free field and a gradient field), as well as the use of the fixed-point argument and elliptic regularity in the proof of Proposition A (the current statement and proof of which is due essentially to me). I thank my departmental colleague, John Holmes, for discussions on PDE theory.

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