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MODNET
Model Theory and Applications

MI.1: Study of hierarchy of n-simplicity and progress report on profinite structures

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Report on Workpackage MI: Pure Model Theory

In the following, members of Network are identified by an asterisk (*) when first mentioned; external experts and collaborators who were identified as having a close involvement with the project in the original proposal are identified by a double asterisk (**).

Task I.1: Theoretical stability and simplicity

Report on Task I.1.f: Study the hierarchy stable $\Rightarrow$ ω-simple $\Rightarrow$ $(n+1)$-simple $\Rightarrow$ n-simple $\Rightarrow$ simple

In early versions of his paper [**6**] Kolesnikov defined graded strengthenings of simplicity, called n-simplicity (where simple = 1-simple), and gave a series of examples to show that the hierarchy

$$\text{stable} \Rightarrow \omega\text{-simple} \Rightarrow (n+1)\text{-simple} \Rightarrow n\text{-simple} \Rightarrow \cdots \Rightarrow \text{simple}$$

is strict. However, it was pointed out by Wagner* (Lyon I) that this definition is not preserved under adding imaginaries. The search for a better definition led Tristram de Piro* (Modnet ER, Camerino), Byunghan Kim and Jessica Young in [**3**] to study the notion of *n-amalgamation* already introduced by Hrushovski** (Jerusalem) in [**4**] under the name of $\mathcal{P}(n)^-$-amalgamation. Namely, given sets A_I for all $I \subset n$ and elementary maps $\pi_J^I : A_I \to A_J$ for $I \subseteq J$ such that $\pi_I^I = id_{A_I}$ and $\pi_K^J \circ \pi_J^I = \pi_K^I$ for all $I \subseteq J \subseteq K$, satisfying

- $\{\pi_I^{\{i\}}(A_{\{i\}}) : i \in I\}$ is independent over $A_\emptyset$, and
- $A_I = \mathrm{bdd}(\pi_I^{\{i\}}(A_{\{i\}}) : i \in I)$ for all $I \subset n$,

we can find A_n and elementary embeddings π_n^I for $I \subset n$ such that the above conditions still hold. Then 3-amalgamation is equivalent to the independence theorem, so holds in any simple theory. T has *complete n-amalgamation* if it has m-amalgamation for all $m \leq n$. They prove that Kolesnikov's examples show that the hierarchy is again strict, and that n-amalgamation does not necessarily imply m-amalgamation for $m < n$. They prove that the random graph has complete ω-amalgamation, as do stable theories, provided the base set $A_\emptyset$ is a model. Additionally, Hrushovski [**5**] shows that every pseudo-algebraically closed structure has complete ω-amalgamation, and Chatzidakis* (Paris 7) and Hrushosvki [**2**] show complete ω-amalgamation for the theory of existentially closed fields with an automorphism in characteristic zero. Furthermore, Hrushovski gives an example of a stable theory which does not have 4-amalgamation over algebraically closed sets, and proves that if a stable theory eliminates generalized finite imaginaries, then it has 4-amalgamation.

De Piro, Kim and Young then prove a hyperimaginary group configuration theorem for simple theories with complete 4-amalgamation over models. Recall that Ben Yaacov, Tomašić (former Marie Curie EIF, Lyon I) and Wagner had previously shown in [**1**] an *almost* hyperdefinable group configuration theorem: Given a hyperimaginary tuple $(f_1, f_2, f_3, x_1, x_2, x_3)$ over a hyperimaginary e such that

(1) $f_i \in \mathrm{bdd}(f_j, f_k, e)$,
(2) $x_i \in \mathrm{bdd}(f_j, x_k, e)$, and
(3) all other triples and all pairs are independent over e,

then there is an almost hyperdefinable group. So the added assumption of complete 4-amalgamation yields a group with better definability properties (hyperdefinable, i.e. the elements are classes modulo a type-definable equivalence relation) rather than merely *almost* hyperdefinable.

Task I.3: Topological Methods in model theory

Intermediate Report on Task I.3.a: Develop the model theory of the recent notion of a profinite structure, find examples

Profinite structures and groups were introduced by Ludomir Newelski* (Wrocław) [**10, 11**] in analogy with stability; he conjectured that a *small* (with only countably many orbits under the automorphism group on n-tuples for all n) profinite group have an open abelian subgroup. This was shown by Wagner [**12**] for a particular case (m-stable, which corresponds to superstability). Conversely, Krupiński* (Wrocław) proves that infinite products of finite abelian groups of bounded exponent, with the inverse system given by the subgroups with trivial first n coordinates, are small and even m-stable and m-normal (the analogue of one-based) [**7**]; he then generalizes this to arbitrary profinite abelian groups of finite exponent with inverse system indexed by ω. He also obtains similar results for such groups with added structure (i.e. smaller automorphism group) [**8**].

In joint work [**9**] with Wagner, he then studies small profinite rings. They prove that such rings have an open Jacobson radical, which is nil of finite nilexponent; they conjecture in analogy to the group case that it must be nilpotent, and even have an open null ideal. The latter is proven in the m-stable case. They also give an example of a small profinite ring with nowhere dense annihilator, and showe that any expansion of a small profinite additive (abelian) group to a small profinite ring with open annihilator is still small.

References

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