RETURNING TO SEMI-BOUNDED SETS

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ABSTRACT. An o-minimal expansion of an ordered vector space by bounded predicates is called a semi-bounded structure. It is shown that every sufficiently saturated such structure is either linear (hence a reduct of an ordered vector space) or, after a modification of the language, it has an elementary substructure in which every interval admits a definable real closed field.

As a result certain questions about definably compact groups can be reduced to either ordered vector spaces or expansions of real closed fields. Using the known results in these two settings, the number of torsion points in definably compact abelian groups in expansions of ordered groups is given. Pillay's Conjecture for such groups follows.

1. INTRODUCTION

An expansion of an ordered abelian group or an ordered vector space by bounded predicates is sometimes called *a semi-bounded structure* (a combination of semi-linear and bounded). The definable sets in such a structure are called semi-bounded sets. Structural results about semi-bounded sets can be found in [21], [17], [22],[13], [5] (in the o-minimal setting) and [1] (in arbitrary ordered abelian groups). Some results in [15] apply as well.

In this paper I return to the semi-bounded setting, in order to reduce a question about the torsion points of a definably compact groups in o-minimal expansions of ordered groups to similar results in expansions of real closed fields, [9], and in ordered vector spaces, [11].

The idea is as follows: Let $\mathcal{M} = \langle M, <, +, \cdots \rangle$ be a semi-bounded structure which is assumed to be not linear (see [13]). By the Trichotomy Theorem, [19], a real closed field is defined on some open fixed interval $I \subseteq M$. An interval $J \subseteq M$ will be called *short* if it is in definable bijection with I; otherwise it is called *long*. The structure $\mathcal{M}$ will be called *short* is every bounded interval in M is short.

As will be observed, every definably compact group in a short model is contained in the cartesian product of some bounded interval and

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therefore definable in an o-minimal expansion of a real closed field. Hence, all results about definably compact groups in expansions of real closed fields hold when the model is short.

Given a sufficiently saturated arbitrary semi-bounded nonlinear structure $\mathcal{M}$, let D be the collection of all short elements in $\mathcal{M}$ (those elements a such that $(0, |a|)$ is short). One can modify the structure $\mathcal{M}$ to a new o-minimal structure $\widehat{\mathcal{M}}$, with the same universe, basically by extending all partial 0-definable linear maps defined on long intervals to global linear maps, and at the same time restricting $dcl(\emptyset)$, such that every definable set in the original $\mathcal{M}$ is still definable in $\widehat{\mathcal{M}}$. Having done that, the set D becomes an elementary substructure of $\widehat{\mathcal{M}}$.

Now, every $\mathcal{M}$-definable group is definable in $\widehat{\mathcal{M}}$ and because $\widehat{\mathcal{M}}$ has a short elementary substructure D, one can transfer the Edumndo-Otero result, [9], about the torsion points of definable groups in expansions of real closed fields to groups definable in $\widehat{\mathcal{M}}$.

Together with the result of Eleftheriou and Starchenko, [11], on definable groups in ordered vector spaces, one obtains (see Theorem 7.6 below):

Theorem 1.1. *If G is a definably connected, definably compact abelian group in an o-minimal expansion of an ordered group then for every k,*

$$Tor_k(G) = (\mathbb{Z}/k\mathbb{Z})^n.$$

Since this is the only missing ingredient for proving Pillay's Conjecture for definable groups in o-minimal expansions of groups, one may conclude the conjecture in this setting as well (see Section 8).

Remark 1.2. *The treatment of semi-bounded sets suggested here does not make use of the known structure theorems for definable sets in semi-bounded structures (see [17] and [5]), where the analysis is given in terms of bounded sets an unbounded intervals. Instead, bounded sets are replaced by those bounded sets that are contained in D^n and unbounded intervals are replaced by long intervals. At the end of the paper several conjectures are made about possible structure theorems for definable sets and groups semi0-bounded structures.*

Notation The letters $\mathcal{M}, \mathcal{N}, \mathcal{D}$ are used for structures whose universe, respectively, is M, N, D.

the Trichotomy Theorem might have on topological properties of o-minimal expansions of ordered groups (questions which I was not able to answer).

2. The basic definition and properties

As is shown by Edmundo in [5], semi-boundedness has several equivalent definitions. Here I use the following:

Definition 2.1. *A* semi-bounded structure $\mathcal{M} = \langle M, <, +, \cdots \rangle$ *is an o-minimal expansion of an ordered group without poles. Namely, there is no definable bijection between a bounded interval and an unbounded interval. Note that this is a property which is preserved in elementarily equivalent structures.*

Example 2.2. *(1) Every ordered vector space is semi-bounded.*

(2) The expansion $\mathbb{R}_{bdd}$ of the ordered group of real numbers by restricted multiplication is a semi-bounded structure in which every interval is short. In fact, very bounded semi-algebraic set is definable in $\mathbb{R}_{bdd}$.

(3) Any elementary extension of $\mathbb{R}_{bdd}$ is still semi-bounded, but only intervals of finite size are in definable bijection with $(0, 1)$, hence (see 3.3 below) only those intervals admit a real closed field structure.

2.1. **Expansions of ordered groups.** Given $\mathcal{M}$ an o-minimal expansion of an ordered group, there are three possibilities for the structure on $\mathcal{M}$:

(a) $\mathcal{M}$ **is linear** which, by [13], is equivalent to saying that $\mathcal{M}$ is a reduct of an ordered vector space over an ordered division ring.

(b) $\mathcal{M}$ **is nonlinear** and therefore, by the Trichotomy Theorem, [19], a real closed field whose ordering agrees with that of $\mathcal{M}$, is definable on some interval $(-a, a)$. There are two sub-cases to consider:

(b1) $\mathcal{M}$ **is semi-bounded**.

(b2) $\mathcal{M}$ **is not semi-bounded**. In this case, one can endow the whole structure $\mathcal{M}$ with a definable real closed field R (but $+$ might not be the addition of the field). Indeed, this is claimed in [19], but the reference there is not precise, so I spell out the argument: Assume that $\sigma : (b_1, b_2) \to (c, +\infty)$ is a definable map with $\lim_{t \to b_2} \sigma(t) = +\infty$. Without loss of generality, $b_2 - b_1 < a$.

Using translation, it can be assumed that $b_1 = 0$ and $b_2 < a$. However, being inside a real closed field, the intervals $(0, a)$ and $(0, b_2)$ are

in definable bijection, so (c, ∞) (and therefore also $(0, \infty)$) is isomorphic to the positive elements of R. This is clearly enough to get a real closed field on the whole of $\mathcal{M}$.

2.2. **Model theoretic preliminaries.** *Assume now that $\mathcal{M}$ is an o-minimal expansion of an ordered group, which is semi-bounded.*

An immediate corollary of this assumption is: If $f : (a, b) \to M$ is a definable function on a bounded interval then f is bounded on (a, b) and therefore the limit of $f(t)$ as t tends to either a or b exists in M.

Proposition 2.3. *If $\mathcal{M} \prec \mathcal{N}$ and $\mathcal{M}_1$ is the convex hull of $\mathcal{M}$ in $\mathcal{N}$ then $\mathcal{M}_1 \prec \mathcal{N}$.*

Proof. Without loss of generality, the language contains a constant for every element of M. It is sufficient to see that $dcl_{\mathcal{N}}(M_1) = M_1$. Equivalently, for every M-definable function $F(\bar{x})$ in $\mathcal{N}$, and every $\bar{a}$ from M_1, $F(\bar{a}) \in M_1$. Use induction of the number of variables in F.

Assume that $F(\bar{w}, y)$ is of $n+1$ variables, $n \geq 0$, and $\bar{a}$ and b are from M_1 such that $(\bar{a}, b) \in dom F$. Let $f_{\bar{w}}(y) = F(\bar{w}, y)$. By partitioning the graph of F, we may also assume that for every $\bar{w}$, the domain of $f_{\bar{w}}$ is either empty, or it is an open (bounded or unbounded) interval. Also, without loss of generality, every $f_{\bar{w}}$ is monotonely increasing (the decreasing case is handled similarly).

Assume first that $dom f_{\bar{a}} = M$. In this case, Since b is in the convex hull of M, there are $b_1 < b < b_2$, $b_1, b_2 \in M$, and hence $f_{\bar{a}}(b_1) \leq f_{\bar{a}}(b) \leq f_{\bar{a}}(b_2)$. Since $f_{\bar{a}}(b_1), f_{\bar{a}}(b_2) \in dcl_{\mathcal{N}}(\bar{a})$ one may use induction to conclude that they are in M_1, so by convexity so is $f_{\bar{a}}(b)$.

If $dom f_{\bar{a}} = (c, +\infty)$, for $c \in M$ then c is in $dcl_{\mathcal{N}}(\bar{a})$ hence, by induction it is in M_1. One can now find $b_2 \in M$ such that $c < b < b_2$ and proceed as before. The remaining case is handled similarly. $\square$

Recall that for ordered structures $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is said to be *Dedekind complete in $\mathcal{N}$* if for every element $n \in N$, if $m_1 < n < m_2$ for some $m_1, m_2 \in M$ then n has a standard part in M. Namely, there exists $m \in M$ with no element of M strictly between n and m. Note that if $\mathcal{M}_1$ is convex in $\mathcal{N}$ then it is clearly Dedekind complete in it. The following powerful theorem of Marker and Steinhorn [14] will be used below:

Theorem 2.4. *If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ which is moreover Dedekind complete in $\mathcal{N}$ then for every N-definable set $X \subseteq N^k$, the set $X \cap M^k$ is definable in $\mathcal{M}$.*

Corollary 2.5. *Assume that $F : S \times (a, b) \to M$ is a definable map such that for every $s \in S$, the map $f_s(x) = F(s, x)$ is a bijection between*

the bounded interval (a, b) and $(0, d_s)$ for some $d > 0$. Then there is an $m \in M$ such that for every $s \in S$, $d_s < m$.

Proof. If not, then in an elementary extension $\mathcal{N}$ of $\mathcal{M}$, there exist $n \in \mathcal{N}$ which is greater than all elements of $\mathcal{M}$, and $s \in S$ such that f_s is a definable bijection between (a, b) and $(0, n)$. Let $\mathcal{M}_1$ be the convex hull of $\mathcal{M}$. Then by Proposition 2.3, $\mathcal{M}_1$ is an elementary substructure of $\mathcal{N}$, which is obviously Dedekind complete in $\mathcal{N}$.

Let Γ be the intersection of the graph of f_s with $M_1 \times M_1$. By Theorem 2.4, Γ is definable in $\mathcal{M}_1$ and it is still the graph of a definable function. Moreover, because f_s was a bijection, for every $y > 0$ in $\mathcal{M}_1$ there exists $x \in (a, b) \subseteq M_1$ such that $f_s(x) = y$. Therefore there exists in $\mathcal{M}_1$ a surjective map between a sub-interval of (a, b) and the interval $(0, +\infty)$. This is impossible because $\mathcal{M}_1$ and $\mathcal{M}$ are elementarily equivalent. $\square$

3. SHORT AND LONG INTERVALS

Here $\mathcal{M}$ is assumed to be semi-bounded and in addition nonlinear.

Fix an element, call it $1 > 0$, such that a real closed field, whose universe is $(0, 1)$ and whose ordering agrees with the $\mathcal{M}$-ordering, is definable in $\mathcal{M}$. Assume from now on that $1 \in dcl(\emptyset)$.

Definition 3.1. Two open intervals (a, b) and (c, d) are called *equivalent* if there exists a definable bijection between them.

An element $a \in M$ is called *short* if either $a = 0$ or $(0, |a|)$ and $(0, 1)$ are equivalent; otherwise it is called *tall*. An interval (a, b) is called *short* if $b - a$ is small, otherwise it is called *long*.

The following lemma can be proved using standard o-minimal arguments, together with the fact that every definable function on a bounded interval has a limit at the endpoints of the interval.

Lemma 3.2. *If (a, b) and (c, d) are equivalent intervals then there exists a definable and continuous, strictly monotone bijection between them (if the intervals are bounded one can always choose the bijection to be increasing).*

Corollary 3.3. *For any interval $I \subseteq M$, I is short if and only if I admits a definable real closed field whose ordering agrees with that of $\mathcal{M}$.*

Proof. If I is short then, by the last lemma it has a definable ordered preserving bijection with $(0, 1)$ so admits a definable real closed field. For the converse, if I admits a real closed field structure, then after translation one may assume that either $(0, 1) \subseteq I$ or $I \subseteq (0, 1)$. In

both cases one gets an interval inside another real closed field so the two are in definable bijections. (Actually, by [16], the fields on $(0, 1)$ and I and are also definably isomorphic but this will not be required here).   □

Lemma 3.4.   (1) *If I is a short interval then it is definably bijective with any subinterval of I. In particular, if a is short and $0 < |b| < |a|$ then b is short.*
     (2) *If (a, b) and (b, c) are short interval then so is (a, c).*
     (3) *If a and b are short elements then so are $a + b$ and $-a$.*

Proof. (1) By the last lemma, I admits a reals closed field structure whose ordering agrees with the $\mathcal{M}$-ordering. In real closed field any two 1-dimensional open intervals are definably bijective.

(2) Since (a, b) is in bijection with $(0, 1)$ is it also in bijection with $(0, 1/2)$, and similarly, (b, c) is in bijection with $(1/2, 1)$.

(3) This is immediate from (2).   □

Lemma 3.5. *Assume that $f : X \to M$ is a definable function whose domain X is a definably connected set, contained in a cartesian product of short intervals. Then $f(X)$ is contained in a short interval.*

Proof. If not, then by definable choice there is a definable curve in X which is in bijection with a long interval in M. Using projections one gets a bijection between short and long intervals. Contradiction.   □

Proposition 3.6. *If $\mathcal{M}$ is $|T|^+$-saturated then the set D of all short elements in $\mathcal{M}$ is a proper convex subgroup of M. In particular, it is not definable.*

Proof. By 3.4, it is left to see that $D \neq M$, and here saturation is important since without it this might fail (consider the reals with restricted multiplication). Assume towards contradiction that $D = M$.

Consider the type $p(x)$ which says, for every uniformly definable family of injections from $(0, 1)$ into M, that none of these maps is a bijection between $(0, 1)$ and $(0, x)$. By our assumptions, this type is inconsistent, hence there are finitely many definable families of injections from $(0, 1)$ into M such that for every $x \in M$, one such injection gives a bijection between $(0, 1)$ and $(0, x)$. In particular, there exists an $a \in M$ and a definable family of bijections $f_s : (0, 1) \to (0, s)$, for all $s > a$. This contradicts Corollary 2.5.   □

An immediate corollary of the non-definability of D is:

Lemma 3.7. *Let $\{I_s : s \in S\}$ be a uniformly definable family of intervals in a $|T|^+$-saturated $\mathcal{M}$. If all intervals are short then there exists a short $a \in M$ such that the length of every I_s is at most a. If all*

intervals are long then there exists a tall $b \in M$ such that the length of every I_s is not less than b.

4. Affine and linear functions

Here $\mathcal{M}$ is a semi-bounded non-linear structure.

Some of the results in this section, such as 4.3 and 4.7, were proved in [15] for unbounded intervals instead of long ones.

Definition 4.1. A function $f : (a,b) \to M$ is called *linear* on (a,b) if for every $x, y \in (a,b)$, if $x + y \in (a,b)$ then $f(x) + f(y) = f(x+y)$. The function is *affine* if for some (all) $c \in (a,b)$, the function $f(c+x) - f(c)$ is linear on $(a-c, b-c)$.

$f : (a,b) \to M$ is called *locally linear* (affine) if for every $x \in (a,b)$ there exists a neighborhood on which f is linear (affine).

The following is standard:

Lemma 4.2. *If $f : (a,b) \to M$ is definable and locally linear (affine) then f is linear (affine) on (a,b).*

Lemma 4.3. *If (a,b) is an interval in M $(a, b \in M \cup \{\pm\infty\})$ and $f : (a,b) \to M$ is 0-definable then there are $a = a_0 < \cdots < a_n = b$ in $dcl(\emptyset)$ such that whenever $I = (a_i, a_{i+1})$ is long the restriction of f to I is affine.*

Proof. The function f can be assumed to be continuous and strictly increasing. The set of all x such that f locally affine near x is definable, and therefore there is a 0-definable partition $a = a_0 < \cdots < a_n = b$ such that on each (a_i, a_{i+1}) either f is locally affine (hence affine on the whole interval) or f is nowhere affine. It is sufficient to see that whenever the latter occurs then the interval must be short. Assume towards a contradiction that f is nowhere affine on (a_i, a_{i+1}) and that the interval is long. Notice that the interval remains long in any elementary extension hence one may assume that $\mathcal{M}$ is sufficiently saturated.

Consider the map $g(x) = f(x+1) - f(x)$, defined on the long interval $J = (a_i, a_{i+1} - 1)$. The function g is continuous and, by our assumption on f, it is positive everywhere. The interval J can be partitioned into finitely many sub-intervals such that g is either constant or strictly monotone on each sub-interval.

The following is again obtained by a standard o-minimal argument.

Claim 4.4. *If g is constant on a sub-interval J' then f is affine on J'.*

Because f is assumed to be nowhere affine, it follows that g is strictly monotone on each sub-interval of J and by 3.4, at least one of these intervals, which is denoted by J again, is long.

Claim 4.5. *There is $d \in D$ such that for every $x \in J$, $g(x) < d$.*

Proof. Indeed, consider the family of maps, $h_x : (0,1) \rightarrow M$, $x \in J$, given by $h_x(t) = f(x+t) - f(x)$. This is a definable family of strictly increasing continuous bijections between $(0,1)$ and the interval $(0, g(x))$, hence (clearly, all intervals $(0, g(x))$ are short) by Lemma 3.7, there exists a bound $d \in D$ such that $g(x) < d$ for all $x \in J$, thus proving the claim.

It now follows that the map g, which is injective on J, sends J into the interval $(0, d)$. This is impossible because J is long while $(0, d)$ is short. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Two affine functions $f_1 : I \rightarrow M$ and $f_2 : J \rightarrow M$ are said to be *equivalent* if the associated linear functions $f_1(a + x) - f_1(a)$ and $f_2(b+x) - f(b)$, $a \in I$, $b \in J$, have the same germ at 0.

Remark 4.6. *Note that two linear functions, defined on the same open interval I are equivalent if and only if they agree one at least one nonzero element in their common domain (see for example Proposition 4.1 in [13]).*

As in the case for unbounded intervals, one can prove that there is no infinite definable family of non-equivalent linear functions on long intervals:

Lemma 4.7. *If $\{f_s : s \in S\}$ is a 0-definable family of linear functions, $f_s : (0, a_s) \rightarrow M$ then there are finitely many 0-definable linear functions $\lambda_1, \ldots, \lambda_k$, and a short $b \in M$, such that for every $s \in S$,*
(i) Either $|I_s| < b$, or
(ii) For some $i = 1, \ldots, k$, the function f_s is the restriction of λ_i to I_s (in particular, I_s is contained in $dom(\lambda_i)$).

Proof. The equivalence relation on linear functions induces a definable equivalence relation $\sim$ on S and by definable choice there exists a definable set of representatives $S_1 \subseteq S$ for the $\sim$-classes.

For every $r \in S_1$, let $J_r = \bigcup_{s \sim r} I_s$, and let $\lambda_r = \bigcup_{s \sim r} f_s$ (this makes sense because of the equivalence). Our goal is to show that there is a finite set $F \subseteq S_1$ such that for all $r \in S_1 \setminus F$, the interval J_r is short. Indeed, if that is proved then, by 3.7 there is an upper bound b on the length of all J_r, $r \in S_1 \setminus F$, and therefore $|I_s| < b$ for all $s \sim r \in (S_1 \setminus F)$.

Assume towards contradiction that there are infinitely many $r \in S_1$ for which I_r is long. By continuity arguments (applied to the endpoints of J_s) one may find an infinite definable $S_2 \subseteq S_1$ and a tall ℓ such that for every $r \in S_2$, $(0, \ell) \subseteq J_r$. Since the equivalence class of a linear function is determined by its value at a single non-zero element,

it is possible to re-parameterize the family $\{\lambda_r : r \in S_2\}$ by $\lambda_r(\ell)$ and so assume that S_2 is an open interval in M.

Fixing a generic $r_0 \in S_2$ then, by continuity, for r sufficiently close to r_0 the element $a = \lambda_r(\ell) - \lambda_{r_0}(\ell)$ is a short element. The function $\lambda_r(t) - \lambda_{s_0}(t)$ is now a linear function (hence continuous and monotone) sending the long interval $(0, \ell)$ onto the short interval $(0, a)$. Contradiction.

It was therefore shown that for all but finitely many $r \in S_1$, the domain of λ_r is a short interval, whose length is bounded by some short $b \in D$. It is left to see that this finite set of r's is 0-definable. This can be done by considering the 0-definable set of intervals $\{J_r : r \in S_1\}$. If all J_r's are short there is nothing to do. Otherwise, what was shown so far implies that there are only finitely many J_r's of maximal length (possibly infinite). This set is clearly 0-definable so can be omitted, consider the remaining J_r's and repeat the process, until there are no remaining long J_r's in the family. $\qquad\square$

Remark 4.8. *In the notation of the last proof, it is possible that S_1 will be infinite, namely that there will be an infinite family of nonequivalent linear maps, all defined on short intervals. This will imply the definability of local multiplication over the group $\langle M, + \rangle$ but does not contradict semi-boundedness.*

The following lemma will not be used in the subsequent arguments. It is included here for a possible future use.

Lemma 4.9. *Assume that $C \subseteq M^{n+1}$ is an open cell, C_1 the projection of C on the first n coordinates. If $F : C \to M$ is a 0-definable function then there are finitely many 0-definable linear functions $\lambda_1, \ldots, \lambda_k$, each defined on a long interval, and for every $x \in C_1$, there is a partition of the interval C_x as follows: $a_0(x) < a_1(x) < \cdots < a_r(x)$ (r depending on x), and for every i, either*
(i) The interval $(a_i(x), a_{i+1}(x))$ is short, or
(ii) The function $f_x(y) = F(x, y)$ is affine on $(a_i(x), a_{i+1}(x))$ and the map $t \mapsto f_x(a_i(x) + t) - f_x(a_i(x))$ is the restriction of one of the $\lambda'_j s$.

Proof. The initial partition of every C_x is given by Lemma 4.3. For every x, consider all intervals in the partition of C_x on which f_x is nowhere affine. This is a definable family of short intervals, hence by 3.7, there is a short upper bound b on the length of all of these intervals.

The remaining intervals in C_x are those on which f_x is affine and now consider the family of all f_x, restricted to these intervals, as x varies in C_1 (namely, for every $x \in C_1$ there might be finitely many

such functions). By translation, one may assume that each such function is linear. Applying 4.7 one obtains finitely many definable linear functions $\lambda_1, \ldots, \lambda_k$ and a short element b, such that every interval in this family is either of length less than b or is a restriction of some λ_i, $i = 1, \ldots, k$. This implies the lemma.   □

5. Changing the language

Assume now that $\mathcal{M}$ is semi-bounded, non-linear and $|T|^+$-saturated.

Let Λ be the collection of all 0-definable linear functions whose domain is a long interval of the form $(0, a_\lambda)$. For every 0-definable $X \subseteq D^n$ in $\mathcal{M}$, let R_X be an n-place predicate symbol and let $\mathcal{L}_D$ be the collection of all those predicates.

Let

$$\widetilde{\mathcal{L}} = \{<, +, 1\} \cup \mathcal{L}_D \cup \{\lambda : \lambda \in \Lambda\},$$

where each λ is a unary function symbol. Let $\widetilde{\mathcal{M}}$ be the corresponding $\widetilde{\mathcal{L}}$-structure whose universe is M and all other symbols in the language interpreted naturally (with λ taken to be 0 outside $(0, a_\lambda)$).

Obviously, every 0-definable set in $\widetilde{\mathcal{M}}$ is 0-definable in $\mathcal{M}$. The converse is almost true, in the following sense:

Theorem 5.1. *Let $\widetilde{\mathcal{M}}_C$ be the expansion of $\widetilde{\mathcal{M}}$ by a new constant symbol for every element in $dcl_{\mathcal{M}}(\emptyset)$. Then, every 0-definable set in the structure $\mathcal{M}$ is definable in $\widetilde{\mathcal{M}}_C$.*

Proof. This will be done by induction in a usual o-minimal method. It is sufficient to show that every 0-definable $f : U \to M$, where U is an open cell M^n, is definable in $\widetilde{\mathcal{M}}_C$.

Definition 5.2. Let $U \subseteq M^n$ be an open set, $f : U \to M$ a definable function. For $S \subseteq \{1, \ldots, n\}$, the function f is S-bounded if if for all $i \in S$ there exists $d \in D$ such that $\pi_i(U) \subseteq [-d, d]$ (where π_i is the projection onto the i-th coordinate). In particular, every f is $\emptyset$-bounded.

Note that if $S = \{1, \ldots, n\}$ and f is 0-definable in $\mathcal{M}$ and S-bounded then its domain is contained in D^n and by 3.5, its image is contained in a short interval, so after translation by an element of $dcl_{\mathcal{M}}(\emptyset)$, the function is $\widetilde{\mathcal{M}}_C$-definable. Using this notion it is sufficient to prove the following claim:

Claim 5.3. *For every 0-definable function $f : U \to M$. If f is S-bounded, for some $S \subsetneq \{1, \ldots, n\}$ and $i \notin S$ then f can be defined using finitely many $\widetilde{\mathcal{M}}_C$-definable sets, together with finitely many 0-definable functions in $\mathcal{M}$ which are $S \cup \{i\}$-bounded.*

Once the claim is proved, then by proceeding to handle the $S \cup \{i\}$-bounded functions one can eventually reach $\{1, \ldots, n\}$-bounded functions, thus proving the theorem.

Proof of Claim 5.3.: Use induction on n:

For $n = 1$: As usual, $dom f$ can be assumed to be either M or an interval (a, b) with $b \in M \cup \{+\infty\}$ (in case $dom f = (-\infty, b)$ f can be replaced by $f(-x)$). The function f can also be assumed to be weakly monotone, and either nowhere affine on its domain or affine on its whole domain.

First, replace f by $\widetilde{f}(t) = f(a_1 + t) - f(a_1)$, with $a = 0$ if $dom f = M$ and $a_1 = a$ if $dom f = (a, b)$ (note that "$f(a_1)$" makes sense because f extends continuously to a_1). Hence, $dom \widetilde{f}$ is either M, or $(0, b - a)$. In either case, $\widetilde{f}$ is 0-definable in $\mathcal{M}$ and $\widetilde{f}(0) = 0$.

If $dom f$ is short then f is 1-bounded, which implies that it is $\widetilde{\mathcal{M}}$-definable. If $dom f$ is long then, by 4.3, $\widetilde{f}(x)$ must be linear and 0-definable in $\mathcal{M}$, therefore it equals $\lambda(x)$ for some $\lambda \in \Lambda$.

In both cases, f is clearly defined using $\widetilde{f}$, $+$, and $a_1 \in dcl(\emptyset)$, hence it is $\widetilde{\mathcal{M}}_C$-definable.

The $n + 1$ case: Without loss of generality, $i = n + 1 \notin S$. By standard o-minimal methods one may assume the following:
(1) The domain of f is an open cell $C \in M^{n+1}$ whose projection in M^n is denoted by C_1:

$$C = \{(x, y) \in C_1 \times M : h_1(x) < y < h_2(x)\},$$

for 0-definable $h_1, h_2 : C \to M \cup \{\pm\infty\}$ such that $h_1 < h_2$ on C_1.
(2) For every $x \in C_1$, the following hold:
(a) The fiber C_x is either M, or of the form $(h_1(x), h_2(x))$ for $h_1(x) \in M$, and $h_2(x) \in M \cup \{+\infty\}$, uniformly in x. (Indeed, if C_x is of the form $(-\infty, b)$ then $f(x, y)$ can be replaced by $f(-x, y)$)).
(b) The function $f_x(t) = f(x, t)$ is continuous and is either constant, strictly increasing in t, or strictly decreasing in t, uniformly in x.
(c) Either, for every $x \in C_1$ the function f_x is nowhere affine, or for every $x \in C_1$ the function f_x is affine on its domain.

(d) If $C_x \neq M$ then the limit of $f_x(t)$ as t tends to $h_1(x)$ (the lower bound of C_x), exists in M, call it $\sigma(x)$.

Actually, all except 2(d) can be achieved in any o-minimal structure. The semi-boundedness assumption gives 2(d) as well.

First replace f by a function $\widetilde{f}$ defined as follows: If $C_x = M$ for every $x \in C_1$, let

$$\widetilde{f}(x,t) = f(x,t) - f(x,0).$$

If $C_x = (h_1(x), h_2(x))$ for $h_1(x) \neq -\infty$, let

$$\widetilde{f}(x,t) = f(x, h_1(x) + t) - f(x, h_1(x)).$$

(by 2(d), f_x can be extended to $h_1(x)$).

The new domain of $\widetilde{f}$, which will still be called C, is either $C_1 \times M$ or

$$\{(x,y) : x \in C_1, \ 0 < y < h_2(x) - h_1(x)\}$$

(where $+\infty - h_1(x)$ is taken to be $+\infty$), so $\widetilde{f}$ is still S-bounded. In both cases, $\widetilde{f}_x(0) = 0$ for every $x \in C_1$.

By induction, $h_1, h_2, f(x,0)$ and $f(x, h_1(x))$ are $\widetilde{\mathcal{M}}_C$-definable. Also, f can clearly be recovered from $\widetilde{f}$ using $h_1(x)$, $f(x, h_1(x))$ and $+$, so it is sufficient to show that $\widetilde{f}$ can be defined using finitely many $\widetilde{\mathcal{M}}_C$-definable sets, together with finitely many 0-definable functions in $\mathcal{M}$ which are $S \cup \{n+1\}$-bounded.

Case 1 For every $x \in C_1$, the function $f_x(t) = f(x,t)$ is nowhere affine.

In this case, by 4.3, every interval $(0, h_2(x) - h_1(x))$ is short and hence there exists an upper bound $b \in D$ to the length of all C_x. Namely the domain of $\widetilde{f}$ is contained in $C_1 \times (0, b)$, so f is $S \cup \{n+1\}$-bounded.

Case 2 For every $x \in C_1$ the function f_x is affine on its domain.

It follows that every $\widetilde{f}_x$ is linear. By Lemma 4.7, there exists a short element b and there are finitely many functions $\lambda_1, \ldots, \lambda_k \in \Lambda$ such that for every $x \in C_1$, either $|C_x| < b$, or $\widetilde{f}_x$ is a restriction of one of the λ_i's to C_x.

By further partitions (using $\lambda_1, \ldots, \lambda_k$), it can be assumed that either for every $x \in C_1$, $\widetilde{f}_x$ is the restriction of some λ_i (same λ_i uniformly in x), or for every $x \in C_1$, $C_x \subseteq (0, b)$.

In the first case, $\widetilde{f}$ is definable using C and functions in Λ, so by induction it is $\widetilde{\mathcal{M}}_C$-definable. In the second case, the domain of $\widetilde{f}$ is contained in $C_1 \times (0, b)$ so it is $S \cup \{n+1\}$-bounded. $\square$

Lemma 5.1 shows in particular that if a structure $\mathcal{M}$ has no poles then every definable set is defined using the ordered group structure, partial (or global) 0-definable linear functions, and finitely many bounded sets. This shows that the "no poles" definition of semi-boundedness implies the one from the introduction. The opposite implication is proved using automorphisms (see the proof of Theorem 1.2 in [17]). The equivalence of the two definitions was originally established by Edmundo in [5].

6. EXTENDING PARTIAL LINEAR MAPS TO GLOBAL ONES

For $\lambda \in \Lambda$, denote by $\widehat{\lambda}$ the corresponding equivalence class of the linear function, and let $\widehat{\Lambda}$ be the collection of all those equivalence classes. Notice that $\widehat{\Lambda}$ is a ring under point-wise addition. Moreover, because the image of a long interval under a linear function is also long, $\widehat{\Lambda}$ is closed under composition and inverse composition, therefore it is an ordered division ring. Actually, as in Corollary 9.3 in [19], since it was assumed that $\mathcal{M}$ is not a linear structure, a real closed field R is definable in a neighborhood of 0, and therefore the compositional group $\widehat{\Lambda} \setminus \{0\}$ can be embedded in $GL_1(R)$ which is commutative. It follows that $\langle \widehat{\Lambda}, +, \circ \rangle$ is actually an ordered field.

Lemma 6.1. *There exists an expansion of $\widetilde{\mathcal{M}}$ to an o-minimal structure*

$$\widehat{\mathcal{M}} = \langle M, <, +, \{R_X \in \widetilde{\mathcal{L}}_D\}, \{\widehat{\lambda} : \lambda \in \Lambda\} \rangle,$$

in which every $\widehat{\lambda} : M \to M$ is a global linear map, extending all corresponding λ's in Λ (and all other symbols in $\widetilde{\mathcal{L}}$ are interpreted as before).

Proof. The first step is to expand the structure

$$M_\Lambda = \langle M, <, +, \{\lambda : \lambda \in \Lambda\} \rangle,$$

to a full ordered vector space $V = \langle M, <, +, \{\widehat{\lambda} : \lambda \in \Lambda\} \rangle$ over the field $\widehat{\Lambda}$, where every partial linear map λ is extended to a global linear map $\widehat{\lambda} : M \to M$. The existence of such a V is exactly the content of Theorem 6.1 in [13]: Indeed, although there is a linearity assumption in that theorem, the proof itself is done in the setting of an o-minimal expansion of an ordered group by partial linear functions, as given here. Also, because $\mathcal{M}$ is already saturated there is no need to go to an elementary extension in the current setting.

Having V as above, define

$$\widehat{\mathcal{M}} = \langle M, <, +, \{R_X \in \widetilde{\mathcal{L}}_D\}, \{\widehat{\lambda} : \lambda \in \Lambda\} \rangle,$$

with the original interpretation of every R_X. The goal is to show that $\widehat{\mathcal{M}}$ is o-minimal.

Consider the following (see Proposition 5.1 in [13]):

Proposition 6.2. *Let V be an ordered vector space over a field $\widehat{\Lambda}$, $I = [-a, a]$ a closed interval in V, and let*

$$\mathcal{V} = \langle V, <, +, \{\widehat{\lambda} : \lambda \in \Lambda\}, \{P : P \in \mathcal{P}\}\rangle$$

be an expansion of V by some collection $\mathcal{P}$ of subsets of I^n, for various n. Assume also:
(i) $\mathcal{P}$ contains all those a-definable sets in the ordered vector space V.
(ii) $\mathcal{P}$ is closed under definability in I, namely, every 0-definable set in the structure $\mathcal{I} = \langle I, \{P : P \in \mathcal{P}\}\rangle$ is already in $\mathcal{P}$.
   Then $\mathcal{V}$ eliminates quantifiers.

Let us see first why this theorem implies that $\widehat{\mathcal{M}}$ is o-minimal. It is clearly enough to consider finitely many predicates from $\widehat{\mathcal{L}}$ so, by taking the projection of each such R_X into D, it is possible to find $a \in dcl_{\widetilde{\mathcal{M}}}(\emptyset) \cap D$, such that all R_X's are contained in $[-a, a]$ for some $a \in D$.

Let $\mathcal{P}$ be the collection of all $\emptyset$-definable subsets of I^n in $\widetilde{\mathcal{M}}$, as n varies.

Claim 6.3. *$\mathcal{P}$ satisfies assumption (i) and (ii) of Proposition 6.2.*

Proof. (i) Every a-definable subset of I^n in the ordered vector space is already in $\mathcal{P}$:

The problem is that V has linear function which do not exist in $\widetilde{\mathcal{M}}$. However, by quantifier elimination in ordered vector spaces, every a-definable set on V^n is a boolean combination of solutions to:

(1)
$$\widehat{\lambda}_1(x_1) + \cdots + \widehat{\lambda}_k(x_k) + \widehat{\lambda}_{k+1}(a) = 0 \; ; \; \widehat{\lambda}_1(x_1) + \cdots + \widehat{\lambda}_k(x_k) + \widehat{\lambda}_{k+1}(a) > 0,$$

for $\widehat{\lambda}_i \in \widehat{\Lambda}$.

On elements of D, $\widehat{\lambda}_i = \lambda_i$ and therefore these equalities and inequalities are already definable in $\widetilde{\mathcal{M}}$ and hence are in $\mathcal{P}$.

For (ii), because every $\mathcal{I}$-definable set is already $\widetilde{\mathcal{M}}$-definable it is clear, by the definition of $\mathcal{P}$, that it is closed under definability in $\mathcal{I}$. End of Claim 6.3.

Now that the assumptions of Proposition 6.2 are established, one may conclude that the structure

$$\widehat{\mathcal{M}}_{\mathcal{P}} = \langle M, <, +, \{P : P \in \mathcal{P}\}, \{\widehat{\lambda} : \lambda \in \Lambda\}\rangle$$

(which expands $\widetilde{\mathcal{M}}$) has Quantifier elimination.

Claim 6.4. $\widehat{\mathcal{M}}_{\mathcal{P}}$ *is o-minimal.*

Proof. By quantifier elimination, every 0-definable set in $\widehat{\mathcal{M}}_{\mathcal{P}}$ is a boolean combination of terms inequalities in the ordered vector space structure, and formulas of the form

$$(t_1(x_1, \ldots, x_n), \ldots, t_k(x_1, \ldots, x_n)) \in X,$$

for some $\mathcal{I}$-definable $X \subseteq I^k$ and $t_1, \ldots, t_k$ terms in the ordered vector space language. It is clearly sufficient to handle this last type of formulas, which gives rise to 1-variable formulas:

$$\left(\widehat{\lambda}_1(x) + a_1, \ldots, \widehat{\lambda}_k(x) + a_k\right) \in X,$$

for $a_1, \ldots, a_k \in M$. Because $\widehat{\lambda}(x) + a = \widehat{\lambda}(x + \lambda^{-1}(a))$, every such formula defines a set of the form:

$$B = \{x \in M : (\widehat{\lambda}_1(x + b_1), \ldots, \widehat{\lambda}_k(x + b_k)) \in X\},$$

for $b_1, \ldots, b_k \in M$. Now let

$$A = \{(x_1, \ldots, x_k) \in M^k : (\widehat{\lambda}_1(x_1), \ldots, \widehat{\lambda}_k(x_k)) \in X\}.$$

It may be assumed that none of the $\widehat{\lambda}_i$ is 0. Because $X \subseteq I^k$ (and I is short) the set A is also contained in some J^k, for some short J, and therefore definable in the original $\widetilde{\mathcal{M}}$. The set B is now the set of all $x \in M$ such that $(x, \ldots, x) \in A - (b_1, \ldots, b_k)$. This set is also definable in $\widetilde{\mathcal{M}}$ and therefore it is a finite union of intervals.

The structure $\mathcal{M}_{\mathcal{P}}$ is therefore o-minimal and as a result $\widehat{\mathcal{M}}$ is o-minimal as well. $\square$

Remark 6.5. *Proposition 6.2 above is exactly Proposition 5.1 from [13]. However, it was pointed out by Belegradek, [1], that the proof of that proposition contained a serious gap. The gap was then fixed by Belegradek himself, using an idea of Hrushovski, to yield a similar, but slightly different result. The two results are discussed in Appendix,*

For every $\widehat{\lambda} \in \widehat{\Lambda}$, $\widehat{\lambda}(D) \subseteq D$, hence the set $D \subseteq M$ is an $\widehat{\mathcal{L}}$- substructure of $\widehat{\mathcal{M}}$, which is denoted by $\widehat{\mathcal{D}}$.

Lemma 6.6. *The structure $\widehat{\mathcal{D}}$ is an elementary substructure of $\widehat{\mathcal{M}}$.*

Proof. This is a repetition of the proof of Theorem 1.2 from [17]. By o-minimality, it is sufficient to prove that $dcl_{\widehat{\mathcal{M}}}(D) = D$. Equivalently, it will be shown that for every $a \in M \setminus D$, there exists an automorphism σ of $\widehat{\mathcal{M}}$, fixing D point-wise, such that $\sigma(a) \neq a$.

Fix $a \in M \setminus D$. Because D is a $\widehat{\Lambda}$-subspace of M, it has a (non-definable) complement D^c in M such that $M = D \oplus D^c$, as an ordered vector space. If one now takes $\sigma(d) = d$ for every $d \in D$, and $\sigma(y) = 2y$ for every $y \in D^c$ then σ is an automorphism of the ordered vector space V whose fixed elements are exactly the elements of D. Because every other atomic relation of $\widehat{\mathcal{M}}$ is contained in D^n for some n, σ is clearly an automorphism of $\widehat{\mathcal{M}}$ fixing D point-wise and moving a. It follows that $dcl_{\widehat{\mathcal{M}}}(D) = D$ and therefore $\widehat{\mathcal{D}}$ is an elementary substructure of $\widehat{\mathcal{M}}$. $\qquad\qquad\square$

7. Definable groups in semi-bounded structures

There are several papers on definable sets and groups which are definable in o-minimal expansions of ordered groups (rather than real closed fields). The main difficulty there is the lack of a triangulation theorem and therefore the development of the basic topological tools is much more difficult. In [2] and [7] sheaf Cohomology for such structures has started to emerge. In [8] the authors use this Cohomology to give an upper bound for the number of torsion points in abelian definable groups. In [6] other properties of groups in the semi-bounded setting are developed.

Here is a simple observation:

Lemma 7.1. *If G is a definably compact group in a semi-bounded structure then every chart in the atlas of G is bounded.*

Proof. If not then there exists a definable curve in one of the charts U of M which is unbounded. Because there are no definable poles, there is a definable injection $\sigma : (a, +\infty) \to U$ whose image is unbounded. Because G is definably compact this map has a limit point g in G (in the G-topology) as t tends to ∞. This limit point belongs to another chart U but now it is easy to obtain a definable injection from an unbounded interval to a bounded interval. Contradiction. $\qquad\square$

7.1. Definable groups in short models.

Definition 7.2. Let $\mathcal{M}$ be an o-minimal semi-bounded structure which is not linear. $\mathcal{M}$ is called *short* if every element in M is a short element.

It follows that if $\mathcal{M}$ is a short model then every definably compact group in $\mathcal{M}$ is definable in some o-minimal expansion of a real closed field. Indeed, all the charts of G must be bounded so there exists an in interval I such that all charts are contained in I^n for some n. Because $\mathcal{M}$ is short I admits a definable real closed field.

This in turn implies, using the (heavy) theorem of Edmundo and Otero [9]:

Corollary 7.3. *If G is a definably compact, definably connected abelian n-dimensional group in a short model then for every $k \in \mathbb{N}$,*

$$Tor_k(G) = (\mathbb{Z}/k\mathbb{Z})^n.$$

7.2. **Uniformity in parameters.** Because not every definable group in o-minimal expansion of group can necessarily be embedded, as a topological group, in M^n (or at least, it is not known whether this is so), there is some subtlety in showing that definable connectedness and definable compactness, with respect to the group topology, are definable properties in parameters.

In this section $\mathcal{M}$ can be any o-minimal expansion of a group.

Lemma 7.4. *Let $\{G_s : s \in S\}$ be a uniformly definable family of abelian groups. Then:*
(i) The set of s for which G_s is definably connected is definable.
(ii) The set of s for which G_s is definably compact is definable.

Proof. (i) It is known, [20], that G_s is not definably connected if and only if there exists $n \in \mathbb{N}$ such that the image of G_s under $g \mapsto ng$, call it nG_s, is different than G_s. By compactness there exist a bound $N \in \mathbb{N}$ such that whenever $nG_s \neq G_s$ for some n then necessarily there exists such an n with $n \leq N$. But now, G_s is definably connected if and only if $N!G_s \neq G_s$.

(ii) Without loss of generality every G_s has the same dimension n. By Pillay's theorem on groups, [20], there exists, uniformly in s, a definable family of atlases and maps for the family of groups. Namely, there is some k, and there exists an definable family of open subsets of M^n, $\{U_{i,s} : s \in S, i = 1, \ldots, k\}$, together with a definable family of bijections $\phi_{i,s} : U_{i,s} \to G_s$, such that $G_s = \bigcup_{i=1}^k \phi_{i,s}(U_{i,s})$ for every $s \in S$, the transition maps are continuous, and such that the group operations on G_s are continuous when read through the charts. By 7.1, it may be assumed that each $U_{i,s}$ is a bounded subset of M^n.

For every $\epsilon > 0$ in M, let $U_{i,s}^\epsilon$ be the set of all elements in $U_{i,s}$ whose distance (using the maximum norm) from the boundary of $U_{i,s}$ is greater than ϵ. This is easily seen to be an open set as well. The following claim is based on an observation of Elefethriou:

Claim 7.5. *For every $s \in S$, the group G_s is definably compact if and only if there exists an $\epsilon > 0$ such that*

$$G_s = \bigcup_{i=1}^{k} \phi_{i,s}(U_{i,s}^{\epsilon}).$$

Proof. If G_s is definably compact then the negation of the condition yields a definable curve $\gamma : (0, a) \to G_s$, such that for every t,

$$\gamma(t) \in G_s \setminus \bigcup_{i=1}^{k} \phi_{i,s}(U_{i,s}^{t}).$$

If $g \in G_s$ is the limit of $\gamma(t)$ as t tends to 0 (which exists by definable compactness) then for some $i = 1, \ldots, k$, $\phi_{i,s}^{-1}(g) \in U_{i,s}$, therefore for all sufficiently small $\epsilon > 0$, $\phi_{i,s}(g) \in U_{i,s}^{\epsilon}$. This easily leads to a contradiction.

For the converse, if there exists an ϵ as above, then any definable curve γ in G_s will be eventually contained in one of the $\phi_{i,s}(U_{i,s}^{\epsilon})$, and because $U_{i,s}^{\epsilon}$ is bounded the curve $\phi_{i,s}^{-1}(\gamma(t))$ has a limit in $x \in M^n$, which must be in $U_{i,s}$. The element $\phi_{i,s}(x) \in G_s$ is the limit of $\gamma(t)$. $\square$

7.3. **Torsion of definably compact groups.**

Theorem 7.6. *Let G be a definably compact, definably connected, abelian group in an o-minimal expansion $\mathcal{M}$ of an ordered group. Then for every $k \in \mathbb{N}$, we have*

$$Tor_k(G) = (\mathbb{Z}/k\mathbb{Z})^n.$$

Proof. By Elefetheriou-Starchenko [11], the result holds for groups definable in ordered vector spaces over ordered division rings, and hence for all linear expansions of ordered groups. By Edmundo-Otero the result holds in those expansions which are not semi-bounded (see discussion in Section 2.1)

One may therefore assume that $\mathcal{M}$ is semi-bounded. Consider the structure $\widehat{\mathcal{M}}$ as given in Theorem 6.1, and its elementary sub-structure $\widehat{D}$ (which is a short model).

The group G is definable in the structure $\widehat{\mathcal{M}}$, possibly over a set of parameters s. Namely, $G = G_s$ for some D-definable family $\{G_s : s \in S\}$ of definable groups, in the structure $\widehat{\mathcal{M}}$. By 7.4, one may assume that for every $r \in S(D)$, the group $G_r(D)$ is definably connected, definably compact abelian group.

Because $\widehat{D}$ is a short model, given $k \in \mathbb{N}$, for every $r \in S(D)$, $Tor_k(G_r(D)) = (\mathbb{Z}/k\mathbb{Z})^n$. This is clearly a first order property of $\widehat{D}$, hence it is true in $\widehat{\mathcal{M}}$ as well and in particular for $G = G_s$. $\square$

8. Pillay's Conjecture

As is pointed out in [12] (see Remark 4 at the end of Section 8), the presence of an ambient real closed field is used twice in the proof of Pillay's Conjecture:

1. In order to apply Theorem 2.1 from [18] to a definably compact group G one needs to know that closed subsets of G are closed and bounded. This is true if G, with its group topology, is a subspace of M^n, which in expansions of real closed field this can always be achieved, but not known in expansions of groups. The following idea was suggested by Eleftheriou:

Using Claim 7.5, there are finitely many pairs of bounded open sets $V_1 \subseteq U_1, \ldots, V_k \subseteq U_k$, subsets of M^n, such that for each i, $Cl(V_i) \subseteq U_i$ (closure taken in M^n) and such that

$$G = \bigcup_i \phi_i(U_i) = \bigcup_i \phi_i(V_i).$$

Given any closed set $X \subseteq G$, each set $\phi_i^{-1}(X) \cap Cl(V_i)$ is closed and bounded in M^n. Using Theorem 2.1 in [18], this is sufficient to prove the result needed in that paper:

If $X \subseteq G$ is a definable closed set and $\mathcal{M}_0$ is a small model then the set of $\mathcal{M}_0$-conjugates of X is finitely consistent if and only if X has a point in $\mathcal{M}_0$.

2. The second missing ingredient in the proof of Pillay's Conjecture iss Theorem 7.6, which is now proved.

It therefore follows that Pillay's conjecture holds in expansions of ordered groups.

9. some open questions

9.1. **The structure of definable sets.** In [17] and [5], structure theorems for definable sets in semi-bounded structures are given. The conjecture below is a natural strengthening of those results.

Conjecture 1 *If $\mathcal{M}$ is semi-bounded then every definable subset of M^n can be written as a finite union of sets of the form:*

$$C + \{\Sigma_{i=1}^k(\lambda_{i,1}(t_i), \cdots, \lambda_{i,n}(t_i)) : t_1 \in I_1, \ldots, t_k \in I_k\},$$

for a definable $C \subseteq D^n$, $\lambda_{i,j} \in \Lambda$ and $I_1, \ldots, I_k$ long (possibly unbounded) intervals.

9.2. **Definable groups in semi-bounded structures.**

It was shown by Edmundo, Eleftherious and Onshuus, [6], that every definable group in a semi-bounded structure has a definable normal subgroup which is definably isomorphic to $\langle M^n, + \rangle$, such that the quotient is definably isomorphic to a bounded group (namely, a group whose universe is a bounded set in M^n). Because of the above conjectured structure theorem and because definable functions are linear outside short intervals, the following conjecture seems reasonable:

Conjecture 2 *Let G a definable group in a semi-bounded structure. Then there exists a definable normal $A \subseteq G$, with A is definably isomorphic to a semi-linear group, such that the quotient G/A is definably isomorphic to to a group contained in D^n.*

The conjecture, if true, will allow to analyze every definable group in an o-minimal expansion of ordered groups in terms of semi-linear groups and groups definable in expansions of real closed fields.

9.3. *A general transfer principle.*

The arguments used to prove Theorem 7.6 can clearly be used to transfer other results from o-minimal expansions of real closed fields to o-minimal expansions of groups. This suggests a possible general transfer principle between o-minimal expansions of fields and of groups. The following conjecture is modeled after another transfer principle, suggested by L. van den Dries in [4] (and proved false in the original setting):

Let $\phi(R_1, \ldots, R_n, f_1, \ldots, f_k)$ be a sentence in a language $\mathcal{L}$ expanding the language of ordered sets, with $R_1, \ldots, R_n, f_1, \ldots, f_k$ all relation and function symbols that are different than $<$.

Conjecture 3 *Assume that $\phi(R_1, \ldots, R_n, f_1, \ldots, f_k)$ holds in every o-minimal $\mathcal{L}$-expansion of a real closed field, where $<$ is interpreted as the natural ordering of the field.*

Then $\phi(R_1, \ldots, R_n, f_1, \ldots, f_k)$ holds in every o-minimal $\mathcal{L}$-expansion of an ordered group that is not linear, where $<$ is interpreted as the natural ordering.

The arguments presented here show that it is enough to prove the above for short models.

10. Appendix

I now return to Proposition 6.2 (Proposition 5.1 from [13]). As was pointed out in [1], the proof for that theorem contained an error. The error was fixed in Belegradek's paper, using an idea of Hrushovski. However, the new result (Fact 0.1 in [1]), reads as follows:

Fact 10.1. *Let V be an ordered vector space over an ordered division ring D, a a nonnegative element in V and*

$$\mathcal{V} = \langle V, <, +, \{\lambda : \lambda \in D\}; \{P : P \in \mathcal{P}\} \rangle$$

an expansion of V by a collection $\mathcal{P}$ of relations on $I = [-a, a]$. Supposes that every relation on $[-a, a]$ which is a-definable in $\mathcal{V}$ belongs to $\mathcal{P}$. Then the structure $\mathcal{M}_{\mathcal{P}}$ admits elimination of quantifiers.

To see that Fact 10.1 implies Proposition 6.2 it is left to prove:

Every a-definable subset of I^n in the structure $\mathcal{V}$ is already definable in $\mathcal{I} = \langle I, <, \{P : P \in \mathcal{P}\} \rangle$.

Proof. It is sufficient to prove that every automorphism of $\mathcal{I}$ can be extended to an automorphism of $\mathcal{V}$ which fixes a. Let $\sigma : I \to I$ be such an $\mathcal{I}$-automorphism. Because $0 \in I$ is definable, σ is necessarily order preserving. As was shown in Claim 6.3, every subset of I^n that is definable in the ordered vector space V is already $\mathcal{I}$-definable therefore σ, as a vector space automorphism, can be extended to a vector space automorphism of V which necessarily fixes a. However, all atomic relations in $\mathcal{V}$ which are not part of the ordered vector space are part of the $\mathcal{I}$ structure, and therefore σ is a $\mathcal{V}$-automorphism as well.

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