Weyl groups of small groups of finite Morley rank

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November 7, 2007

1 Introduction

Infinite groups of finite Morely rank have little truly geometric structure; however, their algebraic properties are remarkably reminiscent of algebraic groups. The strongest conjecture to this effect is the *Cherlin-Zil'ber algebraicity conjecture* which postulates that an infinite simple group of finite Morley rank is a linear algebraic group over an algebraically closed field.

There are a number of partial results towards this conjecture and a complete proof in the *even & mixed type* cases, read potentially characteristic two. In other cases, much recent work has followed two themes : an analysis of the minimal simple groups where Bender's method is well understood [Bur07], and the analysis of torsion using genericity arguments [BBC07, BC08], as well as work involving both techniques [BCJ07, Del07, AB08].

A major part of the combined thread is the analysis of the Weyl group $W := N(T)/C^\circ(T)$ of G associated to some maximal *decent* torus T. Here a decent torus is merely the smallest definable subgroup containing some divisible abelian torsion subgroup, such as a p-torus $\mathbb{Z}(p^\infty)^n$. In fact the reader my may always replace decent torus by p-torus. One may speak of the Weyl group of G because maximal tori are conjugate in a group of finite Morley rank [Che05].

In the present article, we show that the Weyl group is cyclic in a minimal connected simple group of finite Morley rank.

Theorem 3.1. *Let G be a minimal connected simple group of finite Morley rank, and let T be a decent torus of G. Then the Weyl group $W := N(T)/C(T)$ is cyclic.*

We also give a condition on the *Prüfer p-rank* of a torus that admits p as a divisor of $|W|$. The Prüfer p-rank $\mathrm{pr}_p(T)$ is merely the maximal n such that $\mathbb{Z}(p^\infty)^n$ is a subgroup of T.

Theorem 4.1. *Let G be a minimal connected simple group of finite Morley rank, and let T be a nontrivial p-torus of G. Suppose that p is a prime divisor of $|W_T|$. Then $p-1$ divides the Prüfer p-rank $\mathrm{pr}_p(T)$ of T, and p is not the minimal prime divisor of $|W|$ unless $p = 2$.*

*Supported by NSF postdoctoral fellowship DMS-0503036.

Our primary concern here is groups of *degenerate type*, where Sylow 2-subgroups are trivial [BBC07]. The second authors thesis work covers the *odd type* case (see Fact 3.2 below) and the afore mentioned classification in even & mixed types covers those cases.

As an introduction, we first prove a far stronger yet easier result, but under strong number theoretic hypotheses.

2 Tameness Wagner style

The Weyl group of an algebraic group is the quotient of the normalizer of a maximal algebraic torus by the centralizer of the same torus. In general, groups of finite Morley rank need not contain any algebraic torus, but frequently have so-called decent tori. A *decent torus* is a divisible abelian group which is the definable hull of its torsion. The group $W_T := N(T)/C(T)$ is the Weyl group associated to a given torus T. Here one may use only the connected component of the centralizer by the following.

Fact 2.1 ([AB08, Theorem 1])**.** *Let T be a decent torus of a connected group H of finite Morley rank, Then $C(T)$ is connected.*

We naturally say "the Weyl group of G" when T is a maximal decent torus; this is well defined by conjugacy of maximal decent tori [Che05].

Lemma 2.2. *Let G be a minimal connected simple group of finite Morley rank, and let T be a nontrivial decent torus of G. Then the Weyl group $W_T := N(T)/C(T)$ associated to T is naturally isomorphic to a subgroup of the Weyl group W of G.*

Here we'll use the fact that solvable groups have trivial Weyl groups (see also [AB08, Lemma 5.11]).

Fact 2.3 ([AB08, Lemma 6.6])**.** *Let H be a connected solvable group of finite Morley rank and let K be a definable connected subgroup of H such that $[N_H(K) : K] < \infty$. Then $N_H(K) = K$.*

Proof of Lemma 2.2. Let S be a maximal decent torus containing T. Then $S \leq C^\circ(T) \lhd N(T)$. So W_T is a section of $W := N(T)/C(T)$ by a Frattini argument using [Che05]. Of course the kernel here is the Weyl group $N_{C(T)}(S)/C_{C(T)}(S)$ of S inside $C(T)$. But $C(T)$ is solvable by Fact 2.1. So this kernel is trivial by Fact 2.3, as desired. □

A similar argument shows that, in a minimal connected simple group, the Weyl group is isomorphic to $W_Q := N(Q)/Q$ where Q is a Carter subgroup containing the maximal torus S. In a minimal simple group, one need not specify the Carter subgroup Q as they are all conjugate [Fré08] but this need not hold in general.

In the present article we examine some variations on the following result.

Theorem 2.4. *Let G be a minimal connected simple group of finite Morley rank and degenerate type, and let T be a nontrivial decent torus of G. Then either*

1. *the Weyl group $W := N(T)/C(T)$ is trivial, or*
2. *G interprets a bad field in characteristic p where p is at most the minimal prime divisor of $|W|$.*

Here a bad field $(k, H, +, \cdot)$ is a field k of finite Morley rank with a proper nontrivial definable subgroup H of it's multiplicative subgroup. Such fields exist in characteristic zero but are quite unlikely in positive characteristic p for number theoretic reasons. In this second situation, there are only finite many primes of the form $\frac{p^n-1}{p-1}$ by [Wag97]. Moreover the asymptotics in [HW] hold :

$$(p^n - 1)_\pi \asymp p^{\alpha n}$$

Here π denotes the set of primes π appearing in the bad field's multiplicative subgroup, and α denoted the ratio $\mathrm{rk}(T)$ So our theorem says that Weyl groups are unlikely degenerate type.

To prove the theorem, we consider a minimal connected simple group G, and a nontrivial decent torus T of G. By Lemma 2.2, we may assume that T is a maximal decent torus.

Let p be the minimal prime divisor of $W := N(T)/C(T)$. Then G contains p-unipotence by the following important fact.

Fact 2.5 ([BC08, Theorem 5])**.** *Let H be a connected group of finite Morley rank. Suppose the Weyl group associated to a maximal decent torus is nontrivial and has odd order, with p the smallest prime divisor of its order. Then H contains a unipotent p-subgroup.*

Hence there is a Borel subgroup B of G with $U_p(B) \neq 1$. So consider some B-minimal $A \leq U_p(B)$.

Suppose first that $C_B(A) < B$. By the Zilber field theorem, there is a field k interpretable in B with $A \cong k_+$ and $B/C_B(A) \hookrightarrow k^*$. If G has degenerate type, k is our bad field in characteristic p, as desired.

So suppose otherwise that $C_B(A) = B$. Then $B \cap B^g = 1$ for $g \notin N(B)$ by the Jaligot Lemma :

Fact 2.6 ([Bur07, Lemma 2.1])**.** *Let B_1, B_2 be two distinct Borel subgroups of G satisfying $U_{p_i}(B_i) \neq 1$ for some prime p_i $(i = 1, 2)$. Then $F(B_1) \cap F(B_2) = 1$.*

It follows that $\bigcup B^G$ is generic in G by the genericity argument :

Fact 2.7 ([BBC07, Lemma 4.1])**.** *Let G be a group of finite Morley rank, H a definable subgroup of G, and X a definable subset of G. Then $\mathrm{rk}(\bigcup X^G) = \mathrm{rk}(G)$ whenever*

$$\mathrm{rk}(X \setminus \bigcup_{g \notin H} X^g) \geq \mathrm{rk}(H).$$

But now B contains a maximal decent torus of G by the following.

Fact 2.8 ([BC08, Theorem 1]). *Let G be a connected group of finite Morley rank and a be a generic element of G. Then $d(a)$ contains a maximal decent torus of G.*

So we may assume that $B = C^\circ(T)$, and p divides $N(B)/B$, contradicting the following.

Fact 2.9 ([AB08, Lemma 4.3]). *Let B be a Borel subgroup of G such that $U_p(B) \neq 1$ for some prime p. Then $p \not| [N_G(B) : B]$.*

This concludes the proof of Theorem 2.4.

3 Cyclicity of Weyl groups

Theorem 2.4 completes our picture of Weyl groups in minimal connected simple groups of degenerate type, *if* there are no bad fields in positive characteristic. However, one can still prove something without such strong hypotheses.

Theorem 3.1. *Let G be a minimal connected simple group of finite Morley rank, and let T be a decent torus of G. Then the Weyl group $W_T := N(T)/C(T)$ is cyclic.*

We may start the proof with a non-trivial decent torus T which by Lemma 2.2 can be taken maximal. Again recall that $C(T)$ is connected by Fact 2.1.

We dispense with the odd type case by using the second author's thesis work. As observed after the proof of Lemma 2.2, letting Q be a Carter subgroup of G containing T, we find $W \cong W_Q := N(Q)/Q$.

Fact 3.2 ([Del07, see Théorème p.89]). *Let G be a minimal connected simple group of finite Morley rank of odd type, and let Q be a Carter subgroup of G containing a Sylow$^\circ$ 2-subgroup. Then $|N(Q)/Q| = 1, 2, 3$.*

So $|W| = 1, 2, 3$ if such a G has odd type. We may therefore assume that G has degenerate type by the Even Type Theorem [ABC07].

The real starting point for our analysis is an earlier result about Weyl groups in minimal connected simple groups.

Fact 3.3 ([AB08, Proposition 5.1 & Corollary 5.7]). *W is a metacyclic Frobenius complement.*

A Frobenius complement is the stabilizer of a point in a Frobenius group; whch is a transitive permutation group on a finite set, such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. Such groups have a quite restrictive structure, which is described by the following.

Fact 3.4 ([Gor80, 10.3.1 p. 339]). *Let W be a Frobenius complement. Then*

1. *Any subgroup of W of order pq, p and q primes, is cyclic.*

2. *Sylow p-subgroups of W are either cyclic or possibly generalized quaternion if $p = 2$.*

Of course metacyclicity follows for any Frobenius complement inside a degenerate type group [Gor80, 7.6.2 p. 258].

Consider a minimal non-cyclic subgroup Ξ of W. Then Ξ is metacyclic but not cyclic while every proper subgroup of Ξ is cyclic. In particular every proper subgroup of Ξ is abelian yet Ξ itself is not.

Claim 3.5. *There are prime numbers p and $q|p-1$ and elements $\alpha, \beta \in \Xi$ with $\alpha^p = 1$, $\beta^{q^m} = 1$, and $\beta^q \neq 1$ such that :*

- $\Xi = \langle \alpha, \beta \rangle$;
- $[\alpha, \beta^q] = 1$;
- $\Xi' = \langle \alpha \rangle$.

Proof. Say $\Xi/C_1 \cong C_2$ where C_1 and C_2 are cyclic groups. Let $\alpha \in \Xi$ be a generator of C_1, and let β be an element of Ξ whose image modulo C_1 generates C_2. Of course $[\alpha, \beta] \neq 1$ since Ξ is not abelian.

By minimality of Ξ, α and β must have prime powers as orders, say p^n and q^m respectively. It follows that $p \neq q$ because Sylow p-subgroups are cyclic.

Also notice that by minimality:

$$[\alpha^p, \beta] = [\alpha, \beta^q] = 1 \qquad (\dagger)$$

So $\langle \beta \rangle \simeq \mathbb{Z}/q^m\mathbb{Z}$ normalizes $C_1 = \langle \alpha \rangle \simeq \mathbb{Z}/p^n\mathbb{Z}$. As X is non-abelian, the action is non-trivial. Recall that any group automorphism of $\mathbb{Z}/p^n\mathbb{Z}$ as a group is also a ring automorphism. Our action of β on $\langle \alpha \rangle$ by conjugation therefore becomes a multiplicative action on $\mathbb{Z}/p^n\mathbb{Z}$ by some $1 \neq \lambda \in \mathbb{Z}/p^n\mathbb{Z}$.

By ($\dagger$), the action of λ^q is trivial, i.e. $\lambda^q \equiv 1 \pmod{p^n}$. On the other hand, b acts trivially on $\langle \bar{a^p} \rangle$, so $p\lambda \equiv p \pmod{p^n}$. This means $p^{n-1}|\lambda - 1$. Hence there is an integer $k \in \{1, \ldots, p-1\}$ such that $\lambda \equiv 1 + kp^{n-1} \pmod{p^n}$. Raising to the qth power, we find $1 \equiv \lambda^q \equiv (1 + kp^{n-1})^q \pmod{p^n}$.

Now, if n were ≥ 2, we'd have $1 \equiv 1 + qkp^{n-1} \pmod{p^n}$ because all other terms are divisible by at least $p^{2(n-1)}$, which is trivial in $\mathbb{Z}/p^n\mathbb{Z}$. So, if $n \geq 2$, we derive $p|qk$, which is absurd. Therefore $n = 1$, and α has order exactly p.

Now Fact 3.4(1) and non-abelianness of Ξ together imply that β^q must be non-trivial.

Clearly $\langle \alpha \rangle$ contains $\Xi' \neq 1$. As $[\alpha, \beta] \neq 1$, everything is proved. □

Our next step is to lift Ξ isomorphically to a subgroup $X \leq N(T)$, which will require some attention.

Let a be a p-element of $N(T) \setminus C^\circ(T)$ whose image in $\Xi \leq W$ is α. [1] Such liftings exist by the usual torsion lifting principle [BN94, Ex. 11 p. 93; Ex. 13c p. 72]. However, such a wild lifting of b need not normalize our $\langle a \rangle$.

[1] Let's be consistent about the use of notation blocks.

Claim 3.6. *$C(T)$ has $p^\perp$-type.*

Proof. Otherwise we suppose towards a contradiction that $U_p(C(T)) \neq 1$. Then $C(T)$ lies inside a unique Borel subgroup B by Fact 2.6. In particular B is normalized by the Weyl element representative a. As $U_p(B) \neq 1$, it follows that $a \in B$ using Fact 2.9, but this contradicts Fact 2.3. $\square$

In consequence, [BC08, Theorem 4] says that any definable subgroup of $N(T)$ satisfies conjugacy of Sylow p-subgroups

We let T_p denote the maximal p-torus in T, which might be trivial.

Claim 3.7. *There is a q-element b_0 lifting β and normalizing $\langle a \rangle T_p$.*

Proof. Let $\pi : N(T) \to N(T)/C(T)$ denote the quotient map, and let $H = \pi^{-1}(\Xi)$ and $K = \pi^{-1}(\langle \alpha \rangle)$. Of course $K \lhd H$ by Claim 3.5. Also, $\langle a \rangle T_p$ is clearly a Sylow p-subgroup of K. As Sylow p-subgroups of K are K-conjugate, a Frattini argument yields $H = K \cdot N_H(\langle a \rangle T_p)$. So $N(\langle a \rangle T_p)$ meets the coset β. Again torsion lifting yields a q-element b_0 in $N(\langle a \rangle T_p)$ whose image is β. $\square$

Claim 3.8. *The group $X_0 := \langle a, b_0 \rangle$ is finite.*

Proof. As $\langle a \rangle T_p$ is locally finite, so is $(\langle a \rangle T_p) \cdot \langle b_0 \rangle \geq \langle a, b_0 \rangle$. $\square$

Our X_0 needn't yet be isomorphic to Ξ; however, it's elements retain their original orders.

Lemma 3.9. *Let G be a minimal connected simple group of finite Morley rank, let T be a nontrivial maximal torus, and let $\bar{x} \in N(T)/C(T)$. Then any lifting of $\bar{x}$ to a torsion element $x \in N(T) \setminus C(T)$ has order $|\bar{x}|$. In other words $\langle x \rangle \cap C(T) = 1$ whenever $x \in N(T) \setminus C(T)$ is torsion.*

Here we use the that minimal simple groups are covered by their Borel subgroups.

Fact 3.10 ([AB08, Corollary 4.4])**.** *Let G be a minimal connected simple group of finite Morley rank. Any torsion element x of G lies inside any Borel subgroup of G which contains $C^\circ(x)$.*

Proof of Lemma 3.9. Consider a lifting x of finite order with $x^n \in C(T)$. Suppose towards a contradiction that $x^n = 1$. Fix some Borel subgroup B containing $C^\circ(x^n) \geq C^\circ(x)$. Since x is torsion, $x \in B$ by Fact 3.10. But $T \leq C^\circ(x^n) \leq B$ too, contradicting Fact 2.3. $\square$

In particular a has order p and b_0 order q^m.

To prove the isomorphism, we require the following observation.

Lemma 3.11. *Let T be a torus in a minimal connected simple group, and let $x \in N(T) \setminus C(T)$. Then the function $\varphi : T \to$ given by $t \mapsto [t, x]$ is a surjective endomorphism.*

Proof. The map φ is obviously a group homomorphism. Now $\ker \varphi = C_T(x)$. If the latter is infinite, then there is a non-trivial subtorus $1 \neq \tau \leq T$ such that $x \in C(\tau) = C^\circ(\tau)$ by Fact 2.1. Hence $C^\circ(\tau)$ contains the group $T \cdot \langle x \rangle$, a contradiction to Fact 2.3. This proves that $C_T(x)$ is finite, hence φ is surjective. □

Claim 3.12. *There is a q-element b lifting β and normalizing $\langle a \rangle$.*

Proof. A translation tb_0 of b_0 by some $t \in T$ has the same q^m, by Lemma 3.9. So we search for a t for which b normalizes $\langle a \rangle$.

Since β normalizes $\langle \alpha \rangle$ but does not centralize it, there is an integer $1 < k < p$ with $\alpha^\beta = \alpha^k$. So there is an element $s \in C(T)$ such that $a^{b_0} = sa^k$, and clearly $s \in X_0$ has finite order. Since b_0 normalizes $\geq aT_p$ and a normalizes T_p, we actually have $s \in T_p \leq T$.

By Lemma 3.11, there is a $t \in T$ be such that $[t, a^{-1}] = s^{-b_0{}^{-1}}$. Set $b := tb_0$. Then still $\pi(b) = \beta$. As $b^{q^m} \in T$ has finite order, b has order q^m by Lemma 3.9. Furthermore,

$$a^b = (a^t)^{b_0} = ([t, a^{-1}]a)^{b_0} = [t, a^{-1}]^{b_0} a^{b_0} = s^{-1}sa^k. \quad \square$$

So now $X := \langle a, b \rangle \leq N(T)$ is an isomorphic group lifting Ξ.

We next push our subgroup X into some Borel subgroup of G, after first extracting one more consequence of Borel covering (Fact 3.10).

Lemma 3.13. *Let G be a minimal connected simple group of finite Morley rank. If U is a finite subgroup of G with a cyclic Sylow r-subgroup R that meets $Z(U)$, then U lies inside some Borel subgroup of G; in fact any Borel subgroup containing $C^\circ(z)$ for some $z \in Z(U)$.*

Proof. Fix $z \in Z(U)$ of order r, and let B be a Borel containing $C^\circ(z)$. Consider some $x \in U$. If r divides $|x|$ then z is a power of $y := x$. If r does not divide $|x|$ then z is a power of $y := xz$. In either case $C^\circ(y) \leq C^\circ(z)$. So $y \in B$ by Fact 3.10. Hence $x \in \langle y \rangle$ lies inside B too. □

Claim 3.14. *There is a Borel subgroup B of G containing X.*

Proof. Suppose first that $a^p \neq 1$. By Claim 3.5, we find that $Y := \langle a^p, b \rangle$ is cyclic, and a^p is central in X. So B exists by Lemma 3.13.

Suppose alternatively that $a^p = 1$. Then, since $[a, b] \neq 1$, one has $b^q \neq 1$ by Fact 3.4 (i). Again $Y := \langle a, b^q \rangle$ is cyclic by Claim 3.5, and now p^q is central in X. So once again B exists by Lemma 3.13. □

Of course X determines the roles of a and b inside B.

Claim 3.15. $X' = \langle a \rangle \leq U_p(B)$ *and* $b \notin F(B)$

Proof. Let S_p be a Sylow p-subgroup of B' containing a. By [BN94, Theorem 9.29 & Corollary 6.20], $S_p = U_p(B) * T_p$ where T_p is some maximal p-torus of $F(B)$. Of course T_p is central in B by [BN94, Theorem 6.16]. So $a \notin T_p$ since $[a, b] \neq 1$. The cylic group $X' = \langle a \rangle$ is the Sylow p-subgroup of X. So $a \in U_p(B)$.

Also $b \notin F(B)$ since X is non-nilpotent. □

We now find an honestly toral alternative to b who still centralizes some unipotent element related to a. In fact such an element must centralize an infinite unipotent subgroup by the following lemma.

Lemma 3.16. *Let B be a Borel subgroup with $U_p(B) \neq 1$, and $t \in B$. If $C_{U_p(B)}(t) \neq 1$ then it is infinite.*

Here one uses the following fact about relatively prime actions.

Fact 3.17 ([ABCC01], [Bur04, Fact 3.3])**.** *Let $H = KT$ be a group of finite Morley rank. Suppose that T is a solvable π-group of bounded exponent and that K is a definable abelian normal $\pi^\perp$-subgroup of H. Then $H = [H, T] \oplus C_H(T)$.*

Proof of Lemma 3.16. Let $x \in C_{U_p(B)}(t)^\#$ and let $Z_i := Z_i^\circ(U_p(B))$ for $i \in \mathbb{N}$. As $U_p(B)$ is nilpotent and connected, there is some integer i such that $x \in Z_{i+1} \setminus Z_i$. In particular the action of t on the *connected* abelian quotient $Y := Z_{i+1}/Z_i$ has some centralization. By Fact 3.17, $C_Y(t) \cong Y/[Y, t]$ is connected and infinite, and hence $C_Y(t)$ is infinite. As now $U_p(C_Y(t)) \neq 1$, it follows that $U_p(C_B(t)) \neq 1$. □

Claim 3.18. *There is a toral q-element t of B with an infinite centralizer in either $U_p(B)$ or possibly $U_q(B)$. In particular B is the only Borel containing $C_B^\circ(t)$.*

Proof. Fix some Sylow q-subgroup S_q of B containing b. By [BN94, Theorem 9.29 & Corollary 6.20], $S_q = T_q * U_q(B)$, where T_q is a q-torus of B. We thus have $T_q \neq 1$ because S_q cannot centralize $a \in U_p(B)$ while $U_q(B)$ does. So we may assume $U_q(B) = 1$ since otherwise any element of T_q suffices. In particular $b \in T_q$ is toral.

By Claims 3.5, 3.12, and 3.15, b^q centralizes $a \in U_p(B)^\#$. So our main conclusion follows from Corollary 3.16. Now our last conclusion follows from Fact 2.6. □

At this point, the structure surrounding B solidifies considerably.

Fix $g \in G$ such that $t \in T^g$. Set $T_1 := T^g$. Of course $T_1 \leq B$ by Claim 3.18.

Claim 3.19. *There is no Borel subgroup of G both containing T and meeting $N(T) \setminus C^\circ(T)$.*

Proof. Suppose there is a Borel subgroup B_0 of G containing both T and some $c \in N(T) \setminus C(T)$. As B_0 is connected and solvable, $N_{B_0}(T) = C_{B_0}(T)$ by Fact 2.3. So c centralizes T, a contradiction. □

Claim 3.20. *$a^g \notin N(B)$.*

Proof. Here $a^g \notin B$ because $a \notin B^{g^{-1}}$ by Claim 3.19. So $a^g \notin N(B)$ by Fact 2.9. □

Lemma 3.21. *The Prüfer p-rank $\mathrm{pr}_q(T_1)$ of T_1 is at least 3.*

Proof. Suppose first that $\mathrm{pr}_q(T_1) = 1$. Then $a^g \in N(T_1)$ normalizes $\langle t \rangle$. As B is the only Borel subgroup containing $C^\circ(t)$, a^g also normalizes B, in contradiction with Claim 3.20.

Suppose next that $\mathrm{pr}_q(T_1) = 2$. We have $p > q+1$ because $q|p-1$ and $q \neq 2$. So there is no injective homomorphisms from $\mathbb{Z}/p\mathbb{Z}$ to $\mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z})$. So a^g must centralize t, again contradicting Claim 3.20. □

We now derive our final contradiction. By the following, there is an elementary abelian q-group $E_0 \leq E := \Omega_1(T)$ with $m_q(E_0) \geq 2$ such that $C_{U_p(B)}(E_0) \neq 1$.

Fact 3.22 ([Bur04, Fact 3.7])**.** *Let q be a prime number. Let H be a solvable $q^\perp$-group of finite Morley rank. Let E be a finite elementary abelian q-group acting definably on H. Then*

$$H = \langle C_H(E_0) : E_0 \leq E, [E : E_0] = q \rangle.$$

By Fact 3.22 again, there is a $v \in E_0^\#$ such that $C_{U_p(B^a)}(v) \neq 1$ too. But now $U_p(C^\circ(v))$ meets both $U_p(B)$ and $U_p(B^a)$ nontrivially, a contradiction to Fact 2.6.

This concludes the proof of theorem 3.1.

4 Possible Prüfer p-ranks

We're aware, thanks to Fact 2.5, that a nontrivial Weyl group W produces p-unipotence for some divisor p of $|W|$; making p-tori seem unlikey. Here we show that such a p-torus has only a limited selection of the Prüfer p-ranks.

Theorem 4.1. *Let G be a minimal connected simple group of finite Morley rank, and let T be a nontrivial p-torus of G. Suppose that p is a prime divisor of $|W_T|$. Then $p-1$ divides the Prüfer p-rank $\mathrm{pr}_p(T)$ of T, and p is not the minimal prime divisor of $|W|$ unless $p = 2$.*

The proof entirely relies on the following observation.

Lemma 4.2. *Let T be a p-torus and α be an automorphism of order p of T. Then:*

- *T has Prüfer-rank at least $p-1$;*
- *if T has Prüfer-rank p and lives inside a group of finite Morley rank in which α is a definable automorphism of T, then $C_T(\alpha)$ is infinite.*

Here we apply the Tate module construction of a complex representation for our Weyl group.

Fact 4.3 ([Ber01, BB04, §3.3]). *Let T be a p-torus of Prüfer p-rank n. Then* $\mathrm{End}(T)$ *can be faithfully represented as the ring* $M_{n\times n}(\mathbb{Z}_p)$ *of* $n \times n$ *matrices over the p-adic integers* $\mathbb{Z}_p$.

Proof of Lemma 4.2. Recall that $1+X+\cdots+X^{p-1}$ is irreducible in $\mathbb{Q}_p[X]$. In the Tate module, α is represented by a matrix $M \in \mathrm{GL}_d(\mathbb{Z}_p) \leq \mathrm{GL}_d(\mathbb{Q}_p)$ where $d = \mathrm{pr}_p(T)$. As $M^p = 1$, the minimal polynomial μ of M must divide $X^p - 1$. Since $\alpha \neq \mathrm{Id}$, we have $M \neq \mathrm{Id}$, so $\mu \neq X - 1$. Therefore $\mu \neq X - 1$ divides $X^p - 1 = (X-1)(1+X+\cdots+X^{p-1})$. As $1+X+\cdots+X^{p-1}$ is irreducible over $\mathbb{Q}_p$, it follows that $1+X+\cdots+X^{p-1}$ divides μ.

So the minimal polynomial has degree at least $p-1$. By the Cayley-Hamilton Theorem, so must the characteristic polynomial; but the latter has degree d, whence $d \geq p-1$. This proves the first claim.

Also, if $d = p$, then the characteristic polynomial is $(1+X+\cdots+X^{p-1})(X-a)$ for some element $a \in \mathbb{Q}_p$, which is an eigenvalue of M. So a has multiplicative order p in $\mathbb{Q}_p$, and this proves $a = 1$. Let x be an eigenvector for the eigenvalue 1. We may assume that x lies in $\mathbb{Z}_p^d \setminus \mathbb{Z}^d$.

Now projecting, we deduce that at every stage $(\mathbb{Z}/p^n\mathbb{Z})^d$, α centralizes an element x_n of order p^n. By compactness, α centralizes some element of infinite order. Therefore $C_T(\alpha)$ is infinite. □

We now begin the proof of Theorem 4.1. Let T be a p-torus where p divides $|W_T|$. Let g be a p-element of $N(T)$ that has order p in W_T, i.e. $g^p \in C(T)$.

Claim 4.4. *$C_T(g)$ is finite.*

Proof. If not, then $\tau = C_T^\circ(g)$ is a non-trivial subtorus of T. In particular, Fact 2.1 says that $C(\tau)$ is a connected solvable group. Since it contains the p-group $T \cdot \langle g \rangle$, we find $g \in C(T)$, in contradiction with Fact 2.3. □

Let $T_0 \leq T$ be a subtorus of Prüfer p-rank exactly 1. Let $T_1 = T_0 + T_0^g + \cdots + T_0^{g^{p-1}}$. This is a non-trivial g-invariant p-torus of Prüfer rank at most p.

Claim 4.5. *T_1 has Prüfer rank $p-1$.*

Proof. If either $\mathrm{pr}_p(T_1) < p-1$ or $\mathrm{pr}_p(T_1) = p$, then Lemma 4.2 says that g centralizes a non-trivial subtorus of T_1. But this contradicts Claim 4.4. □

Claim 4.6. *There is a g-invariant subtorus $T' < T$, possibly trivial, with Prüfer-rank* $\mathrm{pr}_p(T') = \mathrm{pr}_p(T) - (p-1)$.

Proof. Since we're working with an automorphism of order p, the intersection $T_1 \cap T'$ may be non-trivial but remains finite. The rank computation follows as in Maschke's theorem. [2] □

[2] I've not actually 100% sure about this comment anyway, change as you like.

It follows, by induction on $\mathrm{pr}_p(T)$, that $T^{(k)} = 0$ eventually. So $\mathrm{pr}_p(T) = k(p-1)$ as desired.

This proves the first part of Theorem 4.1.

We prove the second as follows.

Proposition 4.7. *Let G be a minimal connected simple group of finite Morley rank, let T be a decent torus of G, and let $a \in N(T) \setminus C(T)$ be a p-element. Then either $\mathrm{pr}_p(T) = 0$ or $U_p(C(a)) = 1$.*

Of course both are impossible together by [BC08, Theorem 3].

Proof. Otherwise suppose that both $\mathrm{pr}_p(T) = 0$ and $U_p(C(a)) = 1$ hold. As $U_p(C(a)) \neq 1$, $C^\circ(a)$ lies inside a unique Borel subgroup B_a by Fact 2.6. Let T_p be the p-torus of T, and its nontrivial. Then there is some nontrivial $z \in Z(\langle a \rangle T_p) \cap T_p$ by [BN94, ???]. So z normalizes $C^\circ(a)$, and hence B_a. Hence $z \in B_a$ by Fact 2.9, and $U_p(C_{B_a}(z)) \neq 1$ too. But $T \leq C(z)$. So $U_p(C(T_p)) \neq 1$. Again, by Fact 2.6, $C(T_p)$ lies inside a unique Borel subgroup B_z, and $a \in B_z$ by Fact 2.9. But $N_{B_z}(T) = C_{B_z}(T)$ by Fact 2.3, a contradiction. □

Corollary 4.8. *Let G be a minimal connected simple group whose Weyl group W has odd order. Then G has no divisible p-torsion for p the minimal prime divisor of $|W|$.*

The corollary is a direct consequence of the following variation on Fact 2.5 above.

Fact 4.9 ([BC08, Corollary 5.3])**.** *Let G be a minimal connected simple group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then any r-element representing an element of order r in W centralizes a unipotent r-subgroup.*

This proves the second part of Theorem 4.1.

Acknowledgments

The authors would like to thank Greg Cherlin and Tuna Altınel for their helpful comments and advice.

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