Fields interpretable in rosy theories

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Abstract

We are working in a monster model $\mathfrak{C}$ of a rosy theory T. We prove the following theorems, generalizing the appropriate results from the finite Morley rank case and o-minimal structures. If R is a $\bigvee$-definable integral domain of positive, finite $\mathrm{U}^{\mathrm{þ}}$-rank, then its field of fractions is interpretable in $\mathfrak{C}$. If A and M are infinite, definable, abelian groups such that A acts definably and faithfully on M as a group of automorphisms, M is A-minimal and $\mathrm{U}^{\mathrm{þ}}(M)$ is finite, then there is an infinite field interpretable in $\mathfrak{C}$. If G is an infinite, solvable but non nilpotent-by-finite, definable group of finite $\mathrm{U}^{\mathrm{þ}}$-rank and T has NIP, then there is an infinite field interpretable in $\langle G, \cdot \rangle$.

In the last part, we study infinite, superrosy, dependent fields. Using measures, we show that each such field K satisfies $K = K^n - K^n$ for every $n \geq 1$.

0 Introduction

An important goal in model theory is to obtain, in a definable way, classical algebraic structures in theories satisfying some general model theoretic or algebraic assumptions. There is a long history of results of this kind, e.g. different versions of the group configuration theorem (originally proved by E. Hrushovski, see [6, Chapters 5 and 7]), getting fields from definable actions of abelian groups in the finite Morley rank case [8, Chapter 3] or in o-minimal structures [5], or getting fields from solvable, non nilpotent-by-finite groups in the finite Morley rank case [8, Corollary 3.20] and from any non abelian-by-finite groups in o-minimal structures [5, Corollary 5.1].

The goal of this paper is to generalize some of the results about the existence of an infinite field to a general context of rosy theories. We also try to understand the structure of superrosy fields satisfying NIP.

Let $\mathfrak{C}$ be a monster model of a rosy theory T. We always work in $\mathfrak{C}^{eq}$.

At the beginning of Section 2, we generalize [5, Lemma 4.1]. Namely, we prove

Theorem 1 *Let R be a $\bigvee$-definable integral domain of positive, finite $U^{þ}$-rank. Then the field of fractions of R, call it F, is interpretable in $\mathfrak{C}$. Moreover, there is a $\bigvee$-definable ring embedding of R onto a subring of F with the same $U^{þ}$-rank as F.*

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In [5], this was proved in the o-minimal context using essentially certain topology on $\bigvee$-definable rings, which was defined by means of the standard o-minimal topology. In our case, we give a very general proof, which just uses some properties of $\mathrm{U}^{\mathrm{þ}}$-rank, and that is why it works in any situation where we have a nice notion of dimension, for example in stable or simple theories (the dimension there is SU-rank).

In the further part of Section 2, we prove some variants of [8, Theorem 3.7] and [5, Lemma 4.2] in our general rosy context. For example, we prove

Theorem 2 *Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M is A-minimal and $U^{þ}(M)$ is finite. Then there is an infinite field interpretable in $\mathfrak{C}$.*

The proofs of these results use Zilber's Indecomposables Theorem and chain conditions in the finite Morley rank case, and the topology in the o-minimal case. Neither of these tools are present in our situation. Our proofs rely on Theorem 1 and some tricks eliminating applications of topology or chain conditions on intersections of uniformly definable groups.

In Section 3, we prove the main result of this paper.

Theorem 3 *Let G be a group of finite $U^{þ}$-rank definable in $\mathfrak{C}$ and suppose that T has NIP. Assume that G is solvable-by-finite but not nilpotent-by-finite. Then there is an infinite field interpretable in $\langle G, \cdot \rangle$.*

Similarly to the finite Morley rank case, in the proof of Theorem 3 we use Theorem 2. But the proof in our context is different.

In the last section we prove some partial results concerning the following conjecture formulated in [1].

Conjecture 4 *Each infinite, superrosy field with NIP is either a real closed or an algebraically closed field.*

In particular, we show that each such field has the property $K = K^n - K^n$ for every $n \geq 1$. In our proofs we use absolute connected components and Kiesler measures.

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1 Preliminaries

Let $\mathfrak{C}$ be a monster model of a theory T. We work in $\mathfrak{C}^{eq}$. Our general assumption in this paper is that T is rosy, but often, we only need a nice notion of dimension. In the rosy (or rather superrosy) context such a notion of dimension is $\mathrm{U}^{\mathrm{þ}}$-rank.

The most interesting examples of rosy theories are simple theories and o-minimal theories.

We are not going to repeat the basic definitions of rosiness, þ-forking and $\mathrm{U}^{\text{þ}}$-rank. The fundamental paper about these notions is [4]. For the basic theory of rosy groups, the reader is referred to [1] where a short exposition of rosy theories and þ-forking is also given. Let us only recall here that if T is rosy, then þ-independence, denoted by $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}^{\text{þ}}$, is a ternary relation on subsets of $\mathfrak{C}^{eq}$, which satisfies all the properties of forking independence in simple theories except for the Independence Theorem. Using $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}^{\text{þ}}$, we define $\mathrm{U}^{\text{þ}}$-rank in the same way as U-rank is defined in stable theories by means of $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$. $\mathrm{U}^{\text{þ}}$-rank has most of the nice properties that U-rank has in stable theories, e.g. it satisfies Lascar Inequalities. If D is an A-definable set, then $\mathrm{U}^{\text{þ}}(X) := sup\{\mathrm{U}^{\text{þ}}(d/A) : d \in D\}$. Of course, if this supremum is finite, then it is just the maximum. It turns out that if D is a definable group and T is rosy, then the supremum is also attained [1, Remark 1.20].

The following fact follows from a standard application of Lascar Inequalities.

Fact 1.1 *If K is a definable field and T is superrosy, then for every $n > 0$, $[K^* : (K^*)^n] < \omega$ and, if $char(K) = p \neq 0$, then the range of the function $f : K \to K$ defined by $f(x) = x^p - x$ is a subgroup of finite index in K^+.*

Another property of T that we sometimes assume is the non independence property (NIP), also called T being dependent.

Definition 1.2 *We say that T has the NIP if there is no formula $\varphi(x, y)$ and sequence $\langle a_i \rangle_{i<\omega}$ such that for every $w \subseteq \omega$ there is b_w such that $\models \varphi(a_i, b_w)$ iff $i \in w$.*

We will need the following consequence of NIP proved by Shelah (for the proof see e.g. [2, Theorem 6.1]).

Fact 1.3 *If G is a (type-)definable group and T has NIP, then G^{00} (the smallest type-definable subgroup of bounded index) exists.*

Another important for this paper consequence of NIP and rosiness is (see [1, Proposition 1.7]):

Fact 1.4 *Suppose T is rosy and has NIP. Any group G definable in T has icc, i.e. the uniform chain condition on intersections of uniformly definable groups.*

We also have the following easy observation [1, Proposition 4.1].

Fact 1.5 *Let K be an infinite, definable field. If $(K^+)^{00}$ exists, then $(K^+)^{00} = K$. In particular, if T has NIP, then $(K^+)^{00} = K$.*

Recall that a (global) Kiesler measure on a definable set D is a finitely additive probability measure on definable subsets of D. We say that a Kiesler measure μ on X is definable (over A) if for each formula $\varphi(x, y)$ and closed subset C of $[0, 1]$, $\{b \in \mathfrak{C} : \mu(\varphi(D, b)) \in C\}$ is type-definable (over A). We say that a definable group G is definably amenable if there is a left invariant Kiesler measure on G.

In the last section, we will need the following fact [3, Lemma 5.8].

Fact 1.6 *Assume that T has NIP and G is a definably amenable, definable group. Then there is a left invariant, definable Kiesler measure on G.*

Now let us recall the definition of a $\bigvee$-definable ring from [5].

Definition 1.7 *We say that a ring $\langle R, \cdot, + \rangle$ is a $\bigvee$-definable (or rather $\bigvee$-interpretable) ring if $R = \bigcup_{i \in I} X_i$ where X_i's are A-definable subsets of some sort of $\mathfrak{C}^{eq}$ for some set A, for every $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$, and the restrictions of addition and multiplication to $X_i \times X_j$ are definable functions.*

The assumption that for every $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$ is purely cosmetic, because we can always extend the family $\{X_i : i \in I\}$ by adding all unions of finitely many X_i's.

By the compactness theorem, if D is a definable subset of X, then it is covered by finitely many X_i's (so, if fact, by one X_i) and hence addition and multiplication restricted to D are definable.

We define similarly $\bigvee$-definable groups or just sets. $\bigvee$-definable groups and rings occur naturally as subgroups and subrings generated by definable subsets of definable groups and rings.

If $X = \bigcup_{i \in I} X_i$ is a $\bigvee$-definable set, then $\mathrm{U}^{\mathrm{þ}}(X)$ is defined as the supremum of $\mathrm{U}^{\mathrm{þ}}$-ranks of the X_i's. If this supremum is finite, then $\mathrm{U}^{\mathrm{þ}}$-ranks of all X_i's are finite and $\mathrm{U}^{\mathrm{þ}}(X)$ is just the maximum of $\mathrm{U}^{\mathrm{þ}}$-ranks of the X_i's. Notice that in a rosy theory, by the compactness theorem, $\mathrm{U}^{\mathrm{þ}}(X)$ is the supremum of $\mathrm{U}^{\mathrm{þ}}$-ranks of all definable subsets of X.

Definition 1.8 *If $R_1 = \bigcup_{i \in I} X_i$ and $R_2 = \bigcup_{j \in J} Y_j$ are $\bigvee$-definable rings, then a homomorphism $f : R_1 \to R_2$ is called $\bigvee$-definable if its restriction to each X_i is definable.*

Another notion needed is G-minimality. If G is a definable group acting definably on another definable group H by automorphisms, then we say that H is G-minimal if H is infinite and does not have infinite, proper, definable subgroups invariant under the action of G.

2 Getting fields from $\bigvee$-definable rings and definable actions of abelian groups

In this section, we generalize some results from Section 4 of [5] and [8, Theorem 3.7]. The main obstacle here in comparison with the o-minimal case is that we do not have a nice topology, and with the finite Morley rank case, that we do not have Zilber's Indecomposables Theorem.

We work in a monster model $\mathfrak{C}$ of a rosy theory T.

The following theorem was proved in the o-minimal context [5, Lemma 4.1] using a nice topology on $\bigvee$-definable rings. Here we give a very general proof, which works in any context in which we have a nice notion of dimension, e.g. in simple and in o-minimal structures.

Theorem 2.1 *Let R be a $\bigvee$-definable integral domain of positive, finite $U^{\flat}$-rank. Then the field of fractions of R, call it F, is interpretable in $\mathfrak{C}$. Moreover, there is a $\bigvee$-definable ring embedding of R onto a subring of F with the same $U^{\flat}$-rank as F.*

Proof. Since R is $\bigvee$-definable, we have that $R = \bigcup_{i \in I} X_i$ where all X_i's are sets definable over some set A, for any $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$, and the restrictions of addition and multiplication to any $X_i \times X_j$ are definable. So for every $r \in R$ the map $f_r : R \to R$ given by $f_r(x) = rx$ restricted to any X_i is definable.

Now let $D := X_i$ be such that $\mathrm{U}^{\flat}(D) = \mathrm{U}^{\flat}(R)$.

Claim 1 *For any $a, b \in R \setminus \{0\}$, $(Da - Da) \cap (Db - Db) \neq \{0\}$.*

Proof of Claim 1. Consider the function $f : D \times D \to R$ defined by

$$f(r_1, r_2) = r_1 a + r_2 b.$$

As $f_a \restriction D$, $f_b \restriction D$ and $+$ restricted to any definable subset of R are definable, so is f. But $\mathrm{U}^{\flat}(D \times D) = 2\mathrm{U}^{\flat}(D) > \mathrm{U}^{\flat}(D) = \mathrm{U}^{\flat}(R)$, hence by Lascar inequalities we easily get that f is not injective. Thus, there are two distinct pairs $(r_1, r_2), (r_1', r_2') \in D \times D$ such that $r_1 a + r_2 b = r_1' a + r_2' b$. So $(r_1 - r_1')a = r_1 a - r_1' a = r_2' b - r_2 b = (r_2' - r_2)b$. We know that $a, b \neq 0$ and at least one of the elements $r_1 - r_1'$ and $r_2' - r_2$ is nonzero. Hence the element $r_1 a - r_1' a$ is nonzero and, of course, it belongs to $(Da - Da) \cap (Db - Db)$. □

Choose any $a \in R \setminus \{0\}$ and put $X = Da - Da$.

Claim 2 *For any $r_1, r_2 \in R \setminus \{0\}$, $r_1 X \cap r_2 X \neq \{0\}$.*

Proof of Claim 2. Since R is commutative, $r_i X = r_i Da - r_i Da = D(r_i a) - D(r_i a)$. As r_1, r_2 and a are nonzero, we see that $r_1 a, r_2 a \in R \setminus \{0\}$. So by Claim 1, we get $r_1 X \cap r_2 X = (D(r_1 a) - D(r_1 a)) \cap (D(r_2 a) - D(r_2 a)) \neq \{0\}$. □

The rest of the proof that the field of fractions of R is interpretable is the same as in the proof of [5, Lemma 4.1]. Namely, the fraction field F equals $(R \times (R \setminus \{0\}))/\sim$ where $(r_1, s_1) \sim (r_2, s_2) \iff r_1 s_2 = r_2 s_1$. By Claim 2, F can be indentified with $(X \times (X \setminus \{0\}))/\sim_X$ where $\sim_X$ is the restriction of $\sim$ to $X \times (X \setminus \{0\})$, and obviously $\sim_X$ is definable. Since addition and multiplication in R restricted to any definable subset are definable, we easily get that addition and multiplication in F are definable. Hence F is an interpretable field. If we fix a nonzero $r_0 \in R$, then the map $r \mapsto (rr_0, r_0)/\sim$ gives us a $\bigvee$-definable embedding of R into F.

The fact that $\mathrm{U}^{\flat}(F) = \mathrm{U}^{\flat}(R)$ requires an extra explanation. Since R is $\bigvee$-definably embeddable in F, we easily get that $\mathrm{U}^{\flat}(R) \leq \mathrm{U}^{\flat}(F)$. Let B be a set containing $A \cup \{a\}$ and such that addition and multiplication restricted to any X_i are definable over B. Then F is interpretable over B. Now consider $(r_1, r_2) \in X \times (X \setminus \{0\})$

such that $U^{\flat}([(r_1,r_2)]_{\sim_X}/B) = U^{\flat}(F)$. We need to show that $U^{\flat}([(r_1,r_2)]_{\sim_X}/B) \leq U^{\flat}(R)$. Since the function $r \mapsto (r_1 r, r_2 r)$ from $R \setminus \{0\}$ to $R \times (R \setminus \{0\})$ is a $\bigvee$-definable injection and its range is contained in the $\sim$-class of (r_1,r_2) (computed in $R \times (R \setminus \{0\})$), there is $j \in I$ such that $X \subseteq X_j$ and $U^{\flat}$-rank of the $\sim_{X_j}$-class of (r_1,r_2) (treated as a subset of $X_j \times (X_j \setminus \{0\})$) is at least $U^{\flat}(R)$. Since $(X \times (X \setminus \{0\}))/{\sim_X}$ can be B-definably identified with $(X_j \times (X_j \setminus \{0\}))/{\sim_{X_j}}$, we can work in $(X_j \times (X_j \setminus \{0\}))/{\sim_{X_j}}$. Let $d = [(r_1,r_2)]_{\sim_{X_j}} \in (X_j \times (X_j \setminus \{0\}))/{\sim_{X_j}}$. So there is $(r_1', r_2') \sim_{X_j} (r_1,r_2)$ in $X_j \times (X_j \setminus \{0\})$ such that $U^{\flat}((r_1',r_2')/B,d) \geq U^{\flat}(R)$. Since $d \in dcl(r_1',r_2',B)$, by Lascar Inequalities, we get $2U^{\flat}(R) \geq U^{\flat}((r_1',r_2')/B) = U^{\flat}((r_1',r_2'),d/B) \geq U^{\flat}((r_1',r_2')/B,d) + U^{\flat}(d/B) \geq U^{\flat}(R) + U^{\flat}(d/B)$. So $U^{\flat}(d/B) \leq U^{\flat}(R)$. ■

Notice that in the above theorem the assumption that R is of positive $U^{\flat}$-rank is necessary. Indeed, if $\mathfrak{C}$ is a real closed field, then $\mathbb{Q}$ is a $\bigvee$-definable field of $U^{\flat}$-rank 0 and it is not interpretable in $\mathfrak{C}$.

Now we are going to generalize some classical results about getting fields from definable actions of abelian groups [8, Theorem 3.1], [5, Lemma 4.2]. As in the o-minimal case, we cannot apply the method from the finite Morley rank case because we do not have Zilber's Indecomposables Theorem. But, as in [5], we can apply Theorem 2.1. Once again, we give here general proofs which omit any applications of o-minimal topology or chain conditions.

Theorem 2.2 *Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M is A-minimal and $U^{\flat}(M)$ is finite. Then there is an infinite field interpretable in $\mathfrak{C}$.*

Proof. For $a \in A$ we define $Fix(a) = \{m \in M : am = m\}$ and for $m \in M$ we put $Stab(m) = \{a \in A : am = m\}$. Of course, $Fix(a)$ and $Stab(m)$ are definable subgroups of M and A, respectively.

Claim 1 *There are $m_1, \dots, m_n \in M$ such that $Stab(m_1) \cap \dots \cap Stab(m_n) = \{e\}$.*

Proof of Claim 1. For every $a \in A \setminus \{e\}$, $Fix(a)$ is a proper, definable subgroup of M invariant under the action of A. So by A-minimality of M, $Fix(a)$ is finite. Hence for any infinite subset S of M, $\bigcap_{m \in S} Stab(m) = \{e\}$. Thus, by the compactness theorem, there are $m_1, \dots, m_n \in M$ such that $Stab(m_1) \cap \dots \cap Stab(m_n) = \{e\}$. □

Let R be the ring of endomorphisms of M generated by A. Then R is commutative.

Notice that every $r \in R$ is determined by $(r(m_1), \dots, r(m_n))$. If not, then there is $r \in R \setminus \{0\}$ such that $r(m_1) = \dots = r(m_n) = 0$. Since R is commutative, we get that $ker(r)$ is a proper, definable and invariant under the action of A subgroup of M containing $\{m_1, \dots, m_n\}$. So $Am_1 + \dots + Am_n \subseteq ker(r)$. On the other hand, by choice of $m_1, \dots, m_n$, we get that the function $a \mapsto (am_1, \dots, am_n)$ is an injection from A to M^n. So there is i such that Am_i is infinite, and hence $ker(r)$ is infinite. This contradicts the assumption that M is A-minimal.

Having the above observation, we get the following in a rather standard way.

Claim 2 *The ring R is $\bigvee$-definable, contained in M^n with the addition inherited from M^n, and $0 < U^\flat(R) < \omega$.*

Proof of Claim 2. Let $H = \langle A(m_1, \dots, m_n)\rangle$. By the above observation, the function $f : R \to H \subseteq M^n$ defined by $f(r) = (r(m_1), \dots, r(m_n))$ is a bijection. Of course, $H := \bigcup_{i<\omega} X_i$ where $X_i = \pm A(m_1, \dots, m_n) \pm \dots \pm A(m_1, \dots, m_k)$ (i-many times). So H is a $\bigvee$-definable subgroup of M^n.

By the definition of f, we see that for any $r_1, r_2 \in R$ we have $f(r_1 + r_2) = f(r_1) + f(r_2)$ ($+$ on the left hand side is the addition in R and $+$ on the right hand side is the addition in M^n).

Now we define multiplication, $*$, on H to make f a ring isomorphism, i.e. $f(r_1) * f(r_2) := f(r_1 r_2)$ for all $r_1, r_2 \in R$. We leave as an easy exercise to check that $* : H \times H \to H$ is $\bigvee$-definable, i.e. for any $i, j < \omega$, $* : X_i \times X_j \to M^n$ is definable. Of course, $0 < \mathrm{U}^\flat(Am_i) \leq \mathrm{U}^\flat(H) \leq \mathrm{U}^\flat(M^n) < \omega$. □

The next claim has the same proof as in the finite Morley rank case.

Claim 3 *R is an integral domain.*

Proof of Claim 3. Take any $r_1, r_2 \in R$ such that $r_1 r_2 = 0$. If $r_2 \neq 0$, then $ker(r_2)$ is a proper, definable subgroup of M invariant under the action of A. So by A-minimality of M, $ker(r_2)$ is finite. So $rng(r_2)$ is an infinite, definable subgroup of M invariant under the action of A. Thus $rng(r_2) = M$. So we get $r_1 = 0$. □

By Claims 2, 3 and Theorem 2.1, we get an infinite field interpretable in $\mathfrak{C}$. ■

Assuming that M does not have nontrivial, proper, definable subgroups invariant under the action of A, we get even more specific information about our interpretable field.

Proposition 2.3 *Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M does not have any nontrivial, proper, definable subgroups invariant under the action of A and $U^\flat(M)$ is finite. Then for every nonzero $m \in M$ there is a field K definable in $\mathfrak{C}$ whose underlying additive group is $\langle M, +\rangle$, and $\langle A, \cdot\rangle$ is definably embeddable in K^* by sending $a \in A$ to am. After the embedding, the action of A on M becomes the scalar multiplication.*

Proof. Let R be the ring of endomorphisms of M generated by A. We easily see that every nonzero $r \in R$ is an automorphism of M. Indeed, since $ker(r)$ is a proper, definable subgroup of M invariant under the action of A, it must be trivial. So $rng(r)$ is an infinite, definable subgroup of M invariant under the action of A, and hence it is equal to M.

Choose a nonzero $m \in M$. We conclude that every element $r \in R$ is determined by $r(m)$. So by the proof of Theorem 2.2, R is $\bigvee$-definable (after the indentification of every $r \in R$ with $r(m) \in M$), contained in M with the addition inherited from M, and the field of fractions, F, of R is interpretable in $\mathfrak{C}$. More precisely, $F = (X \times (X \setminus \{0\}))/\sim$ where X is a definable subset of R.

The rest is the same as in the last paragraph of the proof of [5, Lemma 4.2]. Every element $(\alpha, \beta)/\sim \in F$ can be identified with the automorphism $\alpha\beta^{-1}$ of M. So F is a field of automorphisms of M. We easily see that the action of F on M is definable. As above, we show that every element $k \in F$ is determined by $k(m)$. Hence F can be definably embedded into M by sending $k \in F$ to $k(m)$. The range of this map is a definable field, say K, whose additive group is a subgroup of M invariant under A, so it must be M. Of course, A is definably embeddable in K^* by sending $a \in A$ to am. The fact that after this embedding the action of A on M coincides with the field multiplication is trivial. ■

Using Proposition 2.3, we obtain the following strengthening of Theorem 2.2.

Corollary 2.4 *Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M is A-minimal and $U^{\flat}(M)$ is finite. Then there is an infinite field K interpretable in $\mathfrak{C}$ whose underlying additive group is M/M_0 for some finite subgroup M_0 of M invariant under A, and A/A_0 is definably embeddable in K^* for some finite subgroup A_0 of A. In fact, the action of A on M induces a faithful and definable action of A/A_0 on M/M_0 by automorphisms, and after the embedding this action becomes the scalar multiplication.*

Proof. By Proposition 2.3, in order to prove the corollary, it is enough to find a finite subgroup M_0 of M which is invariant under the action of A, and a finite subgroup A_0 of A such that A/A_0 acts faithfully and definably on M/M_0 as a group of automorphisms and M/M_0 does not have nontrivial, proper, definable subgroups invariant under A/A_0.

Define $M_0 = \{m \in M : [A : Stab(m)] < \omega\}$. Of course, M_0 is a subgroup of M invariant under A. We claim that M_0 is finite (and hence definable). If not, there is an infinite, countable set S contained in M_0. Then $\bigcap_{m \in S} Stab(m)$ is a nontrivial (in fact, of bounded index) subgroup of A. So there is a nontrivial $a \in \bigcap_{m \in S} Stab(m)$, which means that $S \subseteq Fix(a)$, a contradiction with the fact that $Fix(a)$ is finite.

Since M_0 is invariant under A, the action of A on M induces an action of A on M/M_0. It is easy to see that A acts on M/M_0 by automorphisms.

Define A_0 as the set of those $a \in A$ which act as the trivial automorphism on M/M_0. Then A_0 is a subgroup of A. We claim that it is finite. Indeed, by Claim 1 in the proof of Theorem 2.2, there are $m_1, \dots, m_n \in M$ such that $Stab(m_1) \cap \dots \cap Stab(m_n) = \{e\}$. So every $a \in A$ is determined by $(am_1, \dots, am_n)$. On the other hand, if a induces the trivial automorphism of M/M_0, then $am_1 \in m_1 + M_0, \dots, am_n \in m_n + M_0$. Since M_0 is finite, we get only finitely many possibilities for $a \in A$ inducing the trivial automorphism of M/M_0, i.e A_0 is finite.

Summarizing, we get that A/A_0 acts faithfully and definably on M/M_0 as a group of automorphisms. It remains to check that M/M_0 does not have nontrivial, proper, definable subgroups invariant under A/A_0. Consider any definable subgroup G of M/M_0 invariant under A/A_0 and let $M_1 < M$ be the preimage of G under the quotient map. We see that M_1 is a definable subgroup of M invariant under A. So either $M_1 = M$, and then $G = M/M_0$, or M_1 is finite. In the second case, for any $m \in M_1$ the orbit $Am \subseteq M_1$ is finite so $[A : Stab(m)] < \omega$, i.e. $m \in M_0$; hence $M_0 = M_1$, which means that G is trivial. ■

3 Getting fields in solvable non-nilpotent groups

In this section we prove the main result of the paper.

Theorem 3.1 *Let G be a group of finite $U^þ$-rank definable in a monster model of a rosy theory satisfying NIP. Assume that G is solvable-by-finite but not nilpotent-by-finite. Then there is an infinite field interpretable in $\langle G, \cdot \rangle$.*

Before we start to prove the theorem, let us show the following general lemma and a standard remark.

Lemma 3.2 *Suppose P and Q are infinite abelian groups, P acts on Q by automorphisms and for every $p \in P \setminus \{e_P\}$ and $q \in Q \setminus \{e_Q\}$, $p \cdot q \neq q$. Then $Q \rtimes P$ is solvable but not nilpotent-by-finite.*

Proof. Solvability is obvious. Suppose for a contradiction that $Q \rtimes P$ is nilpotent-by-finite. Then there are subgroups P_1 and Q_1 of finite index in P and Q, respectively, such that the restriction of the action of P on Q gives us an action by automorphisms of P_1 on Q_1 satisfying the property $(\forall p \in P_1 \setminus \{e_P\})(\forall q \in Q_1 \setminus \{e_Q\})(p \cdot q \neq q)$, and moreover $Q_1 \rtimes P_1$ is nilpotent. So wlog $Q \rtimes P$ is nilpotent. To get a contradiction, it is enough to show that $Z(Q \rtimes P)$ is trivial.

We can identify Q with $Q \times \{e_P\} < Q \rtimes P$ and P with $\{e_Q\} \times P < Q \rtimes P$. After this identification $Q \rtimes P = QP$. Let e be the neutral element of $Q \rtimes P$. By assumption, for all $p \in P \setminus \{e\}$ and $q \in Q \setminus \{e\}$ we have $pqp^{-1}q^{-1} = (p \cdot q)q^{-1} \neq qq^{-1} = e$.

Take any $qp \in Z(Q \rtimes P)$ where $p \in P$ and $q \in Q$. Then $qpq(qp)^{-1}q^{-1} = e$ so $pqp^{-1}q^{-1} = e$. By the last paragraph, we get $p = e$ or $q = e$. But once again using the last paragraph, we also see that $P \cap Z(Q \times P) = Q \cap Z(Q \rtimes P) = \{e\}$. So $p = e$ and $q = e$. Thus we have proved that $Z(Q \rtimes P) = \{e\}$. ■

Remark 3.3 *(i) Let G be a group such that all definable quotients of definable subgroups of G have icc on centralizers. Assume that G is solvable-by-finite. Then G has a definable, solvable subgroup H of finite index, and H has a normal sequence $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ such that each quotient H_{i+1}/H_i is abelian and each H_i is definable.*
(ii) Let G be a group such that all definable quotients of definable subgroups of G have icc on centralizers. Assume that N is a nilpotent subgroup of G. Then G has a definable nilpotent subgroup H containing N. Thus, the upper central series of H consists of definable subgroups of G.

Proof. (i) By a standard trick, there is a normal, solvable subgroup L of finite index in G. Then the derivative sequence of L, call it $\{e\} = L_0 \lhd L_1 \lhd \cdots \lhd L_n = L$, consists of normal subgroups of G. Now we define a sequence $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n$ of definable, normal subgroups of G with abelian quotients, and such that $H_i > L_i$ for every $0 \leq i \leq n$.

H_0 is defined as $\{e\}$. Suppose $H_0, \ldots, H_i$ satisfying all the above assumptions have been constructed. Then we define $H_{i+1} = \pi_i^{-1}[Z(C(L_{i+1}H_i/H_i))]$ where $\pi_i : G \to G/H_i$ is the natural quotient map. Using icc on centralizers, one can easily check that this construction works.

Now $H := H_n$ together with the sequence $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ have the desired properties.

(ii) The proof is the same as in the stable case [8, Theorem 3.17], by induction on the nilpotent class. If N is abelian, then $H = Z(C(N))$ works. For the induction step, let $Z = Z(C(Z(N)))$. By icc on centralizers, Z is definable, abelian, it is centralized by N and $Z(N) < Z$. Hence $NZ/Z \cong N/(N \cap Z)$ is a nilpotent subgroup of $C(Z)/Z$ of a smaller class of nilpotency than N so by induction hypothesis, there is a definable, nilpotent subgroup of $C(Z)/Z$ containing NZ/Z. Then its preimage under the natural quotient map is a definable, nilpotent subgroup of G containing N. ■

Proof of Theorem 3.1. By Remark 3.3, we can assume that G is solvable and it has a normal sequence consisting of definable subgroups with abelian quotients. It is also clear that we can assume that $(G, \cdot)$ is our monster model.

The proof is by induction on $\mathrm{U}^{\text{þ}}(G)$. In fact, in the paragraph below we will show that our assumptions on G imply that $\mathrm{U}^{\text{þ}}(G) \geq 2$. The fact that G is infinite follows immediately from the assumption that G is not nilpotent-by-finite.

Assume that the theorem is true for groups of $\mathrm{U}^{\text{þ}}$-rank smaller that $\mathrm{U}^{\text{þ}}(G)$. By icc on centralizers, we can assume that G is centralizer connected. If $Z(G)$ is infinite, then since G is not nilpotent-by-finite, $G/Z(G)$ is a non nilpotent-by-finite, solvable group of $\mathrm{U}^{\text{þ}}$-rank smaller than $\mathrm{U}^{\text{þ}}(G)$. So by induction hypothesis, we get an infinite interpretable field (notice that if $\mathrm{U}^{\text{þ}}(G) = 1$, then $[G : Z(G)] < \omega$, a contradiction). So we need to consider the case when $Z(G)$ is finite. Dividing out by $Z(G)$, we can assume that G is also centerless. This implies that G does not have nontrivial, finite, normal subgroups. Indeed, if $H \lhd G$ is finite, then $[G : C(h)] < \omega$ for every $h \in H$. But G is centralizer connected so $H \subseteq Z(G) = \{e\}$. So by solvability of G, there is an infinite, definable, abelian, normal subgroup H of G (notice that if $\mathrm{U}^{\text{þ}}(G) = 1$, then $[G : H] < \omega$, a contradiction; so we have proved that $\mathrm{U}^{\text{þ}}(G) \geq 2$).

Now choose a definable subgroup G_0 of finite index in G and an infinite, definable, abelian, normal subgroup H of G_0 with minimal possible $\mathrm{U}^{\text{þ}}$-rank (ranging over all such pairs (G_0, H)). Wlog $G = G_0$. We can also assume that H is centralizer connected in G.

Since G is not nilpotent-by-finite, $G/C(H)$ is infinite. As $\mathrm{U}^{\text{þ}}(G/C(H)) < \mathrm{U}^{\text{þ}}(G)$, by induction hypothesis, we can assume that $G/C(H)$ is nilpotent-by-finite. Using icc on centralizers and Remark 3.3, we can replace G be a definable subgroup of finite index so that $G/C(H)$ becomes nilpotent and centralizer connected. This implies that $Z(G/C(H))$ is infinite.

Put $A = Z(G/C(H))$ and $A_0 = \pi^{-1}[A]$ where $\pi : G \to G/C(H)$ is the natural quotient map. Then A_0 is a definable, normal subgroup of G and $A = A_0/C(H)$ is an infinite, abelian group interpretable in $\langle G, \cdot \rangle$. Moreover, A acts faithfully and definably on H by automorphisms: $aC(H) * h = h^a$ for $aC(H) \in A$ and $h \in H$.

Claim 1 *For every $a \in A_0 \setminus C(H)$, $C(a) \cap H = \{e\}$.*

Proof of Claim 1. Let $B = C(a) \cap H$. It is enough to show that $B \lhd G$ (because H was chosen to have minimal possible positive $\mathrm{U}^{\mathrm{þ}}$-rank, H is centralizer connected in G and G does not have nontrivial, finite, normal subgroups). Take any $g \in G$. We need to show $B^g = B$. Of course, $B^g = C(a^g) \cap H$. Since $A = Z(G/C(H))$, $a^g C(H) = (aC(H))^g = aC(H)$. So $a^g = ac$ for some $c \in C(H)$. Thus $B^g = C(a^g) \cap H = C(ac) \cap H = C(a) \cap H = B$. □

By Claim 1, $P := A$ and $Q := H$ satisfy the assumptions of Lemma 3.2 so we conclude that $R := H \rtimes A$ is an interpretable group which is solvable but not nilpotent-by-finite. We also have $\mathrm{U}^{\mathrm{þ}}(R) = \mathrm{U}^{\mathrm{þ}}(H) + \mathrm{U}^{\mathrm{þ}}(A)$. So if $\mathrm{U}^{\mathrm{þ}}(A) < \mathrm{U}^{\mathrm{þ}}(G/H)$, then $\mathrm{U}^{\mathrm{þ}}(R) < \mathrm{U}^{\mathrm{þ}}(H) + \mathrm{U}^{\mathrm{þ}}(G/H) = \mathrm{U}^{\mathrm{þ}}(G)$ and hence we are done by induction hypothesis. Therefore, we can assume that $\mathrm{U}^{\mathrm{þ}}(A) = \mathrm{U}^{\mathrm{þ}}(G/H)$. But $\mathrm{U}^{\mathrm{þ}}(A) = \mathrm{U}^{\mathrm{þ}}(A_0) - \mathrm{U}^{\mathrm{þ}}(C(H)) \leq \mathrm{U}^{\mathrm{þ}}(G) - \mathrm{U}^{\mathrm{þ}}(H) = \mathrm{U}^{\mathrm{þ}}(G/H)$ and equality holds iff $[G : A_0] < \omega$ and $[C(H) : H] < \omega$. So we get $[C(H) : H] < \omega$, and we can assume that $G = A_0$.

By NIP, H^{00} exists.

Claim 2 H^{00} is definable.

Proof. Take any $a \in G \setminus C(H)$. We claim that $C(a)$ is infinite. If not, then $\mathrm{U}^{\mathrm{þ}}(a^G) = \mathrm{U}^{\mathrm{þ}}(G)$ so $(aH)^G$ is an infinite subset of G/H. But since $G/C(H)$ is abelian and $[C(H) : H] < \omega$, we get a contradiction.

On the other hand, by Claim 1, $C(a) \cap H = \{e\}$. So if we put $G_1 = HC(a)$, then $[G_1 : H]$ is infinite.

By NIP, G_1^{00} exists. Notice that $H^{00} = G_1^{00} \cap H$. The inclusion ($\subseteq$) is obvious. To prove ($=$), assume for a contradiction that $H^{00} \subsetneq G_1^{00} \cap H$. By the definition of G_1 and the fact that $C(a) \cap H = \{e\}$, we get that $H^{00}C(a)$ is a type-definable subgroup of G_1 of bounded index, not containing G_1^{00}, a contradiction.

Since $[G_1 : G_1 \cap C(H)] \geq \omega$, there is $b \in G_1^{00} \setminus C(H)$. By Claim 1, for every $c \in bH$ we have $\mathrm{U}^{\mathrm{þ}}(c^H) = \mathrm{U}^{\mathrm{þ}}(H) = \mathrm{U}^{\mathrm{þ}}(bH)$. As bH is closed under conjugations by elements of H, we get that $bH = c_1^H \cup \dots \cup c_n^H$ for some $c_1, \dots, c_n \in bH$. We also know that $G_1^{00} \lhd G_1$ so $bH \cap G_1^{00} = c_{i_1}^H \cup \dots \cup c_{i_k}^H$ for some $1 \leq i_1 < \dots < i_k \leq n$. Thus, $bH \cap G_1^{00}$ is definable. On the other hand, since $b \in G_1^{00}$, by the last paragraph, we get $bH \cap G_1^{00} = b(H \cap G_1^{00}) = bH^{00}$. Therefore, H^{00} is definable. □

By Claim 2, replacing H by H^{00} (and repeating all arguments preceding Claim 2 for this new H), we can assume that $H = H^{00}$.

Claim 3 *H does not have nontrivial, proper, definable subgroups invariant under the action of A.*

Proof of Claim 3. Suppose H_1 is a definable subgroup of H invariant under A. Since $A = G/C(H)$, we get $H_1 \lhd G$. So, by minimality of $\mathrm{U}^{\text{þ}}(H)$, H_1 is either finite or of finite index in H. On the other hand, we know that G does not have nontrivial, finite, normal subgroups and $H = H^{00}$. Hence $H_1 = \{e\}$ or $H_1 = H$. □

By Claim 3, we see that $M := H$ and A satisfy the assumptions of Theorem 2.2 (or even Proposition 2.3) so an infinite, interpretable field exists. ■

In [1], Theorem 3.1 was proved in the case of $\mathrm{U}^{\text{þ}}(G) = 2$ but under a much stronger assumption that G has hereditarily fsg (finitely satisfiable generics). In fact, under this assumption there was proved even more, namely:

Fact 3.4 *Assume that G has NIP, hereditarily fsg, $U^{\text{þ}}(G)$=2 and G is not nilpotent-by-finite. Then, after possibly passing to a definable subgroup of finite index and quotienting by its finite center, G is (definably) the semidirect product of the additive and multiplicative groups of an algebraically closed field F interpretable in $\langle G, \cdot \rangle$, and moreover $G = G^{00}$.*

Analyzing carefully the proof of Theorem 3.1 and modifying it a little bit, we obtain the following strengthening of Theorem 3.1 in the $\mathrm{U}^{\text{þ}}$-rank 2 case.

Corollary 3.5 *Let G be a group of $U^{\text{þ}}$-rank 2 definable in a monster model of a rosy theory satisfying NIP. Assume that G is solvable-by-finite but not nilpotent-by-finite. Then, after possibly passing to a definable subgroup of finite index and quotienting by its finite center, G is (definably) the semidirect product of the additive group and a finite index subgroup of the multiplicative group of a field K interpretable in $\langle G, \cdot \rangle$.*

Proof. By the proof of Theorem 3.1, we know that there is no group of $\mathrm{U}^{\text{þ}}$-rank 0 or 1 satisfying the assumptions of Theorem 3.1. Therefore, under the assumption $\mathrm{U}^{\text{þ}}(G) = 2$, the proof of Theorem 3.1 necessarily leads us to the last paragraph and produces a field using Proposition 2.3. So for any nontrivial $h \in H$ we get an interpretable field, say K, whose additive group is $\langle H, \cdot \rangle$ and such that the map $f : G/C(H) \to K^*$ given by $f(gC(H)) = gC(H) * h = h^g$ is a definable embedding of $G/C(H)$ into K^*, and after this embedding the action of $G/C(H)$ on H coincides with the field multiplication. Since $\mathrm{U}^{\text{þ}}(K^*) = 1 = \mathrm{U}^{\text{þ}}(G/C(H))$, the image of $G/C(H)$ by f is a finite index subgroup of K^*, call it L.

Claim 1 *Without loss of generality we can assume that $G = HB$ where B is a definable, abelian group of $U^{\text{þ}}$-rank 1, $H \cap B = \{e\}$ and $C(H) = H$.*

Proof of Claim 1. Since $[C(H) : H]$ is finite, we can choose $a \in G \setminus C(H)$. By the first paragraph of the proof of Claim 2 and by Claim 1 in the proof of Theorem 3.1, we get that $C(a)$ is infinite and $C(a) \cap H = \{e\}$. Thus $\mathrm{U}^{\text{þ}}(C(a)) = 1$ and so $C(a)$ is nilpotent-by-finite. Using Remark 3.3 and considering the centralizer connected component of $C(a)$, we get that $C(a)$ has a definable abelian subgroup B of finite index. Since $\mathrm{U}^{\text{þ}}(HB) = 2$, we can assume $G = HB$. In order to finish the proof of

Claim 1, it is enough to show the following

Subclaim $C(H) = H$.

Proof of Subclaim. It is enough to show that $C(H) \cap B = \{e\}$. So we will be done if we show that for any $b \in B \setminus C(H)$, $C(b) \cap C(H) = \{e\}$. We proceed in the same way as in the proof of Claim 1 in the proof of Theorem 3.1.

Take any $b \in B \setminus C(H)$. Let $C = C(b) \cap C(H)$. It is enough to show that $C \lhd G$. Indeed, if $C \lhd G$, then since G does not have nontrivial, finite, normal subgroups, we get that either $C = \{e\}$ or C is infinite. In the second case, $[C(H) : C]$ is finite. This implies that $[H : H \cap C]$ is finite. Since $H = H^{00}$, we get $H \subseteq C$ and so $b \in C(H)$, a contradiction.

Take any $g \in G$. We need to show $C^g = C$. Of course, $C^g = C(b^g) \cap H$. Since $G/H \cong B$ is abelian, $b^g H = (bH)^g = bH$. So $b^g = bc$ for some $c \in H$. Thus $C^g = C(b^g) \cap C(H) = C(bc) \cap C(H) = C(b) \cap C(H) = C$. □

By Claim 1, $G/C(H) = G/H = BH/H \cong B$ and we see that B is definably isomorphic to L by sending $b \in B$ to h^b. We also easily see that the action of L on H by the field multiplication is the same as the action of B on H by conjugation. So $G = HB$ is definably isomorphic to $K^+ \rtimes L$. ■

4 Superrosy dependent fields

The main motivation in this section is Conjecture 4. To prove this conjecture it is enough to show that each infinite, superrosy field K with NIP and containing $\sqrt{-1}$ is algebraically closed. In fact, it suffices to show that for every natural number $n > 0$, $K^n = K$, and if p is the characteristic of K, then the function $f : K \to K$ defined by $f(x) = x^p - x$ is onto (because then we can apply the standard Macintyre's proof [8, Theorem 3.1]). The fact that f is onto follows from Facts 1.1 and 1.5.

In this paper we will prove a weaker condition than $K^n = K$, namely that for every natural number $n > 0$, $K = K^n - K^n$. In particular, if n is odd or if $\sqrt[2^k]{-1}$ exists in K where 2^k is the largest power of 2 dividing n, then $K = K^n + K^n$. We also prove other results of this kind. The main idea involved here is to apply definable measures.

Let us start from a general fact.

Proposition 4.1 *Let K be any field and $G = K \rtimes K^*$ (i.e. $(k_1, k_2) \cdot (k_1', k_2') = (k_1 + k_2k_1', k_2k_2')$). Then G is amenable and there is a finitely additive, probabilistic measure on K which is invariant under additive and non-zero multiplicative translations.*

Proof. Of course, G is solvable so it is amenable, i.e. there is a finitely additive, probabilistic, left (two-sided) invariant measure $\mathbf{m}$ on G.

Define a function $\mu : \mathcal{P}(K) \to [0, 1]$ by

$$\mu(A) = \mathbf{m}(A \times K^*).$$

It is obvious that μ is a finitely additive, probabilistic measure on K. Now we will check that μ is additively and multiplicatively invariant. In the additive case, for every $A \subseteq K$ and $k \in K$ we have:

$$\mu(k+A) = \mathbf{m}((k+A) \times K^*) = \mathbf{m}((k,1) \cdot (A \times K^*)) = \mathbf{m}(A \times K^*) = \mu(A).$$

In the multiplicative case, for every $A \subseteq K$ and $k \in K^*$ we have:

$$\mu(kA) = \mathbf{m}((kA) \times K^*) = \mathbf{m}((0,k) \cdot (A \times K^*)) = \mathbf{m}(A \times K^*) = \mu(A). \qquad \blacksquare$$

Note that if K is definable in a monster model $\mathfrak{C}$ of a theory satisfying NIP, then since $G := K \rtimes K^*$ is also definable in $\mathfrak{C}$ and G is amenable, by Fact 1.6, there is a definable, left invariant Keisler measure on G. Using this measure as $\mathbf{m}$ in the above proof we get:

Corollary 4.2 *Let K be any field definable in a monster model of a theory satisfying NIP. Then there is a definable Keisler measure on K invariant under additive and non-zero multiplicative translations.*

Definition 4.3 *Suppose G is a definably amenable group (with left invariant Keisler measure μ) definable in a monster model of any theory T. We say that a definable set $X \subseteq G$ is μ-generic if $\mu(X) > 0$. We say that a type (or its set of realizations) is μ-generic if the conjunction of any finitely many formulas in this type defines a set whose intersection with G is μ-generic.*

It is obvious that in every definably amenable group non-μ-generic sets form an ideal and hence every partial μ-generic type can be extended to a global μ-generic type. In particular, at least one global μ-generic type exists.

The following proposition (except for point (i)) is a variant of [2, Corollary 4.3] for μ-generics.

Proposition 4.4 *Let G be a definably amenable group definable in a monster model of a theory satisfying NIP. Then*
(i) there is a definable, left invariant Keisler measure μ on G,
(ii) there are only boundedly many global μ-generic types,
(iii) for every global type p, $Stab(p) \subseteq G^{00}$,
(iv) for every definable set $X \subseteq G$, $Stab_\mu(X) := \{g \in G : \mu(gX \triangle X) = 0\}$ is a type-definable subgroup of bounded index in G; in particular, $G^{00} \subseteq Stab_\mu(X)$,
(v) for every global μ-generic type p, $Stab(p) = G^{00}$.

Proof. (i) is just Fact 1.6.
(ii) is true for an arbitrary Keisler measure and it follows from [2, Corollary 3.4].
(iii) is true whenever G^{00} exists; it follows from the fact that a partial type defining some translate of G^{00} is in p.
(iv). The fact that $Stab_\mu(X)$ is a subgroup follows from left invariance of μ. The fact that $Stab_\mu(X)$ is type-definable follows from definability of μ. Finally, the fact that

the index of $Stab_\mu(X)$ in G is bounded follows from [2, Corollary 3.4] and the observation that for every $g,h \in G$ we have: $gStab_\mu(X) = hStab_\mu(X)$ iff $\mu(gX \triangle hX) = 0$.
(v) Since μ is left invariant, gp is μ-generic for every $g \in G$. So $\bigcap\{Stab_\mu(X) : X \in p\} \subseteq Stab(p)$. Hence we are done by (iii) and (iv). ■

From now on, let K be an infinite field definable in a monster model of a theory T satisfying NIP. By Corollary 4.2, we can find a definable Keisler measure invariant under additive and non-zero multiplicative translations; we denote it by μ.

Proposition 4.5 *If X is a definable (or type-definable) μ-generic subset of K, then*
(i) for every $k \in K$, $(k + X) \cap X$ is μ-generic and of the same measure as X,
(ii) $K = X - X$,
(iii) for every $k \in (K^*)^{00}$, $kX \cap X$ is μ-generic and of the same measure as X,
(iv) $(K^*)^{00} \subseteq XX^{-1}$.

Proof. Items (ii) and (iv) follow from (i) and (iii), respectively. Item (i) follows from Fact 1.5 and Proposition 4.4(iv). Item (iii) follows from Proposition 4.4(iv). ■

From now on, assume that T is additionally superrosy.

Corollary 4.6 *For every natural number $n > 0$ we have $K = K^n - K^n$. Hence, if n is odd or if $\sqrt[2^k]{-1}$ exists in K where 2^k is the largest power of 2 dividing n, then $K = K^n + K^n$.*

Proof. By Fact 1.1, K^* is a union of finitely many cosets of $(K^*)^n$. So at least one and hence all the cosets of $(K^*)^n$ are μ-generic (and of the same measure). So by Proposition 4.5(ii) we get $K = K^n - K^n$. From this the second part follows immediately. ■

In fact, we get even more, namely, for every $k \in K$, $(k + K^n) \cap K^n$ is μ-generic and of the same measure as K^n. We also get that every element of K can be written as a difference $x - y$ where $x, y \in (K^*)^{00} \subseteq \bigcap_{n>0} K^n$. Notice also that this property (and, in particular, the conclusion of Corollary 4.6) holds in every finite extension of K.

Conjecture 4.7 *Assume $\sqrt{-1}$ exists in K. Then for every natural number n and $a \in K^*$ we have that $K = K^n - aK^n$.*

If we proved the above conjecture (even only for prime numbers n), we could apply the proof of [7, Theorem 4.6] to get that the Brauer group of K is trivial (assuming that $\sqrt{-1} \in K$). Corollary 4.6 is a weaker result than Conjecture 4.7.

Now using our measure μ we will easily conclude that certain particular formulas have the order property.

Proposition 4.8 *For every natural number $n > 0$, if $K^n \neq K$, then the formula $(\exists z)(x - y = z^n)$ has the order property.*

Proof. We need to find sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ such that $a_i - b_j \in K^n \iff i > j$.

For the base step it is enough to choose any $b_0 \in K$ and $a_0 \notin b_0 + K^n$. In order to do that, we need to know that $b_0 + K^n \neq K$. Take any $a \notin K^n$. Since by the comment right after Corollary 4.6, $\mu((b_0 + K^n) \setminus K^n) = 0$, we get $\mu((b_0 + K^n) \cap aK^n) = 0$. As $\mu(K^n) = \mu(aK^n) > 0$, it follows that $aK^n \setminus (b_0 + K^n) \neq \emptyset$.

Suppose we have chosen $(a_i)_{i\leq m}$ and $(b_i)_{i\leq m}$ satisfying the desired property. Then it is enough to choose any $a_{m+1} \in \bigcap_{i\leq m}(b_i + K^n)$ and $b_{m+1} \notin \bigcup_{i\leq m+1}(a_i - K^n)$. In order to do that we need to show that $\bigcap_{i\leq m}(b_j + K^n) \neq \emptyset$ and $\bigcup_{i\leq m+1}(a_i - K^n) \neq K$. By the comment right after Corollary 4.6, we get that $\bigcap_{i\leq m}(b_i + K^n)$ is μ-generic so it is nonempty. To show $\bigcup_{i\leq m+1}(a_i - K^n) \neq K$, we use a similar argument as in the base induction step. ■

Proposition 4.9 *Assume $a \notin -K^n$. Then there is an indiscernible (over a) sequence $(a_i)_{i\in\omega}$ such that:*
(i) if $1 - a \in K^n$, then $a_i - aa_j \in K^n \iff i \geq j$,
(ii) if $1 - a \notin K^n$, then $a_i - aa_j \in K^n \iff i > j$.
In particular, the formula $(\exists z)(x - ay = z^n)$ has the order property.

Proof. By compactness it is enough to construct a sequence satisfying (i) or (ii). For the base step we choose any $a_0 \in (K^*)^n$.

Suppose we have chosen $(a_i)_{i\leq m}$. Now it is enough to show that the set $A := ((K^*)^n \cap \bigcap_{i\leq m}(aa_i + K^n)) \setminus \bigcup_{i\leq m}(\frac{1}{a}a_i - \frac{1}{a}K^n)$ is non-empty and to choose any $a_{m+1} \in A$.

In order to see that $A \neq \emptyset$, notice that $(K^*)^n \cap \bigcap_{i\leq m}(aa_i + K^n)$ is μ-generic and is contained in K^n. On the other hand, $\mu(\bigcup_{i\leq m}(\frac{1}{a}a_i - \frac{1}{a}K^n) \setminus -\frac{1}{a}K^n) = 0$, so $\mu(K^n \cap \bigcup_{i\leq m}(\frac{1}{a}a_i - \frac{1}{a}K^n)) = 0$ (as $-\frac{1}{a}K^n \cap K^n = \{0\}$). Hence $A \neq \emptyset$. ■

Proposition 4.10 *If n is odd, then for every $a \in K$ the formula $\phi(x, y) := (\exists z)(x - ay = z^n)$ does not have the strict order property.*

Proof. We will show that there are no $k, l \in K$ such that $\phi(K, k)$ is a proper subset of $\phi(K, l)$. Suppose $\phi(K, k) \subseteq \phi(K, l)$, i.e. $ak + K^n \subseteq al + K^n$. Then $-ak - K^n \subseteq -al - K^n$, so $-ak + K^n \subseteq -al + K^n$, so $al + K^n \subseteq ak + K^n$. ■

It is well-known that if a formula ϕ has the order property and does not have the independence property, then a conjunction of finitely many instances (or their negations) of ϕ has the strict order property. In Propositions 4.8 and 4.9 we have found (assuming that $K \neq K^n$ for some n) some particular formulas with the order property about which, by Proposition 4.10, we know that they do not have the strict order property (if n is odd). Maybe a more complicated computation (involving somehow superrosiness) could show that also finite conjunctions of instances of those formulas do not have the strict order property, and then we would get that $K = K^n$ (at least for odd n's).

References

[1] C. Ealy, K. Krupiński, A. Pillay. *Superrosy dependent groups having finitely satisfiable generics*, Annals of Pure and Applied Logic, to appear.

[2] E. Hrushovski, Y. Peterzil, A. Pillay. *Groups, measures and the NIP*, Journal of AMS, to appear.

[3] E. Hrushovski, A. Pillay. *On NIP and invariant measures*, preprint.

[4] A. Onshuus. *Properties and consequences of thorn-independence*, Journal of Symbolic Logic 71, no. 1, 1-21, 2006.

[5] Y. Peterzil, S. Starchenko. *Definable homomorphisms of abelian groups in o-minimal structures*, Annals of Pure and Applied Logic 101, 1-27, 2000.

[6] A. Pillay, *Geometric stability theory*, Clarendon Press, Oxford, 1996.

[7] A. Pillay, T. Scanlon, F. Wagner. *Supersimple fields and division rings*, Mathematical Research Letters 5, 473-483, 1998.

[8] B. Poizat. *Stable Groups*, Mathematical Surveys and Monographs, 87, American Mathematical Society, Providence, 2001.

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