Stability, NIP, and NSOP; Model Theoretic Properties of Formulas via Topological Properties of Function Spaces

Karim Khanaki

Faculty of Fundamental Sciences, Arak University of Technology,
P.O. Box 38135-1177, Arak, Iran; e-mail: khanaki@arakut.ac.ir

School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran; e-mail: khanaki@ipm.ir

Abstract. We study and characterize stability, NIP and NSOP in terms of topological and measure theoretical properties of classes of functions. We study a measure theoretic property, 'Talagrand's stability', and explain the relationship between this property and NIP in continuous logic. Using a result of Bourgain, Fremlin and Talagrand, we prove the 'almost and Baire 1 definability' of types assuming NIP. We show that a formula $\phi(x, y)$ has the strict order property if and only if there is a convergent sequence of continuous functions on the space of ϕ-types such that its limit is not sequentially continuous. We deduce from this a theorem of Shelah and point out the correspondence between this theorem and the Eberlein-Šmulian theorem.

Keywords: Talagrand's stability, independence property, strict order property, continuous logic, relative weak compactness, angelic space.
AMS subject classification: 03C45, 03C90, 46E15, 46A50

Contents

1. Introduction

In [Ros74] Rosenthal introduced independence property for families of real-valued functions and used this property for proving a dichotomy in Banach space theory: a Banach space is either 'good' (every bounded sequence has a weak-Cauchy subsequence) or 'bad' (contains an isomorphic copy of l^1). After this and another work of Rosenthal [Ros77], Bourgain, Fremlin and Talagrand [BFT78] found some topological and measure theoretical criteria for independence property and proved that the space of functions of the first Baire class on a Polish space is angelic; a topological notation which the terminology was introduced by Fremlin. This theorem turn out that a set of continuous functions on a Polish space is either 'good' (its closure is precisely the set of limits of its sequences) or 'bad' (its closure contains non-measurable functions). In fact these dichotomies correspond to the NIP/IP dichotomy in continuous logic; see Fact 3.10 below.

In this paper we propose an generalization of Shelah's dividing lines for classification of first order theories which deals with real-valued formulas instead of 0-1 valued formulas. The principal aim of this paper is to study and characterize some model theoretic properties of formulas, such as OP, IP and SOP, in terms of topological and measure theoretical properties of function spaces. This study enables us to obtain new results and to reach a better understanding of the known results.

Let us give the background and our own point of view. In Shelah's stability theory, the set-theoretic criteria largely pass over in favor of definitions which mention ranks or combinatorial properties of a particular formula. There are known interactions between some of these combinatorial properties and some topological properties of function spaces. As an example, a formula $\phi(x, y)$ has the order property (OP) if there exist $a_i b_i, i < \omega$ such that $\phi(a_i, b_j)$ holds if and only if $i < j$. One can assume that ϕ is a 0-1 valued function, such that $\phi(a, b) = 1$ iff $\phi(a, b)$ holds. Then ϕ has the order property iff there exist a_i, b_j such that $\lim_i \lim_j \phi(a_i, b_j) = 1 \neq 0 = \lim_j \lim_i \phi(a_i, b_j)$. Thus failure of the order property, or stability, is equivalent to failure of two different double limits. Using a crucial result essentially due to Grothendieck, the latter is a topological property of a family of functions; see Fact 2.8 below. Similarly, using the result of Bourgain, Fremlin and Talagrand mentioned above, one can obtain some topological and measure theoretical characterizations of NIP formulas. Therefore, it seems reasonable that one studies real-valued formulas and hopes to obtain new classes of functions (formulas) and develop a sharper stability theory by making use of topological properties of function spaces instead of only combinatorial properties of formulas.

The following is a summary of the results of this paper. We work in continuous logic which is an extension of classical first order logic; thus our results hold in the latter case. Let L be a language and M an L-structure. A maximal set of formulas in $L(M)$ which is finitely satisfiable in a model of $Th(M)$ is called a complete type. The set of all complete types, denoted by $S(M)$, is a compact Hausdorff space. For every formula $\psi(x)$, there is a continuous function $f_\psi : S(M) \to \mathbb{R}$ which is an extension of the interpretation of formula $\psi(x)$ in M. By Grothendieck's criterion, it is well known that a subset A of real-valued functions on M (or $S(M)$) is stable iff it is relatively (pointwise) compact in the space of all continuous functions on $S(M)$. Equivalently, a formula $\phi(x, y)$ is stable in M iff the set A of all functions $\phi^a : p \mapsto \phi(p, a)$, where $a \in M$, is relatively compact in the space of continuous functions. By the result of Bourgain, Fremlin and Talagrand (BFT), one can continue this pattern for the NIP case; $\phi(x, y)$ is NIP on M iff the pointwise closure of A is a subset of measurable functions for all Radon measures on $S(M)$; see Theorem 3.13 below. In fact, the role played by Grothendieck's criterion in stable theories is mirrored in NIP theories by the BFT result. Using this, we can prove various forms of definability of types for NIP models; see Theorems 3.16 and 3.20 below. Amongst the results, we give some characterizations of NIP in terms of measure, topology and measure algebra. Then we study the strict order property (SOP) and show that a formula $\phi(x,y)$ has the SOP if there are a_i's such that the sequence $\phi(x,a_i)$ is pointwise convergent but its limit is not sequentially continuous. We deduce from this a theorem of Shelah; a theory is unstable iff it has the IP or the SOP. Finally, we point out the correspondence between Shelah's theorem and the well known compactness theorem of Eberlein and Šmulian.

This is not the end of the story if one defines a notation of non-forking extension in NIP theories such that it satisfies symmetry and transitivity. Moreover, one can study sensitive families of functions, dynamical systems and chaotic maps and their connections with stability theory. We will study them in a future work.

It is worth recalling another lines of research. After the preparation of the first version of this paper, we came to know that simultaneously in [Iba14] and [Sim14b] the relationship between NIP and Rosenthal'dichotomy were noticed in the contexts of $\aleph_0$-categorical structures in continuous logic and classical first order setting, respectively. Before them, the relationship between NIP in integral logic and Talagrand's stability was studied in [Kha14].

This paper is organized as follows: In the second section, we briefly review continuous logic and stability. In the third section, we study Talagrand's stability and its relationship with NIP in logic, and give some characterizations of NIP in terms of measure and topology. The result of Bourgain, Fremlin and Talagrand used in this section for proving of definability of types in NIP theories. In the fourth section, we study the SOP and point out the correspondence between Shelah's theorem and the Eberlein-Šmulian theorem. In the first appendix, we give a proof of Grothendieck's criterion and get a result which is used in the paper. In the second appendix, we remind definability of types of stable formulas.

Acknowledgements. I am very much indebted to professor David H. Fremlin for his kindness and his helpful comments. I am grateful to Márton Elekes for valuable comments and observations, particularly Example 4.5 below. This research is partially supported by a grant from IPM, (No. 93030032).

---

2. Continuous Logic

In this section we give a brief review of continuous logic from [BU10] and [BBHU08]. Results stated without proof can be found there. The reader who is familiar with continuous logic can skip this section.

2.1. **Syntax and semantics.** A *Language* is a set L consisting of constant symbols and function/relation symbols of various arities. To each relation symbol R is assigned a bound $\flat_R \geqslant 0$ and we assume that its interpretations is bounded by $\flat_R$. It is always assumed that L contains the metric symbol d and $\flat_d = 1$. We use $\mathbb{R}$ as value space and its common operations $+, \times$ and scalar products as connectives. Moreover to each relation symbol R (function symbol F) is assigned a modulus of uniform continuity Δ_R (Δ_F). We also use the symbols 'sup' and 'inf' as quantifiers.

Let L be a language. *L-terms* and their bound are inductively define as follows:

- Constant symbols and variables are terms
- If F is a n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $F(t_1, \ldots, t_n)$ is a term.
  All L-terms are constructed in this way.

Definition 2.1. *L-formulas and their bounds are inductively defined as follows:*

- *Every $r \in \mathbb{R}$ is an atomic formula with bound $|r|$.*
- *If R is a n-ary relation symbol and $t_1, \dots, t_n$ are terms, $R(t_1, \dots, t_n)$ is an atomic formula with bound $\flat_R$.*
- *If ϕ, ψ are formula and $r \in \mathbb{R}$ then $\phi + \psi, \phi \times \psi$ and $r\phi$ are formula with bound resp $\flat_\phi + \flat_\psi, \flat_\phi \flat_\psi, |r|\flat_\phi$.*
- *If ϕ is a formula and x is a variable, $\sup_x \phi$ and $\inf_x \phi$ are formulas with the same bound as ϕ.*

Definition 2.2. A *perstructure* in L is pseudo-metric space (M, d) equipped with:

- for each constant symbol $c \in L$, an element $c^M \in M$
- for each n-ary function symbol F a function $F^M : M^n \to M$ such that
$$d_n^M(\bar{x}, \bar{y}) \leqslant \Delta_F(\epsilon) \Longrightarrow d^M(F^M(\bar{x}), F^M(\bar{y})) \leqslant \epsilon$$
- for each n-ary relation symbol R a function $R^M : M^n \to [-\flat_R, \flat_R]$ such that
$$d_n^M(\bar{x}, \bar{y}) \leqslant \Delta_R(\epsilon) \Longrightarrow |R^M(\bar{x}) - R^M(\bar{y})| \leqslant \epsilon.$$

If M is a prestructure, for each formula $\phi(\bar{x})$ and $\bar{a} \in M$, $\phi^M(\bar{a})$ is defined inductively starting from atomic formulas. In particular, $(\sup_y \phi)^M(\bar{a}) = \sup_{b \in M} \phi^M(\bar{a}, b)$. Similarly for $\inf_y \phi$.

Proposition 2.3. *Let M be an L-prestructure and $\phi(\bar{x})$ a formula with $|\bar{x}| = n$. Then $\phi^M(\bar{x})$ is a real-valued function on M^n with a modulus of uniform continuity Δ_ϕ and $|\phi^M(\bar{a})| \leqslant \flat_\phi$ for every $\bar{a}$.*

Interesting prestructures are those which are *complete* metric spaces. They are called *L-structures.* Every prestructure can be easily transformed to a complete L-structure by first taking the quotient metric and then completing the resulting metric space. By uniform continuity, interpretations of function and relation symbols induce well-defined function and relations on the resulting metric space.

2.2. **Compactness, types, stability.** Let L be a language. An expression of the form $\phi \leqslant \psi$, where ϕ, ψ are formulas, is called a *condition.* The equality $\phi = \psi$ is called a condition again. These conditions are called closed if ϕ, ψ are sentences. A *theory* is a set of closed conditions. The notion $M \models T$ is defined in the obvious way. M is then called a model of T. A theory is *satisfiable* if has a model.

An ultraproduct construction can be defined. The most important application of this construction in logic is to prove the Łoś theorem and to deduce the compactness theorem.

Theorem 2.4 (Compactness Theorem)**.** *Let T be an L-theory and $\mathcal{C}$ a class of L-structures. Suppose that T is finitely satisfiable in $\mathcal{C}$. Then there exists an ultraproduct of structures from $\mathcal{C}$ that is a model of T.*

There are intrinsic connections between some concepts from functional analysis and continuous logic. For example, types are well known mathematical objects, *Riesz homomorphisms.* To illustrate this, there are two options; Gelfand representation of C^*-algebras, and Kakutani representation of M-spaces. We work in a real-valued logic, so we use the latter.

Suppose that L is an arbitrary language. Let M be an L-structure, $A \subseteq M$ and $T_A = Th(M, a)_{a \in A}$. Let $p(x)$ be a set of $L(A)$-statements in free variable x. We shall say that $p(x)$ is a *type over A* if $p(x) \cup T_A$ is satisfiable. A *complete type over A* is a maximal type over A. The collection of all such types over A is denoted by $S^M(A)$, or simply by $S(A)$ if the context makes the theory T_A clear. The *type of a in M over A*, denoted by $\mathrm{tp}^M(a/A)$, is the set of all $L(A)$-statements satisfied in M by a. If $\phi(x, y)$ is a formula, a *ϕ-type* over A is a maximal consistent set of formulas of the form $\phi(x,a) \geqslant r$, for $a \in A$ and $r \in \mathbb{R}$. The set of ϕ-types over A is denoted by $S_\phi(A)$.

We now give a characterization of complete types in terms of functional analysis. Let $\mathcal{L}_A$ be the family of all interpretations ϕ^M in M where ϕ is an $L(A)$-formula with a free variable x. Then $\mathcal{L}_A$ is an Archimedean Riesz space of measurable functions on M (see [Fre04]). Let $\sigma_A(M)$ be the set of Riesz homomorphisms $I : \mathcal{L}_A \to \mathbb{R}$ such that $I(\mathbf{1}) = 1$. The set $\sigma_A(M)$ is called the *spectrum* of T_A. Note that $\sigma_A(M)$ is a weak* compact subset of $\mathcal{L}_A^*$. The next proposition shows that a complete type can be coded by a Riesz homomorphism and gives a characterization of complete types. In fact, by Kakutani representation theorem, the map $S^M(A) \to \sigma_A(M)$, defined by $p \mapsto I_p$ where $I_p(\phi^M) = r$ if $\phi(x) = r$ is in p, is a bijection.

Proposition 2.5. *Suppose that M, A and T_A are as above.*

(i) The map $S^M(A) \to \sigma_A(M)$ defined by $p \mapsto I_p$ is bijective.

(ii) $p \in S^M(A)$ if and only if there is an elementary extension N of M and $a \in N$ such that $p = tp^N(a/A)$.

We equip $S^M(A) = \sigma_A(M)$ with the related topology induced from $\mathcal{L}_A^*$. Therefore, $S^M(A)$ is a compact and Hausdorff space. For any complete type p and formula ϕ, we let $\phi(p) = I_p(\phi^M)$. It is easy to verify that the topology on $S^M(A)$ is the weakest topology in which all the functions $p \mapsto \phi(p)$ are continuous. This topology sometimes called the *logic topology.* The same things are true for $S_\phi(A)$.

Definition 2.6. A formula $\phi(x,y)$ is called *stable in a structure* M if there are no $r > s$ and infinite sequences $a_n, b_n \in M$ such that for all $i > j$: $\phi(a_i,b_j) \geqslant r$ and $\phi(a_j,b_i) \leqslant s$. A formula ϕ is *stable in a theory* T if it is stable in every model of T. If ϕ is not stable in M we say that it has the *order property* (or short the OP). Similarly, ϕ has the OP in T if it is not stable in some model of T.

It is easy to verify that $\phi(x,y)$ is stable in M if whenever $a_n, b_m \in M$ form two sequences we have

$$\lim_n \lim_m \phi(a_n,b_m) = \lim_m \lim_n \phi(a_n,b_m),$$

provided both limits exist.

Lemma 2.7. *Let $\phi(x,y)$ be a formula. Then the following are equivalent:*

(i) The formula ϕ is stable.

(ii) For each $\epsilon > 0$, in any model of T there is no infinite sequence $(a_ib_i : i < \omega)$ satisfying for all $i < j$: $|\phi(a_i,b_j) - \phi(a_j,b_i)| \geqslant \epsilon$.

(iii) For each $\epsilon > 0$, there exists a natural number N such that in model of T there is no finite sequence $(a_ib_i : i < N)$ satisfying:

$$\text{for all } i < j < k : \qquad |\phi(a_j,b_i) - \phi(a_j,b_k)| \geqslant \epsilon.$$

By the following result of Grothendieck [Gro52], stability of a formula $\phi(x,y)$ is equivalent to relatively weakly compactness of a family of functions. Because we believe that the proof of this result is informative and useful we mention it in Appendix A. In every thing that follows, if X is a topological space then $C_b(X)$ denotes the Banach space of bounded real-valued functions on X, equipped with the supremum norm. A subset $A \subseteq C_b(X)$ is relatively weakly compact if it has compact closure in the weak topology on $C_b(X)$. If X is a compact space, then we write $C(X)$ instead of $C_b(X)$.

Fact 2.8 (Grothendieck's Criterion)**.** *Let* M *be a structure and* $\phi(x, y)$ *a formula. Then the following are equivalent:*

(i) $\phi(x, y)$ *is stable in* M*.*
(ii) The set $A = \{\phi(x, b) : S_x(M) \to \mathbb{R} \mid b \in M\}$ *is relatively weakly compact in* $C(S_x(M))$*.*

---

3. NIP

In this section we study Talagrand's stability and its relationship to NIP in continuous logic. Then, we give some characterizations of NIP in terms of topology and measure, and deduce various forms of definability of types for NIP models.

3.1. **Independent family of functions.**

In [Ros74] Rosenthal introduced the independence property for families of real-valued functions and used it for proving his dichotomy. As we will see shortly, this notation corresponds to a generalization of the IP for real-valued formulas.

Definition 3.1. A family F of real-valued functions on a set X is said to be *independent* (or has the *independence property*, short IP) if there exist real numbers $s < r$ and a sequence $f_n \in F$ such that for each $k \geqslant 1$ and for each $I \subseteq \{1, \dots, k\}$, there is $x \in X$ with $f_i(x) \leqslant s$ for $i \in I$ and $f_i(x) \geqslant r$ for $i \notin I$. In this case, sometimes we say that every finite subset of F is shattered by X. If F has not the independence property then we say that it has the *dependent property* (or the NIP).

We have the following remarkable topological characterizations of this property. More details and several equivalent presentations can be found in Theorem 2.11 from [GM14].

Fact 3.2. *Let* X *be a compact space and* $F \subseteq C(X)$ *a bounded subset. The following conditions are equivalent:*

(i) F *does not contain an independent subsequence.*
(ii) Each sequence in F *has a pointwise convergent subsequence in* $\mathbb{R}^X$*.*
(iii) F *does not contain a subsequence equivalent to the unit basis of* l^1*, i.e.* F *does not contain a subsequence such that its closed linear span in* $l^\infty(X)$ *be linearly homeomorphic to the Banach space* l^1*.*

Lemma 3.3. *Let* $\{f_n\}$ *be a bounded sequence of continuous functions on a topological space* X*. Let* Y *be a dense subset of* X*. Then* $\{f_n\}$ *is an independent sequence on* X *if and only if the sequence of restrictions* $\{f_n|_Y\}$ *is an independent sequence on* Y*.*

Definition 3.4. We say that a (bounded) family F of real-valued function on a set X has the *relatively sequentially compactness* (short RSC) if for every sequence in F has a pointwise convergent subsequence in $\mathbb{R}^X$.

As we will see shortly, the following statement is a generalization of a model theoretic fact, i.e. IP implies OP.

Fact 3.5. *Let* X *be a compact space and* $F \subseteq C(X)$ *a bounded subset. If* F *is relatively weakly compact in* $C(X)$*, then* F *has the RSC.*

3.2. **Talagrand's stability.** Historically, Talagrand's stability (see Definition 3.6 below), which we call the almost dependence property, arose naturally when Talagrand and Fremlin were studying pointwise compact sets of measurable functions; they found that in many cases a set of functions was relatively pointwise compact because it was almost dependent (see Fact 3.7 below). Later did it appear that the concept was connected with Glivenko-Cantelli classes in the theory of empirical measures, as explained in [Tal87]. In this subsection we study this property and show that it is the 'correct' counterpart of NIP in integral logic (see [Kha14]). Then, we point out the connection between NIP in continuous logic and this property.

Definition 3.6 (Talagrand's stability)**.** Let $A \subseteq C(X)$ be a pointwise bounded family of real-valued continuous functions on X. Suppose that μ is a measure on X. We say that A has the *μ-almost dependence property* (or short *μ-almost NIP*), if A is a stable set of functions in the sense of Definition 465B in [Fre06], that is, whenever $E \subseteq M$ is measurable, $\mu(E) > 0$ and $s < r$ in $\mathbb{R}$, there is some $k \geqslant 1$ such that $(\mu^{2k})^* D_k(A, E, s, r) < (\mu E)^{2k}$ where

$$D_k(A, E, s, r) = \bigcup_{f \in A} \big\{w \in E^{2k} : f(w_{2i}) \leqslant s,\ f(w_{2i+1}) \geqslant r \text{ for } i < k\big\}.$$

We say that A has the *universal almost dependence property* (or short *universal almost NIP*), if A has the μ-almost NIP for all Radon measures μ on X.

Now we invoke the first result connecting this notion, as pointed out by Fremlin in [Fre06, 465D].

Fact 3.7. *Let X be a compact Housdorff space and $A \subseteq C(X)$ be a pointwise bounded family of real-valued continuous functions from X. Suppose that μ is a Radon measure on X. If A has the μ-almost NIP, then the poinwise closure of A, denoted by $cl_p(A)$, has the μ-almost NIP and every element in $cl_p(A)$ is μ-measurable.*

In [Fre75] Fremlin obtained a remarkable result, it is called Fremlin's dichotomy: a set of measurable functions on a perfect measure space is either 'good' (relatively countably compact for the pointwise topology and relatively compact for the topology of convergence in measure) or 'bad' (with neither property). We recall that a subset A of a topological space X is *relatively countably compact* if every sequence of A has a cluster point in X.

Fact 3.8 (Fremlin's dichotomy)**.** *Let (X, Σ, μ) be a perfect σ-finite measure space, and $\{f_n\}$ a sequence of real-valued measurable functions on X. Then*

either $\{f_n\}$ has a subsequence which is convergent almost everywhere
or $\{f_n\}$ has a subsequence with no measurable cluster point in $\mathbb{R}^X$.

Let (X, Σ, μ) be a finite Radon measure on compact a space X, and $A \subseteq \mathcal{L}^0$ a bounded family of real-valued measurable functions on X. Then we say that A satisfies condition (M), if for all $s < r$ and all k, the set $D_k(A, X, r, s)$ be measurable (this applies, in particular, if A is countable).

Proposition 3.9. *Let (X, Σ, μ) be a finite Radon measure on a compact space X, and $A \subseteq \mathcal{L}^0$ a bounded family of real-valued measurable functions on X. Then (i) $\Rightarrow$ (ii). If A satisfies condition (M), then (ii) $\Rightarrow$ (i). (i) $\Rightarrow$ (iii), but (iii) $\not\Rightarrow$ (i) and (iii) $\not\Rightarrow$ (ii).*

(i) A has the μ-almost NIP.
(ii) There do not exist measurable set E, $\mu(E) > 0$ and $s < r$ in $\mathbb{R}$, such that for each n, and almost all $w \in E^n$, for each subset I of $\{1, \dots, n\}$, there is $f \in A$ with

$$f(w_i) < s \text{ if } i \in I \text{ and } f(w_i) > r \text{ if } i \notin I.$$

(iii) Every sequence in A has a subsequence which is convergent μ-almost everywhere.

Proof. (i) $\Rightarrow$ (ii) is evident.

(M)$\wedge$(ii) $\Rightarrow$ (i) is Proposition 4 in [Tal87].

Let $\{f_n\}$ be any sequence in A, and take an arbitrary subsequence of it (still denoted by $\{f_n\}$). Let $\mathcal{D}$ be a non-principal ultrafilter on $\mathbb{N}$, and then define $f(x) = \lim_{\mathcal{D}} f_i(x)$ for all $x \in X$. (By the assumption, there is a real number r such that $|h| \leqslant r$ for each $h \in A$, and therefore f is well defined.) Since A has the μ-almost NIP and $f \in \mathrm{cl}_p(\{f_n\})$, the function f is measurable (see Fact 3.7). So every subsequence of $\{f_n\}$ has a measurable cluster point. Fremlin's dichotomy now tell us that $\{f_n\}$ has a subsequence which is convergent almost everywhere. Thus we see that (i) $\Rightarrow$ (iii).

(iii) $\not\Rightarrow$ (i)$\vee$(ii): In [SF93] Shelah and Fremlin found that in a model of set theory there is a separable pointwise compact set A of real-valued Lebesgue measurable functions on the unit interval which it is not μ-almost NIP. Thus we see that (iii) $\not\Rightarrow$ (i). Since the set A is separable, it satisfies condition (M) and therefore (ii) fails. $\square$

Professor Fremlin kindly pointed out to us that Shelah's model, described in their paper [SF93], in fact deals with the point that there is a countable set of *continuous* functions which is relatively pointwise compact in $\mathcal{L}^0(\mu)$ for a Radon measure μ, but that it is not μ-almost NIP. Of course this will not happen if a set be relatively pointwise compact in $\mathcal{L}^0(\mu)$ for every Radon measure μ:

Fact 3.10 (BFT Criterion)**.** *Let X be a compact Hausdorff space, and $F \subseteq C(X)$ be uniformly bounded. Then the following are equivalent.*

(i) F has the NIP (see Definition 3.1 above).
(ii) F is relatively compact in $\boldsymbol{M}_r(X)$, i.e. every $f \in cl_p(F)$ is μ-measurable for every Radon measure μ on X.
(iii) F has the RSC (see Definition 3.4 above).
(iv) Each sequence in F has a subsequence which is convergent μ-almost everywhere for every Radon measure μ on X.
(v) For each Radon measure μ on X, each sequence in F has a subsequence which is convergent μ-almost everywhere.

Proof. The equivalence (i)–(iii) is the equivalence (ii) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (iv) of Theorem 2F of [BFT78]. (See also Fact 3.2 above.)

Fremlin's dichotomy and the equivalence (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (iv) of Theorem 2F of [BFT78] imply (v) $\Leftrightarrow$ (i) $\Leftrightarrow$ (iv). $\square$

We will see that the BFT criterion in NIP theories plays a role similar to the role played by Grothendieck's criterion in stable theories.

3.3. **NIP formulas.** In [She71] Shelah introduced the independence property (IP) for 0-1 valued formulas; a formula $\phi(x,y)$ has the IP if for each n there exist $b_1,\ldots,b_n$ in the monster model such that each nontrivial Boolean combination of $\phi(x,b_1),\ldots,\phi(x,b_n)$ is satisfiable. By some set-theoretic considerations, a formula $\phi(x,y)$ has IP if and only if $\sup\{|S_\phi(A)| : A \text{ of size } \kappa\} = 2^\kappa$ for some infinite cardinal κ. Although this property was introduced for counting types, its negation (NIP) is a successful extension of local stability and also an active domain of research in classical first order logic and another areas of mathematics. The following generalization of NIP (in the frame work of continuous logic) also has a natural topological presentation.

Definition 3.11. *Let M be a structure, and $\phi(x,y)$ a formula. The following are equivalent and in any of the cases we say that $\phi(x,y)$ is NIP on $M \times M$ (or on M).*

(i) For each sequence $\phi(a_n, y)$ in the set $A = \{\phi(a, y) : S_y(M) \to \mathbb{R} \mid a \in M\}$, where $S_y(M)$ is the space of all complete types on M in the variable y, and $r > s$ there is some $I \subseteq \mathbb{N}$ such that

$$\Big\{y \in S_y(M) : \big(\bigwedge_{i\in I} \phi(a_n, y) \leqslant s\big) \wedge \big(\bigwedge_{i\notin I} \phi(a_n, y) \geqslant r\big)\Big\} = \emptyset.$$

(ii) Every sequence $\phi(a_n, y)$ in A has a convergent subsequence, equivalently A has the RSC.

With some natural adaptations of classical logic one can give more equivalent notations of NIP using elementary extensions and indiscernible sequences.

Remark 3.12. Let M be a structure and $\phi(x, y)$ a formula. The space $S_{\phi(y)}(M)$ of all $\phi(y)$-types on M is the quotient of $S_y(M)$ given by the family of functions $\{\phi(a, y) : a \in M\}$ (see [BU10], Fact 4.8). So in Definition 3.11 above, $S_y(M)$ can be replace by $S_{\phi(y)}(M)$.

Theorem 3.13. *Let M be an L-structure, $\phi(x; y)$ a formula, $A = \{\phi(a, y) : a \in M\}$ and $\tilde{A} = \{\phi(x, b) : b \in M\}$. Then the following are equivalent:*

(i) ϕ is NIP on M.

(ii) $\tilde{A}$ has the μ-almost NIP for all Radon measures on M.

Proof. (i) $\Rightarrow$ (ii): By the compactness theorem of continuous logic, since $\phi(x, y)$ is NIP, there is some integer n such that no set of size n is shattered by $\phi(x, y)$. We note that by Proposition 465T of [Fre06], the conditions (i) and (ii) of Proposition 3.9 are equivalent. So if $E \subseteq M$, $\mu(E) > 0$, $r > s$, then for each $(a_1, \ldots, a_n) \in E^n$ there is a set $I \subseteq \{1, \ldots, n\}$ such that

$$\Big\{y \in S_y(M) : \big(\bigwedge_{i\in I} \phi(a_i, y) \leqslant s\big) \wedge \big(\bigwedge_{i\notin I} \phi(a_i, y) \geqslant r\big)\Big\} = \emptyset,$$

where $S_y(M)$ is the space of all complete types on M in the variable y. Since $M \subseteq S_y(M)$, the set $\tilde{A}$ has the μ-almost NIP for every Radon measure μ on M.

(ii) $\Rightarrow$ (i): Suppose that $\tilde{A}$ has the μ-almost NIP for every Radon measure μ on M. Since M is dense in $\tilde{X} = S_x(M)$, every Radon measure on X has a unique extension to a Radon measure on $\tilde{X}$ and every Radon measure on M is a restriction of a Radon measure on $\tilde{X}$. So $\tilde{A}$ has the μ-almost NIP for every Radon measure μ on $\tilde{X}$. Thus, by Fact 3.7, $\tilde{A}$ is relatively compact in $\mathbf{M}_r(\tilde{X})$ (the space of all μ-measurable functions on $\tilde{X}$ for each Radon measure μ on $\tilde{X}$). By the BFT criterion, for each sequence $\phi(x, a_n)$ in $\tilde{A}$, and $r < s$, there is some $I \subseteq \mathbb{N}$ such that

$$\Big\{x \in S_x(M) : \big(\bigwedge_{i\in I} \phi(x, a_i) \leqslant s\big) \wedge \big(\bigwedge_{i\notin I} \phi(x, a_i) \geqslant r\big)\Big\} = \emptyset.$$

Thus the dual formula $\tilde{\phi}(y; x) := \phi(x; y)$ is NIP on M. So, by applying the direction (i) $\Rightarrow$ (ii) to the formula $\tilde{\phi}$, we see that $\tilde{\tilde{A}} = A$ has the μ-almost NIP for every Radon measure μ on M (equivalently on $X = S_y(M)$). Thus, again by the BFT criterion, we conclude that $\phi(x, y)$ is NIP on M. $\square$

In fact the proof of the previous result says more:

Corollary 3.14 (NIP duality)**.** *Under the above assumptions, ϕ is NIP if and only if $\tilde{\phi}$ is NIP. Similarly, A has the μ-almost NIP (for every Radon measure μ) if and only if $\tilde{A}$ has the μ-almost NIP (for every Radon measure μ). Consequently, ϕ is NIP if and only if A has the μ-almost NIP (for every Radon measure μ).*

The previous results also show that why the μ-almost NIP is the 'correct' notation of NIP in integral logic (see [Kha14]).

3.4. **Almost definable types.** It is well known that every type on a stable model is definable (see Appendix B below). Here we want to give a counterpart of this fact for NIP theories. In [Kha14] it is shown that if a formula ϕ (in integral logic) has the μ-almost NIP on a model M, then every type in $S_\phi(M)$ is μ-almost definable. Here we say a function $\psi : X \to \mathbb{R}$ on a topological space X is *universally measurable*, if it is μ-measurable for every probability Radon measure μ on X. We say that a universally measurable function $\psi : S_{\phi(y)}(M) \to \mathbb{R}$ *defines* type $p \in S_{\phi(x)}(M)$ if $\psi(p,b) = \psi(b)$ for all $b \in M$, and in this case we say that p is *universally definable*.

The following is a translation of the BFT criterion:

Proposition 3.15. *Let* M *be a structure and* $\phi(x,y)$ *a formula. Then the following are equivalent:*

(i) *ϕ is NIP on M.*
(ii) *Every $p \in S_{\phi(x)}(M)$ is definable by a universally measurable relation $\psi(y)$ over $S_{\phi(y)}(M)$.*

Proof. (i) $\Rightarrow$ (ii): Let $A = \{\phi^a(y) : S_{\phi(y)}(M) \to \mathbb{R} \mid a \in M\}$. By NIP, A is relatively compact in $M_r(S_{\phi(y)}(M))$ (see the BFT criterion). Suppose that $p_{a_i} \to p \in S_{\phi(x)}(M)$ where p_{a_i} is realized by $a_i \in M$. (We note that the set of all types realized in M is dense in $S_{\phi(x)}(M)$.) Thus $\phi^{a_i} \to \psi$ pointwise where ψ is universally measurable, and ψ defines p.

(ii) $\Rightarrow$ (i): Suppose that $\phi^{a_i} \to \psi$ pointwise. We can assume that $p_{a_i} \to p \in S_{\phi(x)}(M)$. Suppose that p is definable by a universally measurable relation φ, so we have $\psi = \varphi$ on M. Since every Radon measure μ on $S_{\phi(y)}(M)$ is a complete measure and $\mu\{b : \psi \neq \varphi\} = 0$ (because M is of full outer measure in $S_{\phi(y)}(M)$), so ψ is measurable for all Radon measures on $S_{\phi(y)}(M)$. Again by the BFT criterion, ϕ is NIP. □

Here we are going to give some characterizations of NIP. First we need some definitions. Let ψ be a measurable function on $(S_\phi(M),\mu)$ where μ is a probability Radon measure on $S_\phi(M)$. Then ψ is called an *almost ϕ-definable relation over M* if there is a sequence $g_n : S_\phi(M) \to \mathbb{R}$, $|g_n| \leqslant |\phi|$, of continuous functions such that $\lim_n g_n(b) = \psi(b)$ for almost all $b \in S_\phi(M)$. (We note that by the Stone-Weierstrass theorem every continuous function $g_n : S_\phi(M) \to \mathbb{R}$ can be expressed as a uniform limit of algebraic combinations of (at most countably many) functions of the form $p \mapsto \phi(p,b)$, $b \in M$.) An almost definable relation $\psi(y)$ over M defines $p \in S_\phi(M)$ if the set $A = \{b \in S_\phi(M) : \phi(p,b) = \psi(b)\}$ is measurable and $\mu(A) = 1$, and in this case we say that p is (μ-)*almost definable*. Suppose that every type p in $S_\phi(M)$ is almost definable by a measurable function ψ^p. Then, we say that p is *almost equal to* q, denoted by $p \equiv q$, if $\psi^p = \psi^q$ almost everywhere. Define $[p] = \{q \in S_\phi(M) : p \equiv q\}$ and $[S_\phi](M) = \{[p] : p \in S_\phi(M)\}$. Then $[S_\phi](M)$ has a natural topology which is defined by metric $d([p],[q]) = \int |\psi^p - \psi^q| d\mu$ for $p, q \in S_\phi(M)$.

When measuring the size of a structure we will use its density character (as a metric space), denoted $\|M\|$, rather than its cardinality. Similarly, we measure the size $[S_\phi](M)$ by its metric density $\|[S_\phi](M)\|$.

Theorem 3.16 (Almost definability of types). *Let T be a theory and $\phi(x,y)$ a formula. Then the following are equivalent:*

(i) *ϕ is NIP.*
(ii) *For every model M and measure μ on $S_\phi(M)$, every type $p \in S_\phi(M)$ is μ-almost definable, and $\|[S_\phi](M)\| \leqslant \|M\|$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $p \in S_\phi(M)$ is definable by a universally measurable relation ψ on $S_\phi(M)$. Let μ be a Radon measure on $S_\phi(M)$. Then there is a sequence g_n of continuous functions on $S_\phi(M)$ such that $g_n \to \psi$ in $L^1(\mu)$ (see [Fol99, 7.9]), and hence a subsequence (still denoted by g_n) that converges to ψ almost everywhere. So p is almost definable. Moreover, by the Stone-Weierstrass theorem, $\|C(S_\phi(M))\| \leqslant \|M\|$. Now, since $C(S_\phi(M))$ is dense in $L^1(\mu)$ (again see [Fol99, 7.9]), $\|L^1(\mu)\| \leqslant \|M\|$. By definition, $\|[S_\phi](M)\| \leqslant \|M\|$ and the proof is completed.

(ii) $\Rightarrow$ (i): Let $p \in S_\phi(M)$. Suppose that $p_{a_i} \to p$ where p_{a_i} is realized by $a_i \in M$. Then the function $\psi(y) = \lim_i \phi(a_i, y)$ is measurable for all Radon measures on $S_\phi(M)$. Indeed, by definition, for each Radon measure μ, there is a measurable function ψ_μ such that $\psi_\mu(b) = \phi(p, b)$ μ-almost everywhere. Since μ is Radon (and so is complete), and $\psi = \psi_\mu$ almost everywhere, ψ is μ-measurable (see [Fol99, 2.11]). Then, by Proposition 3.15, the proof is completed. □

Here we mention another characterization of NIP in term of measure algebra. For this, a notation is needed. Let $\phi(x, y)$ be a formula. Then the measure algebra generated by ϕ on $S_\phi(M)$ is the measure algebra generated by all sets of the forms $\phi(x, a) \geqslant r$ and $\phi(x, b) \leqslant s$ where $a, b \in M$ and $r, s \in \mathbb{R}$. ($\phi(x, a) \geqslant r$ is the set $\{x \in S_\phi(M) : \phi(x, a) \geqslant r\}$. Similarly, $\phi(x, b) \leqslant s$ is defined.) One can assume that all r, s are rational numbers. Now, a straightforward translation of the proof for classical first order theories, as can be found in [Kei87, Theorem 3.14], implies that:

Fact 3.17. *Let T be a theory and $\phi(x, y)$ a formula. Then the following are equivalent:*
(i) ϕ is NIP.
(ii) For every saturated enough model M, each Radon measure on $S_\phi(M)$ has a countably generated measure algebra (which is the measure algebra generated by ϕ).

3.5. **Baire 1 definable types.** More results can be reached, if one works in a separable model. Let X be a Polish space. A function $f : X \to \mathbb{R}$ is of Baire class 1 if it can be written as the pointwise limit of a sequence of continuous functions. The set of Baire class 1 functions on X is denoted by $B_1(X)$. The following is another criterion for NIP (see [BFT78], Corollary 4G).

Fact 3.18 (BFT Criterion for Polish spaces)**.** *Let X be a Polish space, and $A \subseteq C(X)$ pointwise bounded set. Then the following are equivalent:*
(i) A is relatively compact in $B_1(X)$.
(ii) A is relatively sequentially compact in $\mathbb{R}^X$, or A has the RSC.

Fremlin's notation of an angelic topological space is as follows: a regular Hausdorff space X is *angelic* if (i) every relatively countably compact set is relatively compact, (ii) the closure of a relatively compact set is precisely the set of limits of its sequences. The following is the principal result of Bourgain, Fremlin and Talagrand (see [BFT78], Theorem 3F).

Fact 3.19. *If X is a Polish space, then $B_1(X)$ is angelic under the topology of pointwise convergence.*

Let M be a structure and $\phi(x, y)$ a formula. A Baire class 1 function $\psi : S_\phi(M) \to \mathbb{R}$ defines $p \in S_\phi(M)$ if $\phi(p, b) = \psi(b)$ for all $b \in M$. We say p is Baire 1 definable if some Baire class 1 function ψ defines it.

Theorem 3.20 (Baire 1 definability of types)**.** *Let $\phi(x, y)$ be a NIP formula on a separable model M. Then every $p \in S_{\phi(x)}(M)$ is Baire 1 definable by a $\tilde{\phi}$-Baire definable relation $\psi(y)$ over M, where $\tilde{\phi}(y, x) = \phi(x, y)$.*

Proof. The proof is an easy consequence of Fact 3.18. Suppose that $p_{a_i} \to p \in S_{\phi(x)}(M)$ where $a_i \in M$. If a type $p \in S_{\phi(y)}(M)$ is realized in M, then it can be seen as a function ϕ^a from M to

$\mathbb{R}$ by $\phi^a : b \mapsto \phi(a,b)$. (We recall that the set of all types realized in M is dense in $S_{\phi(y)}(M)$.) Since ϕ is NIP, the set $\hat{A} = \{\phi(a,y) : S_y(M) \to \mathbb{R} : a \in M\}$ is relatively sequentially compact, and particularly the set $A = \{\phi^a : a \in M\}$ is relatively sequentially compact. Now by Fact 3.18, since M is Polish, so A is relatively compact in $B_1(M)$. Thus, there is a $\psi \in B_1(M)$ such that $\phi^{a_i} \to \psi$, so p is definable by a Baire class 1 function. Moreover, since $B_1(M)$ is angelic, there is some sequence $\phi^{a_n}, a_n \in M$ such that $\phi^{a_n} \to \psi$. $\square$

Remark 3.21. Suppose that M, ϕ^{a_n} and ψ are as above. Since ϕ is NIP, the sequence $\widehat{\phi}^{a_n} : S_y(M) \to \mathbb{R}$, defined by $p \mapsto \phi(a_n, p)$, has a convergent subsequence (still denoted by $\widehat{\phi}^{a_n}$), and $\lim_n \widehat{\phi}^{a_n} = \widehat{\psi}$. Clearly, $\widehat{\psi}|_M = \psi$.

Corollary 3.22. *Let ϕ and M be as above. Then $|S_\phi(M)| \leqslant 2^{\aleph_0}$.*

Proof. M is separable, so $|C(X)| \leqslant 2^{\aleph_0}$. Thus, there are at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ sequences of continuous real-valued functions on M, and therefore $B_1(M) \leqslant 2^{\aleph_0}$. $\square$

---

4. SOP

In [She71] Shelah introduced the strict order property as complementary to the independence property: a theory has OP iff it has IP or SOP. In functional analysis, the Eberlein-Šmulian theorem states that a subset of a Banach space is not relatively weakly compact iff it has a sequence without any weakly convergent subsequence or it has a weakly convergent sequence in which the limit is not continuous. In fact there is a corresponding between the Eberlein-Šmulian theorem and Shelah's result above. To determine this correspondence, we first give a topological description of the strict order property, and then study the above dividing line. Let us start with the classical logic.

4.1. **0-1 valued formulas.**

In classical model theory a formula $\phi(x,y)$ has the *strict order property* (or short SOP) if there exists a sequence $(a_i : i < \omega)$ in the monster model $\mathcal{U}$ such that for all $i < \omega$,

$$\phi(\mathcal{U}, a_i) \subsetneqq \phi(\mathcal{U}, a_{i+1}).$$

We can assume that $\phi(x,y)$ is a 0-1 valued function on $\mathcal{U}$ such that $\phi(a,b) = 1$ iff $\models \phi(a,b)$. Then $\phi(x,y)$ has the strict order property if and only if there are sequences $(a_i, b_j : i,j < \omega)$ in $\mathcal{U}$ such that for each $b \in \mathcal{U}$, the sequence $\{\phi(b,a_i)\}_i$ is increasing – therefore the pointwise limit $\psi(x) := \lim_i \phi(x,a_i)$ is well-defined – and $\phi(b_j,a_j) < \phi(b_j,a_{j+1})$ for all $j < \omega$.

Now, suppose that $\phi(x,a_i)$'s are continuous functions on $S_\phi(\mathcal{U})$, the space of all complete ϕ-types. Suppose that ϕ has not the SOP, and $\phi(x,a_i) \nearrow \psi(x)$. Then $\psi : S_\phi(\mathcal{U}) \to \{0,1\}$ is continuous, because there is a k such that $\phi(x,a_k) = \phi(x,a_{k+1}) = \cdots$. Conversely, suppose that $\phi(x,a_i) \nearrow \psi(x)$ and ψ is continuous. It is a standard fact that a increasing sequence of continuous functions which converges to a continuous function converges uniformly (Dini's Theorem). Therefore, our sequence is eventually constant, because the logic is 0-1 valued.

Therefore, it is natural to conjecture that the SOP in classical logic (or continuous logic) is equivalent to the existence of a pointwise convergent sequence (not necessary increasing) of continuous functions such that its limit is not continuous. Our next goal is to convince the reader that by a technical consideration this is indeed the case.

4.2. **Real-valued formulas.** Similar to the classical logic, we say a formula $\phi(x,y)$ in continuous logic has the *strict order property* (SOP) if there exists a sequence $(a_ib_i : i<\omega)$ in the monster model $\mathcal{U}$ and $\epsilon>0$ such that for all $i<j$,

$$\phi(\mathcal{U},a_i)\leqslant\phi(\mathcal{U},a_{i+1}) \quad \text{and} \quad \phi(b_j,a_i)+\epsilon<\phi(b_i,a_j).$$

The acronym SOP stands for the strict order property and NSOP is its negation.

We note that every formula of the form $\psi(y_1,y_2)=\sup_y(\phi(x,y_1)\dot{-}\phi(x,y_2))$ defines a continuous pre-ordering (see [Ben13] for the definition), in analogy with formulae of the form $\psi(y_1,y_2)=\forall x(\phi(x,y_1)\to\phi(x,y_2))$ in classical logic.

We will observe that the following notation corresponds to NSOP on the model-theoretic side.

Definition 4.1. Let X be a topological space and $F\subseteq C(X)$. We say that F has the *continuous sequential closure property* (or short CSCP, or shorter SCP) if for each pointwise convergent sequence $\{f_n\}\subseteq F$, $f_n\to f$, and each sequence $\{a_m\}\subseteq X$, whenever $\lim_m f(a_m)$ exists, then $\lim_m f(a_m)=f(a)$ where a is a cluster point of $\{a_m\}$.

As we will see shortly, the following statement is a generalization of a well known model theoretic fact, i.e. SOP implies OP.

Fact 4.2. *Let X be a compact space and $F\subseteq C(X)$ a bounded subset. If F is relatively weakly compact in $C(X)$, then F has the SCP.*

In Proposition A.3 below, we will see that a family F has the SCP iff its sequential closure, denoted by $\mathrm{scl}(F)$, is a subset of $C(X)$. The next results is another application of Grothendieck's criterion:

Theorem 4.3. *Let X be a compact space and $A\subseteq C(X)$ be bounded. Then A is a relatively weakly compact in $C(X)$ iff it has RSC and SCP.*

Proof. First we show that $cl_p(A)\subseteq C(X)$ if every sequence of A has a convergent subsequence in $\mathbb{R}^X$ and the limit of every convergent sequence of A is sequentially continuous. Suppose that A has RSC and SCP. Let $\{f_n\}_n\subseteq A$ and $\{a_m\}_m\subseteq X$, and suppose that the double limits $\lim_m\lim_n f_n(a_m)$ and $\lim_n\lim_m f_n(a_m)$ exist. Let a be a closure point of $\{a_m\}_m$. By RSC, there is a convergent subsequence f_{n_k} such that $f_{n_k}\to f$. Therefore $\lim_m\lim_{n_k} f_{n_k}(a_m)=\lim_m f(a_m)$ and $\lim_{n_k}\lim_m f_{n_k}(a_m)=\lim_{n_k} f_{n_k}(a)=f(a)$. By SCP, $\lim_m f(a_m)=f(a)$. Since the double limits exist, it is easy to verify that $\lim_m\lim_n f_n(a_m)=\lim_m\lim_{n_k} f_{n_k}(a_m)$ and $\lim_n\lim_m f_n(a_m)=\lim_{n_k}\lim_m f_{n_k}(a_m)$. So A has the double limit property and thus it is relatively weakly compact in $C(X)$, equivalently it is stable. The converse is not really important, and is just Facts 3.5 and 4.2. $\square$

Proposition 4.4. *If the set $\{\phi(x,a):a\in\mathcal{U}\}$ has the SCP, then $\phi(x,y)$ is NSOP.*

Proof. Suppose, if possible, that $\{\phi(x,a):a\in\mathcal{U}\}$ has the SCP and ϕ is SOP. By SOP, there are $(a_ib_i:i<\omega)$ in the monster model $\mathcal{U}$ and $\epsilon>0$ such that $\phi(\mathcal{U},a_i)\leqslant\phi(\mathcal{U},a_{i+1})$ and $\phi(b_j,a_i)+\epsilon<\phi(b_i,a_j)$ for all $i<j$. Let b be a closure point of $\{b_i\}_{i<\omega}$. By SCP, $\phi(S_\phi(\mathcal{U}),a_i)\nearrow\psi$ and ψ is continuous. But $\lim_i\lim_j\phi(b_j,a_i)+\epsilon\leqslant\lim_i\lim_j\phi(b_i,a_j)$ and by continuity $\psi(b)+\epsilon\leqslant\psi(b)$. $\square$

The following example shows that the converse does not hold in Analysis. It is dedicated to us by Márton Elekes.

Example 4.5. Let X be the Cantor set. Let $H=\{0\}\cup(X\cap(2/3,1))$. (We note that H is Δ^0_2, i.e. it is F_σ and G_δ at the same time, but neither open nor closed.) Then it is easy to see that there exists a sequence H_n of clopen subsets of X such that if f_n is the characteristic function of H_n and f is the characteristic function of H then $f_n \to f$ pointwise. Let $A = \{f_n : n < \omega\}$. Then all f_n are continuous, uniformly bounded (even 0-1 valued), the pointwise closure is $A \cup \{f\}$ (which are all Baire class 1 functions), and all monotone sequences in A are eventually constant: indeed, if there were a true monotone subsequence then its limit would be the characteristic function of an open or a closed set, but A is neither open nor closed. Also, we note that A has the RSC but it is not relatively weakly compact in $C(X)$.

Again we give a topological presentation of a model theoretic property.

Proposition 4.6. *Suppose that T is a theory. Then the following are equivalent:*

(i) T is NSOP.

(ii) For each indiscernible sequence $(a_n)_{n<\omega}$ and formula $\phi(x, y)$, if the sequence $(\phi(x, a_n))_{n<\omega}$ converges then its limit is continuous.

Proof. (i) $\Rightarrow$ (ii): Suppose that there are an indiscernible sequence $(a_n)_{n<\omega}$ and a formula $\phi(x, y)$ such that the sequence $(\phi(x, a_n))_{n<\omega}$ converges but its limit is not continuous. Since the limit is not continuous, $\tilde{\phi}(y, x) = \phi(x, y)$ has OP on $\{a_n\}_{n<\omega} \times S_\phi(\mathcal{U})$. Since every sequence in $\{\phi(x, a_n)\}_{n<\omega}$ has a convergent subsequence, $\tilde{\phi}(y, x)$ is NIP on $\{a_n\}_{n<\omega} \times S_\phi(\mathcal{U})$. Now, a straightforward translation of the proof for classical first order theories, as can be found in [Poi00] and [Sim14a], implies that there is a formula $\psi(y_1, y_2)$ in T defining a pre-order with infinite chains, thus T is SOP.

(ii) $\Rightarrow$ (i): Suppose that the formula $\phi(x, y)$ has SOP as witnessed by a sequence $(a_n b_n : n < \omega)$. Then the formula $\psi(y_1, y_2) = \sup_y(\phi(x, y_1) \dot{-} \phi(x, y_2))$ defines a continuous pre-order for which the sequence $(a_n : n < \omega)$ forms an infinite chain. Replace $(a_n)_{n<\omega}$ by an indiscernible sequence $(c_n)_{n<\omega}$, and return to $\phi(x, y)$. Therefore, $\phi(x, y)$ has SOP as witnessed by the sequence $(c_n b_n : n < \omega)$. Now, $\phi(S_\phi(\mathcal{U}), c_n) \nearrow \varphi$ but φ is not continuous. $\square$

Corollary 4.7 ([She71]). *Suppose that T is NIP and NSOP. Then T is stable.*

Proof. Let $\phi(x, y)$ be a formula, $(a_n)_{n<\omega}$ an indiscernible sequence, and $(b_n)_{n<\omega}$ an arbitrary sequence. Suppose that the double limits $\lim_m \lim_n \phi(a_m, b_n)$ and $\lim_n \lim_m \phi(a_m, b_n)$ exist. By NIP and NSOP, $\lim_m \lim_n \phi(a_m, b_n) = \lim_n \lim_m \phi(a_m, b_n)$. (Compare Theorem 4.3.) By Grothendieck's criterion, T is stable. $\square$

4.3. **Theorems of Eberlein-Šmulian and Shelah.** The well known compactness theorem of Eberlein and Šmulian says that weakly compactness and weakly sequentially compactness are equivalent on a Banach space. Since $C(X)$ is a Fréchet-Urysohn space for any compact space X, i.e. the closure of a relatively compact set is precisely the set of limits of its sequences, the Eberlein-Šmulian theorem is equivalent to the following expression: 'relatively weakly compactness and relatively weakly sequentially compactness are equivalent on $C(X)$'. Now, we show the correspondence between Shelah's theorem and the Eberlein-Šmulian theorem.

Proposition 4.8. *Suppose that X is a space of the form $S_\phi(M)$ and $A = \{\phi(a, y) : a \in M\}$ where M is a saturated enough model of a theory T and $\phi(x, y)$ a formula. Then the following are equivalent.*

(i) **The Eberlein-Šmulian theorem:** *For every $A \subseteq C(X)$, the following statements are equivalent:*

  (a) The weak closure of A is weakly compact in $C(X)$.

  (b) Each sequence of elements of A has a subsequence that is weakly convergent in $C(X)$.

(ii) **Shelah's theorem:** *The following statements are equivalent:*

(a′) T is stable.
(b′) T has the NIP and the NSOP.

Proof. First, we note that by the Grothendieck's criterion, (a) ⇔ (a′).

(i) ⇒ (ii): Suppose that (a′) holds, i.e. for every formula $\phi(x,y)$ and model M, the set $A = \{\phi(a,y) : a \in M\}$ is relatively weakly compact. Thus by (a)→(b), T has the NIP and the NSOP. Indeed, by (b), each sequence of elements of A has a subsequence that is weakly convergent in $C(X)$, i.e. ϕ is NIP on M. Now suppose that (f_n) is a convergent sequence of the form $(\phi(a_n,y))$ where (a_n) is an indiscernible sequence. Again by (b), $f = \lim f_n$ is continuous, i.e. ϕ is NSOP on M. Since every formula in every model is NIP and NSOP, T is so. Conversely, suppose that (b′) holds, i.e. T has the NIP and the NSOP. Then by NIP, each sequence $(\phi(a_n,y))$ of elements of A has a subsequence that is weakly convergent to a $\varphi \in \mathbb{R}^X$. By the NSOP, φ is continuous (see Proposition A.3). Thus by (b)→(a), the set $\{\phi(a_n,y) : n < \omega\}$ is relatively weakly compact, so it has the double limit property, and (a′) holds.

(ii) ⇒ (i): Suppose that (a) holds, i.e. A is relatively weakly compact. By (ii), A has the NIP, and hence each sequence of elements of A has a subsequence that is pointwise convergent in $\mathbb{R}^X$. Since A is relatively weakly compact in $C(X)$, the pointwise limit f of a convergent sequence $\{f_n\} \subseteq A$ is continuous. Conversely, if (b) holds, then similar to what was said above, one can see that T has the NIP and the NSOP. Thus, by (ii), A is relatively weakly compact in $C(X)$. □

To summarize:

$$\text{Logic:} \quad \text{Stable} \iff \text{NIP} + \text{NSOP}$$

$$\text{Analysis:} \quad \text{WAP} \iff \text{RSC} + \text{SCP}$$

Of course, the Eberlein-Šmulian theorem is proved for arbitrary Banach spaces, but it follows easily from the case $C(X)$ (see [Fre06], Theorem 462D). On the other hand, we note that for a compact space X, countably compactness and sequentially compactness are equivalent in $C(X)$.

Before we defined angelic topological spaces. Roughly an angelic space is one for which the conclusions of the Eberlein-Šmulian theorem hold. By the previous observations one can say that 'first order logic is angelic.'

Appendix A. Grothendieck's criterion

Because we believe that the proof of Grothendieck's criterion is essential for any understanding of some results of this paper we mention it. The following proof is presented in [KN63, 8.18] for the case of compact spaces. First, we recall a useful fact.

Fact A.1. *Let X be a compact topological space, and A a subset of $C(X)$. Then A is weakly compact in $C(X)$ iff it is norm-bounded and pointwise compact.*

Proof. Use the Lebesgue's Dominated Convergence Theorem (see [Fre06], Proposition 462E). □

Fact A.2 (Grothendieck's criterion)**.** Let X be an arbitrary topological space, $X_0 \subseteq X$ a dense subset. Then the following are equivalent for a subset $A \subseteq C_b(X)$:

(i) The set A is relatively weakly compact in $C_b(X)$.

(ii) The set A is bounded, and for every sequences $\{f_n\} \subseteq A$ and $\{x_n\} \subseteq X_0$, we have

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m),$$

whenever both limits exist.

Proof. (i) $\Rightarrow$ (ii): If $\overline{A} \subseteq C_b(X)$ is weakly compact, then $I[K] \subseteq \mathbb{R}$ must be compact, therefore bounded, for every $I \in C_b(X)^*$; by the Uniform Boundedness Theorem, $\overline{A}$ is norm-bounded.

Suppose that $f_n \in A$ and $x_n \in X_0$ form two sequences and the limits $\lim_n \lim_m f_n(x_m)$ and $\lim_m \lim_n f_n(x_m)$ exist. Let f in $C_b(X)$ and x in X be cluster points of $\{f_n\}$ and $\{x_m\}$. (Note that since all the maps $f \mapsto f(x)$, where $x \in X$, are bounded linear functionals on $C_b(X)$, the pointwise topology is coarser than the weak topology; so a weak convergent net is pointwise convergent. Thus, every sequence in A has a cluster point in $C_b(X)$ with respect to the poinwise topology. Indeed, let $\mathcal{D}$ be an ultrafilter on $\mathbb{N}$, then $f(x) = \lim_{\mathcal{D}} f_n(x)$ is a cluster point of $\{f_n\}$.) Thus,

$$\lim_n \lim_m f_n(x_m) = \lim_n f_n(x) = f(x) = \lim_m f(x_m) = \lim_m \lim_n f_n(x_m).$$

(ii) $\Rightarrow$ (i): First, we assume that X is compact. Then by Grothendieck's Lemma, we must show that the poinwise closure of A, denoted by $\overline{A}$, is compact. Since A is bounded (i.e. there is an r such that $|f| \leqslant r$ for all $f \in A$) and by Tychonoff's theorem $[-r, r]^X$ is compact, it suffices to show that $\overline{A} \subseteq C(X)$. Let $f \in \overline{A}$. Suppose that f is not continuous at a point x in X. Then there is a neighborhood U of $f(x)$ such that each neighborhood of x contains a point y of X_0 with $f(y)$ not belonging to U. Take any f_1 in A; then there is an x_1 in X_0 such that $|f_1(x) - f_1(x_1)| < 1$ and $f(x_1) \notin U$. Take f_2 in A so that $|f_2(x_1) - f(x_1)| < 1$ and $|f_2(x) - f(x)| < 1$. Now choose x_2 in X_0 such that $|f_i(x) - f_i(x_2)| < 1/2$ $(i = 1, 2)$ and $f(x_2) \notin U$. Then take f_3 in A so that $|f(x_j) - f_3(x_j)| < 1/2$ and $|f(x) - f_3(x)| < 1/2$. Proceeding in this way, one obtains sequences $\{f_n\}_n$ and $\{x_m\}_m$ in A and X_0 such that, for each n, $|f_i(x) - f_i(x_n)| < 1/n$ $(i = 1, 2, \ldots, n)$, $|f(x_j) - f_{n+1}(x_j)| < 1/n$ $(j = 1, 2, \ldots, n)$, $|f(x) - f_{n+1}(x)| < 1/n$, and $f(x_n) \notin U$. Then $\lim_n \lim_m f_n(x_m) = \lim_n f_n(x) = f(x)$, and $\lim_n f_n(x_m) = f(x_m) \notin U$. Since it is possible to take a subsequence of $\{x_m\}_m$ so that the corresponding subsequence of $\{f(x_m)\}_m$ converges to a point outside of U, the assumption that f is not continuous contradicts the iterated limit condition of (ii).

Finally, we assume that X is an arbitrary topological space. Write $\mathbf{X}$ for the set of all Riesz homomorphisms $x : C_b(X) \to \mathbb{R}$ such that $x(\mathbf{1}) = 1$, with its weak* topology. Then $\mathbf{X}$ is compact and we have a natural map $f \mapsto \widehat{f} : C_b(X) \to \mathbb{R}^{\mathbf{X}}$ defined by setting $\widehat{f}(x) = x(f)$ for $x \in \mathbf{X}, f \in C_b(X)$. The map $f \mapsto \widehat{f}$ is a homeomorphism between $C_b(X)$, with its weak topology, and its image $\widehat{C_b(X)}$ in $C(\mathbf{X})$, with the topology of pointwise convergence. Moreover, the image of the natural map $x \mapsto \widehat{x} : X \to \mathbf{X}$, defined by $\widehat{x}(f) = f(x)$, is dense in $\mathbf{X}$. Since $\widehat{A} \subseteq C(\mathbf{X})$ is relatively weakly compact, $A \subseteq C_b(X)$ also is. $\square$

The next result illustrates the reason why we say 'continuous sequential closure' property.

Proposition A.3. *Let X be a compact topological space and $F \subseteq C(X)$. Then the following are equivalent.*

(i) F has the SCP.
(ii) The sequential clusure of F is a subset of $C(X)$, i.e. scl$(F) \subseteq C(X)$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $\{g_n\} \subseteq F$ and $g_n \to f$ pointwise, but that f is not continuous. Then by an argument similar to the above, there are $\{x_m\} \subseteq X$, $\{f_n\} \subseteq \{g_n\}$, and a neighborhood U of $f(x)$ such that $\lim_n \lim_m f_n(x_m) = \lim_n f_n(x) = f(x)$, and $\lim_n f_n(x_m) = f(x_m) \notin U$. Take a subsequence $\{x_{m_k}\}_k$ so that the subsequence $\{f(x_{m_k})\}_k$ converges to a point outside of U, and let y be a closure point of $\{x_{m_k}\}$. Since $g_n \to f$ so that $f_n \to f$, and thus by the SCP, $\lim_k f(x_{m_k}) = f(y)$. Moreover, $\lim_n \lim_m f_n(x_m) = \lim_n \lim_k f_n(x_{m_k}) = \lim_n f_n(y) = f(y) \notin U$. This contradicts $\lim_n \lim_m f_n(x_m) = \lim_n f_n(x) = f(x) \in U$. The converse is evident. $\square$

Appendix B. Stability and definability of types

Almost all the combinatorial content of stability in classical model theory is contained in the theorem of definability of types (see [Pil96]). In [Ben14] Itaï Ben Yaacov gave a proof of definability types of a formula stable in a specific structure (in contrast with stability of a formula in all models of a theory) using Grothendieck's criterion. (It is worth pointing out that the proofs of definability of types in this paper based on the idea from his article.) Since a formula ϕ is stable in a theory T if it is stable in every model of T, definability of types for stable formulae follows. We include the proof for the sake of completeness (more details can be found in [Ben14]).

Fact B.1 (Definability of types)**.** *Suppose that ϕ is stable in M. Then every type in $S_\phi(M)$ is definable.*

Proof. Let $X = S_\phi(M)$ and $p(x) \in X$. Since the set $X_0 \subseteq X$ of all complete types realized in M is dense, there is some net $a_i \in M$ such that $\lim_i tp_\phi(a_i/M) = p$. Since ϕ is stable, by the Grothendieck's criterion, the set $A = \{\phi^a : p \mapsto \phi(a,p) \ |a \in M\}$ is relatively pointwise compact in $C(X)$. Therefore there is a $\psi \in C(X)$ such that $\lim_i \phi^{a_i}(y) = \psi(y)$. Clearly, $\psi(y)$ is a $\tilde{\phi}$-definable relation over M (see [BU10], Fact 6.4), and for $b \in M$ we have $\phi(p,b) = \lim_i \phi(a_i,b) = \psi(b)$. Also, ψ is unique because X_0 is dense. $\square$

Because $C(X)$ is angelic, and therefore the closure of a relatively compact subset is precisely the set of limits of its sequences, it is possible that one finds a sequence $\phi(a_n,y)$ such that $\lim_n \phi(a_n,y) = \psi(y)$. A classical result of Mazur asserts that if (f_n) is a bounded sequence of continuous functions on X which poinwise converges to a *continuous* function f, there exists a sequence $g_n \in \mathrm{conv}((f_k)_{k\geqslant n})$ which *uniformly* converges to f. (Here $\mathrm{conv}((h_k))$ denotes the set of convex combinations of the h_k's.) Therefore ψ can be written as a uniform limit of formulae of the form $\frac{1}{n}\sum_{i<n}\phi(a_i,y)$. This is another proof of Theorem B4 of [BU10].

References

[Ben14] I. Ben-Yaacov, Model theoretic stability and definability of types, after A. Grothendiek, Bulletin of Symbolic Logic, (2014), 20, pp 491-496.

[Ben13] I. Ben-Yaacov, On theories of random variables, Israel J. Math. 194 (2013), no. 2, 957-1012

[BBHU08] I. Ben-Yaacov, A. Berenstein, C. W. Henson, A. Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis, vol. 2 (Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, eds.), London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008.

[BU10] I. Ben-Yaacov, A. Usvyatsov, Continuous first order logic and local stability, Transactions of the American Mathematical Society **362** (2010), no. 10, 5213-5259.

[BFT78] J. Bourgain, D. H. Fremlin, and M. Talagrand. Pointwise compact sets of baire-measurable functions. American Journal of Mathematics, 100(4):pp. 845886, 1978.

[Fol99] G. B. Folland, *Real Analysis, Modern Techniquess and Their Appilcations*, 2nd ed. Wiley, New York (1999).

[Fre75] D. H. Fremlin, Pointwise compact sets of measurable functions, Manuscripta Math. 15 (1975) 219- 242.

[Fre04] D. H. Fremlin, *Measure Theory*, vol.3, (Measure Algebras, Torres Fremlin, Colchester, 2004).

[Fre06] D. H. Fremlin, *Measure Theory*, vol.4, (Topological Measure Spaces, Torres Fremlin, Colchester, 2006).

[GM14] E. Glasner, M. Megrelishvili, Eventual nonsensitivity and dynamical systems, arXiv 2014.

[Gro52] A. Grothendieck, Critères de compacité dans les espaces fonctionnels généraux, American Journal of Mathematics **74** ,168-186, (1952).

[Iba14] T. Ibarlucía, The dynamical hierachy for Roelcke precompact Polish groups, arXiv:1405.4613v1, 2014.

[Kei87] H. J. Keisler, Measures and forking, Annals of Pure and Applied Logic 45 (1987), 119-169.

[KN63] J. L. Kelley, I. Namioka, *Linear topological spaces*, New York (1963).

[Kha14] K. Khanaki, *Amenability, extreme amenability, model-theoretic stability, and NIP in integral logic*, ArXiv 2014

[Pil96] A. Pillay, Geometric stability theory, Oxford logic guides, Clarendon Press, 1996.

[Poi00] B. Poizat, A course in model theory: An introduction to contemporary mathematical logic, Springer, New York, (2000).

[Ros74] H. P. Rosenthal, *A characterization of Banach spaces containing* l^1, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2411-2413.

[Ros77] H. P. Rosenthal. *Pointwise compact subsets of the first baire class*. Amer. Jour. of Math., 99(2):362378, 1977.

[Sim14a] P. Simon, A guide to NIP theories, lecture note (2014).

[Sim14b] P. Simon, Rosenthal compacta and NIP formulas, arXiv 2014

[She71] S. Shelah, Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory, Annals of Mathematical Logic, vol. 3 (1971), no. 3, pp. 271-362.

[SF93] S. Shelah, D. H. Fremlin, Pointwise compact and stable sets of measurable functions, J. Symbolic Logic 58 (1993) 435-455.

[Tal87] M. Talagrand, The Glivenko-Cantelli problem, *Ann. of Probability* 15 (1987) 837-870.