LOVELY PAIRS AND DENSE PAIRS OF O-MINIMAL STRUCTURES

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Abstract. We study the theory of Lovely pairs of þ-rank one theories, in particular O-minimal theories. We show that the class of $\aleph_0$-saturated dense pairs of O-minimal structures studied by van den Dries [6] agrees with the corresponding class of lovely pairs. We also prove that the theory of lovely pairs of O-minimal structures is super-rosy of rank $\leq \omega$.

1. Introduction

This paper brings together results on dense pairs by van den Dries [6] and lovely pairs of rank one simple theories developed by Vassiliev [14] using the framework of rosy theories. In [14] Vassiliev studies lovely pairs of a SU-rank one simple theory T and, provided T eliminates the quantifier $\exists^\infty$, shows that the theory of the lovely pairs of T exists and it is simple. The first goal of this paper is the generalize Vassiliev's work and show that the theory of lovely pairs of þ-rank one rosy theories exists provided the original theory eliminates the quantifier $\exists^\infty$.

In [6] van den Dries studies dense pairs of O-minimal theories that expand the theory of ordered abelain groups, generalizing the classical work of Robinson on the completeness of the theory of real closed fields with a predicate for a real dense closed subfield [13]. The author shows that the theory of dense pairs is complete and gives a description of definable sets. It is well known that dense O-minimal theories eliminate the quantifier $\exists^\infty$ and are of þ-rank one (see section 5 of [10]), so we can consider the corresponding theory of lovely pairs. In this paper we show that the theory of lovely pairs of O-minimal theories expanding the theory of ordered abelian groups agrees with the corresponding theory of dense pairs. Part of the goals of this paper is to extend the ideas presented in [6] to the more general framework of lovely pairs.

Berenstein, Ealy and Gunaydin showed in [4] that the theory of dense pairs of O-minimal theories that expand the theory of ordered abelain groups is super-rosy of rank $\leq \omega$. The tools used in the proof depended mainly on the description of definable sets given by van den Dries in [6].

A key idea throughout this paper is the notion of *smal set* and *smal closure*. For $(M, P(M))$ a lovely pair, and $X \subset M$ definable, we say that X is small if it is a subset of an image of a cartesian power of $P(M)$ under a definable function. For $a \in M$ and $B \subset M$, we write $a \in \mathrm{scl}(B)$ if there are $g_1, \ldots, g_n \in P(M)$ such that $a \in \mathrm{dcl}(B, g_1, \ldots, g_n)$; that is, if a belongs to small subset defined over B. In this paper we follow the ideas of van den Dries (used for dense pairs) to describe

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definable subsets of lovely pairs of O-minimal structures in terms of small sets. Using this description of definable sets we prove that the theory of dense pairs of O-minimal structures is super-rosy; the proof follows the steps of the corresponding result in [4].

Finally, following ideas of Buechler and Vassiliev [2, 14], we prove that the rank of a generic type is determined by the local geometry of the underlying O-minimal structure:

Main Theorem Let M be an O-minimal structure, let $P(M) \preceq M$ and assume that $(M, P(M))$ is a lovely pair. Given $a \in M$ the following holds:

(1) If M is trivial in a neighborhood of a, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 1$ and equality holds if a is not algebraic.
(2) If M induces the structure of an ordered vector space over an ordered division ring in a neighborhood of a, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 2$ and equality holds if $a \notin \mathrm{scl}(\emptyset)$.
(3) If M induces the structure of an O-minimal expansion of a real closed field in a neighborhood of a, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq \omega$ and equality holds if $a \notin \mathrm{scl}(\emptyset)$.

This paper is divided as follows. In the second section we study lovely pairs of þ-rank one structures that eliminate the quantifier $\exists^\infty$. In the third section we study the definable sets of a lovely pair. In section four we give a more detailed analysis of definable sets of a lovely pair of O-minimal structures. In section five we prove that the theory of lovely pairs of O-minimal structures is super-rosy of rank $\leq \omega$. In section six we prove the Main Theorem.

We assume throughout this paper that the reader is familiar with the basic ideas of rosy theories presented in [10], [1]. We follow the notation from [4], we write capital leters such as C, D, X, Y for definable sets and sometimes we write $C_{\vec{b}}$ to emphazise that C is definable over $\vec{b}$. We may write $\vec{b} \in C_{\vec{y}}$ to mean that $\vec{b}$ is a tuple of the same arity as $\vec{y}$ whose components belong to C.

2. Lovely pairs of þ-rank one structures

We begin by translating to the setting of þ-rank one theories, the definitions used by Vassiliev in [14]. Let T be a þ-rank one theory (see [10]) with quantifier elimination in a language L. By symmetry of thorn forking, in any model of T acl has the exchange property and defines a pregeometry. Examples of such theories includes strongly minimal theories, SU-rank one simple theories with quantifier elimination and O-minimal theories. Let P be a new unary predicate and let $L_P = L \cup \{P\}$. Let T' be the L_P-theory of all structures (M, P), where $M \models T$ and $P(M)$ is an L-algebraically closed subset of M. Let T_{pairs} be the theory of elementary T-pairs, that is, the theory of structures of the form $(M, P(M))$ where $P(M) \preceq M$ and $M \models T$.

Notation 2.1. *Let $(M, P(M)) \models T'$ and let $A \subset M$. We write $P(A)$ for $P(M) \cap A$.*

Notation 2.2. *Throughout this section independence means acl_L-independent, where acl_L means algebraic closure in the sense of L. We write $\mathrm{tp}(\vec{a})$ for the L-type of a.*

Definition 2.3. We say that a structure $(M, P(M))$ is a *lovely pair of models* of T if

(1) $(M, P(M)) \models T'$

(2) (Density property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in P(M)$ such that $a \models q$.
(3) (Extension property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in M$, $a \models q$ and $a \notin \mathrm{acl}(A \cup P(M))$.

Equivalently, we could follow the approach from [3] and define, for $\kappa \geq |T|^+$, the class of κ-lovely pairs, replacing the condition $A \subset M$ *is algebraically closed and finite dimensional* in the clauses (2) and (3) above for $A \subset M$ *is algebraically closed of dimension* $\leq \kappa$.

Note that if $(M, P(M))$ is a lovely pair, the extension property implies that M is $\aleph_0$-saturated. If $(M, P(M))$ is a κ-lovely pair, the extension property implies that M is κ-saturated and that $M \setminus P(M)$ is non-empty. Assume now that T is an O-minimal theory and that $a, b \in M$ are such that $a < b$; then the partial type $a < x < b$ is non-algebraic and by the density property it is realized in $P(M)$. Thus, the density property implies that $P(M)$ is dense in M.

Lemma 2.4. *Any lovely T-pair is an elementary T-pair.*

Proof. We apply the Tarski-Vaught test. Let $(M, P(M))$ be a lovely T pair, let $\varphi(x, \vec{y})$ be an L-formula and let $\vec{b} \in P(M)_{\vec{y}}$. Assume that there is $a \in M$ such that $M \models \varphi(a, \vec{b})$. If a is algebraic over $\vec{b}$, since $P(M)$ is algebraically closed we get $a \in P(M)$. If a is not algebraic over $\vec{b}$, the type $\mathrm{tp}(a/\vec{b})$ is not algebraic and by the density property there is $a' \in P(M)$ such that $a' \models \mathrm{tp}(a/\vec{b})$; in particular, $M \models \varphi(a', \vec{b})$. $\square$

We follow now section 3 of [3]. The existence of κ-lovely pairs follows from [3, Lemma 3.5]. The proof presented there does not use the Independence Theorem, in fact it only uses transitivity and the existence of non-forking extensions. Exchanging the word independence for þ-independence gives a proof in our setting.

Definition 2.5. Let A be a subset of a lovely pair $(M, P(M))$ of models of T. We say that A is *P-independent* if A is independent from $P(M)$ over $P(A)$.

Lemma 2.6. *Let $(M, P(M))$ and $(N, P(N))$ be lovely pairs of modells of T. Let $\vec{a}$, $\vec{b}$ be finite tuples of the same length from M, N respectively, which are both P-independent. Assume that $\vec{a}$, $\vec{b}$ have the same quantifier free L_P-type. Then $\vec{a}$, $\vec{b}$ have the same L_P-type.*

Proof. It is a similar argument to the one presented in [3, Lemma 3.8]. $\square$

The previous result has the following consequence:

Corollary 2.7. *All lovely pairs of T are elementary equivalent.*

We write T_P for the common complete theory of all lovely pairs of T.

To axiomatize T_P we follow the ideas of [14, Prop 2.15]. Here we need an additional hypothesis, namely, we assume that T eliminates $\exists^\infty$. It follows from a result of Hrushovski (Lemma 4.2 [9]) that a supersimple theory of SU-rank 1 satisfies this property. It is also well known, by uniform finiteness, that a dense O-minimal theory eliminates this quantifier. Recall that whenever T eliminates $\exists^\infty$ the expression *the formula $\varphi(x, \bar{b})$ is nonalgebraic* is first order.

Theorem 2.8. *Assume T eliminates $\exists^\infty$. Then the theory T_P is axiomatized by:*

(1) T'
(2) *For all L-formulas $\varphi(x,\vec{y})$*
$\forall\vec{y}(\varphi(x,\vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x,\vec{y})\wedge x\in P))$.
(3) *For all L-formulas $\varphi(x,\vec{y})$, $m\in\omega$, and all L-formulas $\psi(x,z_1,\dots,z_m,\vec{y})$ such that for some $n\in\omega$ $\forall\vec{z}\forall\vec{y}\exists^{\leq n}x\psi(x,\vec{z},\vec{y})$ (so $\psi(x,\vec{y},\vec{z})$ is always algebraic in x)*
$\forall\vec{y}(\varphi(x,\vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x,\vec{y})\wedge x\notin P)\wedge$
$\forall w_1\dots\forall w_m\in P\neg\psi(x,w_1,\dots,w_m,\vec{y}))$

The second scheme of axioms corresponds to the density property and the third scheme to the extension property.

Proof. Let T_0 be the theory axiomatized by the scheme of axioms described above.

Claim Any lovely T-pair is a model of T_0.

Let $(M,P(M))$ be a lovely T-pair. Clearly it is a model of T'. Now let $\varphi(x,\vec{y})$ be a formula, let $\vec{b}\in M_{\vec{y}}$ and assume that $\varphi(x,\vec{b})$ is non-algebraic. Let $B=\text{acl}(\vec{b})$ and let $p(x)$ be a non algebraic L-type over B extending $\varphi(x,\vec{b})$. Since $(M,P(M))$ is a lovely pair, by the density property $p(x)$ is realized in $P(M)$ and thus the second axiom holds. Now assume that $\psi(x,\vec{z},\vec{y})$ is a formula such that there is n with the property that for all $\vec{z}$, $\vec{y}$ there are at most n realizations of $\psi(x,\vec{z},\vec{y})$. Let $\varphi(x,\vec{y})$ be a formula, let $\vec{b}\in M_{\vec{y}}$ be such that $\varphi(x,\vec{b})$ is non-algebraic. Let $B=\text{acl}(\vec{b})$ and let $p(x)$ be a non algebraic L-type over B extending $\varphi(x,\vec{b})$. By the extension property there is $c\in M$ realizing p and independent from $P(M)$ over B. Let $\vec{d}\in P(M)_{\vec{z}}$, then c is not algebraic over $\vec{d}\vec{b}$, so $M\models\neg\psi(c,\vec{d},\vec{b})$ and the third axiom holds.

Claim Any $\aleph_0$-saturated model of T_0 is a lovely pair.

Let $(M,P(M))\models T_0$ be $\aleph_0$-saturated and let $A\subset M$ be algebraically closed and finite dimensional. Let $p(x)$ be a non-algebraic L-type over A. First consider the L_P partial type $p(x)\wedge P(x)$. By the second axiom this partial type is finitely realizable and by $\aleph_0$-saturation it is realized in $(M,P(M))$. Thus $(M,P(M))$ satisfies the density property. Now consider the partial type $p(x)\cup\{\forall\vec{w}\in P\neg\psi(x,\vec{w},\vec{a}) : \psi$ is as in (3), $\vec{a}\in A_{\vec{y}}\}$. By the third axiom this type is finitely realizable in $(M,P(M))$ and by $\aleph_0$-saturation it is realized in $(M,P(M))$. Thus $(M,P(M))$ satisfies the extension property. □

We now compare lovely pairs with the dense pairs studied by van den Dries in [6]. We start by recalling some definitions from that paper:

Let $L=\{<,0,1,+,-,\dots\}$ be a language and let T be an O-minimal L-theory that extends the theory of ordered abelian groups with a positive element 1.

Definition 2.9. A *dense pair* is an elementary pair (so $P(M)\preceq M$) such that $P(M)\neq M$ and $P(M)$ is dense in M.

Clearly any lovely T pair $(M,P(M))$ is a dense pair. It is proved in [6, Theorem 2.5] that the common theory of dense pairs is complete and thus it coincides with T_P. The study of T_P can be seen as a generalization of van den Dries' work on dense pairs of O-minimal structures.

3. Definable sets

Fix T a þ-rank one theory that eliminates quantifiers and eliminates $\exists^\infty$ and let $(M,P(M)\models T_P$. Our next goal is to obtain a description of definable subsets of M and $P(M)$ in the language L_P.

We start by considering the L_P-definable subsets of M, we follow the ideas from [3, Corollary 3.11]. We will extend the language adding new relation symbols. Let L'_P be L_P together with new relation symbols $R_\varphi(\vec{y})$ for each L-formula $\varphi(\vec{x},\vec{y})$. Let T'_p be the theory T_P together with the sentences $\forall \vec{y}(R_\varphi(\vec{y}) \leftrightarrow \exists \vec{x}(P(\vec{x}) \wedge \varphi(\vec{x},\vec{y})))$. Since T_p is a complete theory so is T'_p. We will show that T'_p has quantifier elimination. We should point out that this result is also proved in [6, Theorem 2.5] for dense pairs of O-minimal structures that extends the theory of ordered abelian groups.

Lemma 3.1. *Let $(M, P(M))$, $(N, P(N))$ be lovely pairs. Let $\vec{a}$, $\vec{b}$ be tuples of the same arity from M, N respectively. Then the following are equivalent:*

(1) *$\vec{a}$, $\vec{b}$ have the same quantifier-free L'_P-type.*
(2) *$\vec{a}$, $\vec{b}$ have the same L_P-type.*

Proof. Clearly (ii) implies (i). Assume (i). Since L has quantifier elimination, $\mathrm{tp}(\vec{a}) = \mathrm{tp}(\vec{b})$. Since T is super-rosy, there is $A \subset P(M)$ finite such that $\mathrm{tp}(\vec{a}/P(M))$ does not þ-fork over A. Let $q(\vec{z},\vec{a})$ be the L-type of A over $\vec{a}$. Since the quantifier free L'_P type of $\vec{a}$ agrees with the quantifier free L'_P type of $\vec{b}$, $q(\vec{z},\vec{b})$ does not þ-fork over $P(N)$. Since $(N, P(N))$ is a lovely pair, $q(\vec{z},\vec{b})$ is realized in $P(N)$, say by B.

Claim $\vec{b}$ is þ-free from $P(N)$ over B.

Say $\vec{b} = (b_1, \ldots, b_n)$ and assume that for some $k \leq n$, $(b_1, \ldots, b_k)$ are B independent and $\vec{b} \in \mathrm{acl}(B, b_1, \ldots, b_k)$. If the claim does not hold, $\dim(\vec{b}/B \cup P(N)) < k$ say $b_k \in \mathrm{acl}(b_1, \ldots, b_{k-1}, B, P(N))$. Let $d_1, \ldots, d_m \in P(N)$ such that $b_k \in \mathrm{acl}(b_1, \ldots, b_{k-1}, B, d_1, \ldots, d_m)$. Since the quantifier free L'_P type of $\vec{a}, A$ agrees with the quantifier free L'_P type of $\vec{b}, B$, there are $c_1, \ldots, c_m \in P(M)$ such that $a_k \in \mathrm{acl}(a_1, \ldots, a_{k-1}, A, d_1, \ldots, d_m)$, a contradiction.

Also note that $\vec{a}A$, $\vec{b}B$ have the same quantifier free L_P-type, so the result follows from Lemma 2.6. □

Now we are interested in the L_P-definable subsets of $P(M)$. For this material we follow the presentation from [6, Theorem 2].

Proposition 3.2. *Let $(M, P(M))$ be a lovely pair and let $Y \subset P(M)^n$ be L_P-definable. Then there is $X \subset M^n$ L-definable such that $Y = X \cap P(M)^n$.*

Proof. Let $(M_1, P(M_1)) \succeq (M, P(M))$ be κ-saturated where $\kappa > |M| + |L|$ and let $\vec{a}, \vec{b} \in P(M_1)^n$ such that $\mathrm{tp}(\vec{a}/M) = \mathrm{tp}(\vec{b}/M)$. We will prove that $\mathrm{tp}_P(\vec{a}/M) = \mathrm{tp}_P(\vec{b}/M)$ and the result will follow by compactness. Since $\vec{a}, \vec{b} \in P(M_1)^n$, we get that $M\vec{a}$, $M\vec{b}$ are P-independent sets and thus by Lemma 2.6 we get $\mathrm{tp}_P(\vec{a}/M) = \mathrm{tp}_P(\vec{b}/M)$. □

4. More on definable sets: the O-minimal case

Fix T an O-minimal theory that eliminates the quantifier $\exists^\infty$.

Definition 4.1. Let $(M, P(M))$ be a lovely pair of models of T. A definable set $D \subset M^k$ is *small* if and only if there is some m, and an L-definable function $f : M^m \to M^k$ such that $D \subset f(P(M)^m)$. Let F be a cell and let $S \subset F$ be definable. We say S is *large in* F if $F \setminus S$ is small. A definable subset $D \subset M^k$ is *basic small* if it is small and of the form $\exists y_1 \in P \ldots \exists y_n \in P \varphi(\vec{x},\vec{y})$, where $\varphi(\vec{x},\vec{y})$ is an L-formula.

The definition above is what is called $P(M)$-*bound* in [4] and it turns out to be equivalent to the notion of small set from [4] (see Corollary 2.16). Note that if $D_1, D_2 \subset M^k$ are small their union is also small.

We need to refine the description of L_P-definable subsets of M that we obtained in the previous section. In particular, we want to generalize Theorem 4 of [6] to general lovely pairs of O-minimal structures. We will follow the strategy from [6] and we start by reproving Lemma 4.3 of [6]. The proof we present is the one given in [6], we include it for completeness.

Lemma 4.2. *Let $X \subset M$ be small. Then X is a finite union of sets of the form $f(P(M)^m \cap E)$ where E is an L-definable open cell in M^m and $f : E \to M$ is L-definable and continuous.*

Proof. Since X is small, $X \subset f(P(M)^m)$ for some L-definable function f from M^m into M. Thus we may write $X = f(X')$ for some L_P-definable set $X' \subset P(M)^m$. By Proposition 3.2 we have $X' = P(M)^m \cap Y$ for some L-definable $Y \subset M^m$. The rest of the proof is by induction on m. The case $m = 0$ is trivial, as X is either empty or a single point. So assume the result holds for values lower than m and we will prove it for m. We can subdivide Y into smaller cells E so that $f \restriction_E$ is continuous. If E is an open cell in M^m we get the conclusion of the lemma. If E is not open and $\dim(E) = d < m$, there are indices $1 \leq i_1 < i_2 < \cdots < i_d \leq m$ such that the projection map $\pi : M^m \to M^d$, $\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_d})$ is homeomorphism from E into the open cell $E' = \pi(E)$ of M^d. Let μ be the inverse of this map. Then $f(P(M)^m \cap E) = (f \circ \mu)(P(M)^d \cap E' \cap \mu^{-1}(P(M)^m))$ and by Proposition 3.2 there is an L-definable set $F' \subset E'$ such that $P(M)^d \cap E' \cap \mu^{-1}(P(M)^m) = P(M)^d \cap F'$. By the induction hypothesis, $f(P(M)^m \cap E) = (f \circ \mu)(P(M)^d \cap F')$ is of the desired form. □

Lemma 4.3. *Let $C \subset M^k$ be a cell. Then there is a partition $C_1, \ldots, C_n$ of C into cells such that $C_i \cap P(M)^k$ is either empty or a dense subset of C_i.*

Proof. The proof is by induction on k. The result is clear for $k = 0$. Assume now that the result holds for values smaller than or equal to k and we will prove it for $k+1$. First assume that C is the set of realizations of the formula $f(y_1, \ldots, y_k) < x < g(y_1, \ldots, y_k)$ for $\vec{y}$ in a cell D and f, g continuous functions. By induction hypothesis we need to consider two cases. If $D \cap P(M)^k$ is dense in D, then $C \cap P(M)^{k+1}$ is dense in C. If $D \cap P(M)^k$ is empty, then so is $C \cap P(M)^{k+1}$.

Now assume that C is of the form $x = f(y_1, \ldots, y_k)$ for $\vec{y}$ in a cell D and f a continuous function. Then there is $d \leq k$ and there are indices $1 \leq i_1 < i_2 < \cdots < i_d \leq k+1$ such that the projection map $\pi : M^{k+1} \to M^d$, $\pi(x_1, \ldots, x_{k+1}) = (x_{i_1}, \ldots, x_{i_d})$ is homeomorphism from C into the open cell $C' = \pi(C)$ of M^d. Let μ be the inverse of this map. Note that μ is a definable function. Then $P(M)^{k+1} \cap C = \mu(P(M)^d \cap C' \cap \mu^{-1}(P(M)^{k+1}))$ and by Proposition 3.2 there is an L-definable set $F \subset C'$ such that $P(M)^d \cap C' \cap \mu^{-1}(P(M)) = P(M)^d \cap F$. By the induction hypothesis we can find a finite partition F into cells $\{F_j : j \leq n_1\}$ such that either $F_j \cap P(M)^d = \emptyset$ or $F_j \cap P(M)^d$ is dense in F_j. Furthermore, we can extend the partition $\{F_j : j \in J\}$ to a partition $\{C'_i : i \leq n_2\}$ of C' with the same properties. Since μ is a homeomorphism, $\{\mu(C'_j) : j \in J\}$ forms a partition of C into cells. Let $C_j = \mu(C'_j)$. Note that if $C_k \cap P(M)^{k+1} \neq \emptyset$, then $\pi(C_k) \cap P(M)^d \cap \mu^{-1}(P(M)^{k+1}) \neq \emptyset$, so $\pi(C_k) = F_j$ for some j such that

$F_j \cap P(M)^d$ is dense in F_j. Then $\mu(F_j \cap P(M)^d)$ is a dense subset of C_j. Since $P(M)^d \cap F_j \subset P(M)^d \cap C' \cap \mu^{-1}(P(M)^{k+1})$, $\mu(F_j \cap P(M)^d) \subset P(M)^{k+1}$, so $C_j \cap P(M)^{k+1}$ is a dense subset of C_j. □

Now we generalize Lemma 2.15 from [4]:

Proposition 4.4. *Let $D \subset M$ be definable in $(M, P(M))$ over $\vec{d}$. Then there is a partition $-\infty = a_0 < \cdots < a_n = \infty$ and basic small dense sets $S_1, \ldots, S_n$ such that $D \cap [a_{i-1}, a_i]$ is either contained in the set S_i or contains the set $S_i^c \cap [a_{i-1}, a_i]$, and each S_i is defined from $\vec{d}$.*

Proof. We first show the result for sets D defined by formulas of the form $\exists y_1 \ldots \exists y_n P(y_1) \wedge \cdots \wedge P(y_n) \wedge \varphi(y_1, \ldots, y_n, x)$, where $\varphi(y_1, \ldots, y_n, x)$ is a cell.

Assume the cell $\varphi(y_1, \ldots, y_n, x)$ is of the form $f(y_1, \ldots, y_n) < x < g(y_1, \ldots, y_n)$ for $\vec{y}$ in a cell C and f, g continuous functions. Then, after subdividing C if necessary, we obtain two cases. If $P(M)^n \cap C$ is empty, then D is empty. If $P(M)^n \cap C$ is dense in C, then D is an open interval.

Now assume that the cell $\varphi(y_1, \ldots, y_n, x)$ is of the form $x = f(y_1, \ldots, y_n)$ for $\vec{y}$ in a cell C and f is a continuous function, which is either constant, strictly increasing or strictly decreasing. As above, after subdividing C if necessary, we obtain the following cases. If $P(M)^n \cap C$ is empty, then D is empty. If $P(M)^n \cap C$ is dense in C and f is constant, then D is a point. If f is strictly monotone, then D is a dense small subset of $f(C)$.

Clearly if the conclusion of the Proposition holds for a set D, then it also holds for the complement of D. It remains to see what happens with intersections. Assume that D_1, D_2 are definable over $\vec{d}$ and that there is a partition $-\infty = a_0 < \cdots < a_n = \infty$ and basic small dense sets $S_{11}, \ldots, S_{1n}$, $S_{21}, \ldots, S_{2n}$ as prescribed by the Proposition for D_1, D_2 respectively. If $D_1 \cap [a_i, a_{i+1}] \subset S_{i1}$, then $(D_1 \cap D_2) \cap [a_i, a_{i+1}] \subset S_{i1}$. On the other hand, if $D_1 \cap [a_i, a_{i+1}] \supset S_{i1}^c \cap [a_i, a_{i+1}]$, $D_2 \cap [a_i, a_{i+1}] \supset S_{i2}^c \cap [a_i, a_{i+1}]$, then $D_1 \cap D_2 \cap [a_i, a_{i+1}] \supset (S_{i1} \cup S_{i2})^c \cap [a_i, a_{i+1}]$. □

Proposition 4.5. *If $X \subset M$ is L_P-definable and small, then there is a partition $-\infty = b_0 < b_1 < \cdots < b_{k+1} = \infty$ of M such that for each $i = 0, \ldots, k$, either $X \cap (b_i, b_{i+1}) = \emptyset$, or $X \cap (b_i, b_{i+1})$ as well as $(b_i, b_{i+1}) \setminus X$ are dense in (b_i, b_{i+1}). If $X \subset M$ is L_P-definable then there is a partition $-\infty = b_0 < b_1 < \cdots < b_{k+1} = \infty$ of M such that for each $i = 0, \ldots, k$, either $X \cap (b_i, b_{i+1}) = \emptyset$, or $X \cap (b_i, b_{i+1}) = (b_i, b_{i+1})$ or $X \cap (b_i, b_{i+1})$ as well as $(b_i, b_{i+1}) \setminus X$ are dense in (b_i, b_{i+1}).*

Proof. Let $X \subset P(M)$ be small. By Lemma 4.2 we can write X as a finite union of sets $f(P(M)^m \cap E)$ where $E \subset M^m$ is an open cell and f is L-definable continuous function. If X is a single point there is nothing to prove, so we may assume that $f(E)$ is an interval possibly with endpoints. The set $f(P(M)^m \cap E)$ is dense in $f(E)$ and by the extension property $f(E) \setminus f(P(M)^m \cap E)$ is also dense in $f(E)$. The second part of the Proposition follows from the first part and from Proposition 4.4. □

5. Þ-rank

In [4, Theorem 3'] it is shown that:

Theorem 5.1. *Suppose that $(R, +, \ldots)$ is an o-minimal expansion of a group in the language L. Consider the expansion $\mathfrak{R} = (R, P, +, \ldots)$ in the language $L_P = L \cup \{P\}$ where P is a unary predicate such that:*

(1) *$P(R)$ is small, and contained in some interval, $(a,\infty) \subseteq R$, in which it is dense.*
(2) *Each L_P-formula $\psi(x)$ is equivalent to a boolean combination of formulas of the form $\exists \vec{y}\big(P(y_1) \wedge \cdots \wedge P(y_n) \wedge \varphi(x,\vec{y})\big)$ where φ is an L-formula.*
(3) *For each definable $D \subseteq P(R)^k$ there is an L-definable set E such that $D = E \cap P(R)^k$*

Then $\mathfrak{R}$ is rosy of þ-rank less than or equal to ω and þ-rank of $P(R)$ is 1. Moreover, if $\mathfrak{R}$ includes a field structure, þ-rank of $\mathfrak{R}$ equals ω.

Our goal in this section is to prove that for $(M, P(M)) \models T_p$, þ-rank $(x = x) = \omega$ and þ-rank$(P(x)) = 1$. Our approach follows the ideas of [4], but we need to modify slightly the proofs as we do not have necessarily the structure of an ordered abelian group.

Notation 5.2. *For $(M, P(M)) \models T_P$, $a \in M$, $B \subset M$ we write* $\mathrm{tp}(a/B)$ *for the L-type of a over B and* $\mathrm{tp}_P(a/B)$ *for the L_P-type of a over B.*

We start by recalling some technical results from [7] on þ-rank:

Fact 5.3. *Let T be a complete theory and let $N \models T$. If $D \subset N^k$ is definable and has þ-rank α, then D^n has þ-rank at least αn and equality holds if $\alpha = 1$.*

Fact 5.4. *Let T be a complete theory and let $N \models T$. Let $D \subset N^k$, $E \subset N^l$ be definable and assume there is a definable function $f : D \twoheadrightarrow E$. Then if D has þ-rank α, then E has þ-rank $\leq \alpha$. Furthermore, if the fibers are finite , we have equality.*

The following Lemma is a generalization of Lemma 44 in [4].

Lemma 5.5. *If $\varphi(x,\vec{b})$ $\varphi(x,\vec{b})$ is an infinite set definable in L, then $\varphi(x,\vec{b})$ does not þ-divide over the empty set.*

Proof. Assume, for a contradiction, that $\varphi(x,\vec{b})$ does þ-divide over the empty set. So $\mathrm{tp}_P(\vec{b})$ is non-algebraic and there is some $\theta(\vec{y},\vec{c}) \in \mathrm{tp}(\vec{b})$ and some $k \in \mathbb{N}$ such that whenever $\vec{b}_1,\ldots,\vec{b}_k$ are distinct elements of $\theta(M^{\mathrm{eq}}_{\vec{y}},\vec{c})$, we have that $\varphi(x,\vec{b}_1) \wedge \cdots \wedge \varphi(x,\vec{b}_k)$ is inconsistent. Since φ defines an infinite L-definable set, by the O-minimality of $M|_L$, it defines a finite collection of points and open intervals.

We may assume that each $\varphi(x,\vec{b})$ defines a single interval, modifying φ and θ if necessary. Since $(M,<)$ is a linear order, after modifying φ and θ we may assume that $\varphi(x,\vec{b})$ 2-þ-divides. Then $\exists \vec{y}(\theta(\vec{y}) \wedge \varphi(x,\vec{y}))$ defines an infinite union of disjoint open intervals, a contradiction with Proposition 4.5. □

Theorem 5.6. $\mathfrak{M} = (M, P(M))$ is rosy of þ-rank less than or equal to ω and þ-rank of $P(M)$ is 1.

Proof. First we wish to show that the þ-rank of $P(M)$ is 1. For a contradiction, suppose that some formula $\varphi(x,\vec{b})$ which defines a infinite subset of $P(M)$ þ-divides over the empty set, where $\vec{b}$ may come from any sort in $(M, P(M))^{eq}$. Say that k, $\theta(\vec{y},\vec{c})$ are such that $\bigwedge_{i<k} \varphi(x,\vec{b}_i)$ is inconsistent for any k distinct elements $\vec{b}_1,\ldots,\vec{b}_k$ satisfying $\theta(x,\vec{c})$.

Then, by Proposition 3.2 $\varphi(x,\vec{b})$ is of the form $\psi(x,\vec{b}) \cap P(M)$, where $\psi(x,\vec{b})$ is an L formula. Without loss of generality, we may assume that $\psi(x,\vec{b})$ defines a single open interval. By the previous lemma, $\psi(x,\vec{b})$ does not þ-divide, so we may find an infinite set $\{\vec{b}_i : i \in K\}$ such that $\vec{b}_i \in \theta(M^{\mathrm{eq}}_{\vec{y}}, \vec{c})$ and $\bigwedge_{i\in K} \psi(M, \vec{b}_i)$ is nonempty and, hence, contains an open set (d_1, d_2).

Then $\bigwedge_{i\in K} \varphi(M, \vec{b}_i) = \bigwedge_{i\in K}(\psi(M, \vec{b}_i) \cap P(M)) \supset P(M) \cap (d_1, d_2) \neq \emptyset$, so $\bigwedge_{i\in K} \varphi(M, \vec{b}_i)$ is non-empty, a contradiction.

Second, we wish to show that the þ-rank of $x = x$ is no larger than ω. Suppose that $\varphi(x, \vec{b})$ k-þ-divides over the empty set, where, again, $\vec{b}$ may come from any sort in $(M, P(M))^{eq}$. We observe that it suffices to show that $D_{\vec{b}} := \varphi(M, \vec{b})$ must be a small set and by Fact 5.4 and Fact 5.3 we can conclude that it has finite þ-rank. Then we will have shown that any formula, $\varphi(x, \vec{b})$, which þ-divides has finite þ-rank, and, thus, þ-rank$\big(x = x\big) \leq \omega$.

Now assume for a contradiction that $\varphi(x, \vec{b})$ is not a small set. By Proposition 4.4 there is some open interval $I_{\vec{b}}$ such that $D_{\vec{b}}$ is large in $I_{\vec{b}}$. Suppose that $\theta(\vec{y}, \vec{c})$ is such that for any $\vec{b}_1, \ldots, \vec{b}_k$ different realizations of $\theta(\vec{y}, \vec{c})$, one has

$$D_{\vec{b}_1} \cap \cdots \cap D_{\vec{b}_k} = \emptyset.$$

Claim $J := I_{\vec{b}_1} \cap \cdots \cap I_{\vec{b}_k} = \emptyset$.

Otherwise $\mathbb{J}$ is an open interval (d_1, d_2). Let $S_{\vec{b}}$ be a small set such that $D_{\vec{b}} = I_b \setminus S_{\vec{b}}$. Then $(D_{\vec{b}_1} \cap \cdots \cap D_{\vec{b}_k}) \cap (d_1, d_2) = J \setminus (S_{\vec{b}_1} \cup \cdots \cup S_{\vec{b}_k}) \neq \emptyset$ by the extension property.

Thus, if $\psi(x, \vec{b})$ defines $I_{\vec{b}}$, we see that $\psi(x, \vec{b})$ also þ-divides. But since intervals are L-definable, this contradicts the previous lemma. Thus we conclude that þ-rank$\big(x = x\big)$ is no greater than ω.

$\square$

Note that we also proved the following result, that will prove useful later

Corollary 5.7. *If $\varphi(x, \vec{b})$ þ-forks over A, then $\varphi(x, \vec{b})$ defines a small set.*

6. Þ-rank and the trichotomy Theorem

There is strong relationship between the pregeometry associated to an SU-rank one theory and the rank of the associated lovely pair. It was shown by Buechler in [2] that for a strongly minimal theory T, T_P is totally transcendental and that $MR(T_P) = 1$ if T is trivial, $MR(T_P) = 2$ is T is locally modular non-trivial and that $MR(T_P) = \omega$ in all other cases. This result was generalized by Vassiliev in [14], where he showed that for T a simple theory of SU-rank one, $SU(T_P) = 1$ if T is trivial, $SU(T_P) = 2$ is T is locally modular non-trivial and that $SU(T_P) = \omega$ in all other cases.

In [4] Berenstein, Ealy and Günaydin showed that for $T = Th(\mathbb{R}, +, 0, 1, <)$, þ-rank$(T_P) = 2$ and for $T = Th(\mathbb{R}, +, \times, 0, 1, <)$, þ-rank$(T_P) = \omega$. When these results were proved, the authors were interested in showing that there was an analogy between the rank of the dense pairs of O-minimal theories and the ranks of lovely pairs of simple theory of SU-rank one.

Our goal in this subsection is to study, for $(M, P(M))$ a lovely pair and $a \in M$, the relation between the rank of $\mathrm{tp}_P(a)$ and the local L-structure that M induces on a neighborhood of a. According to the Trichotomomy Theorem of Peterzil and Stacherko [11, 12], for any a in M, either a is trivial (in which case we prove that $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a)) = 1$ for a non-algebraic), or there is a convex neighborhood of a where the structure is an ordered vector space over an ordered division ring (in which case we show that $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a)) = 2$ for a sufficiently general), or there is a neighborhood where the structure is that of an expansion of a real closed field (and then $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a)) = \omega$ for generic a).

We start with some preliminary observations.

Proposition 6.1. *Assume that the structure around a is that of an ordered vector space over an ordered division ring $\mathcal{R}$. Then for $B \subset M$ $a \in \mathrm{scl}(B)$ if and only if there is $b \in \mathrm{dcl}(B)$ and $n \in \mathcal{R}^{>0}$ such that $a \in b + P(M)/n$.*

Proof. Right to left is clear. Now assume that $a \in \mathrm{scl}(B)$. By Proposition 4.4, a is contained in S, a basic small set defined over B. Let $\exists \vec{y} \in P(M)^k(\varphi(x, \vec{y}))$ be a formula defining S. For each $\vec{g}$, $\varphi(M, \vec{g})$ is a finite union of points and intervals. However, if for any $\vec{g}$ in $P(M)^k$, $\varphi(M, \vec{g})$ contains a non-empty open interval, then S is not small. Thus, we may reduce to the case where $\varphi(x, \vec{y})$ is $x = f(\vec{y})$, where

$$f(\vec{y}) = b + \sum_{i=1}^{k} \frac{m_i}{n_i} y_i$$

for some $b \in \mathrm{dcl}(B)$, $m_i \in \mathcal{R}$ and $n_i \in \mathcal{R}^{>0}$. Let n be the least common multiple of the n_i. Thus $f(G^k)$ is contained in $b + P(M)/n$, and $a \in b + P(M)/n$. $\square$

We are ready to show our main Theorem for this section:

Main Theorem Let M be an O-minimal structure, let $P(M) \preceq M$ and assume that $(M, P(M))$ is a lovely pair. Given $a \in M$ the following holds:

(1) If M is trivial in a neighborhood of a, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 1$ and equality holds if a is not algebraic.
(2) If M induces the structure of an ordered vector space over an ordered division ring in a neighborhood of a, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 2$ and equality holds if $a \notin \mathrm{scl}(\emptyset)$.
(3) If M induces the structure of an O-minimal expansion of a real closed field in a neighborhood of a, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq \omega$ and equality holds if $a \notin \mathrm{scl}(\emptyset)$.

Proof. (1) Trivial case. If $a \in \mathrm{scl}(\emptyset)$ then by triviality there is $b \in P(M)$ such that $a \in \mathrm{dcl}(b)$. Since þ-$rk(P(M)) = 1$, we get $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 1$. So assume that $a \notin \mathrm{scl}(\emptyset)$ and that $B \subset M$ is such that $\mathrm{tp}(a/B)$ þ-forks over $\emptyset$. Then $a \in \mathrm{scl}(B)$, so $a \in \mathrm{dcl}(B \cup P(M))$. Since M is trivial in a neighborhood of a and $a \notin P(M)$ we get that $a \in \mathrm{dcl}(B)$ so $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a/B)) = 0$ and $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 1$.

(2) Locally modular case. By Proposition 6.1, every small subset of M has þ-rank at most one, and by Corollary 5.7, a þ-forking extension of $\mathrm{tp}(a)$ must include a formula defining a small set. Thus $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 2$. We want to show that for $a \notin \mathrm{scl}(\emptyset)$ we have $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) = 2$.

Let $g \in P(M)$ be such that $\mathrm{tp}(g)$ is non-algebraic and $g \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}} a$. Then we get $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}(a/g)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}(a + g/g))$.

Claim $a + g \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}} g$.

Otherwise, by Corollary 5.7 we would have $a + g \in \mathrm{scl}(g) = \mathrm{scl}(\emptyset)$, and thus $a \in \mathrm{scl}(\emptyset)$, a contradiction.

Thus $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a + g/g)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a + g))$, and it suffices to show that $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a + g)) = 2$.

Consider the chain $\text{tp}_P(a+g/\emptyset) \subset \text{tp}_P(a+g/a) \subset \text{tp}_P(a+g/a,g)$. First note that $\text{tp}_P(a+g/a)$ contains the formula saying $x \in P(M)+a$. This formula is true of $a+g$ and þ-divides over the empty set. Thus, $\text{tp}_P(a+g/a)$ is a þ-forking extension of $\text{tp}_P(a+g)$. Second, note that $\text{tp}_P(a+g/a,g)$ is algebraic, and thus it is a þ-forking extension of $\text{tp}_P(a+g/a)$. We just proved that the chain described above þ-forks at every step, so $\text{U}^{\text{þ}}(\text{tp}(a)) = 2$.

(3) Field case. Assume that in an open neighborhood V of a, M induces a field structure. By Corollary 5.7 for any set B, if $\text{tp}(a/B)$ þ-forks over $\emptyset$ we get that $a \in \text{scl}(B)$. In particular, there are $g_1, \dots, g_n \in P(M)$ such that $a \in \text{dcl}(g_1, \dots, g_n, B)$. By Lascar's inequality this implies $\text{U}^{\text{þ}}(\text{tp}_P(a/B)) \leq n$ and we get that $\text{U}^{\text{þ}}(\text{tp}_P(a)) \leq \omega$.

To show the other direction, let us assume that $a \notin \text{scl}(\emptyset)$ and we show that for every $n \geq 0$, there exists B such that $\text{U}^{\text{þ}}(\text{tp}_P(a/B)) = n$. Let $c_1, \dots, c_n \in V$ be such that $c_1 \notin \text{scl}(a)$, $c_2 \notin \text{scl}(a, c_1), \dots, c_n \notin \text{scl}(a, c_1, \dots, c_{n_1})$ (these elements exist by the extension property). Now let $g_1, \dots, g_n \in P(M)$ be independent from each other and independent from $c_1, \dots, c_n, a$.

Claim $g_i \in \text{dcl}(c_1 g_1 + \dots c_n g_n, c_1, \dots, c_n)$ for $i \leq n$.

Consider the equation $c_1 x_1 + \dots + c_n x_n = c_1 g_1 + \dots c_n g_n$. If there is a solution $(g_1', \dots, g_n')$ in $P(M)^n$ different from $(g_1, \dots, g_n)$ we get $c_1(g_1 - g_1') + \dots + c_n(g_n - g_n') = 0$ and $g_j - g_j' \neq 0$ for some $j \leq n$. Then $c_j \in \text{scl}(c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_n)$ and this is a contradiction.

Let $d = a + c_1 g_1 + \dots c_n g_n$ and $B = \{d, c_1, \dots, c_n\}$. Then a and $c_1 g_1 + \dots c_n g_n$ are interdefinable over B and by the claim both this sets are interdefinable with $\{g_1, \dots, g_n\}$ over B. Thus $\text{U}^{\text{þ}}(\text{tp}(a/B)) = \text{U}^{\text{þ}}(\text{tp}(g_1, \dots, g_n/B))$. On the other hand, $a \notin \text{scl}\{c_1, \dots, c_n\}$, so $d \notin \text{scl}\{c_1, \dots, c_n\}$ and $d \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}} \{c_1, \dots, c_n, g_1, \dots, g_n\}$. This implies that $\text{U}^{\text{þ}}(\text{tp}(g_1, \dots, g_n/B)) = \text{U}^{\text{þ}}(\text{tp}(g_1, \dots, g_n/\{c_1, \dots, c_n\})) = n$ and thus $\text{U}^{\text{þ}}(\text{tp}(a/B)) = n$ as we wanted. □

We end this section with an example of a trivial dense pair.

Lemma 6.2. *The structure* $(\mathbb{R}, <, \mathbb{Q})$ *is a lovely pair.*

Proof. We first show that the Density property holds. Let $A \subset R$ be finite, say $A = \{a_1, \dots, a_k\}$ with $a_1 < a_2 < \dots < a_k$ and let $q \in S_1(A)$ be non-algebraic. Then q is describing an open interval, either $(-\infty, a_1)$, (a_i, a_{i+1}) for some i, or (a_k, ∞). Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there is $c \in P(\mathbb{R}) = \mathbb{Q}$ such that $c \models q$.

Now we show that the Extension property holds. Let $A \subset R$ be finite, say $A = \{a_1, \dots, a_k\}$ with $a_1 < a_2 < \dots < a_k$ and let $q \in S_1(A)$ be non-algebraic. Then q describes an open interval with endpoints in the set A. Since $\mathbb{R} \setminus (A \cup \mathbb{Q})$ is dense in $\mathbb{R}$, we can find a realization of q in $\mathbb{R} \setminus (A \cup \mathbb{Q})$. □

It is easy to check that the pair $(\mathbb{R}, <, \mathbb{Q})$ is an expansion of $(\mathbb{R}, <)$ with a generic predicate (in the sense of Chatzidakis, Pillay [5]). It is proved in [5, Corollary 2.6 part 3] that for such expansions, the algebraic closure in the extended language L_p coincides with the algebraic closure in the language L. In particular, algebraic independence inside the structure $(\mathbb{R}, <, \mathbb{Q})$ satisfies the usual properties of an independence relation for *real elements.* On the other side, Sergio Fratarcangeli showed

[8] that expansions of O-minimal structures with a generic predicate eliminate imaginaries. Thus algebraic independence inside the structure $(\mathbb{R}, <, \mathbb{Q})$ defines an independence relation that extends to an independence relation for all elements in $(\mathbb{R}, <, \mathbb{Q})^{eq}$ and thus T_P is rosy and acl_L-independence oincides with thorn-forking independence in the sense of T_P. Furthermore þ-rank$(Th((\mathbb{R}, <, \mathbb{Q}))) = 1$, as we expected from the Main Theorem.

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