# Definable proper actions and equivariant definable Tietze extension

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## Abstract

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field $R$. Let $G$ be a definable group and $X$ a definable proper definable $G$ set. We prove that $X$ has only finitely many orbit types. We also prove equivariant definable Tietze extension theorem.

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## 1 . Introduction.

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field $R$.

General references on o-minimal structures are [2], [3], and also see [9]. Any definable category is a generalization of the semialgebraic category and the definable category on $\mathcal{R} = (R, +, \cdot, <)$ coincides with the semialgebraic one. It is known in [8] that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers.

Let $G$ be a definable group. A *definable G set* means a pair consisting of a definable set $X$ and a group action $\phi : G \times X \to X$ such that $\phi$ is definable. A definable map between definable sets is called *definably proper* if the inverse image of every definably compact definable set is definably compact. We call a definable $G$ set $X$ a *proper definable G set* if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (gx, x)$ is definably proper.

Let $G$ be a definable group. We can define *orbit types* as well as $G$ is definably compact ([5]).

**Theorem 1.1.** *Let $G$ be a definable group. Then every proper definable $G$ set $X$ has only finitely many orbit types.*

Theorem 1.1 is proved the case where $R$ is the field $\mathbb{R}$ of real numbers ([5]).

The following theorem is an equivariant version of definable Tietze extension theorem [1]

**Theorem 1.2.** *Let $G$ be a definably compact definable group, $X$ a definable $G$ set and $A$ a $G$ invariant definably compact definable subset of $X$. Every $G$ invariant definable function $f : A \to R$ is extensible to a $G$ invariant definable function $F : X \to R$ with $F|A = f$.*

## 2 . Proof of results.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of $f$ ($\subset X \times Y \subset R^n \times R^m$) is a definable set. A group $G$ is a *definable group* if $G$ is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A definable subset $X$ of $R^n$ is *definably compact* if for every definable map $f : (a, b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in $X$, where $(a, b)_R = \{x \in R | a \leq x < b\}, -\infty \leq a < b \leq \infty$. A definable subset $X$ of $R^n$ is definably compact if and only if $X$ is closed and bounded ([7]). Note that if $X$ is a definably compact definable set and $f : X \to Y$ is a definable map, then $f(X)$ is definably compact.

We say that two homogeneous proper definable $G$ sets are *equivalent* if they are definably $G$ homeomorphic. Let $(G/H)$ denote the equivalence class of $G/H$. The set of all equivalence classes of homogeneous proper definable $G$ sets has a natural order defined as $(X) \geq (Y)$ if there exists a definable $G$ map $X \to Y$. By the definition the reflexivity and the transitivity clearly hold If $(X) = (G/H)$ and $(Y) = (G/K)$, then $(X) \geq (Y)$ if and only if $H$ is conjugate to a definable subgroup of $K$. By a way similar to the proof of 4.1 [5], we have the following lemma.

**Lemma 2.1.** *Let $G$ be a definable group, $H$ a definable subgroup of $G$ and $g \in G$. If $gHg^{-1} \subset H$, then $gHg^{-1} = H$.*

By Lemma 2.1, the anti-symmetry is true.

By a way to similar to the proof of 1.1 [5], we have Theorem 1.1.

Note that every definable subgroup of a definable group is closed ([6]) and a closed subgroup of a definable group is not necessarily definable. For example $\mathbb{Z}$ is a closed subgroup of $\mathbb{R}$ but not a definable subgroup of $\mathbb{R}$.

Recall existence of definable quotient.

**Theorem 2.2** (Existence of definable quotient, 10.2.18 [2])**.** *Let $G$ be a definably compact definable group and $X$ a definable $G$ set. Then the orbit space $X/G$ exists as a definable set, and the orbit map $\pi : X \to X/G$ is definable, surjective and definably proper.*

The following theorem is the topological case of Tietze extension theorem.

**Theorem 2.3** (Tietze extension theorem)**.** *Let $X$ be a normal space and $A$ a closed subset of $X$. Then every continuous map $f : A \to \mathbb{R}$ is extensible to a continuous map $F : X \to \mathbb{R}$ with $F|A = f$.*

The following theorem is the definable case of Tietze extension theorem.

**Theorem 2.4** (Definable Tietze extension theorem, [1])**.** *Let $A$ be a definable closed subset of $R^n$. Then every definable map $f : A \to R$ is extensible to a definable map $F : R^n \to R$ with $F|A = f$.*

A definable map $f : X \to Y$ is *definably closed* if for any definable closed subset $A$ of $X$, $f(A)$ is a definable closed subset of $Y$.

**Theorem 2.5** ([4])**.** *Let $f : X \to Y$ be a definable map. Then $f$ is definably proper if and only if $f$ is definably closed and has definably compact fibers.*

*Proof of Theorem* 1.2. By Theorem 2.1, $X/G$ exists as a definable set in $R^n$ and the projection $\pi : X \to X/G$ is a surjective definable definably proper map. By Theorem 2.4 and $A$ is definably compact, $\pi(A)$ is a definable closed subset of $R^n$. Since $f$ is a $G$ invariant definable map, it induces a definable map $f' : f(A) \to R$ with $f = \pi \circ f'$. By Theorem 2.2, there exists a definable map $F : R^n \to R$ with $F|f(A) = f'$. Hence $H = \pi \circ F$ is the required map. □

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