Jonsson Theories in Positive Logic

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Résumé. Cet article est une introduction générale et élémentaire à la Logique Positive, où seuls sont considérés les énoncés dits h-inductifs, ce qui permet d'étendre aux homomorphismes les notions de Théorie des Modèles classiquement associées aux plongements, et en particulier celle de modèles existentiellement clos primitivement définie par Abraham Robinson, qui deviennent ici les modèles positivement clos. Il diffère de l'exposé de BEN YAACOV & POIZAT 2007, qu'il résume parfois, par son caractère plus schématique, parce qu'il tient compte de résultats récents en ce domaine, et parce qu'il se focalise sur ce que deviennent les théories de Jonsson dans un contexte positif ; son appendice contient des outils utiles pour déterminer les théories h-inductives de structures données, dans une situation concrète.

Қысқа түсіндірме. Бұл мақала позитивті логикаға жалпы және элементарлы кіріспе болып табыла отырып, тек қана h-индуктивті сөйлем қаралатын, сонымен қатар классикалық еңгізулер ұғымдар гомоморфизмдерге дейін кеңейтілген. Дербес жағдайда экзистенцианалды тұйық модельдер ұғымын өз кезінде бірінші А. Робинсон енгізген, мақалада позитивті тұйық модельмен ауыстырылады. Бұл мақала BEN YAACOV & POIZAT 2007 жылы дүние көрген мақаласынан өзгеше айырмашылығы бар, өйткені мазмұны қысылған қысқаша түрде жазылған және осы салада соңғы пайда болған нәтижелерден тұрады, қорыта келе бұл мақала Йонсондық теорияны позитивтендірілуіне бағытталған; нақты мәтінде, берілген структураның h-индуктивті теорияны анықтау үшін қосымша бөлімінде кейбір практикалық құралдарын құрайды.

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Abstract. This paper is a general and elementary introduction to Positive Logic, where only the said h-inductive sentences are under consideration, allowing the extension to homomorphisms of model-theoretic notions which are classically associated to embeddings, in particular the notion of existentially closed models primitively defined by Abraham Robinson, which become here positively closed models. It differs from the exposition paper of BEN YAACOV & POIZAT 2007, that it summarizes sometimes, because it is more concise, because it accounts for recent results in this domain, and because it is oriented towards the positivisation of Jonsson theories. It contains in its appendix some practical tools for the determination of the h-inductive theory of a given structure, in a concrete context.

Аннотация. Эта статья является общим и элементарным введением в позитивную логику, где рассматриваются только h-индуктивные предложения,а также классические понятия вложений расширяются до гомоморфизмов.В частности понятие экзистенциально замкнутой модели, впервые введенной А. Робинсоном, в статье заменяется позитивно замкнутой моделью. Статья отличается от статьи BEN YAACOV & POIZAT 2007 года, потому что данное изложение более сжатое и содержит последние результаты в этой области, и потому что статья ориентирована на позитивизацию Йонсоновских теорий. Статья содержит в приложении некоторые практические инструменты для определения h-индуктивный теории данной структуры, в конкретном контексте.

Keywords. Model Theory, Inductive Limit, Compacity, Jonsson Theory, Amalgams, Back-and-forth Games

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0. Why Positive Logic ?

Abraham Robinson, one of the founding fathers of the Theory of Models, developed in the middle of the last century a now classic body of knowledge : his inductive theories, existentially closed models, model-complete theories, etc. are based on the notions of embedding, existential formula and inductive sentence (for a general reference, see ROBINSON 1956). Note that it is not sufficient to consider only existential sentences ; on the contrary a great role is given to the universal sentences, which are precisely their negations !

To-day, this Robinson's theory appears to be a special case of the Theory of Models for Positive Logic, where general homomorphisms are considered, and not only embeddings. In this logic, we consider only positive existential formulae, and a special kind of inductive sentences which we call h-inductive (for the reason that they are preserved under inductive limits of homomorphisms, and not only under inductive limits of embeddings, as are the inductive sentences in Robinson's setting). More precisely, Robinson's theory corresponds to the special positive case where the language is expanded by relation symbols interpreting the negations of the atomic formulae, transforming these negations into positive beings (the fact that two positive formulae are complementary is expressed by h-inductive axioms).

Under the present view of Positive Logic, Robinson's restriction appears to be highly unadequate, since on one hand all the results obtained by himself and his followers extends in a quite straightforward manner to the general positive case, which in fact demands no more efforts that the robinsonian setting, and on the other this positive case is truly wider, allowing practical applications that Robinson's frame does not permit.

Indeed, a typical feature of Positive Model Theory is that it does not distinguish substantially between definable and infinitely definable sets, since it is innocuous for it to expand the language by the introduction of a new relation symbol for any infinite conjunction of positive formulae, provided that one is only interested in the sufficiently positively saturated models. This cannot be done in Robinson's frame, which would force us to introduce also the negation of this new atomic formula, affecting drastically the model-theoretic properties of the structures under consideration.

Certainly, Abraham Robinson had all the tools at his disposal to develop his theory in the general positive frame ; but he did not, so forcing us to rewrite all of his results with a positive tag. This is an easy, but not a vain, exercise ; an immediate reward is that the positive proofs go smoother : Positive Logic is very direct, as it allows more freedom in the literal sense (for instance, free amalgams exist for homomorphisms, not for embeddings !) ; examples and counter-examples are easy to find in Positive Logic, and experience shows that, after some elaboration, they can be transformed into examples and counterexamples valid in the robinsonian case. In some sense, Positive Logic throws some clarifying light on Robinson's theory itself !

The fact is that we are embarrassed, whenever we extend to the general positive context some notion that was originally introduced in Robinson's setting, to be compelled to mark it with prefixes like pos. (for positive) or h- (for homomorphism), since our intimate belief is that it is the robinsonian special notion that should be marked ; sometimes we drop the positive marks, hoping that no confusion will arise.

As was said above, the motivation of our enterprise is that the model theory of Positive Logic offers really new situations, which occurs naturally even when we work in the classical full First Order logic with negation, and that this negative logic is unable to treat by itself. We admit that, when first seen, these new phenomenons may have a certain power of disturbance on the minds of the logicians which have been educated under Robinson's portico.

H-inductive theories have been sporadically studied (under a different name, or no name at all) since the fifties, notably in the works of the categorical model-theorists of the Province de Québec in the seventies (see MAKKAI & REYES 1977 for a general reference). A remarkable premonition of Positive Logic is found in SHELAH 1975, and later HRUSHOVSKI 1997 and PILLAY 2000 announce its advent ; but a systematic treatment of its model theory has been undertaken only at the beginning of this millenium by Itaï Ben Yaacov (BEN YAACOV 2003, 2003bis, etc.), one of his original motivations being the model-theoric study of quotients by infinitely definable equivalence relations.

The present paper provides an elementary exposition to the theory of models for Positive Logic, and contains the extension of many results previously obtained by Robinson's disciples, in particular a positivisation of Jonsson theories (for a survey of Jonsson theories in robinsonian context, see YESHKEEV 2009) ; in so doing, it summarizes more than sixty years of activity in Model Theory. It also accounts for some very recent works in Positive Logic. The proofs are most of the time sketchy, or omitted when we can provide a reference (mainly from the detailed exposition paper BEN YAACOV & POIZAT 2007) ; in a few cases we quote arguments from published works, when we think that they are essential for a reasonable self-countenance of our paper.

We shall not adopt here a strict yaacobian ideology ; that is, we shall not focus our attention only to the positively saturated models of our theories, nor attempt to give an account of forking in this context (BEN YAACOV 2003ter, 2004, etc.), nor of the most subtle developments of Positive Logic in connection with metric topologies (BEN YAACOV 2005, BEN YAACOV & UTSYATSOV 2010, for a beginning). We shall simply, possibly in a naive way, consider the h-inductives theories and their models as objects of study per se, and even describe some elementary classes or properties, in the language of the full First Order Logic which negation, which are associated to them !

1. Basic definitions

1.1. Homomorphisms and positive formulae

We consider £-structures in a fixed but arbitrary language £ , involving individual constants, functions and relations ; there is always a binary relation symbol = denoting equality. To avoid useless complications, we adopt the usual convention that the underlying set of a structure should be non-void.

By definition, an *homomorphism* between two £-structures M and N is a map h from M to N such that, for every individual constant c , every function symbol f and every relation symbol r of £ , and every tuple $\underline{a}$ of elements of M , the following holds :

- $h(c_M) = c_N$
- $h(f_M(a_1, \dots a_m)) = f_N(h(a_1), \dots h(a_m))$
- if $M \models r_M(a_1, \dots a_n)$ then $N \models r_N(h(a_1), \dots h(a_n))$.

In other words, any atomic formula satisfied by $\underline{a}$ in M is satisfied by $h(\underline{a})$ in N .

When there is an homomorphism from M to N, we say that N is a *continuation of* M . We observe that a continuation of M is nothing but a model of the *positive diagram* $Diag^+(M)$ of M , which is the set of atomic sentences satified by M in the language £(M) obtained by adding to £ individual constants naming the elements of M .

We say that the homomorphism h is an *embedding* if moreover h is injective and, for every tuple $\underline{a}$ in M and every relation symbol r in the language, we have : $M \models r_M(a_1, \dots a_n)$ if and only if $N \models r_N(h(a_1), \dots h(a_n))$; in other words, an atomic formula is satisfied by $\underline{a}$ in M if and only if it is satisfied by $h(\underline{a})$ in N . An *isomorphism* is a bijective embedding. If there is an embedding from M to N , that is, if M is isomorphic to a substructure of N , we say as usual that N is an *extension* of M .

By definition, a *positive formula* is a formula which is obtained from the atomic formulae by the use of ∨ , ∧ and ∃ (Caveat : no universal quantifiers). It can be written in prenex form as $(\exists \underline{x})\ \varphi(\underline{x})$, where φ is positive quantifier-free ; φ in turn can be written as a finite disjunction of finite conjunctions of atomic formulae.

An immediate, but fundamental, observation is the following : *If* h *is an homomorphism from* M *to* N *and* $\underline{a}$ *is a tuple of elements of* M *, then every positive formula satisfied by* $\underline{a}$ *in* M *is satisfied by* $h(\underline{a})$ *in* N . It can be proved easily by induction on the complexity of the formula, or by the consideration of its prenex form.

If every tuple $\underline{a}$ in M satisfies the same positive formulae than its image $h(\underline{a})$ in N , we say that h is a *pure* homomorphism, or an *immersion*. An immersion is in particular an embedding.

Isomorphisms, and more generally elementary embeddings for the full First Order Logic with negation, are obvious immersions ; other examples are the *retractile* homomorphisms h from M to N , for which we can find an homomorphism g from N to M such that g o h is an automorphism of M (see the definition of the local retromorphisms in the Appendix).

We say that M is *positively closed* (in short, pc) in a class Γ of structures if every homomorphism from M to any N in Γ is pure.[3]

We say that M is *h-maximal,* or *diagram maximal* (in short, dm) in Γ if every homomorphism h from M to any N in Γ is an embedding. [4]

If the class Γ contains the *Terminus Structure*, formed by a single element satisfying every atomic formulae, then Terminus is the only pc, and even dm, element of Γ . We observe that Terminus satisfies all the sentences of the form $(\forall \underline{x})(\exists \underline{y})\ \varphi(\underline{x},\underline{y})$, where φ is positive quantifier-free.

More generally, we say that an element M of Γ is *positively universal* in Γ if it is a continuation of every element of Γ . We say that M is the *final,* or *homogeneous-universal,* structure in Γ if moreover, when f is an homomorphism between two structures A and B in Γ , any homomorphism g from A into M can be extended to B via f , that is, there is an homomorphism h from B into M such that g = h o f ; since then any homomorphism from M into M is an automorphism, the final structure, when it exists, is unique up to isomorphism ; it may be not the only pc structure in the class. We say that the structure M is *terminal* in Γ if, for any A in Γ , there is a unique homomorphism from A into M ; in this case, M is final and every homomorphism from M into M is the identity.

We say dually that an element M of Γ is (positively) *prime* in Γ if every element of Γ is a continuation of M ; we say that M is *initial* if moreover, when f is an homomorphism between A and B in Γ , any homomorphism g from M into B factors through A , i.e. there is an homomorphism h from A into M such that g = f o g ; since then any homomorphism from M into M is an automorphism, the initial model, when it exists, is unique. We say that M is the *core model* of Γ if, for every A in Γ , there is a unique homomorphism from M into A ; in this case M is initial, and every homomorphism from M to M is the identity.

We say that a set of sentences has a final, terminal, prime, initial or core model if the class of its models has one. For instance, if Γ is the class of

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[3] Positively closed structures were called *positivement existentiellement closes* in BEN YAACOV & POIZAT 2007, and pec in some other papers.

[4] H-maximal structures have been introduced in KUNGOZHIN 2013 ; their name was chosen because their positive diagram is maximal for a structure continuable in a member of Γ .

groups, considered in the usual language, every group is universal and prime, but the trivial group Terminus is the only final and the only initial group.

A useful fact in Positive Logic is that, provided that the language contains at least one individual constant, any set Ta of atomic sentences has a core model. Indeed, let us assume for simplification that the language £ contains no functions ; the underlying set of the core model will be the quotient of the set of individual constants of £ by the equivalence relation generated (using symmetry and transitivity) by the equations of the kind $c_i = c_j$ which belong to Ta ; and an atomic formula $r(\gamma_1, \ldots \gamma_n)$ is true in it only if $\gamma_1 , \ldots \gamma_n$ are the respective images of some constants $c_1 , \ldots c_n$ such that $r(c_1, \ldots c_n)$ belongs to Ta . When the language contains functions, we consider the free algebra generated by the constants, and quotient it by the equivalence relation generated by symmetry, transitivity and substitution from the equalities of terms present in Ta .

1.2. H-inductive and h-universal theories

An *h-inductive*[5] *sentence* is by definition equivalent to a finite conjunction of sentences each of them declaring that a certain positively defined set is included into another. Such a simple h-inductive sentence has the form $(\forall \underline{x}) [(\exists \underline{y}) \varphi(\underline{x},\underline{y}) \Rightarrow (\exists \underline{z}) \psi(\underline{x},\underline{z})]$, and its prenex form is of the kind $(\forall \underline{u})(\exists \underline{v}) \neg\varphi'(\underline{u}) \vee \psi'(\underline{u},\underline{v})$, where φ , φ' , ψ and ψ' are positive quantifier-free ; note that the existential quantifier spans only the positive part of the disjunction.

It is easy to see that the disjunction or the conjunction of two h-inductive sentences is also h-inductive ; but the conjunction of two simple h-inductive sentences may not be equivalent to a simple one (to find a counter-example is a good exercice in Boolean Calculus, that we leave to our readers !).

In Positive Logic only h-inductive sentences are under consideration.

When we replace φ' by the tautology $(\exists t)\ t = t$, we obtain sentences of the kind $(\forall \underline{u})(\exists \underline{v}) \psi(\underline{u},\underline{v})$, where $\psi(\underline{u},\underline{v})$ is positive quantifier-free, which are therefore h-inductive ; we shall call them *positive inductive,* in spite of the fact that they are not always positive formulae in our sense (when the universal quantifier is present) ; this kind of sentences declares that everybody satisfies some positive condition. In particular, sentences like $(\forall \underline{u}) \psi(\underline{u})$ or $(\exists \underline{v}) \psi(\underline{v})$, where ψ is positive quantifier-free, are h-inductive. We are in the necessity to introduce other sentences than the positive inductive ones, with a somehow negative content, if we wish to reach a final destination other than Terminus.

If the positive part $\psi'(\underline{u},\underline{v})$ of the disjunction is absent, we shall nevertheless consider the sentence as a special case of an h-inductive sentence,

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5 We did not dare to risk hinductive, nor huniversal below !

which we call *h-universal* ; it has the form $(\forall \underline{x})\ [\neg(\exists \underline{y})\ \varphi(\underline{x},\underline{y})]$, or otherwise $(\forall \underline{u})\ \neg\varphi'(\underline{u})$, that is $\neg(\exists \underline{u})\ \varphi'(\underline{u})$.[6] An h-universal sentence declares that a certain positively defined set is empty. It would be unwise to call them positively universal, since they are precisely the negations of the positive sentences ! In particular, the negation of an atomic sentence is h-universal.

The conjunction or the disjunction of two h-universal sentences is equivalent to an h-universal sentence.

If an h-universal sentence is satisfied in a continuation of M , then it is also true in M . One can see that this property characterizes the h-universal sentences among the sentences of the full First Order logic with negation (BEN YAACOV & POIZAT 2007, Lemme 21) ; for a positivist fanatic this reciprocal is not truely important, and in fact we shall not use it.

We remark in passing that an h-inductive theory T , that is, a theory which is axiomatized by h-inductive sentences, expresses in fact the equivalence of certain positive formulae, since $(\forall \underline{x})\ \varphi(\underline{x}) \Rightarrow \psi(\underline{x})$ can be written as $(\forall \underline{x})\ \varphi(\underline{x}) \vee \psi(\underline{x}) \Leftrightarrow \psi(\underline{x})$; in particular, an h-universal sentence declares that a certain positive formula is equivalent to the antilogy, that we consider as positive !

This being said, it is easily seen, writing its boolean part in conjunctive form, that any *universal* sentance $(\forall \underline{x})\ \nu(\underline{x})$, where ν is quantifier-free but possibly involving the negation, is h-inductive (but not always h-universal !). By contrast, $(\exists x)\ r_1(x) \wedge \neg r_2(x)$ and $(\exists x)(\forall y)\ r(x,y)$ are not h-inductive.

A property, and in fact a characteristic property among the sentences of full First Order Logic (see CHANG & KEISLER 1973, ex. 5.2.24, BEN YAACOV & POIZAT 2007, Théorème 23), of the h-inductive sentences is that they are preserved under inductive limits of *homomorphisms* ; for one who knows what these limits are, it is absolutely clear that an h-universal sentence has this preservation property, that the two sentences above have not.

The only utility of inductive limits is to establish the equivalence of the Axiom of Choice to the following **Continuation Principle** : *Every model of an h-inductive theory* T *can be continued into a pc model of this theory.*

Since the truth of an axiom, by definition, cannot be proved, and since the Axiom of Choice is admitted in any domain of Modern Mathematics where it does not produce undesirable disturbances (and Model Theory is such a domain), a confortable position for our reader will be to skip the next section

[6] It is observed in BEN YAACOV & POIZAT 2007 that an h-universal sentence has the form $(\forall \underline{u})\ \varphi'(\underline{u}) \Rightarrow \bot$, where $\bot$ is a 0-ary relation symbol, that is, a propositionnal constant, denoting the antilogy, that has to be added to the language (and provides a positive definition to the empty set) ; in this paper, we shall avoid the use of 0-ary relation symbols, even if they prove to be useful for Morleysation and the proof of the Compactness Theorem.

and admit the Continuation Principle. This next section is included for those who are curious to establish the equivalence of the Continuation Principle to a better known formulation of Zermelo's Axiom of Choice.

We conclude the present section by the description of two kinds of remarkable h-inductives sentences.

The first kind expresses that an (n+1)-ary positive formula $\varphi(\underline{x},y)$ is the graph of an n-ary function : $(\forall \underline{x},y,z)\ \varphi(\underline{x},y) \wedge \varphi(\underline{x},z) \Rightarrow y = z$ together with $(\forall \underline{x})(\exists y)\ \varphi(\underline{x},y)$. Therefore there is no loss of generality if we assume that the language contains no functions, since the substitution of a function by its graph translates a positive formula into a positive one, and vice versa.

The second kind expresses that two positive formulae $\varphi(\underline{x})$ and $\psi(\underline{x})$ are each the negation of the other : $\neg\ (\exists \underline{x})\ \varphi(\underline{x}) \wedge \psi(\underline{x})$ together with $(\forall \underline{x})\ \varphi(\underline{x}) \vee \psi(\underline{x})$; note that the second sentence is *not* h-universal even when $\varphi(\underline{x})$ and $\psi(\underline{x})$ are quantifier-free. Such sentences permit to transform, by a mere expansion of the language, any inductive theory T in Robinson's sense[7] into an h-inductive theory : consider the language £' obtained by adding for each relation symbol r of £ (including the equality symbol =) a new relation symbol r' , and the h-inductive theory T' formed by the axioms saying that r' interprets the complement of r , plus the axioms obtained by replacing in the axioms of T the negations of the atomic subformulae by their new positive expressions. T and T' have essentially the same models, since any model of T can be expanded in a unique way to a model of T' , and reciprocally any model of T' , when considered as an £-structure, is a model of T . Homomorphisms between models of T' correspond to embeddings between models of T , and the pc models of T' corresponds to the ec (existentially closed) models of T .

1.3. Inductive limits and compacity

Consider an ascending sequence of groups, each of them being a subgroup of its follower :

$$G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n \subseteq G_{n+1} \subseteq \ldots ;$$

the union G of the G_n is quite naturally equipped with a structure of group whose each G_n is a subgroup. We call G the *inductive limit* of the sequence.

Similarly, when we have an ascending sequence of £-structures in an arbitrary language £ , each of them being a restriction of the next :

$$M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n \subseteq M_{n+1} \subseteq \ldots$$

[7] A theory axiomatized by sentences of the form $(\forall \underline{x})(\exists \underline{y})\ \nu(\underline{x},\underline{y})$, where ν is quantifier-free, but possibly using negation.

we can define the inductive limit M of the sequence as their common extension to the union of their underlying sets : an atomic formula is satisfied by a tuple of elements of M when it is satisfied in M_n for n large enough.

When we have a sequence of embeddings :

$$M_0 \xrightarrow{e_0} M_1 \xrightarrow{e_1} \ldots \xrightarrow{e_{n-1}} M_n \xrightarrow{e_n} M_{n+1} \xrightarrow{e_{n+1}} \ldots$$

we can assimilate $e_n(M_n)$ to a substructure of M_{n+1} and define the inductive limit in the same way, but it is a little harder to convince oneself that the inductive limit exists also when we have a sequence of homomorphisms :

$$M_0 \xrightarrow{h_0} M_1 \xrightarrow{h_1} \ldots \xrightarrow{h_{n-1}} M_n \xrightarrow{h_n} M_{n+1} \xrightarrow{h_{n+1}} \ldots$$

especially when they are not injective.

In this case, as in fact in the preceding ones, the best way to define the inductive limit is to add to the language £ individual constants naming the elements of the M_n , in a disjoint manner ; the inductive limit is the core model of the theory formed by the positive diagrams of the M_n and the equalities $c = f_n(c)$. The inductive limit is the least common continuation of the M_n .

Since an atomic formula (in particular an equality) is true in the limit M provided it is true finally in the M_n , it is visible that if all the M_n satisfy a certain h-inductive sentence, so does the limit. This is in particular the case for h-universal sentences.

A regrettable complication is that it is not enough to consider limits of countable sequences of homomorphisms. We must define (practically in the same manner) the inductive limit of an arbitrary sequences M_i indexed by a totally ordered set I , with an homomorphism h_{ji} from M_i into M_j whenever $i < j$, such that $h_{kj} \circ h_{ji} = h_{ki}$ if $i < j < k$.

By definition, an *h-inductive* class is a class of £-structure closed under inductive limits of homomorphisms. So is the class of models of any h-inductive theory T (but the full First Order theory of an h-inductive class is not necessarily axiomatized by h-inductives sentences : see Example 5 in section 2.7). And also :

Lemma 1. *If* T *is an h-inductive theory, its pc models, and also its dm models, form h-inductive classes.*

Proof. If a tuple $\underline{a}$ in the limit satisfies a positive formula $\varphi(\underline{a})$, there is in some M_i a tuple $\underline{\alpha}$ which projects on $\underline{a}$ and satisfies $\varphi(\underline{\alpha})$ in M_i . **Fin**

Using her[8] favourite variant of the Axiom of Choice, the reader will have no pain to show that : *In an h-inductive class, every point can be continued into a pc element.* In particular, pc models exists for any consistent h-inductive theory.

In BEN YAACOV & POIZAT 2007, an example was given for the reciprocal, that is, the implication from the Continuation Principle towards the Axiom of Choice. We alter it so that it functions in Robinson's setting as well.

> **Example 1.** Consider a non-void set A and an equivalence relation E between the elements of A ; the language £ contains a predicate[9] $r_a(x)$ for each a in A ; the h-universal theory T consists in the axioms $\neg (\exists x,y)\ r_a(x) \wedge r_b(y)$, for each pair (a,b) of distinct elements of A which are congruent modulo E ; T has a pc model (with one point !), or a dm model, or an ec model, if and only if A has a choice-subset for E . The variant T' of T , obtained by adding to it the axioms $\neg (\exists x)\ r_a(x) \wedge r_b(x)$ whenever $a \neq b$, does not need a set theoric hypothesis to have dm models.

In BEN YAACOV & POIZAT 2007, a straightforward proof of the Compacity Theorem for full First Order Logic, based on inductive limits, was proposed. Since functions can be replaced by their graph, it is harmless to assume that the language contains only relations and individual constants, a thing which simplifies greatly the description of the core models ; we can assume also that at least one constant is present in the language, since by definition structures are non-void. Then the proof proceeds in three stages :

The first (Lemme 2) is quite obvious : if every finite subset of a theory Ta∪Tu , where Ta is composed of atomic sentences and Tu is composed of h-universal sentences, has a model, then the core model of Ta is a model of Tu .

The second step (Lemme 3) consists in proving that, if T is an h-inductive theory, and if Tu is the set of h-universal sentences which are consequence of a finite subset of T , then any pc model of Tu is a model of T . Of course, when the Compacity Theorem is known, Tu is simply the set of h-universal consequences of T .

When T is finitely consistent,[10] Tu is also finitely consistent, and consistent thank to the first step ; at this stage the Compacity Theorem for h-inductive theories is obtained by an application of the Continuation Principle to Tu (this is the only use of the Axiom of Choice in the proof).

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[8] Non-sexist languages like French, or Qazaq, would use here the same possessive for both genders !

[9] We recall that, by definition, a predicate is a unary relation symbol.

[10] By "finitely consistent" we mean that every finite fragment of T has a model.

What remains is to interpret the full First Order Logic into Positive Logic by an expansion of the language called *Positive Morleysation.* It consists in the introduction of a new relation symbol r_φ for each formula φ of the logic with negation ; provided that we do not use directly universal quantifiers in our formulae, but discompose them as $\neg\exists\neg$, the conditions compelling r_φ to be interpreted by φ are h-inductive, so that to any theory T of the full First Order Logic is associated an h-inductive theory T' in the expanded language, which has practically the same models as T ; T' is consistent if and only if T is consistent, and T' is finitely consistent if and only if T is finitely consistent, so that the proof is completed.

The only thing that is altered by the morleysation process is the notion of homomorphism : the homomorphisms between the models of T' corresponds exactly to the elementary embeddings between the models of T .

Morleysation is named after the paper MORLEY & VAUGHT 1962, where this process is applied. In fact, this kind of expansion of the language belongs to the prehistory of the Theory of Models ; for instance, it is considered as a well-known fact in the famous paper of Kurt Gödel (GÖDEL 1930), in the proof of the Completeness Theorem as well as in the proof of the Compacity Theorem, where, as far as validity is concerned, it can be assumed that the sentences are inductive.

Positive Morleysation was also well-known from the group of model-theorists of Montreal in the seventies (see MAKKAI & REYES 1977) ; they were studying the h-inductive sentences under the name of *coherent sentences.*

1.4. Positive saturation

We say that a structure M is *positively* ω-*saturated* if, for every tuple $\underline{a}$ in M , every set $\Phi = \{ ... \varphi_i(x,\underline{a}) , ... \}$ of positive formulae which is finitely satisfiable in M is realized in M . Note that, in this definition, we can replace the variable x by a fixed tuple $\underline{x}$ of variables of an arbitrary length.

Lemma 2. *(i) Any pc model of an h-inductive theory* T *can be continued into a pos.* ω-*saturated pc model of* T .
(ii) Any model of an h-inductive theory T *can be embedded, and even immersed, into a pos.* ω-*saturated one.*

Proof. (i) Consider a pc model M_0 of T , and enumerate the sets of positive formulae Φ_0 , ... Φ_λ , ... in one variable, with parameters in M_0 , which are finitely satisfiable in M_0 ; if we assign distinct variables to the Φ_λ , their union is consistent with $\mathrm{Diag}^+(M) \cup T$, so that each of them is realized in some continuation M_1 of M_0 which is a model of T , that we may take pc. We apply the same process to M_1 that we continue in a model M_2 , and repeat. The inductive limit of the M_n is a pc model of T ; it is pos. ω-saturated, because,

since M_n is pc, every set of positive formulae with parameters in M_n, which is consistent with $Diag^+(M_n) \cup T$, is finitely satisfiable in M_n.
(ii) Given a structure M, the h-inductive sentences with parameters in M satisfied in M form an h-inductive theory $T(M)$, in the langage £(M) where names for the elements of M are added to £ ; consider a positive formula $\varphi(\underline{a},\underline{x})$ whith $\underline{a}$ is M : there is some $\underline{b}$ in M satisfying $\varphi(\underline{a},\underline{b})$ unless $\neg (\exists \underline{x}) \varphi(\underline{a},\underline{x})$ belongs to $T(M)$, so that M is obviously a pc model of $T(M)$. We apply the result above. **End**

We can also define positive κ-saturation for any infinite cardinal κ, and prove a similar result of existence. We have the usual limitations of cardinality : typically, for any κ bigger than the size of the language, we obtain κ^+-saturated models of cardinal less than 2^κ, or a κ-saturated model of cardinal at most (yes ! ; see Subsection 2.7) κ when κ is unaccessible.

2. Universal domains

2.1. Companion theories

We say that two h-inductive theories T and T', in a same language £ , are *companion* if every model of one of them can be continued into a model of the other.

Therefore, for any model M of T, the theory formed by T' and $Diag^+(M)$ is consistent ; by compacity, this means that every h-universal consequence of T' is a consequence of T. By symmetry, we see that T and T' are companion if and only if they have the same h-universal consequences.

Moreover, T and T' are companion if and only if they have the same pc models. Indeed, if this is true, they are companion since any model of T can be continued into a pc model of T. Reciprocally, if Tu is the set of h-universal consequences of T, we have seen, in the proof of the Compacity Theorem, that any pc model of Tu is a model of T, and obviously a pc one ; moreover, since any model of Tu can be continued into a model of T, any pc model of T is also pc for Tu. In other words, T and Tu have the same pc models, and so do T and T' if they are companion.

An h-inductive theory T has a minimal[11] companion, which is Tu, and a maximal (h-inductive) one, which is the h-inductive theory Tk of its pc models, that is, the set of h-inductive sentences which are true in each of its pc models (Tk is not necessarily the full First Order theory of the pc models of T : see Example 5 in 2.7) ; Tk is indeed a companion of T since any model of

[11] By an inclusion of theories, we mean the reverse inclusion for their respective classes of models ; that is, we do not distinguish between a theory and its axiomatizations.

T can be continued into a pc model ; it is called, after KAISER 1969, the h-inductive *Kaiser hull* of T . Any h-inductive theory between Tu and Tk is a companion of T ; for instance, the theory Tm of the dm models of T is such a companion.

We define now a companion that does not appear in Robinson's setting : for a reason that will be explained in Section 2.5, we call the *separant* Ts of the h-inductive theory T the union of Tu and of the positive inductive sentences which are consequences of T . The following example shows that the Kaiser hull Tk can be distinct from its separant Tks .

Example 2. The language £ contains an infinite list of constants c_0 , c_1 , ... c_n , ... and two predicates r(x) and s(x) . The h-universal theory Tu consists in $c_m \neq c_n$ for $m < n$, and $\neg r(c_{2m}) \wedge \neg s(c_{2m+1})$ for all m ; Tu has only one pc model, formed by the constants c_{2m} satisfying s , the constants c_{2m+1} satisfying r , and a unique point γ satisfying $r(\gamma)\wedge s(\gamma)$; when we omit γ , we obtain the second h-maximal model of Tu . The sentences $(\forall x)\, r(x)\vee s(x)$ and $(\exists x)\, r(x)\wedge s(x)$ belong to Tks ; other models of Tks contain several copies of γ (they are locally epimorphic to the pc model for reductions of £ to a finite sublanguage ; see Subsection 3.4 of the Appendix) ; therefore, the sentence $(\forall x,y)\ r(x)\wedge r(y)\wedge s(x)\wedge s(y) \Rightarrow x = y$ is in Tk and is not implied by its separant Tks .

2.2. Negative sufficiency

A direct application of the Compacity Theorem gives the following characterization of the pc models :

Lemma 3. *In a pc model* M *of an h-inductive theory* T *, if a tuple* $\underline{a}$ *of elements does not satisfy some positive formula* $\varphi(\underline{x})$ *, it is because it satisfies another positive formula* $\psi(\underline{x})$ *which is contradictory to it :* T *implies the h-universal sentence* $\neg\, (\exists \underline{x})\, \varphi(\underline{x})\wedge\psi(\underline{x})$.

Proof. The theory formed by $\varphi(\underline{a})$, the positive diagram of M and T is contradictory. **End**

In the other direction, it is obvious that a model having this property is pc.

Therefore, a pc model M of an h-inductive theory is *negatively sufficient,* in short ns, in the sense of the following definition : whenever a tuple $\underline{a}$ of elements of M does not satisfy some positive formula $\varphi(\underline{x})$, then it satisfies another positive formula $\psi(\underline{x})$ that M believes to be contradictory with it, that is to say that M satisfies $\neg\, (\exists \underline{x})\, \varphi(\underline{x})\wedge\psi(\underline{x})$.

Lemma 4. *(i) Any structure can be continued into an ns structure satisfying the same h-universal sentences.*
(ii) Two ns structures satisfying the same h-universal sentences are pc models of the same h-inductive theories.

Proof. (i) Any M can be continued into a pc model N of its own h-universal theory ; since N is a continuation of M , it cannot satisfy more h-universal sentences than M ; and N is ns, because, by Lemma 3, an ns structure is nothing but a pc model of its own h-universal theory,
(ii) Suppose that M is a pc model of the h-inductive theory T , i.e. of its h-universal companion Tu , and satisfies $\neg (\exists \underline{x})\ \varphi(\underline{x}) \wedge \psi(\underline{x})$; $\varphi(\underline{x}) \wedge \psi(\underline{x})$ is in contradiction with $Tu \cup Diag^{+}(M)$, and there is a finite fragment $\delta(\underline{y})$ of $Diag^{+}(M)$ such that Tu implies $\neg (\exists \underline{x},\underline{y})\ \varphi(\underline{x}) \wedge \psi(\underline{x}) \wedge \delta(\underline{y})$, or, in other words, that $\psi(\underline{x}) \wedge (\exists \underline{y})\ \delta(\underline{y})$ contradicts $\varphi(\underline{x})$. Therefore, when M is ns, the fact that it is a pc model of T depends only of its h-universal theory : it means that M is a model of Tu , and that, for any pair of positive formulae $\varphi(\underline{x})$ and $\psi(\underline{x})$ which are contradictory in the sense of M , there is a third one, $\delta(\underline{y})$, such that $\neg (\exists \underline{y})\ \delta(\underline{y})$ is untrue in M and that $\varphi(\underline{x})$ and $\psi(\underline{x}) \wedge \delta(\underline{y})$ are contradictory in the sense of Tu . **End**

If the Theory of Models for the full First Order Logic, with negation, consists in the study of elementary embeddings between structures, then the Theory of Models for Positive Logic consists in the study of immersions between ns structures, which, according to the following corollary, are the same thing that homomorphisms between the pc models of some h-inductive theory.

Corollary 5. *Consider a structure* M *immersed into an ns structure* N *; then*
(i) M *is also negatively sufficient ;*
(ii) M *and* N *satisfy the same h-inductive sentences with parameters in* M *;*
(iii) M *and* N *are pc models of the same h-inductive theories.*

Proof. (i) Suppose that $\underline{a}$ in M does not satisfy the positive formula $\varphi(\underline{x})$, in the sense of M , but equivalently in the sense of N ; in N , but also in M , it satisfies some $\psi(\underline{x})$ such that $\neg (\exists \underline{x})\ \varphi(\underline{x}) \wedge \psi(\underline{x})$ is true in N ; this h-universal sentence, being true in a continuation of M , is also true in M .
(ii) Consider two positive formulae $\varphi(\underline{x},\underline{a})$ and $\psi(\underline{x},\underline{a})$ with parameters $\underline{a}$ in M , and the sentence $(\forall \underline{x})\ \varphi(\underline{x},\underline{a}) \Rightarrow \psi(\underline{x},\underline{a})$. If it is false in M , we can find in it $\underline{\alpha}$ satisfying $\varphi(\underline{x},\underline{a})$ and not $\psi(\underline{x},\underline{a})$; $\underline{\alpha}$ satisfying the same in N , the sentence is also false in N . Reciprocally, if the sentence is false in N , we can find in it a $\underline{\beta}$ satisfying $\varphi(\underline{x},\underline{a}) \wedge \psi'(\underline{x},\underline{a})$, for some $\psi'(\underline{x},\underline{y})$ contradictory to $\psi(\underline{x},\underline{y})$ in the sense of N ; M being immersed in N , we can find also such a

tuple $\underline{\beta}'$ in M ; as observed above, ψ' and ψ are also contradictory in the sense of M , which does not satisfy $(\forall \underline{x})\ \varphi(\underline{x},\underline{a}) \Rightarrow \psi(\underline{x},\underline{a})$.
(iii) They have the same h-universal theory. **End**

A point of caution in conclusion of this section : the notion of negative sufficiency is sensitive to the language £ which is used for the description of the structures ; for instance we have observed, in the proof of Lemma 2, that any structure M is ns in the language £(M) !

2.3. Positively model-complete theories

We say that an h-inductive T is *positively model-complete* if every model of T is pc, that is, if every homomorphism between models of T is pure ; in this case, T is obviously equal to its Kaiser hull.

T is pos. model-complete iff to every positive formula $\varphi(\underline{x})$ is associated another one, $\psi(\underline{x})$, such that T declares that each of them is the negation of the other. Indeed, if $\varphi(\underline{x})$ has no positive negation, then by compacity we can find a model M of T with a tuple $\underline{a}$ not satisfying it, and also satisfying no positive formula contradictory to it ; but $\varphi(\underline{a}) \cup \mathrm{Diag}^+(M) \cup T$ is consistent, so that this model M is not pc.

When T is pos. model-complete, it is also equal to its separant, since the fact that two positive formulae are complementary is expressible within Ts ; and, indeed, it is quite obvious that, if any positive formula has a positive negation, then any h-inductive sentence can be replaced by a positive one !

The following example of pos. model-completeness is given in KUNGOZHIN 2013 : if Tu is a finitely axiomatizable h-universal theory in a finite purely relational language, then the class of its dm models and the class of its pc models are elementary (the former being finitely axiomatizable).

The homomorphisms between models of a pos. model-complete theory respect the satisfaction of the formulae of the full First Order Logic with negation : they are *elementary embeddings.* In a pos. model-complete theory, every formula of the full First Order Logic is equivalent to a positive formula, so that positive model-completeness is stronger than the robinsonian notion (where every formula is only equivalent to an existential one).

The effect of positive Morleysation is to transform any theory of the First Order Logic with negation into a pos. model-complete h-inductive theory !

Example 3. The usual axioms of the theory of commutative rings, in the language (+, -, ., 0, 1) , are positive inductive, with the exception of the essential h-universal axiom $0 \neq 1$; atomic formulae are equivalent to polynomial equations with integer coefficients. Since any non-inversible element is sent to 0 in some quotient of the ring, the dm rings are the fields ; in them, the inequation $x \neq y$ is defined positively by the formula

$(\exists z)\ z.(x\text{-}y) = 1$. The pc rings are the algebraically closed fields : this is the content of Hilbert's Nulstellensatz, which states that if K is an algebraically closed field, then any system of polynomial equations in n variables, with coefficients in K , has a solution in K provided that 1 does not belongs to the ideal generated by these polynomials (if not the system cannot have a solution even in a continuation of K ; note that the obstruction to the existence of a solution is expressed by a positive condition on the coefficients of the system). So the theory T of algebraically closed fields is pos. model-complete. Another way to see that is to eliminate the quantifiers by your favourite method, so that any formula is equivalent modulo T to a positive boolean combination of equations and inequations ; then you replace the inequations by their positive (existential) expressions. The elimination of the quantifiers leads to the elimination of the negation !

2.4. Positive Logic and Robinson's Logic

Any h-inductive T is a fortiori inductive in the sense of Robinson, but there is no general reason why its pc models be ec, nor why its ec models be pc, as shows the example of the empty theory in the language of equality (the pc models are reduced to a point, the ec models are infinite). Also, as we have observed, the positive model-completeness of T is a stronger assumption than its model-completeness in the sense of Robinson ; moreover, the amalgamation property for homomorphism (to be defined in the next section) is not the same thing as the amalgamation property for embeddings : none of them implies the other ; similarly, the JCP defined in 2.6 is not the same thing as the JEP, its robinsonian version.

In other words, when we increase the language to make positive the negations of atomic formulae, we may alter the properties of the theory.

But the following easy lemma shows that, if all the models of T are dm, then its pc models and its ec models are the same ; T is pos. model-complete if and only if it is model-complete ; and of course homomorphisms and embeddings are the same thing. In this case, naming the negations of atomic formulae is a benign operation, and we see that Robinson's inductive theories correspond exactly to the special case of h-inductive theories with the property described above.

Lemma 6. *Consider an h-inductive theory* T *and a positive formula* $\varphi(\underline{x})$ *; if for every homomorphisms* h *between models of* T *,* $\varphi(\underline{a})$ *is satisfied iff* $\varphi(h(\underline{a}))$ *is satisfied, then there is a positive formula which in* T *is the negation of* φ *.*

Proof. If $\varphi(\underline{x})$ has no positive negation, then, as was observed above, we can find a model M of T with some $\underline{a}$ not satisfying $\varphi(\underline{x})$ and satisfying no positive formula contradictory to $\varphi(\underline{x})$ in T ; in this case, by compacity again, $T \cup Diag^+(A) \cup \varphi(\underline{a})$ is consistent, in contradiction with our hypothesis. **Fin**

2.5. Spaces of types, amalgamation and separation

Given an h-inductive T and a tuple of variables $\underline{x} = (x_1, \ldots x_n)$, a *complete* n-*type* is a maximal set of (existential) positive formulae $\varphi(\underline{x})$ which is consistent with T (or with any companion of T !). Every type can be realized in some pc model ; by Lemma 3, every tuple in a pc model realizes a complete type.

We put a topology on the sets $S_n(T)$ of types, by declaring that the type satisfying a given positive formula form a basic *closed* set ; the general closed sets are therefore defined by arbitrary (infinite) conjunctions of formulae. We obtain in this way a compact set (English sense : if every finite subfamily of a family of closed sets has a non-void intersection, then the total intersection of the family is non void), that does not necessarily satisfy Hausdorff separation condition.

The separation of the spaces of types is linked to the following property : we say that the h-inductive theory T has the *Amalgamation Property* (for homomorphisms ; in short, AP)[12] if, whenever we consider two homomorphisms f from A to B and g from A two C , where A , B and C are models of T , then we can find a fourth model D of T , and homomorphisms f' from C to D and g' from B to D closing the diagram : $g' \circ f = f' \circ g$.

If T has the Amalgamation Property, every h-inductive theory lying between T and Tk also has the AP , since B and C can be continued into models of Tk . An another obvious remark : if T has AP and M is a model of T , then $T \cup Diag^+(M)$, which is a theory in the language £(M) , has AP.

If T has the AP , a tuple $\underline{a}$ extracted from any model M of T has a unique destiny : if f and g are two homomorphisms from M into some pc models of T , then $f(\underline{a})$ and $g(\underline{a})$ satisfy the same positive formulae. In particular M has a unique h-maximal completion, that is, a unique dm continuation defined on the same underlying set.

It is high time to justify the name we have given to the separant : consider two types p and q in $S_n(T) = S_n(Tk)$ which are separated by two

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[12] BELKASMI 2014 introduces the notion of *amalgamation bases* for an arbitrary h-inductive theory T , which are its models over which can be amamalgamated any pair of continuations within T . They form an inductive class, containing the pc models (see Subsection 3.2), and their h-inductive theory Tb is a companion of T ; in the case of rings, they are the rings with only one maximal ideal, axiomatized by $(\forall x)(\exists y)\ x.y = 1 \vee (1-x).y = 1$.

disjoint open sets ; open sets being defined by possibly infinite disjunctions of formulae, this means that there are two positive formulae $\varphi(\underline{x})$ and $\psi(\underline{x})$ such that p does not satisfies the first, that q does not satisfies the second, but that there are no types which do not satisfy both ; in other words, the positive inductive axiom $(\forall\underline{x})\ \varphi(\underline{x})\vee\psi(\underline{x})$ is true in every pc model, and therefore belongs to Tk ; we shall say that T separates p and q if we can find formulae $\varphi(\underline{x})$ and $\psi(\underline{x})$ as above such that T , and in fact Ts , implies that $(\forall\underline{x})\ \varphi(\underline{x})\vee\psi(\underline{x})$. The following result generalizes slightly the Théorème 20 of BEN YAACOV & POIZAT 2007, stating that the spaces of types are compact Hausdorff if and only if Tk has the Amalgamation Property.

Théorème 7. *An h-inductive theory* T *has the Amalgamation Property if an only iff it separates each pair of its distinct types ;* T *has the* AP *iff* Ts *has the* AP.

Proof. Suppose that T separates the types, and consider A , B and C as above ; we can assume that B and C are pc ; each $\underline{a}$ in A has no choice for its destiny, so that f(A) and g(A) satisfy the same positive formulae, and the theory $T \cup \mathrm{Diag}^+(B) \cup \mathrm{Diag}^+(C) \cup \{f(a) = g(a) / a \in A\}$ is consistent, being finitely interpretable in C ; a model of it is an amalgam.

Reciprocally, suppose that T does not separate the distinct types p and q ; this means that we can find a model A of T with an $\underline{a}$ satisfying only positive formulae which are common to p and q ; by the Lemme 16 of BEN YAACOV & POIZAT 2007, in some continuation B of A the image of $\underline{a}$ is of type p , and in another continuation C its image has type q ; this is an obstacle to amalgamation. **End**

> **Example 4.** Our Example 3 does not have AP, since in a model of Tk a non-constant point not satisfying s(x) can be sent ad libitum to any of the c_{2m} or to γ ; the space $S_1(Tk)$ is formed by the types of the c_m and the type of γ , and its only proper closed subsets are finite. Let us transform it to obtain Hausdorff topologies. The language £ contains now infinitely many constants c_0 , c_1 , ... c_n , ... , a predicate r(x) , and infinitely many other predicates $s_0(x)$, $s_1(x)$, ... $s_n(x)$, ... ; the axioms of T declare that the c_n are pairwise distinct and does not satisfy r , and that s_n is the negation of $x = c_n$; Ts says in addition that r is not empty, and Tk says moreover that all the points in r are equal. These three companions are distinct ; each of them has AP, since their models can be amalgamated into the unique pc model. Tk is not positively model-complete (neither r(x) nor x=y have a positive negation), but the topologies on the spaces of types are easily seen to be Hausdorff : for instance, the space of 1-types is formed by the c_n , which are isolated, and accumulate to the unique type satisfying r .

In Robinson's setting, a well-known theorem of Lyndon states that if T is model-complete and if the set of its universal consequences has AP, then T eliminates the quantifiers. The positivisation of this result is so strong that it looses any interest : every formula will be either tautological or antilogical !

Proposition 8. *If an h-universal theory has the Amalgamation Property, then it has only one pc model ; this model has only one point.*

Proof. Consider the free algebra M_2 for the functions of the language £ generated by the constant of £ plus two points x and y ; on M_2, put the minimal structure, that is, no atomic formula is satisfied except the trivial equalities of terms $t(x,y,\underline{c}) = t(x,y,\underline{c})$; M_2 can be sent homomorphically into any £-structure, and moreover the choice of the images of x and y is arbitrary. Therefore, M_2 is a prime model for every consistent h-universal theory T .

If T has a pc model with two distincts points a and b , we can send (x,y) to (a,a) on one side, and to (a,b) on the other, making the amalgam impossible since (a,b) satisfies a positive formula uncompatible with equality. So if T has the AP, all of its pc models have only one point, and since any pair of them can be amalgamated over M_2, they must be isomorphic. If e is the unique element of the pc model of T , we have no choice for the interpretation of the individual constants and the functions (if T declares that $(\forall x)\ f(x) \neq x$, we cannot amalgamate !) ; if r is a relation symbol in £ , r(e,e, ... e) is true unless T declares that $\neg\ (\exists \underline{x})\ r(\underline{x})$! **End**

2.6. The Joint Continuation Property

We say that an h-inductive theory T has the *Joint Continuation Property* (in short, the JCP) if any two of its models can be simultaneous continued into a third one. We observe that if T has the JCP, then each of its companions has the JCP. Remark also that if T has a prime model and the AP, it has the JCP.

A positively model-complete T has the JCP if and only iff it is complete in the sense of Full First Order Logic, since in this case the JCP means that any pair of models of T have a common elementary extension.

Proposition 9. *An h-inductive theory has the JCP if and only iff any two of its pc models satisfy the same h-universal sentences if and only if they satisfy the same h-inductive sentences.*

Proof. If M and N are two models of our theory T , they can be continued into pc models M' and N' of T ; if M' and N' have the same h-universal theory, the union of their diagrams is consistent with T .

Reciprocally, suppose that T has the JCP ; any two pc models of T can be simultaneously continued into a third, and satisfy the same h-inductive sentences by Corollary 5. **End**

Therefore, the JCP plays the role devoluted to completeness in Full First Order Logic. It insures the uniqueness of the universal domain, that is, of the big κ-positively saturated pc model of cardinality at most (for a second time, yes !) κ , where κ is unaccessible. The mecreants having no faith in the existence of big cardinals will be consolated with the remark that, if T has the JCP, all the pos. ω-saturated pc models of T are in infinite back-and-forth relation, so that they are elementary equivalent, and even more than that ; or if they are lovers of uniqueness, they can take refuge in special models.

JCP is a quite convenient hypothesis ; when we do not assume it, we have to split the theory into its components, corresponding to the various h-universal theories of its pc models.

2.7. Bounded theories

We say that a h-inductive theory T is *unbounded* if it has pc models of arbitrary large cardinality. A *bounded* (that is, not unbouded) theory may have infinite pc models : in the absence of a positive expression for the negation of equality, this does not contradict the Compacity Theorem !

Consider a positive formula $\varphi(x,y)$ which is in contradiction with the equality $x = y$; a *clique* for this formula is a subset A of a model of T such that any pair of distinct elements of A satisfy $\varphi(x,y) \vee \varphi(y,x)$. If the formula has finite cliques of an arbitrary large number of elements, then, by compactness, it has cliques of any infinite cardinality, and any pc model containing such a clique will be at least that big.

Reciprocally, by the Erdos-Rado Theorem, since in a pc model any pair of distinct elements must have a positive reason to be different, an unbounded theory has necessarily a positive formula, uncompatible with equality, which has unbounded cliques ; and, in fact, a pc model of a bounded theory has no more than $2^{\mathrm{card}(£)}$ elements.

Proposition 10. *A bounded h-inductive theory* T *with the JCP has, up to isomorphy, a unique pc model which is universal, embedding all the other models of* T .

Proof. The theory has a pc model M which embeds all the others ; if M is not unique, it can be properly embedded into a copy M_1 of itself ; M_1 is in turn properly embedded into M_2 , etc. ; when we have constructed them we embed the M_n into a copy M_ω of M , and repeat ad libitum, ending with a pc model of a size higher than the cardinality of M : this is impossible. **End**

In the absence of the AP, this universal pc model may not be final, as shows the example of the theory of at least two distinct constants.

We conclude by two simple examples of bounded theories ; in the first, extracted from POIZAT 2010, the universal pc model is the only pos. ω-saturated pc model ; in the second, on the contrary, all the pc models are ω-saturated.

> **Example 5.** Let M be the segment]0 1[of the rational numbers, equipped with their natural order ; in the language £ of the strict order $<$, since $x \neq y$ can be expressed as $x<y \vee x>y$, we have positive quantifier elimination and the theory Tk(M) is the familiar theory of a dense linear order without endpoints ; in the language £' of the loose order $\leq$, nothing positive can force a cut to be filled by two distinct points, so that the h-inductive theory T'k(M) is bounded, its universal pc (in fact, terminal) model being the real segment [0 1] .
>
> In this example, T'k(M) is not the full first order theory of its pc models, since it is unable to express that the order is dense. Indeed, using Theorem 23 of the Appendix, one sees that any two infinite linear orders satisfy the same h-inductive sentences ; this remains true when we name elements, provided that the segments they bound are infinite.

> **Example 6.** The language £ contains infinitely many relation symbols $e_n(x,y)$ and $e'_n(x,y)$; the axioms of T declare that e'_n is the negation of e_n , that each e_n is an equivalence relation, that e_{n+1} refines e_n , each class modulo the second being cut into two classes modulo the first. A model of T is pc, or dm, if and only if two elements congruent modulo all the e_n are equal.

2.8. Infinite Morleysation, dense sets and minimal language

Ben-Yaacov philosophy is that all the properties of the pc ω-saturated models are recoverable from the spaces of types ; by this, we mean that they can be reconstructed from the spaces of types "up to interpretation". Following MUSTAFIN 1998, we suggest to call *semantic properties* of an h-inductive theory, or more exactly of its Kaiser hull, those properties that depend only on the spaces of types, and not on the language. For instance the Amalgamation Property (for Tk) is semantic, since it means that the spaces of types are Hausdorff ; the JCP is also semantic, since it means that two types are always compatible, in other words that the projection from $S_{m+n}(T)$ to $S_m(T) \times S_n(T)$ is always surjective.

By compacity, a clopen subset of $S_n(T)$ must be defined by a formula, so that, when we deal with full First Order Logic, or equivalently with a pos. model-complete theory, we can recover the theory and its models up to interpretation by assigning a name to every clopen subset of the spaces of types.

But in Positive Logic formulae define closed sets that may not be open, so that we must expect that the canonical language will be associated to the closed sets. This is indeed the case thank to the following process of *infinite morleysation* : if T is an h-inductive theory and $F(\underline{x})$ is a closed set of $S_n(T)$, defined by an infinite conjunction of positive formulae $\varphi_i(\underline{x})$, we add to the language a symbol for F , and to the axioms of T all the sentences $(\forall\underline{x})\ F(\underline{x}) \Rightarrow \varphi_i(\underline{x})$ to form a theory T' ; then the following lemma, reproducing the Lemme 25 of BEN-YAACOV & POIZAT 2007, shows that T and T' have essentially the same ω-saturated pc models :

Lemma 11. *The pos.* ω*-saturated pc models of* T' *are the pos.* ω*-saturated pc models of* T *where* $F(\underline{x})$ *is interpreted as the conjunction of the* $\varphi_i(\underline{x})$.

Proof. In a pc model of T' , $F(\underline{x})$ must be interpreted as the conjunction of the $\varphi_i(\underline{x})$. Since any model of T can be transformed into a model of T' , the types of T are consistent with T' , and therefore an ω-saturated pc model of T' must be also ω-saturated for T ; therefore T and T' have the same spaces of types, so that any ω-saturated pc models can be transformed into an ω-saturated pc model of T' . **End**

T' cannot be pos. model-complete when F is not clopen, so that model-completeness is not a semantic property ; but we can characterize semantically the morleysations of model-complete theories as follows : each S_n have a Hausdorff totally disconnected topology, generated by clopen sets ; equality is clopen ; the projection from S_{n+1} onto S_n is open (that is, the image of an open set is open). The minimal language of the theory, up to interpretation, corresponds to the clopen sets of the spaces of types, and its models to the dense subsets of the pos. ω-saturated pc models, which we define in the next paragraph.

If $\underline{a}$ is a tuple of elements of some pc model M of T , it makes sense to speak of the types *over* $\underline{a}$, as the types of the theory $T(\underline{a})$, in the language $£(\underline{a})$, obtained by adding to T all the positive formulae satisfied by $\underline{a}$. Since M is pc, every finite fragment of such a type is realized in M (if not, it would be in contradiction with a positive formula satisfied by $\underline{a}$). A basic open set being composed of the types that do not satisfy a certain positive formula, that is, of the types satisfying some other positive formula in contradiction with it, we see also that the n-types over M which are realized in M form a dense subset of the space $S_n(M)$. We shall call *dense set* any subset of a pc model of T having this property ; since the notion of pc model is language dependent, it cannot be caracterized intresically by this topological property !

There is another case of existence of a minimal language : the projection on $S_m(T)$ of a clopen subset of $S_{m+n}(T)$ defined by a formula $\varphi(\underline{x},\underline{y})$ is defined by $(\exists \underline{y})\ \varphi(\underline{x},\underline{y})$. Therefore, if we work in Robinson's setting, we have a minimal language corresponding to the projections of the clopen sets, and in this case also the pc models are the dense sets (since the test for existential closedness can be restricted to quantifier-free formulae).

By contrast, there is no minimal language in the following example.

Example 7. We consider the real segment $[0\ \ 1]$ as a structure in the language £ containing a predicate for each of its closed subsets ; let T be its h-inductive theory in this language ; for each real a between 0 and 1 , an axiom of T declares that $(\forall x)\ \ x \leq a \vee a \leq x$, and another that $(\exists x)\ x = a$, so that M is the only pc model of T ; $S_1(T)$ is $[0\ \ 1]$ with the usual topology, and $S_n(T)$ bears the product topology ; note that the diagonal $x = y$, and also the order $x \leq y$, define closed subsets of $S_2(T)$.

To generate the topologies, we can take the sublanguage $£_2$ of finite unions of closed segments with endpoints of the form $m/2^n$; we can also take the sublanguage $£_3$ where the endpoints have the form $m/3^n$. The only subsets of M which are defined in both languages are M and $\varnothing$! Moreover, a subset of M is pc for the language $£_2$ if and only if it contains all the rationnals of the form $m/2^n$; in fact, when the language contains singletons, there are dense subsets which are not pc.

The conclusion is that, in the general situation, the notion of formula, and consequently the notion of pc model, are fragile : we can count on only one secure raft, provided by the pos. ω-saturated pc models. And in the case we find them to much language-dependent, we can jump into the Positive Universes of POIZAT 2006.

In our positive context, a *principal type,* which is the only one to satisfy a certain positive formula, must be distinguished from an *isolated type,* which is the only one not to satisfy a given positive formula $\varphi(\underline{x})$; an isolated type is principal, since it must satisfy a formula $\psi(\underline{x})$ contradictory to $\varphi(\underline{x})$, and is obviously the only type to satisfy $\psi(\underline{x})$, but the converse is not true, even for the minimal language in Robinson's context. The positive adaptation[13] of the Omitting Types Theorem is contained in NURTAZIN 2015, : *If* T *is an h-inductive theory in a countable language, and* $p_1 , \ldots p_n , \ldots$ *is a sequence of non-principal types, there exists a pc model of* T *omitting all of them* ; and various consequences are drawn from it.

We assume that our readers will be grateful to us for closing the section with two very simple examples.

[13] In robinsonian setting, but the extension is obvious.

Example 8. The language £ contains infinitely many constants c_0 , c_1 , ... c_n , ... , a predicate symbol r(x) and a binary relation symbol i(x,y) ; the axioms for T' says that the c_n are pairwise distinct, that they do not satisfy r , that i(x,y) is the negation of the equality and that r is infinite. We observe that this theory is complete in the sense of full First Order Logic, and that it satisfies AP. Its pc models contain no point outside r except the c_i , contrarily to its model which are saturated for the logic with negation. Its spaces of positive 1-types is formed by the types of the c_i , which are isolated, and a type p satisfying r , which is not isolated : it is the accumulation point of the c_i ; this space is Hausdorff, as was expected from the AP.

When we drop r from the language, we obtain the familiar model-complete theory T of infinitely many constants : T and T' have the same space of types, and the pc models of T' are the ω-saturated models of T . This is an example of infinite morleysation.

Example 9. This example is given in Robinson's setting (where languages contains implicitely the negations of the atomic formulae) ; £ has infinitely many individual constants c_0 , c_1 , ... c_n , ... , d_0 , d_1 , ... d_n , ... , e_0 , e_1 , ... e_n , ... and a binary relation symbol r(x,y) ; when r(x,y) is satisfied we say that x and y are partners.

The axioms of T say that r(x,y) is symmetric and irreflexive, that any point has at most one partner, that the constants interprets distinct elements, that d_n and e_n are partners, and that the c_n have no partner.

T is a complete universal theory, and, in a saturated model of T , there are infinitely many points without partner and distinct from the c_i .

In an ec model of T , every element distinct from the c_i has a partner ; T is not model-complete, as the fact that x has no partner cannot be expressed by an existential formula ; T has the Amalgamation Property for embeddings, and one sees easily that its spaces of existential types are Hausdorff.

The inspection of the spaces of existential types shows that T is not an infinite morleysation of a model-complete theory, since r(x,y) defines a clopen subset of $S_2(T)$, whose projection on $S_1(T)$ is not open.

2.9. Positive Jonsson Theories

Jon Barwise (in BARWISE 1982, Ch. 2, Def. 6.1) was apparently the first to name *Jonsson theories* the theories in the full First Order language whose class of models satisfies the hypothesises, formulated (finding some inspiration in FRAISSE 1953) by Bjarni Jonsson (JONSSON 1956, 1960), which allow the construction of an universal domain, that is a unique κ-homogeneous-universal structure of unaccessible cardinality κ . An example of such a Jonsson's class is given by the models of a complete theory in the full First Order Logic with negation, after morleysation.

We extend the notion to our positive context, and call *Positive Jonsson Theory* an h-inductive theory having the JCP and the Amalgamation Property. We recall that the Amalgamation Property for the Kaiser hull of the theory means that the spaces of types are Hausdorff ; for instance, for any structure M , the theory Tk(M) is Hausdorff if and only iff it is Jonsson.

To tell the truth, another condition was added by Jonsson, and subsequently by Barwise, namely the existence of infinite models (to avoid the somehow trivial situation of a finite universal domain). In our case, we should say that the theory is not bounded ; but we do not wish to include this condition in our definition, which seems to us unnatural : on one hand, we believe that bounded theories present some interest, and on the other we do not forget that even unbounded theories may have some bounded part (this happens even in Jonsson settings).

A consequence of the Théorème 1 of POIZAT 2010 is that the final model of a bounded Jonsson theory is its unique pos. $£^+$-saturated pc model ; the two examples given in Section 2.7 are Jonsson.

Given a Jonsson h-inductive theory T , and a cardinal κ bigger than the cardinal of the language £ , one constructs big models M of T which are :

(i) κ-*universal* : any model of T of cardinal strictly less than κ can be continued into M ;

(ii) κ-*homogeneous* : for any homomorphism f from A into B , where A and B are models of T of cardinal strictly less than κ , any homomorphism from A into M can be extented to B .

Their construction is purely set-theoric, but in fact these big models were already known to us :

Theorem 12. *(i) If* T *is a pos. Jonsson theory, a model of* T *is* κ-*universal and* κ-*homogeneous if and only if it is pc and pos.* κ-*saturated.*
(ii) In fact, the JCP and the AP are necessary conditions for the existence of κ-*universal and* κ-*homogeneous models.*

Proof. (i) Suppose that M be κ-homogeneous, and consider a tuple $\underline{a}$ in M not satisfying a certain positive formula $\varphi(\underline{x})$; we can find a substructure A of M , of cardinality less than κ , which is a model of T and contains A ; A can be continued in a pc model B of T , of cardinality less than κ , in which the image of $\underline{a}$ satisfies a positive formula $\psi(\underline{x})$ contradicting $\varphi(\underline{x})$; since the identity on A can be extended to an homomorphism from B into M , $\underline{a}$ satisfies $\psi(\underline{x})$ in M . In others words M is pc. Moreover, if A is a subset of M of cardinality less than κ , that we may assume to be a model of T , any set $\varphi_i(x,\underline{a})$ of pos. formulae with parameters in A , which is finitely satisfiable in

M , is satisfiable in a continuation B of A of cardinality less than κ , and therefore satisfiable in M .

Reciprocally, if M is pc and pos. κ-saturated, it is κ-universal thank to the JCP and κ-homogeneous thank to the AP.

(ii) They are obviously necessary when restricted to structures of cardinal less than κ , and they express nothing but the consistency of some diagrams. **End**

In the Qaragandy School, these big models were called semantic models (especially the unique big model of unaccessible cardinality κ). If the theory Tk is not model-complete, the semantic models are never ω-saturated in the sense of First Order Logic with negation (Jonsson theories with a model-complete Kaiser hull were called *perfect*).

Their common Full First Order Theory was named the *center* of the Jonsson theory T by Tölendi Mustafin (see MUSTAFIN 1998). Given an n-tuple $\underline{a}$ of elements of a model M of T , all the images of $\underline{a}$ in any continuation of M in a semantic model have the same (negative) type in the sense of the center, that was called the *central type* of $\underline{a}$. The central types form a dense subset in the space of n-types of the center.

We have given many example where Tk is not model-complete but is nevertheless complete in the sense of full First Order Logic with negation. We have also seen that this is not the general case ; a possibility to define a completion of Tk is to consider the theory of the *generic models,* which are a special kind of pc models obtained by model-theoric forcing à la Robinson. This forcing is defined in two versions ; in the infinite version, forcing conditions are the model of T , and since the pos. ω-saturated are generic, the forced theory will be the center ; but in the finite version, where the forcing conditions are the finite fragments of the positive diagrams of the models of T [14], we may obtain something different : for instance, in our Example 5, the central models have endpoints, but not the generic ones. For more details on forcing in Jonsson context, we refer to YESHKEYEV 2009.

MUSTAFIN 2002 is consecrated to the companions of a Jonsson theory, and provides many examples and counterexamples. Since the JCP is preserved by companionship, a companion of an h-inductive Jonsson theory T is Jonsson provided that it has the AP ; in particular, every h-inductive theory between T and Tk is Jonsson, and so is its separant Ts .

If T and T' are two companion h-inductive Jonsson theories, the theory $T \vee T'$, formed by sentences $\sigma \vee \sigma'$ where σ is in T and σ' is in T' , is also Jonsson, since its models are the models of T and the models of T' . In consequence, if T has a minimal Jonsson companion, it is necessary the

[14] (and where countability of the language is assumed for the existence of generic models)

intersection of all of its Jonsson companion, which is called the *Jonsson kernel* of T ; this kernel does not have always the AP.

In conclusion of this section, we wish to draw the attention of our readers to one of the most mysterious Jonsson theory, the Theory of Groups.

Example 10. If we consider a group as a structure in the language of multiplication, inverse and unit, the only pc group is Terminus, the group reduced to the unit ; since moreover Terminus can be embedded into any group, all the groups satisfy the same h-universal sentences, and, in fact, Terminus is the only ns group !

To obtain something of interest, we add to the language the negation of the equality, and enter into Robinson's setting : the pc groups become then the familiar ec groups, which are the groups G such that any finite system of equations and inequations, which has a solution in some group embedding G , must have a solution in G .

Thank to the Theorem of Highman, Neumann and Neumann, if G is ec two n-tuples $\underline{a}$ and $\underline{b}$ of elements of G which satisfies the same quantifier-free formulae, that is which satisfy the same equations and inequations, must be conjugated, and therefore have the same type. The existential types are described by equations and inequations ; this does not means quantifier elimination ("every formula is equivalent to a quantifier-free one"), but only that the clopen sets in the space of types are associated to quantifier-free formulae. The pc models are not negatively ω-saturated, because they do not contain unconjugated pairs of elements of infinite order.

In any case, the spaces of types are Hausdorff, and indeed groups can be amalgamated ; Tk is unstable, because the formula $x.y = y.x$, whose negation is by assumption (or convention !) positive, has the Property of Independance. We remark in passing that $x.z = z.y$ defines a clopen set in S_3 whose projection in S_2 is not open.

The universal domains, or, equivalently, the spaces of types, are poorly known. For instance, an isolated type is a finitely generated group whose isomorphy type is determined by a finite number of equations and inequations satisfied by its generating system. It is not known if there are isolated types other than the finite groups (an infinite isolated group would provide a very simple example of a finitely axiomatisable strongly minimal theory ; see MAKOWSKI 1974).

3. Appendix

3.1. Five kinds of homomorphisms

We recall here the three kinds of homomorphisms already defined in the first section, and we add to the list two more new kinds. A map h from A to B , where A and B are £-structures, is said :

- *homomorphism* if for every tuple $\underline{a}$ from A , every atomic formula which is satisfied by $\underline{a}$ is also satisfied by h($\underline{a}$) ; under these conditions, every positive formula satisfied by $\underline{a}$ in A is satisfied by h($\underline{a}$) in B ; when we expand the language £ into £(A) by giving a name to each element of A , an homomorphism from A to another structure is the same thing as a model of its positive diagram $Diag^+(A)$;

- *embedding* if for every tuple $\underline{a}$ from A , $\underline{a}$ and h($\underline{a}$) satisfy the same atomic formulae ; in fact, this is not a truely useful notion ; it is equivalent to homomorphism when we add to the language the negations of the atomic formulae (Robinson's setting) ;

- *immersion* if for every tuple $\underline{a}$ from A , $\underline{a}$ in A and h($\underline{a}$) in B satisfy the same positive formulae ; equivalently, B is a model of Tu(A) , the set of h-universal sentences satisfied by A in the language £(A) ; B is not necessarily a model of Tk(A) , the set of h-inductive sentences of £(A) satisfied by A (since A is obviously a pc model of Tu(A) , Tk(A) is indeed the Kaiser hull of Tu(A)) ; the example of A = {a} and B = {a, b} , in the language reduced to equality, shows that it is even possible that Tu(B) be contradictory with Tk(A) !

- *sub-elementary immersion* if B is a model of Tk(A) ;

- *positively elementary immersion* if B is a pc model of Tu(A) ; in this case we say that B is a positive elementary extension of A . According to Corollary 5, any immersion between ns structures is pos. elementary.

Elementary extension for Positive Logic has been introduced in POIZAT 2006 ; the idea behind it was that, since in the logic with negation any structure is an object of study, being a model of its own theory, why should not it be the case in Positive Logic ? Indeed, provided that we name its elements, any structure is a pc model of its own h-inductive theory.

Note that even in Robinson's setting the three last kinds make sense, and are strictly included into the next.

It is easily seen that the composition g o f of two homomorphisms f from A to B and g from B to C , which belong both to one of these five classes, also belongs to the same class.

We have also some easy partial reciprocals : if $g \circ f$ is an embedding, then f is an embedding ; if $g \circ f$ is an immersion, then f is an immersion ; if $g \circ f$ is positive elementary and g is an immersion, then f and g are pos. elementary (Corollary 5).

3.2. Free amalgams

As was already said, to amalgamate two homomorphisms f from A into B and g from A into C, where A, B and C are £-structures, is to find an £-structure D and homomorphisms f' from C into D and g' from B into D such that $g' \circ f = f' \circ g$. In other words, it is to find a model of $\mathrm{Diag}^+(B) \cup \mathrm{Diag}^+(C) \cup \{ ... f(a) = g(a) , ...\}$. We shall keep these notations while speaking of amalgams.

We can always amalgamate homomorphisms, the simplest and useless way being to mail everything to Terminus. This amalgam respects practically nothing of the theories of the structures !

To preserve the maximum of their h-universal theories, the best is to amalgamate freely, satisfying only the unavoidable atomic formulae : the free amalgam $B \oplus_A C = D$ is simply the core model of the set of atomic sentences described above.

This amalgam is free in the categorical sense, that is, every amalgam of f and g is factorizable through the free amalgam. This is a typical feature of Positive Logic, are there is no free amalgam for embeddings if the language contains relation symbols : when we amalgam embeddings freely, by the next lemma we close the diagram by embeddings ; but the amalgam if free only in the category of homomorphisms (when we replace a relation symbol by its negation, we obtain another amalgam).

The following Lemma results from an observation of Belkasmi, which provides the Lemme 8 of BEN YAACOV & POIZAT 2007 with a better proof, avoiding the use of the Axiom of Choice.

Lemma 13. *If* D *is the free amalgam of* f *and* g *, and if* f *is an embedding, then* f' *is an embedding ; if* f *is an immersion, then* f' *is an immersion.*

Proof. We note A', B' and C' the respective images of A, $B\text{-}A$ and $C\text{-}A$ in the amalgam.

Assume that f is an embedding ; then it imposes on A' no equalities nor other atomic formulae than the ones coming from C.

Assume that f is an immersion, and consider a conjunction $\varphi_1(\underline{x}_1,\underline{a}_1,\underline{c}_1) \wedge ... \wedge \varphi_n(\underline{x}_n,\underline{a}_n,\underline{c}_n)$ of atomic formulae with parameters in $A' \cup C'$, which is satisfied in D. If one of them is the equality of two individual constants, it is also true in C since f' is an injection from C to D ; if it is an equality between two variables, or between a variable and a constant, it can be eliminated by a substitution. So we can assume that none of these formulae is an equality ; some of them are satisfiable by some $\underline{x}_i$ in $C = A' \cup C'$, as it is the case in particular for those which have their parameters in C ; the others are satisfiable only in $A' \cup B'$, and since A is pc in B we can reproduce the $\underline{x}$ which are in B' by elements of A . **End**

Corollary 14. *(i) Embeddings can be amalgamated.*
(ii) Immersions can be amalgamated, i.e. if B *and* C *are models of* Tu(A) *, they can be amalgamated into a model of* $Tu(B) \cup Tu(C)$.

Proof. Take the free amalgam. **End**

The following corollary, which rests on the Axiom of Choice, shows that the pc models of an h-inductive theory with JCP form a Jonsson class (which is not elementary in general) ; it clarifies somehow the construction of its pos. κ-saturated pc models ; it does not say that the h-inductive theory Tk of the pc models of an h-inductive theory T has always the Amalgamation Property !

Corollary 15. *Homomorphisms between pc models of an h-inductive theory* T *can be amalgamated.*

Proof. Since the two homomorphisms are immersions, the free amalgam is a model of Tu ; then we continue it into a pc model of Tu . **End**

Corollary 16. *Positively elementary immersions can be amalgamated.*

Proof. If M' and M" are pos. elementary extensions of M , they can be amalgamated in a third one, N , by the last corollary ; by Corollary 5, since they are immersed in N , they are pos. elementary restrictions of N . **End**

> **Example 11.** We provide here an example of non-amalgamable sub-elementary immersions. Let M be the structure formed by the natural integers, with their loose order $\leq$ (<u>not</u> $<$!) and two predicates that we name after the colours black and white : the numbers congruent to 1 modulo 3 are white and not black, the numbers congruent to 2 modulo 3 are black and not white, and the number divisible by 3 are black and white. There are only two pc models of $T_k(M)$: M itself, and the structure obtained by adding to M a black and white maximum. Using the retromorphic techniques of the Appendix, one sees that we obtain a model M_b of $T_k(M)$ by adding to M a black point b greater than the integers, plus a maximum m which is black and white ; we obtain another model M_w by adding a white point w and a black and white maximum m ; $Tk(M_b) \cup T(M_w)$ is contradictory, since it declares that b is the point preceding the maximum and is not white.

We repeat our slogan for the last time : all what we said makes sense in Robinson's setting, and so does what follows !

3.3. Two asymmetric amalgamation lemmata

We extract the following two beautiful lemmata from the doctoral dissertation of Mohammad Belkasmi (BELKASMI 2011 ; see also BELKASMI 2014).

Lemma 17. *If* f *is an immersion from* A *into* B *and* g *an homomorphism from* A *into* C *, they can be amalgamated in such a way that* f' *be pos. elementary.*

Proof. We amalgamate freely, so that f' is an immersion, and the amalgam is a model of Tu(C) ; then we continue it into a pc model of Tu(C) , that is, a pos. elementary extension of C . **End**

Remark. If f is an immersion from A into B , we have seen that it is possible that Tk(A) be contradictory with Tu(B) . When we take A = C , we see that it is not always possible to amalagamate two immersions into a model of Tu(B) ∪ Tk(C) . Otherwise said, if f is an immersion from A into B and g an immersion from A into C , it is not always possible to amalgamate them in such a way that f' be sub-elementary and g' an immersion, even when g is pos. elementary.

Lemma 18. *If* f *is a sub-elementary immersion from* A *into* B *and* g *an immersion from* A *into* C *, they can be amalgamated in such a way that* g' *be an immersion and* f' *be sub-elementary.*

Proof. We have to show that $Tk(C) \cup Diag^+(B) \cup Tu(B)$ is consistent, that is to say that each of its finite fragment is consistent. Let $\varphi(\underline{a},\underline{b}) \wedge \neg(\exists\, \underline{y})\, \psi(\underline{a},\underline{b},\underline{y})$ be a finite fragment of $Diag^+(B) \cup Tu(B)$; since B satisfies Tk(A) , this last one does not contain $(\forall\, \underline{x})\, \varphi(\underline{a},\underline{x}) \Rightarrow (\exists\, \underline{y})\, \psi(\underline{a},\underline{x},\underline{y})$, so that we can find $\underline{b}'$ in A satisfying $\varphi(\underline{a},\underline{b}') \wedge \neg(\exists\, \underline{y})\, \psi(\underline{a},\underline{b}',\underline{y})$; since A is immersed in C , this is also true in C , so that we find in C an interpretation of this finite fragment. **End**

Corollary 19. *If* N *is a pos. elementary extension of* M *, every model of* Tk(M) *can be immersed into a model of* Tk(N) .

Proof. Indeed, M is immersed into N . **End**

With the help of his two sophisticated amalgamation lemmata, Belkasmi has been able to answer in BELKASMI 2011 a question of POIZAT 2010 : *If* N *is a pos. elementary extension of* M *, then* Tk(N) *is Hausdorff if and only if* Tk(M) *is Hausdorff.* We propose here a slight generalization of his result :

Theorem 20. *Let* M *be a negatively sufficient* £*-structure, which is therefore a pc model of its own h-inductive theory* T *, and* E *be a subset of* M *; then* T *is Hausdorff if and only if* T(E) *is Hausdorff.*

Proof. Assume that T be Hausdorff, and consider an homomorphism from A into B and an homomorphism from A into C , where A , B and C are three models of T(E) , the set of h-inductive sentences with parameters in E which are true in M ; by hypothesis they can be amalgamated into a structure D which is a model of T . Since M is a pc model of T , T(E) contains the complete positive type of E , so that if we continue D into a pc model of T we obtain a model of T(E) .

Suppose now that T(E) be Hausdorff, and consider an homomorphism from A into B and an homomorphism from A into C , where A , B are C three models of T . Since the positive type of E is finitely satisfiable in it, we obtain a sub-elementary immersion of A into a model A' of T∪Tu(E) , and by Lemma 16 we can amalgamate A' and B over A into a positive elementary extension of B , which is therefore a model of T ; when we continue this extension into a pc model B' of T , we obtain a model of T(E) . In the same way, we can amalgamate A' and C over A into a model C' of T(E) . By hypothesis B' and C' can be amalgamated over A' into a model of T(E) , and by the commutativity of the diagram this give an amalgam of B and C over A as models of T . **End**

3.4. Retromorphisms

Whenever quantifications (first order or not !) are used, the method of local back-and-forth, introduced by Roland Fraïssé (the oldest reference seems to be FRAISSE 1953a ; see POIZAT 1985) is a useful device to check that two structures satisfy the same sentences of some kind. We shall adapt briefly this method to the context of Positive Logic ; since the sentences that we consider have only one alternance of quantifiers, they will play the back-and-forth (or more exactly the forth-and-back !) game only once.

We call *local homomorphism* from the £-structure M to the £-structure N any homomorphism from a finitely generated substructure of M into N ; and we say that M and N are *locally homomorphic* if for every tuple $\underline{a}$ in M there is an homomorphism from the structure generated by $\underline{a}$ into N , and reciprocally for any $\underline{b}$ in N there is a local homomorphism from N into M which is defined on $\underline{b}$.

We recall that a structure M is positively ω-saturated if, for any $\underline{a}$ in M , every set of existential formulae $(\exists x)\ \varphi_i(\underline{a},x)$ which is finitely realizable in M is realized in it ; this in in particular the case of ω-saturated structures in the sense of full First Order Logic.

Theorem 21. *(i) Two locally homomorphic structures have the same h-universal theory.*
(ii) The reciprocal is true if the language is finite and contains no function, or if the two structures are positively ω*-saturated.*

Proof. (i) Any h-universal sentence can be writen as $\neg(\exists\ \underline{x})\ \varphi(\underline{x})$, where φ is positive quantifier-free ; it is therefore clear that two locally homomorphic structures satisfy the same existential positive sentences, and the same h-universal sentences, which are their negations.
(i) Assume that M and N satisfy the same h-universal sentences, and consider an n-tuple $\underline{a}$ in M . If the language is finite, there is only a finite number of atomic formulae whose variables are taken in a given m-tuple $\underline{x}$; let $\varphi(\underline{x})$ be the conjunction of those atomic formulae which are satisfied by $\underline{a}$; the sentence $(\exists\ \underline{x})\ \varphi(\underline{x})$ being true in M , it is also true in N , since its negation cannot be false. In the other case, the set of atomic formulae satisfied by $\underline{a}$ is finitely satisfiable in N , and satisfied in it by positive saturation. **End**

A local homomorphism h from M to N is said to be *locally extensible* if, for any tuple $\underline{a}$ in M , we can find a local homomorphism extending h and whose domain contains $\underline{a}$; we say that it is n-*extensible* if we have this forward property only for the tuples $\underline{a}$ of length n . We say that two structures M and N are *locally epimorphic* if, for any $\underline{a}$ in M and any n , $\underline{a}$ is in the image of a local n-extensible homomorphism from N to M , and similarly in the other direction.

Theorem 22. *(i) Two locally epimorphic structures have the same positive inductive theory.*
(ii) The reciprocal is true if the language is finite and contains no function, or if the two structures are ω*-saturated for full First Order Logic.*
(iii) When the two structures are ω*-saturated and locally epimorphic, we can use locally extensible homomorphisms instead of* n*-extensible ones.*

Proof. (i) Suppose that $(\forall\ \underline{x})(\exists\ \underline{y})\ \varphi(\underline{x},\underline{y})$ is true in N , where φ is positive quantifier-free and $\underline{y}$ is of length n , and consider an arbitrary $\underline{a}$ in M of the length of $\underline{x}$; by hypothesis, there is an n-extensible local homomorphism h from N to M and a $\underline{b}$ in N such that $h(\underline{b}) = \underline{a}$; by the satisfaction of the sentence above, there is $\underline{c}$ in N such that $\varphi(\underline{b},\underline{c})$ holds, and, since h can be extended to a local homomorphism defined on $\underline{c}$, $(\exists\ \underline{y})\ \varphi(\underline{a},\underline{y})$ is true in M ; therefore the sentence is also true in M .
(ii) Suppose that the language is finite and without function, and that M and N satisfy the same positive inductive sentences ; given an m-tuple $\underline{x}$ and an n-tuple $\underline{y}$, there is only a finite number of formulae of the form $(\exists\ \underline{y})\ \psi(\underline{x},\underline{y})$ with ψ positive quantifier-free ; consider an arbitrary m-tuple $\underline{a}$ of elements

of M and the conjunction φ(x) of formulae of this kind which are not satisfied by a ; the sentence (∀ x) φ(x) is untrue in M , and also in N , in which we can find b not satisfying φ(x) ; by construction, the mapping from b to a is an n-extensible local homomorphism.

For the general case, when N is ω-saturated, for any a in M , we can find b in N such that any existential formula false of a is false of b .

(iii) If N is ω-saturated and if some a in M is for each n in the image of a local n-extensible homomorphism from N to M , then it is in the image of a local locally extensible homomorphism. **End**

A local homomorphism h from M to N , whose domain is generated by a , is called a *local retromorphism* if, for any b in N , we can find an homomorphism g from the substructure of N generated by b and h(a) such that g o h is the identity on a . Note that h must be a local embedding, but that we do not assume that g is an embedding. It is called a *local* n-*retromorphism* if we have this backward property only for the tuples b of length n .

We say that two structures M and N are *locally retromorphic* if, for any a in M and any n , there is a local n-retromorphism from M to N which is defined on a , and similarly in the other direction.

Theorem 23. *(i) Two locally retromorphic structures have the same h-inductive theory.*
(ii) The reciprocal is true if the language is finite and contains no function, or if the two structures are ω*-saturated for full First Order Logic.*
(iii) When the two structures are ω*-saturated and locally retromorphic, we can use local retromorphisms instead of local n-retromorphisms.*

Proof. (i) Suppose that (∀ x)(∃ y) φ(x) ⇒ ψ(x,y) is true in N , and consider a in M satisfying φ ; by hypothesis, there is a local n-retromorphism h from M to N defined on a , where n is the length of y ; since h(a) satisfy φ , there is a b in N such that ψ(h(a),b) is true, and reversing the way from N to M we see that ψ(a,g(b)) , and therefore (∃ y) ψ(a,y) , are true. So the sentence is also true in M .

(ii) Suppose that the language is finite and without function, and that M and N satisfy the same h-inductive sentences ; given an n-tuple y of variables, and a in M , consider the conjunction φ(x) of atomic formulae satisfied by a , and the disjunction ψ(x) of positive formulae of the form (∃ y) ψ'(x,y) where ψ' is positive without quantifiers, which are false for a (this includes the atomic formulae not satisfied by a) . The axiom (∀ x) φ(x) ⇒ ψ(x) is false in M , so it is also false in N , in which we can find b satisfying φ but not ψ : the local isomorphism sending a to b is an n-retromorphism.

For the general case, when N is ω-saturated, for any $\underline{a}$ in M , we can find $\underline{b}$ in N such that any atomic sentence true of $\underline{a}$ is true of $\underline{b}$, and any positive formula not satisfied by $\underline{a}$ is also not satisfied by $\underline{b}$.
(iii) If N is ω-saturated and if, for some $\underline{a}$ in M , there is for each n a local n-retromorphism from M to N defined on $\underline{a}$, then there is a local retromorphism from M to N defined on $\underline{a}$. **End**

If the language is relational and infinite, we can apply the method of forth-and-back by restricting it to each finite sublanguage ; if there are functions, we may replace them by their graphs, but since the notion of local homomorphism is affected by this substitution the method may be not so easy to use. To check that two structures satisfy the same h-inductive axioms, the best is to replace them by saturated elementary extensions (provided that we can guess what they are), and try to apply the last theorem. At least in the simple cases, this gives a very efficient method.

3.5. Granma Positive Model Theory

The extension of their favourite discipline to the general positive context opens a vast field of new activities for the logicians who have an inclination towards Applied Model Theory : besides the description of the class of Jonsson companions of a given h-inductive Jonsson theory, of its kernel, of the negative theory of its pc models, of its center, or of its forcing completion, many questions concerning h-inductive theories and their pc models can be asked in a concrete context.

Can we describe all the Jonsson h-inductive theories of structures of a certain kind, as was done, in a robinsonian frame, in MUSTAFIN 1998, NURKHAIDAROV 1985 and YESHKEYEV 1995 ?

Given an h-inductive theory T (for instance the theory of rings, or the theory of groups with inequality added, or the same theories in the relational language), can we describe its h-universal companion Tu , or its separant Ts , or the h-inductive theory Tm of its dm models, or the h-inductive theory Tb of its amalgamation bases, or its Kaiser hull Tk ? Can we classify its models, its pc models or its dm models ?

Dually, given a structure, can we describe its h-inductive theory, or its h-universal theory ? If this structure is negatively sufficient, can we describe the family of h-universal theories for which it is pc ?

And last, for the ones who are nostalgic of Good Old Time Model Theory, can we say a word on the decidabilty of the theories emerging in some concrete situation ?

We hope that our appendix will provide some efficient tools to the persons wishing to attack this kind of problems.

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