

# O-MINIMAL EXPANSIONS OF REAL CLOSED FIELDS AND COMPLETENESS IN THE SENSE OF SCOTT

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ABSTRACT. We consider an  $o$ -minimal expansion  $\mathcal{M}_0 = (R_0, <, +, \dots)$  of a real closed field, and a real closed field  $R$ , complete in the sense of D. Scott, containing  $R_0$  as a dense subfield. We show that  $\mathcal{M}_0$  has an elementary extension  $\mathcal{M} = (R, <, +, \dots)$  with domain  $R$ . Moreover, such a structure  $\mathcal{M}$  with domain  $R$  is unique.

## 1. INTRODUCTION

By the Compactness Theorem, it is easy to elementarily embed any expansion  $\mathcal{M}_0 = (R_0, <, +, \dots)$  of a real closed field in an expansion  $\mathcal{M} = (R, <, +, \dots)$  of a *Scott-complete* real closed field, that is complete in the sense of Dana Scott (Definition 1.1). However, we have little control on the size of the elementary extension obtained. For instance, if  $R_0$  is countable, it is possible that the field  $R$  has no countable dense subfield. The main result of this paper shows that any  $o$ -minimal expansion of a real closed field  $R_0$  is elementarily embedded in an expansion of a Scott-complete real closed field, in which  $R_0$  is dense (Theorem 1.2). We note that, since we consider an expansion of a real closed field, not just the field structure, the model completeness of the theory of real closed fields will not help us.

We recall the definition of a *complete* real closed field in the sense of [1].

**Definition 1.1.** – *If  $K$  and  $L$  are ordered fields and  $K \subseteq L$ , then  $K$  is dense in  $L$  if between any two distinct elements of  $L$  there lies an element of  $K$ .*

*A given ordered field is called Scott-complete if it has no proper extension to an ordered field in which the given field is dense.*

The main result of this paper is the following.

**Theorem 1.2.** – *Any  $o$ -minimal expansion of a real closed field  $R_0$  has an elementary extension of domain a Scott-complete field  $R$  in which  $R_0$  is dense. Moreover, for a fixed field  $R$ , this elementary extension is unique.*

## 2. NOTATIONS AND PRELIMINARY RESULTS

For the rest of this paper, we fix an  $o$ -minimal expansion  $\mathcal{M}_1 = (R_0, <, \dots)$  of a real closed field  $R_0$ , and we denote by  $R$  a Scott-complete field in which  $R_0$  is dense: its existence is ensured by Fact 2.1 below.

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**Fact 2.1.** – [1, Theorem 1] *Given any ordered field  $K$ , there is a Scott-complete ordered field  $\hat{K}$  in which  $K$  is dense. Any other Scott-complete ordered field in which  $K$  is dense is isomorphic to  $\hat{K}$  by a unique isomorphism that is the identity on  $K$ .*

In the following, we say that a set  $X$  is *definable* if it is definable in the structure  $\mathcal{M}_1$ .

For each integer  $k$  and each subset  $A$  of  $R^k$ , we denote by  $\bar{A}$  its (topological) closure in  $R^k$ . For each subset  $X$  of  $R_0^k$ , we denote by  $\check{X}$  the union of subsets  $\bar{F}$  for  $F$  a  $R_0$ -closed definable subset contained in  $X$ .

For clarity, we use a very different notation for the closure in  $R_0^k$  of a subset  $X$  of  $R_0^k$ : we denote it by  $\text{cl } X$ . Moreover, we denote by  $\partial X$  its frontier:  $\partial X = \text{cl } X \setminus X$ .

For any element  $x = (x_1, \dots, x_k)$  of  $R^k$ , we consider

$$|x| = \max\{|x_i| \mid i \in \{1, \dots, k\}\}$$

**Remark 2.2.** –

- For any subset  $X$  of  $R_0^k$ , it follows from the definition of  $\check{X}$  that  $X = \check{X} \cap R_0^k$ .
- The proof of Theorem 1.2 will show that the elementary extension of  $\mathcal{M}_1$  of domain  $R$  has the following property:

*for any two integers  $k$  and  $m$ , if  $A$  is any subset of  $R_0^k$  defined by a formula  $\varphi(\bar{x}, \bar{a})$  with free variables  $\bar{x} = (x_1, \dots, x_k)$  and parameters  $\bar{a} \in R_0^m$ , then the subset of  $R^k$  defined by  $\varphi(\bar{x}, \bar{a})$  is  $\bar{A}$ .*

The first lemma is certainly well-known, but we could not find a reference for it.

**Lemma 2.3.** – *Let  $k$  be an integer,  $X$  a subset of  $R_0^k$ , and  $f : X \rightarrow R_0$  be a uniformly continuous map. Then for each  $x \in X$ , the following limit exists:*

$$\lim_{\substack{y \in X \\ y \rightarrow x}} f(y)$$

**PROOF** – Let  $\varepsilon \in R_0^{>0}$ . Since  $f$  is uniformly continuous on  $X$ , then we may associate with  $\varepsilon$  some  $\delta(\varepsilon) \in R_0^{>0}$  such that whenever  $|v - u| < \delta(\varepsilon)$  for  $u$  and  $v$  in  $X$ , we have  $|f(v) - f(u)| < \varepsilon$ . Then for each  $y \in X$  and  $z \in X$  such that  $|y - x| < \delta(\varepsilon)/2$  and  $|z - x| < \delta(\varepsilon)/2$ , we have  $|z - y| < \delta(\varepsilon)$  and  $|f(z) - f(y)| < \varepsilon$ .

Let  $C_\varepsilon^-$  (resp.  $C_\varepsilon^+$ ) be the set of elements  $\alpha \in R_0$  such that  $\alpha < f(y) - \varepsilon$  (resp.  $\alpha > f(y) + \varepsilon$ ) for some  $y \in X$  satisfying  $|y - x| < \delta(\varepsilon)/2$ . Let  $C^-$  (resp.  $C^+$ ) be the union of subsets of the form  $C_\varepsilon^-$  (resp.  $C_\varepsilon^+$ ) for  $\varepsilon \in R_0^{>0}$ .

**Claim 1:** *for each  $a \in C^-$  and each  $b \in C^+$ , we have  $a < b$ . In particular, the set  $C^- \cap C^+$  is empty.*

There exist  $y_1$  and  $y_2$  in  $X$ , and  $\varepsilon_1$  and  $\varepsilon_2$  in  $R_0^{>0}$ , such that  $|y_1 - x| < \delta(\varepsilon_1)/2$ ,  $|y_2 - x| < \delta(\varepsilon_2)/2$ ,  $a < f(y_1) - \varepsilon_1$  and  $b > f(y_2) + \varepsilon_2$ . Let  $\delta = \min\{\delta(\varepsilon_1), \delta(\varepsilon_2)\}$ . We fix  $y \in X$  such that  $|y - x| < \delta/2$ . Then we have  $|f(y) - f(y_1)| < \varepsilon_1 < f(y_1) - a$  and  $|f(y) - f(y_2)| < \varepsilon_2 < b - f(y_2)$ , so we obtain  $a < f(y) < b$ .

**Claim 2:** *for each  $r \in R_0^{>0}$ , there is  $a \in C^-$  and  $b \in C^+$  such that  $b - a < r$ .*

Let  $y \in X$  such that  $|y - x| < \delta(r/3)/2$ . Then for  $a = f(y) - r/2$  and  $b = f(y) + r/2$ , we have  $a \in C_{r/3}^-$  and  $b \in C_{r/3}^+$ , and we obtain  $a \in C^-$ ,  $b \in C^+$  and  $b - a < r$ .

**Conclusion:** By [2, §2, Lemma 2.8] (or [1]), there is a unique  $\omega \in R$  satisfying  $a < \omega < b$  for every  $a \in C^-$  and  $b \in C^+$ . Hence for each  $\varepsilon \in R_0^{>0}$ , there is

$\delta(= \delta(\varepsilon)) \in R_0^{>0}$  such that whenever  $|y - x| < \delta/2$  for  $y \in X$ , we have  $\omega \geq f(y) - \varepsilon$  and  $\omega \leq f(y) + \varepsilon$ , so  $|f(y) - \omega| \leq \varepsilon$ . Now  $\omega = \lim_{\substack{y \in X \\ y \rightarrow x}} f(y)$  exists.  $\square$

**Lemma 2.4.** – *Let  $k$  be an integer. If  $E$  and  $F$  are two closed definable subsets of  $R_0^k$ , then  $\overline{E \cap F} = \overline{E} \cap \overline{F}$ . In particular, if  $\overline{E} \cap \overline{F}$  is non-empty, then  $E \cap F$  is non-empty too.*

PROOF – We have just to prove that  $\overline{E \cap F}$  contains  $\overline{E} \cap \overline{F}$ . Let  $x \in \overline{E} \cap \overline{F}$ . For each  $\varepsilon \in R_0^{>0}$ , we fix  $u_\varepsilon \in E$  such that  $|u_\varepsilon - x| < \varepsilon$ . Let  $B_\varepsilon = \{y \in R_0^k \mid |u_\varepsilon - y| \leq \varepsilon\}$ . Since  $x \in \overline{E} \cap \overline{F}$  and since  $|u_\varepsilon - x| < \varepsilon$ , we have  $x \in \overline{E_\varepsilon} \cap \overline{F_\varepsilon}$  where  $E_\varepsilon = B_\varepsilon \cap E$  and  $F_\varepsilon = B_\varepsilon \cap F$ . Moreover, we note that  $E_\varepsilon$  and  $F_\varepsilon$  are closed and bounded definable subsets of  $R_0^k$ .

We show that  $E_\varepsilon \cap F_\varepsilon$  is non-empty. Let  $f_\varepsilon : E_\varepsilon \times F_\varepsilon \rightarrow R_0$  defined by  $f_\varepsilon(z, z') = |z - z'|$ . It is a definable continuous function, so its image is closed and bounded (see [3, Chapter 6 §1]). For each  $\eta \in R_0^{>0}$ , there exist  $u \in E_\varepsilon$  and  $v \in F_\varepsilon$  such that  $|u - v| < \eta$ , so we have  $f_\varepsilon(u, v) = |u - v| < \eta$ . Since the image of  $f_\varepsilon$  is closed and bounded, this implies that it contains zero. Hence there exist  $a \in E_\varepsilon$  and  $b \in F_\varepsilon$  such that  $f_\varepsilon(a, b) = 0$ . Now we have  $a = b \in E_\varepsilon \cap F_\varepsilon$ , and  $E_\varepsilon \cap F_\varepsilon$  is non-empty.

Since  $|u_\varepsilon - x| < \varepsilon$ , we have  $|y - x| < 2\varepsilon$  for any  $y \in B_\varepsilon$ . Thus, the previous paragraph proves that for each  $\varepsilon \in R_0^{>0}$ , there exists  $y \in E \cap F$  such that  $|y - x| \leq 2\varepsilon$ , so  $x \in \overline{E \cap F}$ , as desired.  $\square$

**Corollary 2.5.** – *For each subset  $X$  of  $R_0^k$ , and each  $x \in \check{X}$ , there is a closed and bounded definable subset  $F$  of  $X$  such that  $x \in \overline{F}$ .*

PROOF – Since  $x \in \check{X}$ , there exists  $x_0 \in X$  such that  $|x - x_0| < 1$ . We consider  $F_1 = \{y \in R_0^k \mid |y - x_0| \leq 1\}$ . Then  $F_1$  is a closed and bounded definable subset of  $R_0^k$ . By density of  $R_0$  in  $R$  and since  $|x - x_0| < 1$ , we have  $x \in F_1$ .

Moreover, by the definition of  $\check{X}$ , there is a closed definable subset  $F_2$  of  $X$  such that  $x \in \overline{F_2}$ . Then  $F = F_1 \cap F_2$  is a closed and bounded definable subset of  $X$ , and Lemma 2.4 gives  $x \in \overline{F}$ .  $\square$

**Proposition 2.6.** – *Let  $k$  be an integer. If  $\{A_1, \dots, A_m\}$  is a partition of  $R_0^k$  into definable subsets, then  $\{\check{A}_1, \dots, \check{A}_m\}$  is a partition of  $R^k$ .*

PROOF – First we show that  $R^k$  is the union of  $\check{A}_1, \dots, \check{A}_m$ . Since each definable subset of  $R_0^k$  has a decomposition into cells (see [3, Chapter 3 §2] for more details), we may assume that  $A_1, \dots, A_m$  are cells.

Let  $x \in R^k$ . We show that  $x \in \check{A}_j$  for some  $j \in \{1, \dots, m\}$ . We may assume  $x \notin R_0^k$ . By finiteness of the partition  $\{A_1, \dots, A_m\}$ , there exists  $r \in R^{>0}$  such that, for any  $s \in ]0, r[$ , the set  $I = \{i \in \{1, \dots, m\} \mid \exists a \in A_i, |x - a| < s\}$  is constant. Since  $R_0$  is dense in  $R$ , the set  $I$  is non-empty, and by the definition of  $I$ , the point  $x$  is in the  $R^k$ -closure  $\overline{A_i}$  of  $A_i$  for each  $i \in I$ .

Let  $d$  be the smallest integer such that there is  $j \in \{1, \dots, m\}$  and a definable subset  $B$  of  $A_j$  of dimension  $d$  with  $x$  contained in  $\overline{B}$ . Let  $B_i = A_i \cap \partial B$  for each  $i \in \{1, \dots, m\}$ . Then for each  $i \in \{1, \dots, m\}$ , the subset  $B_i$  of  $A_i$  is definable (see [3, Chapter 1 §3]) and we have  $\dim B_i \leq \dim \partial B < \dim B = d$  [3, Chapter 4 §1]. By the minimality of  $d$ , the point  $x$  is contained in the  $R^k$ -closure  $\overline{B_i}$  of  $B_i$  for no  $i \in \{1, \dots, m\}$ . Consequently, there is  $t \in R^{>0}$  such that  $|y - x| > t$

for any  $y \in \cup_{i=1}^m B_i$ , and we may choose  $t \in R_0$  as  $R_0$  is dense in  $R$ . Since  $\cup_{i=1}^m B_i = \partial B$  and since  $x \in \overline{B}$ , there exists  $b_0 \in B$  such that  $|b_0 - x| < t/2$ . Let  $B_f = \{b \in B \mid |b - b_0| \leq t/2\}$ . By the choices of  $t$  and  $b_0$ , we have  $x \in \overline{B_f}$  and  $\partial B \cap B_f = \emptyset$ . This implies that the set  $B_f$  is a closed definable subset of  $B$  and that  $x$  belongs to  $\check{A}_j$ , as desired.

We show that  $\check{A}_i \cap \check{A}_j = \emptyset$  for any distinct elements  $i$  and  $j$  of  $\{1, \dots, m\}$ . Otherwise there is a closed definable subset  $F_i$  (resp.  $F_j$ ) of  $A_i$  (resp.  $A_j$ ) such that  $\overline{F_i} \cap \overline{F_j} \neq \emptyset$ . By Lemma 2.4, the set  $F_i \cap F_j$  is non-empty, contradicting  $A_i \cap A_j = \emptyset$ . Thus  $\{\check{A}_1, \dots, \check{A}_m\}$  is a partition of  $R^k$ .  $\square$

### 3. PROOF OF THE MAIN THEOREM

We provide three preparatory results before the final argument.

**Lemma 3.1.** – *Let  $S_0 = \{(u, v) \in R_0^2 \mid u < v\}$  and  $S = \{(u, v) \in R^2 \mid u < v\}$ . Then we have  $S = \check{S}_0$ .*

PROOF – First we show that  $\check{S}_0$  contains  $S$ . Let  $(u, v) \in S$ . Then there exists  $r \in R_0^{>0}$  such that  $v - u > r$ . Let  $(u_0, v_0) \in R_0^2$  such that  $|u - u_0| < r/4$  and  $|v - v_0| < r/4$ , and let  $F = \{(x, y) \in R_0^2 \mid |x - u_0| \leq r/4, |y - v_0| \leq r/4\}$ . In particular, we have  $(u, v) \in \overline{F}$ . Moreover we have  $v_0 - u_0 > r/2$ , so we obtain  $y - x > 0$  for each  $(x, y) \in F$ , and  $F$  is a closed definable subset of  $S_0$ . Consequently  $(u, v)$  belongs to  $\check{S}_0$ , and  $\check{S}_0$  contains  $S$ .

Now we show that  $S$  contains  $\check{S}_0$ . Let  $(u, v) \in \check{S}_0$ . By Corollary 2.5, there is a closed and bounded definable subset  $F_0$  of  $S_0$  such that  $(u, v) \in \overline{F_0}$ . Let  $f : F_0 \rightarrow R_0$  defined by  $f(x, y) = y - x$ . Then  $f$  is a definable continuous function, so its image is closed and bounded. Let  $m = \min\{f(x, y) \mid (x, y) \in F_0\}$ . Then we have  $m > 0$  and we obtain  $(u, v) \in S$ .  $\square$

**Corollary 3.2.** – *Let  $T_0 = \{(u, v) \in R_0^2 \mid u = v\}$  and  $T = \{(u, v) \in R^2 \mid u = v\}$ . Then we have  $T = \check{T}_0$ .*

PROOF – Let  $S_0 = \{(u, v) \in R_0^2 \mid u < v\}$  and  $S_1 = \{(u, v) \in R_0^2 \mid u > v\}$ . Then  $\{S_0, S_1, T_0\}$  is a partition of  $R_0^2$ , and Proposition 2.6 says that  $\{\check{S}_0, \check{S}_1, \check{T}_0\}$  is a partition of  $R^2$ . Now Lemma 3.1 gives  $\check{T}_0 = R^2 \setminus (\check{S}_0 \cup \check{S}_1) = T$ .  $\square$

**Lemma 3.3.** – *If  $G_0$  (resp.  $H_0, K_0$ ) denotes the graph of  $\cdot$  (resp.  $+, -$ ) in  $R_0^3$  (resp.  $R_0^3, R_0^2$ ), then the graph of  $\cdot$  (resp.  $+, -$ ) in  $R^3$  (resp.  $R^3, R^2$ ) is  $\check{G}_0$  (resp.  $\check{H}_0, \check{K}_0$ ).*

PROOF – Since  $\cdot$  is a continuous map over  $R_0$ , its graph  $G_0$  in  $R_0^3$  is closed, and we have  $\check{G}_0 = \overline{G_0}$ . Moreover, since  $\cdot$  is a continuous map over  $R$ , its graph  $G$  in  $R^3$  is closed, and since  $G$  contains  $G_0$ , it contains  $\overline{G_0}$  too. But  $R_0$  is dense in  $R$ , hence for each  $(x, y) \in R^2$  we have  $(x, y, x \cdot y) \in \overline{G_0}$ , and  $G$  is contained in  $\overline{G_0}$ . We conclude that  $G = \overline{G_0} = \check{G}_0$ , as desired.

In the same way, we show that the graph of  $+$  (resp.  $-$ ) in  $R^3$  (resp.  $R^2$ ) is  $\check{H}_0$  (resp.  $\check{K}_0$ ).  $\square$

PROOF OF THEOREM 1.2 – We denote by  $\mathcal{L}_1$  the language of  $\mathcal{M}_1 = (R_0, <, \dots)$ . For each function symbol  $f$  of  $\mathcal{L}_1$  with arity  $k$ , we consider a relation symbol  $S_f$  such that  $S_f^{\mathcal{M}_1}$  is the graph of  $f^{\mathcal{M}_1}$ , and for each constant symbol  $c$  of  $\mathcal{L}_1$ , we consider a

relation symbol  $S_c$  such that  $S_c^{\mathcal{M}_1} = c^{\mathcal{M}_1}$ . We obtain a relational language  $\mathcal{L}$  and a structure  $\mathcal{M}_0 = (R_0, <, \dots)$  in  $\mathcal{L}$ . We have just to prove that there is a unique  $\mathcal{L}$ -structure  $\mathcal{M}$  with domain  $R$  such that  $\mathcal{M}$  is an elementary extension of  $\mathcal{M}_0$ .

We note that, for any integer  $k$ , a subset  $X$  of  $R_0^k$  is definable (in  $\mathcal{M}_1$ ) if and only if it is definable in  $\mathcal{M}_0$ .

**Uniqueness:**

First we assume that the structure  $\mathcal{M}$  exists. Let  $S$  be any relation symbol of arity  $k$  of  $\mathcal{L}$ . Let  $F$  be a closed and bounded definable subset of  $S^{\mathcal{M}_0}$ , and let  $\varphi(\bar{x}, \bar{a})$  be an  $\mathcal{L}$ -formula with free variables  $\bar{x} = (x_1, \dots, x_k)$  and parameters  $\bar{a} \in R_0^m$  such that  $F$  is defined by  $\varphi(\bar{x}, \bar{a})$ . Let  $\tilde{F}$  be the subset of  $R^k$  defined by  $\varphi(\bar{x}, \bar{a})$ . Since  $\mathcal{M}$  is an elementary extension of  $\mathcal{M}_0$ , then  $\tilde{F}$  contains  $F$  and it is closed and bounded in  $R^k$ . Thus  $\tilde{F}$  contains  $\overline{F}$ . By Corollary 2.5, this implies that  $S^{\mathcal{M}}$  contains  $\check{S}^{\mathcal{M}_0}$ .

In the same way, the complementary of  $S^{\mathcal{M}}$  in  $R^k$  contains  $\widehat{R_0^k \setminus S^{\mathcal{M}_0}}$ . Now Proposition 2.6 gives  $S^{\mathcal{M}} = \check{S}^{\mathcal{M}_0}$ , so, if it exists such a structure  $\mathcal{M}$ , then it is unique.

**Existence:**

We consider the  $\mathcal{L}$ -structure  $\mathcal{M} = (R, <, \dots)$  where for each relation symbol  $S$  of arity  $k$  of  $\mathcal{L}$ , we define  $S^{\mathcal{M}}$  by  $S^{\mathcal{M}} = \check{S}^{\mathcal{M}_0}$ . By Lemmas 3.1 and 3.3 and Corollary 3.2, it is sufficient to show that  $\mathcal{M}$  is an elementary extension of  $\mathcal{M}_0$ .

First we note that for each relation symbol  $S$  of arity  $k$  of  $\mathcal{L}$ , we have  $S^{\mathcal{M}_0} = \check{S}^{\mathcal{M}_0} \cap R_0^k$  (Remark 2.2), so  $\mathcal{M}_0$  is a substructure of  $\mathcal{M}$ .

**Claim 1:** *if  $A$  is a definable subset of  $R_0^k$  for an integer  $k$ , then we have  $\check{A}_0 = R^k \setminus \check{A}$  where  $A_0 = R_0^k \setminus A$ .*

Since  $\{A, A_0\}$  is a partition of  $R_0^k$ , then Proposition 2.6 says that  $\{\check{A}, \check{A}_0\}$  is a partition of  $R^k$ .

**Claim 2:** *if  $A$  and  $B$  are two definable subsets of  $R_0^k$  for an integer  $k$ , then we have  $\check{A} \cap \check{B} = \widehat{\check{A} \cap \check{B}}$ .*

It is sufficient to prove that  $\widehat{\check{A} \cap \check{B}}$  contains  $\check{A} \cap \check{B}$ . Let  $x \in \check{A} \cap \check{B}$ . Then there exist a closed definable subset  $E$  of  $A$  and a closed definable subset  $F$  of  $B$  such that  $x \in \overline{E} \cap \overline{F}$ . Now  $x$  belongs to  $\overline{E} \cap \overline{F} \subseteq \widehat{\check{A} \cap \check{B}}$  by Lemma 2.4, and we obtain  $\check{A} \cap \check{B} = \widehat{\check{A} \cap \check{B}}$ .

**Claim 3:** *for any two integers  $k$  and  $l$ , if  $A$  and  $B$  are definable subsets of  $R_0^k$  and  $R_0^l$  respectively, then we have  $\check{A} \times \check{B} = \widehat{\check{A} \times \check{B}}$ .*

Let  $x \in \check{A} \times \check{B}$ . We show that  $x$  belongs to  $\widehat{\check{A} \times \check{B}}$ . We have  $x = (u, v)$  for  $u \in \check{A}$  and  $v \in \check{B}$ . Then there are two closed definable subsets  $F$  and  $G$  of  $A$  and  $B$  respectively such that  $u \in \overline{F}$  and  $v \in \overline{G}$ . For each  $\varepsilon \in R^{>0}$ , there  $f \in F$  and  $g \in G$  such that  $|u - f| < \varepsilon$  and  $|v - g| < \varepsilon$ , so  $|(u, v) - (f, g)| < \varepsilon$ . Consequently,  $x = (u, v)$  belongs to  $\overline{F \times G} \subseteq \widehat{\check{A} \times \check{B}}$ .

By Corollary 2.5, for any  $x \in \widehat{\check{A} \times \check{B}}$ , there is a closed and bounded definable subset  $H$  of  $A \times B$  such that  $x \in \overline{H}$ . If  $H_1$  (resp.  $H_2$ ) denotes the image of  $H$  by the projection  $\pi_1 : A \times B \rightarrow A$  (resp.  $\pi_2 : A \times B \rightarrow B$ ), then by the continuity of the projections maps, the set  $H_1$  (resp.  $H_2$ ) is closed and bounded, and  $H$  is

contained in  $H_1 \times H_2$ . Now  $\overline{H}$  is contained in  $\overline{H_1 \times H_2}$ . But  $\overline{H_1} \times \overline{H_2}$  is closed in  $R^{k+l}$  and it contains  $H_1 \times H_2$ , so it contains  $\overline{H_1 \times H_2}$ . Hence  $\overline{H}$  is contained in  $\overline{H_1} \times \overline{H_2}$ , and  $\check{A} \times \check{B}$  contains  $\check{A} \times \check{B}$ .

**Claim 4:** let  $\pi : R^{k+1} \rightarrow R^k$  be the projection on the first  $k$  coordinates for an integer  $k$ . If  $A$  is a definable subset of  $R_0^{k+1}$ , then we have  $\pi(\check{A}) = \widehat{\pi(A)}$ .

First we note that the restriction  $\pi|_{R_0^{k+1}} : R_0^{k+1} \rightarrow R_0^k$  of  $\pi$  to  $R_0^{k+1}$  is definable and continuous. In particular, the set  $\pi(A)$  is definable.

Let  $x \in \check{A}$ . By Corollary 2.5, there is a closed and bounded definable subset  $F$  of  $A$  such that  $x \in \overline{F}$ . Then, for each  $\varepsilon \in R^{>0}$ , there exists  $y \in F$  such that  $|y - x| < \varepsilon$ , so we have  $|\pi(y) - \pi(x)| < \varepsilon$ , and thus we obtain  $\pi(x) \in \overline{\pi(F)}$ . But the restriction  $\pi|_{R_0^{k+1}}$  is definable and continuous, so  $\pi(F)$  is a closed and bounded definable subset contained in  $\pi(A)$ . Hence  $\pi(x)$  belongs to  $\widehat{\pi(A)}$ , and  $\widehat{\pi(A)}$  contains  $\pi(\check{A})$ .

Let  $x \in \widehat{\pi(A)}$ . We show that  $x$  belongs to  $\pi(\check{A})$ . By definable choice [3, Chapter 6 §1], there is a definable map  $f : \pi(A) \rightarrow R_0$  such that  $\{(\alpha, f(\alpha)) \mid \alpha \in \pi(A)\}$  is contained in  $A$ . By cell decomposition [3, Chapter 3 §2], there are finitely many cells  $C_1, \dots, C_s$  of  $\pi(A)$  such that  $\pi(A) = \cup_{i=1}^s C_i$  and  $f$  is continuous on  $C_i$  for each  $i$ . By Proposition 2.6, we have  $x \in \check{C}_r$  for some  $r \in \{1, \dots, s\}$ . By Corollary 2.5, there is a closed and bounded definable subset  $G$  of  $C_r$  such that  $x \in \overline{G}$ . But  $f$  is continuous on  $G$ , so the graph  $\Gamma$  of its restriction  $f|_G : G \rightarrow R_0$  to  $G$  is a closed and definable subset of  $A$ . Moreover, the continuity of  $f$  on  $G$  implies its uniform continuity on  $G$  [3, Chapter 6 §1], hence the following limit exists by Lemma 2.3:

$$u = \lim_{\substack{y \in G \\ y \rightarrow x}} f(y)$$

Now  $(x, u)$  belongs to  $\overline{\Gamma}$ , and since  $\Gamma$  is a closed and definable subset of  $A$ , we obtain  $(x, u) \in \check{A}$  and  $x \in \pi(\check{A})$ .

**Claim 5:** if  $\varphi(\overline{x}, \overline{a})$  is an atomic formula with free variables  $\overline{x} = (x_1, \dots, x_k)$  and parameters  $\overline{a} = (a_1, \dots, a_m)$  in  $R_0^m$ , and if  $A$  is the definable subset of  $R_0^k$  defined by  $\varphi(\overline{x}, \overline{a})$ , then  $\check{A}$  is the  $\mathcal{M}$ -definable subset of  $R^k$  defined by  $\varphi(\overline{x}, \overline{a})$ .

Let  $S$  be a relation symbol such that  $\varphi(\overline{x}, \overline{a}) = S(\overline{x}, \overline{a})$ . By the definition of  $S^{\mathcal{M}}$ , we have  $S^{\mathcal{M}} = \check{S}^{\mathcal{M}_0}$ . Therefore, if  $B = S^{\mathcal{M}_0}$  is the subset of  $R_0^{k+m}$  defined by  $\varphi(\overline{x}, \overline{y})$ , if  $C = R_0^k \times \{\overline{a}\}$ , and if  $\pi : R^{k+m} \rightarrow R^k$  is the projection on the first  $k$  coordinates, then we have  $A = \pi(B \cap C)$ . In the same way, the  $\mathcal{M}$ -definable subset of  $R^k$  defined by  $\varphi(\overline{x}, \overline{a})$  is  $\pi(S^{\mathcal{M}} \cap (R^k \times \{\overline{a}\}))$ . Since by Claim 3 we have  $\check{C} = \check{R}_0^k \times \check{\{\overline{a}\}} = R^k \times \{\overline{a}\}$ , and since by Claims 2 and 4 we have  $\check{A} = \pi(\check{B} \cap \check{C})$ , Claim 5 is proven.

**Claim 6:** let  $\varphi(\overline{x}, y, \overline{a})$  be a formula with free variables  $\overline{x} = (x_1, \dots, x_k)$  and  $y$ , and parameters  $\overline{a} \in R_0^m$ . Let  $A$  be the subset of  $R_0^{k+1}$  defined by  $\varphi(\overline{x}, y, \overline{a})$ , and  $B$  be the subset of  $R_0^k$  defined by  $\exists y \varphi(\overline{x}, y, \overline{a})$ . If the subset of  $R^{k+1}$  defined by  $\varphi(\overline{x}, y, \overline{a})$  is  $\check{A}$ , then the subset of  $R^k$  defined by  $\exists y \varphi(\overline{x}, y, \overline{a})$  is  $\check{B}$ .

This follows from Claim 4.

**Claim 7:** let  $\varphi(\overline{x}, \overline{a})$  be a formula with free variables  $\overline{x} = (x_1, \dots, x_k)$  and parameters  $\overline{a} \in R_0^m$ . Let  $A$  be the subset of  $R_0^k$  defined by  $\varphi(\overline{x}, \overline{a})$ , and  $B$  be the

subset of  $R_0^k$  defined by  $\neg\varphi(\bar{x}, \bar{a})$ . If the subset of  $R^k$  defined by  $\varphi(\bar{x}, \bar{a})$  is  $\check{A}$ , then the subset of  $R^k$  defined by  $\neg\varphi(\bar{x}, \bar{a})$  is  $\check{B}$ .

This follows from Claim 1.

**Claim 8:** let  $\varphi(\bar{x}, \bar{a})$  and  $\phi(\bar{x}, \bar{a})$  be formulas with free variables  $\bar{x} = (x_1, \dots, x_k)$  and parameters  $\bar{a} \in R_0^m$ . Let  $A$  be the subset of  $R_0^k$  defined by  $\varphi(\bar{x}, \bar{a})$ , and  $B$  be the subset of  $R_0^k$  defined by  $\phi(\bar{x}, \bar{a})$ . If the subset of  $R^k$  defined by  $\varphi(\bar{x}, \bar{a})$  is  $\check{A}$  and the one defined by  $\phi(\bar{x}, \bar{a})$  is  $\check{B}$ , then the subset of  $R^k$  defined by  $\varphi(\bar{x}, \bar{a}) \wedge \phi(\bar{x}, \bar{a})$  is  $\widehat{A \cap B}$ .

This follows from Claim 2.

**Conclusion:** it follows from Claims 5, 6, 7 and 8, and from Remark 2.2, that the structure  $\mathcal{M}$  is an elementary extension of  $\mathcal{M}_0$ .  $\square$

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