Some model theory of Polish structures

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Abstract

We introduce a notion of Polish structure and, in doing so, provide a setting which allows the application of ideas and techniques from model theory, descriptive set theory, topology and the theory of profnite groups. We define a topological notion of independence in Polish structures and prove that it has some nice properties. Using this notion, we prove counterparts of some basic results from geometric stability theory in the context of small Polish structures. Then we prove some structural theorems about compact groups regarded as Polish structures: each small, nm-stable compact G-group is solvable-by-finite; each small compact G-group of finite $\mathcal{N}M$-rank is nilpotent-by-finite. Examples of small Polish structures and groups are also given.

0 Introduction

We propose a new, model theoretic, approach to study classical descriptive set theoretic objects, like Polish G-spaces or Borel G-spaces. More generally, we are going to study Polish structures which are defined as follows.

Definition 0.1 *A Polish structure is a pair (X, G) where G is a Polish group acting faithfully on a set X so that the stabilizers of all singletons are closed subgroups of G. We say that (X, G) is small if for every $n \in \omega$, there are only countably many orbits on X^n under the action of G.*

Notice that the assumption that G acts faithfully on X is purely cosmetic as we can always divide G by the maximal subgroup acting trivially on X.

Particular cases of Polish structures are profinite structures (X and G are profinite and the action is continuous) introduced by Newelski in [21, 22] and then considered also by Wagner [15, 27] and by myself [10, 11, 12, 15], and, more generally, compact structures [13] (X is a compact metric space, G is a compact group and the action is continuous). More generally, Polish G-spaces (X is Polish and the action is continuous) and Borel G-spaces (X is a standard Borel space and the action is Borel-measurable) are also examples of Polish structures.

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In Section 1 we introduce the notion of nm-independence and we prove that it has some nice properties, as forking independence in stable theories. We show that nm-independence generalizes the notion of m-independence introduced by Newelski for profinite structures. However, the proof that nm-independence is transitive is rather complicated and it uses some descriptive set theory, whereas transitivity of m-independence follows immediately from the definition. Similarly to the case of profinite structures, in order to get the existence of nm-independent extensions, we need to assume smallness. In fact, in many general results we could just assume the existence of nm-independent extensions (as in [13]), but we prefer to assume smallness since it is more natural and easier to check in concrete examples.

In Section 2 we consider counterparts of some basic notions from geometric stability theory and we prove fundamental results about them.

In Section 3 we give examples of small Polish structures which are not profinite. In particular, we prove that the pseudo-arc considered with the full group of homeomorphisms is a small Polish structure.

In Section 4 we study small topological groups regarded as Polish structures. We prove the existence of generics in a large class of such groups. Then we study the structure of small compact G-groups (Definition 4.1). We give examples showing that counterparts of some conjectures about small profinite groups are false in our wider context. Then we prove the main structural results: each small, nm-stable compact G-group is solvable-by-finite, and each small compact G-group of finite $\mathcal{NM}$-rank is nilpotent-by-finite. We finish with a list of open questions.

The general goal of this paper, as in [13] for the case of compact structures, is to make an attempt to apply stability theory ideas to various classical mathematical objects. A similar motivation appears in [21, 22] where Newelski was considering small profinite structures. However, Polish structures (particularly Polish G-spaces) seem to be more interesting than profinite structures from the point of view of descriptive set theory or topology. Even when we look at profinite groups, it seems more natural to consider them together with the full group of topological automorphisms rather than with the group of automorphisms preserving a distinguished inverse system. Moreover, in my opinion, Polish structures yield a more adequate generalization of profinite structures than compact structures. This is because each small compact structure is profinite, so instead of smallness we should assume here the existence of m-independent extensions; but then it is not easy to find interesting examples of such compact structures which are not profinite. In contrast, we have many natural examples of small Polish structures which are not profinite.

Having established all the basic notions and results, the natural next step would be to try and prove the counterparts of some deep results from stability theory, e.g. a group configuration theorem. Such results were proved by Newelski for small profinite structures [22]. In [13] I noticed that most of them can be generalized to the case of compact [profinite] structures satisfying the existence of m-independent extensions. In small Polish structures the situation is more complicated, and it is even not clear how to formulate the appropriate conjectures.

There are also certain open questions about the existence of small profinite structures with some model theoretic properties (e.g. $\mathcal{M}$-gap conjecture [21, 22, 27]). I think it would be interesting to find counterexamples for them in the wider context of small Polish structures.

The notions introduced in this paper (e.g. nm-independence, $\mathcal{N}M$-rank) may also turn out to be new tools to deal with purely descriptive set theoretic or topological problems.

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1 Independence relation

In this section we define a notion of independence, which we call nm-independence (read non-meager independence), and we study its properties. We also prove that it coincides with Newelski's m-independence in compact structures.

If (X,G) is a Polish structure and $A \subseteq X$, then by G_A we denote the pointwise stabilizer of A. For $a \in X^n$ we define $o(a/A) = \{f(a) : f \in G_A\}$ (the orbit of a over A).

Let us recall the definition of m-independence.

Definition 1.1 *Let (X,G) be a compact structure, a be a finite tuple and A, B finite subsets of X. We say that a is m-independent from B over A (written $a \overset{m}{\perp}_A B$) if $o(a/AB)$ is open in $o(a/A)$. We say that a is m-dependent on B over A (written $a \overset{m}{\not\perp}_A B$) if $o(a/AB)$ is nowhere dense in $o(a/A)$.*

We cannot use the above definition for a Polish structure (X,G) as there is no topology on X. Even if we assumed that (X,G) is a Polish G-space, orbits could be weird, e.g. meager in their relative topologies, and then m-independence would not have nice properties in this context.

The idea to avoid the above obstacle is to define a relation of independence in terms of the Polish group G.

Definition 1.2 *Let (X,G) be a Polish structure, a be a finite tuple and A, B finite subsets of X. Let $\pi_A : G_A \to o(a/A)$ be defined by $\pi_A(g) = ga$. We say that a is nm-independent from B over A (written $a \overset{nm}{\perp}_A B$) if $\pi_A^{-1}[o(a/AB)]$ is non-meager in $\pi_A^{-1}[o(a/A)]$. Otherwise, we say that a is nm-dependent on B over A (written $a \overset{nm}{\not\perp}_A B$).*

One can also define o-independence just replacing the word 'non-meager' by 'open' in the above definition. Some of the results will work for both notions of independence. However, the proof of the existence of nm-independent extensions in small Polish structures does not work for o-independence. In Section 3 we will show that the pseudo-arc is an example of a small Polish structure without the existence of o-independent extensions. That is why nm-independence is a more appropriate notion of independence.

Notice that nm and o in Definition 1.2 come from topological properties 'non-meager' and 'open', whereas m in Definition 1.1 comes from the word 'multiplicity'.

Notation If T is a topological space and $U, V \subseteq T$, then $U \subseteq_{nm} V$ means that U is a non-meager subset of V and $U \subseteq_o V$ means that U is an open subset of V. When we write $* = nm$, it means that $*$ stands for non-meager; similarly, $* = o$ means that $*$ stands for open. Later we will also use this notation for $* \in \{c, m, nwd\}$ where c stands for closed, m for meager and nwd for nowhere dense.

Proposition 1.3 *Let (X, G) be a Polish structure, a be a finite tuple and A, B finite subsets of X. Assume $* = nm$ or $* = o$. Then TFAE:*

(1) $a \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{A} B$,

(2) $G_{AB}G_{Aa} \subseteq_* G_A$,

(3) $G_{AB}/G_{Aa} \subseteq_* G_A/G_{Aa}$.

Proof. (1) $\Leftrightarrow$ (2). An easy computation shows that $\pi_A^{-1}[o(a/AC)] = G_{AC}G_{Aa}$ for any $C \subseteq X$. Hence $\pi_A^{-1}[o(a/AB)] = G_{AB}G_{Aa}$ and $\pi_A^{-1}[o(a/A)] = G_A$. Now the desired equivalence follows directly from the definition of $*$-independence.
(2) $\Rightarrow$ (3). Let $\pi : G_A \to G_A/G_{Aa}$ be the quotient map. Then π is continuous and open.

If $* = o$, the implication follows from the fact that π is open. So consider the case $* = nm$. Suppose G_{AB}/G_{Aa} is a meager subset of G_A/G_{Aa}, i.e. G_{AB}/G_{Aa} is covered by a countable union $\bigcup_{i\in\omega} D_i$ of closed and nowhere dense subsets of G_A/G_{Aa}. Then $G_{AB}G_{Aa} = \pi^{-1}[G_{AB}/G_{Aa}] \subseteq \bigcup_{i\in\omega} \pi^{-1}[D_i]$. Since π is continuous and open, we get that $\pi^{-1}[D_i]$, $i \in \omega$, are closed and nowhere dense. So $G_{AB}G_{Aa}$ is a meager subset of G_A.
(3) $\Rightarrow$ (2). If $* = o$, it follows from continuity of π. So consider the case $* = nm$. Since π is continuous and G_{AB} is Polish, we see that G_{AB}/G_{Aa} is analytic so it has the Baire property [7, Theorem 21.6], i.e. $G_{AB}/G_{Aa} = D \triangle U$ where D is meager and U is open in G_A/G_{Aa}. Assume now that $G_{AB}/G_{Aa} \subseteq_{nm} G_A/G_{Aa}$. Then U is nonempty. We have $G_{AB}G_{Aa} = \pi^{-1}[D] \triangle \pi^{-1}[U]$. Since π is continuous and open, we get that $\pi^{-1}[D]$ is meager and $\pi^{-1}[U] \neq \emptyset$ is open in G_A. Hence $G_{AB}G_{Aa} \subseteq_{nm} G_A$. ■

Definition 1.4 *Let (X, G) be a Polish structure and A be a finite subset of X. We define the algebraic closure of A (written $Acl(A)$) as the set of all elements of X with countable orbits over A. The strong algebraic closure of A (denoted by $acl(A)$) is the set of all elements of X with finite orbits over A. The definable closure of A (written $dcl(A)$) is the set of all elements of X which are fixed by G_A. If A is infinite, we define*

$Acl(A) = \bigcup\{Acl(A_0) : A_0 \subseteq A$ is finite$\}$, $acl(A) = \bigcup\{acl(A_0) : A_0 \subseteq A$ is finite$\}$ and $dcl(A) = \bigcup\{dcl(A_0) : A_0 \subseteq A$ is finite$\}$.

In Polish structures Acl plays a similar role to acl in compact [profinite] structures.

Theorem 1.5 *In any Polish structure (X, G) $*$-independence, where $* = nm$ or $* = o$, has the following properties.*

(0) (Invariance) $a \mathop{\smile\!\!\!\!\!\!|}^*_A B \iff g(a) \mathop{\smile\!\!\!\!\!\!|}^*_{g[A]} g[B]$ whenever $g \in G$ and $a, A, B \subseteq X$ are finite.

(1) (Symmetry) $a \mathop{\smile\!\!\!\!\!\!|}^*_C b$ iff $b \mathop{\smile\!\!\!\!\!\!|}^*_C a$ for every finite $a, b, C \subseteq X$.

(2) (Transitivity) $a \mathop{\smile\!\!\!\!\!\!|}^*_B C$ and $a \mathop{\smile\!\!\!\!\!\!|}^*_A B$ iff $a \mathop{\smile\!\!\!\!\!\!|}^*_A C$ for every finite $A \subseteq B \subseteq C \subseteq X$ and $a \subseteq X$.

(3) For every finite $A \subseteq X$, $a \in Acl(A)$ iff for all finite $B \subseteq X$ we have $a \mathop{\smile\!\!\!\!\!\!|}^*_A B$.

Proof. Only (2) requires a serious argument, but let us first check (0), (1) and (3) for completeness.
(0) We know that $a \mathop{\smile\!\!\!\!\!\!|}^*_A B$ iff $G_{AB}G_{Aa} \subseteq_{nm} G_A$, which in turn is equivalent to $g^{-1}G_{AB}gg^{-1}G_{Aa}g \subseteq_{nm} g^{-1}G_A g$ and hence to $G_{g[A]g[B]}G_{g[A]g(a)} \subseteq_{nm} G_{g[A]}$. The last condition is equivalent to $g(a) \mathop{\smile\!\!\!\!\!\!|}^*_{g[A]} g[B]$.
(1) $a \mathop{\smile\!\!\!\!\!\!|}^*_A b \iff G_{Ab}G_{Aa} \subseteq_* G_A \iff G_{Aa}G_{Ab} \subseteq_* G_A \iff b \mathop{\smile\!\!\!\!\!\!|}^*_A a$.
(3) ($\Rightarrow$). $a \in Acl(A)$ means that $o(a/A)$ is countable. Hence $[G_A : G_{Aa}] \leq \omega$. So G_{Aa} is non-meager in G_A by the Baire category theorem. Since G_{Aa} is a closed subgroup of G_A, we get $G_{Aa} \subseteq_o G_A$, and so $G_{AB}G_{Aa} \subseteq_o G_A$ for any finite $B \subseteq X$. This implies that $a \mathop{\smile\!\!\!\!\!\!|}^*_A B$ for all finite $B \subseteq X$.
($\Leftarrow$). Suppose that $a \mathop{\smile\!\!\!\!\!\!|}^*_A B$ for every finite $B \subseteq X$. Then $a \mathop{\smile\!\!\!\!\!\!|}^*_A Aa$. Hence $G_{Aa} \subseteq_* G_A \Longrightarrow G_{Aa} \subseteq_o G_A \Longrightarrow [G_A : G_{Aa}] \leq \omega \Longrightarrow a \in Acl(A)$.

In order to prove transitivity, we need first to prove some purely descriptive set theoretic lemmas, which seem to be interesting in their own rights.

Lemma 1.6 *Suppose that H_1 and H_2 are closed subgroups of a Polish group H. Let $H_3 = H_1 \cap H_2$. Then H_1H_2 is a Borel subset of H. Moreover, for every $A_1 = A_1H_3$ a Borel subset of H_1 and $A_2 = H_3A_2$ a Borel subset of H_2 we have that A_1A_2 is a Borel subset of H_1H_2.*

Proof. Define $f : H_1 \times H_3 \times H_2 \to H$ by $f(h_1, h_3, h_2) = h_1h_3h_2$. By [7, Theorem 12.17] we can choose a Borel set S_1 meeting every coset from H_1/H_3 in exactly one point and a Borel set S_2 meeting every coset from $H_3\backslash H_2$ in exactly one point.

Let f_0 be the function f restricted to $S_1 \times H_3 \times S_2$. Notice that f_0 is an injection. Indeed, if $s_1, s_1' \in S_1$, $s_2, s_2' \in S_2$, $h, h' \in H_3$ and $s_1hs_2 = s_1'h's_2'$, then $s_1'^{-1}s_1 = h's_2's_2^{-1}h^{-1} \in H_1 \cap H_2 = H_3$. Hence $s_1 = s_1'$. Similarly $s_2 = s_2'$. So $h = h'$.

Since $S_1 \times H_3 \times S_2$ is a Borel subset of the Polish space $H_1 \times H_3 \times H_2$, we get that $f_0[S_1 \times H_3 \times S_2] = S_1H_3S_2 = H_1H_2$ is a Borel subset of H.

To prove the second part of the lemma, first notice that $A_1' := A_1 \cap S_1$ and $A_2' := A_2 \cap S_2$ are Borel subsets of H_1 and H_2, respectively. Hence $f_0[A_1' \times H_3 \times A_2']$ is a Borel subset of H. On the other hand, $f_0[A_1' \times H_3 \times A_2'] = A_1' H_3 A_2' = A_1 H_3 A_2 = A_1 A_2$. Hence $A_1 A_2$ is a Borel subset of H, which implies that it is a Borel subset of $H_1 H_2$. ■

Lemma 1.7 *Suppose that H_1 and H_2 are closed subgroups of a Polish group H such that $H_1 H_2$ is non-meager in its relative topology. Let $H_3 = H_1 \cap H_2$. Then for every $A_1 = A_1 H_3 \subseteq_o H_1$ and $A_2 = H_3 A_2 \subseteq_o H_2$ we have $A_1 A_2 \subseteq_o H_1 H_2$.*

Proof. Let us define a function $\psi : H_1 H_2 \to H_1/H_3 \times H_3 \backslash H_2$ by

$$\psi(h_1 h_2) = (h_1 H_3, H_3 h_2)$$

where $h_1 \in H_1$ and $h_2 \in H_2$.

We check that ψ is well-defined. Suppose $h_1 h_2 = h_1' h_2'$ where $h_1, h_1' \in H_1$ and $h_2, h_2' \in H_2$. Then $h_1'^{-1} h_1 = h_2' h_2^{-1} \in H_1 \cap H_2 = H_3$. Hence $h_1 H_3 = h_1' H_3$ and $H_3 h_2' = H_3 h_2$.

To finish the proof, it is enough to show that ψ is continuous. The following Claim follows from Lemma 1.6.

Claim 1 ψ is Baire measurable.

By Claim 1 and [7, Theorem 8.38] there is a set $C \subseteq H_1 H_2$ which is a countable intersection of dense open subsets of $H_1 H_2$ such that $\psi \restriction C$ is continuous.

Claim 2 ψ is continuous.

Proof of Claim 2. Suppose for a contradiction that there are $h_1^i \in H_1$ and $h_2^i \in H_2$, $i \in \omega$, such that for some $h_1 \in H_1$ and $h_2 \in H_2$ we have:

1. $h_1^i h_2^i \longrightarrow h_1 h_2$,
2. $(h_1^i H_3, H_3 h_2^i) \not\longrightarrow (h_1 H_3, H_3 h_2)$.

Notice that (1) and (2) hold for $hh_1^i, h_2^i g, hh_1, h_2 g$ instead of h_1^i, h_2^i, h_1, h_2 for arbitrary $h \in H_1$ and $g \in H_2$. Hence to get a contradiction, it is enough to find $h \in H_1$ and $g \in H_2$ such that $hh_1 h_2 g \in C$ and $hh_1^i h_2^i g \in C$ for all $i \in \omega$. So we will be done if we prove the following:

Subclaim $(\forall h_1 h_2 \in H_1 H_2)(\forall^* (h,g) \in H_1 \times H_2)(hh_1 h_2 g \in C)$.

Proof of Subclaim. Since $C^* := \{h_1 h_2 \in H_1 H_2 : (\forall^* (h,g) \in H_1 \times H_2)(hh_1 h_2 g \in C)\}$ is invariant under multiplication by H_1 on the left and by H_2 on the right, it is enough to show that

$$(*) \qquad (\exists h_1 h_2 \in H_1 H_2)(\forall^* (h,g) \in H_1 \times H_2)(hh_1 h_2 g \in C).$$

Since C is comeager in $H_1 H_2$, we have

$$(\forall (h,g) \in H_1 \times H_2)(\forall^* h_1 h_2 \in H_1 H_2)(hh_1 h_2 g \in C).$$

So by the Kuratowski-Ulam theorem [7, Theorem 8.41] we get

$$(\forall^* h_1h_2 \in H_1H_2)(\forall^*(h,g) \in H_1 \times H_2)(hh_1h_2g \in C).$$

Since H_1H_2 is non-meager in its relative topology, we get $(*)$ and we are done. ■

Lemma 1.8 *Suppose that H_1 and H_2 are closed subgroups of a Polish group H such that H_1H_2 is non-meager in its relative topology. Let $H_3 = H_1 \cap H_2$. Then for every analytic sets $A_1 = A_1H_3 \subseteq_{nm} H_1$ and $A_2 = H_3A_2 \subseteq_{nm} H_2$ we have $A_1A_2 \subseteq_{nm} H_1H_2$.*

Proof. Since A_1 and A_2 are analytic subsets of Polish spaces H_1 and H_2, by [7, Proposition 14.4, Theorem 21.6], we see that $A_1/H_3 \times H_3\backslash A_2$ has the Baire property as a subset of $H_1/H_3 \times H_3\backslash H_2$. Using [7, Lemma 8.43] and the argument from the proof of Proposition 1.3, we get that $A_1/H_3 \times H_3\backslash A_2 \subseteq_{nm} H_1/H_3 \times H_3\backslash H_2$.

Let us keep the notation from the proof of Lemma 1.7, where it is shown that ψ is continuous, which we use. By the above paragraph, it is enough to prove that $\psi^{-1}[B]$ is non-meager if B is non-meager and has the Baire property.

Suppose for a contradiction that $\psi^{-1}[B]$ is meager. Since B is non-meager and has the Baire property, there exists a nonempty open set U and nowhere dense closed sets F_n, $n \in \omega$, such that $B \supseteq U \setminus \bigcup_n F_n$. Then

$$\psi^{-1}[B] \supseteq \psi^{-1}[U] \setminus \bigcup_n \psi^{-1}[F_n].$$

Since $\psi^{-1}[B]$ is meager and $\psi^{-1}[U]$ is open, there is $n_0 \in \omega$ with $\psi^{-1}[F_{n_0}]$ non-meager. Since $\psi^{-1}[F_{n_0}]$ is closed, it has a nonempty interior, so countably many of its translates of the form $h_1\psi^{-1}[F_{n_0}]h_2$ with $h_1 \in H_1$ and $h_2 \in H_2$ cover H_1H_2. Then the same translates of F_{n_0} cover $H_1/H_3 \times H_3\backslash H_2$ contradicting nowhere density of F_{n_0}. ■

To prove (2) in Theorem 1.5, we will need the following corollary of Lemmas 1.7 and 1.8.

Corollary 1.9 *Suppose that H_1 and H_2 are closed subgroups of a Polish group H such that H_1H_2 is non-meager in its relative topology. Then*
(i) for every $A_1 \subseteq_o H_1$ we have $A_1H_2 \subseteq_o H_1H_2$,
(ii) for every analytic set $A_1 \subseteq_{nm} H_1$ we have $A_1H_2 \subseteq_{nm} H_1H_2$.

Proof. Apply Lemmas 1.7 and 1.8 for $A_2 = H_2$. ■

Now we are returning to the proof of Theorem 1.5.

Proof of (2) in Theorem 1.5. First consider the case $* = o$. We need to prove that

$$G_CG_{Ba} \subseteq_o G_B \wedge G_BG_{Aa} \subseteq_o G_A \iff G_CG_{Aa} \subseteq_o G_A.$$

($\Rightarrow$). Assume $G_C G_{Ba} \subseteq_o G_B \wedge G_B G_{Aa} \subseteq_o G_A$. Define $H := G_A, H_1 := G_B, H_2 := G_{Aa}, A_1 := G_C G_{Ba}$. Then $H_1 H_2 = G_B G_{Aa} \subseteq_o G_A$, so it is non-meager in its relative topology. Moreover, $A_1 \subseteq_o H_1$. Hence by Corollary 1.9(i), we get

$$G_C G_{Aa} = G_C G_{Ba} G_{Aa} = A_1 H_2 \subseteq_o H_1 H_2 = G_B G_{Aa} \subseteq_o G_A.$$

So $G_C G_{Aa} \subseteq_o G_A$.
($\Leftarrow$). Assume $G_C G_{Aa} \subseteq_o G_A$. Then of course $G_B G_{Aa} \subseteq_o G_A$. On the other hand, taking the intersection with G_B, we get that $G_C G_{Ba} \subseteq_o G_B$.

Now consider the case $* = nm$. We need to prove that

$$G_C G_{Ba} \subseteq_{nm} G_B \wedge G_B G_{Aa} \subseteq_{nm} G_A \iff G_C G_{Aa} \subseteq_{nm} G_A.$$

($\Rightarrow$). The proof is similar to the case $* = o$. We only need to check the last implication, namely $G_C G_{Aa} \subseteq_{nm} G_B G_{Aa} \subseteq_{nm} G_A$ implies $G_C G_{Aa} \subseteq_{nm} G_A$.

Suppose for a contradiction that there are closed and nowhere dense subsets D_i, $i \in \omega$, of G_A such that $G_C G_{Aa} \subseteq \bigcup_{i\in\omega} D_i$. By the assumption that $G_C G_{Aa} \subseteq_{nm} G_B G_{Aa}$, there is $i \in \omega$ such that $D_i \cap G_B G_{Aa}$ has a non-empty interior in $G_B G_{Aa}$. So countably many translates (by elements of G_B on the left and by elements of G_{Aa} on the right) of D_i cover $G_B G_{Aa}$. Since D_i is nowhere dense in G_A, we get that $G_B G_{Aa}$ is meager in G_A. This is a contradiction.
($\Leftarrow$). Assume $G_C G_{Aa} \subseteq_{nm} G_A$. Then of course $G_B G_{Aa} \subseteq_{nm} G_A$. It remains to prove that $G_C G_{Ba} \subseteq_{nm} G_B$.

By Proposition 1.3, we get $G_C/G_{Aa} \subseteq_{nm} G_A/G_{Aa}$. Moreover, by [7, Theorem 12.17], we can find a Borel subset B of G_C (and hence of G_A) meeting every left coset of $G_{Aa} \cap G_C$ in exactly one point. Then the continuous function $f : G_A \to G_A/G_{Aa}$ given by $f(g) = gG_{Aa}$ is injective on B, so by [7, Theorem 15.1], $f[B] = G_C/G_{Aa}$ is a Borel subset of G_A/G_{Aa}. We conclude that $G_C/G_{Aa} = D \triangle U$ where D is meager and analytic (even Borel) and $U \neq \emptyset$ is open in G_A/G_{Aa}.

Let $\pi : G_A \to G_A/G_{Aa}$ be the quotient map. Define $D' = \pi^{-1}[D]$ and $U' = \pi^{-1}[U]$. Then we have that $G_C G_{Aa} = D' \triangle U'$, D' is meager and analytic [7, Proposition 14.4] and $U' \neq \emptyset$ is open in G_A. Moreover, $D' G_{Aa} = D'$, $U' G_{Aa} = U'$ and $G_C \subseteq G_B$. Hence $U' \cap G_B \neq \emptyset$.

Suppose for a contradiction that $G_C G_{Ba}$ is meager in G_B. Since $G_C G_{Ba} = (G_C G_{Aa}) \cap G_B = (D' \triangle U') \cap G_B = (D' \cap G_B) \triangle (U' \cap G_B)$ and $U' \cap G_B \neq \emptyset$ is open in G_B, we get that $D' \cap G_B \subseteq_{nm} G_B$. We also know that $D' \cap G_B$ is analytic. Moreover, $G_B G_{Aa} \subseteq_{nm} G_A$ so it is non-meager in its relative topology. Hence by Corollary 1.9(ii), we get $(D' \cap G_B) G_{Aa} \subseteq_{nm} G_B G_{Aa} \subseteq_{nm} G_A$. So, in the same way as in the proof of ($\Rightarrow$), we get $(D' \cap G_B) G_{Aa} \subseteq_{nm} G_A$. Hence $D' G_{Aa} \subseteq_{nm} G_A$, which means that $D' \subseteq_{nm} G_A$, a contradiction. ■

In order to get the existence of nm-independent extensions we assume smallness.

Theorem 1.10 (Existence of nm-independent extensions) *Let (X, G) be a small Polish structure. Then for all finite $a \subseteq X$ and $A \subseteq B \subseteq X$, there is $b \in o(a/A)$ such that $b \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$.*

Before the proof let us show the following remark.

Remark 1.11 *A Polish structure (X,G) satisfies the existence of $*$-independent extensions, where $*=nm$ or $*=o$, iff for all finite $a \subseteq X$ and $A \subseteq B \subseteq X$ there exists $f \in G_A$ such that $G_B f G_{Aa} \subseteq_* G_A$.*

Proof. The existence of $*$-independent extensions is equivalent to the fact that for every finite $a \subseteq X$ and $A \subseteq B \subseteq X$ there is $b \in o(a/A)$ such that $G_B G_{Ab} \subseteq_* G_A$, which in turn is equivalent to the conclusion of the remark. ■

Proof of Theorem 1.10. Consider a, A, B as in the theorem. Let $\{a_i : i \in I\}$ be a countable set of representatives of all orbits over B contained in $o(a/A)$. Then $o(a/A) = \bigcup_{i \in I} o(a_i/B)$. Take $f_i \in G_A$, $i \in I$, such that $f_i(a) = a_i$.

Claim $G_A = \bigcup_{i \in I} G_B f_i G_{Aa}$.

Proof of Claim. Consider any $f \in G_A$. Then $f(a) \in o(a_i/B)$ for some $i \in I$. Hence there is $g \in G_B$ such that $g(f(a)) = f_i(a)$. Then $f_i^{-1} g f \in G_{Aa}$, so $f \in g^{-1} f_i G_{Aa} \subseteq G_B f_i G_{Aa}$. Hence $G_A \subseteq \bigcup_{i \in I} G_B f_i G_{Aa}$. The opposite inclusion is obvious. □

By the Claim and the Baire category theorem, there is $i \in I$ such that $G_B f_i G_{Aa} \subseteq_{nm} G_A$. So the proof is completed by Remark 1.11. ■

We see that the above application of the Baire category theorem works only for nm-independence. In Section 3 (Remark 3.5) we will see that the pseudo-arc is an example of a small Polish structure without the existence of o-independence extensions.

One justification for our definition of nm-independence is the fact that it satisfies all the fundamental properties (Theorems 1.5 and 1.10) necessary to develop a counterpart of basic geometric stability theory. Another justification is given by the next corollary, which shows that in compact [profinite] structures nm-independence coincides with m-independence.

Theorem 1.12 *Let (X,G) be a Polish structure such that G acts continuously on a separable metrizable space X. Let $a, A, B \subseteq X$ be finite. Assume that $o(a/A)$ is non-meager in its relative topology (e.g. it is Polish). Then $a \mathop{\smile\!\!\!\!\!\!\,|}^{*}_{A} B \iff o(a/AB) \subseteq_* o(a/A)$ where $*=o$ or $*=nm$.*

Proof. Let $\pi : G_A/G_{Aa} \to o(a/A)$ be defined by $\pi(gG_{Aa}) = ga$. Since $o(a/A)$ is non-meager in its relative topology, by Effros' theorem [2, Theorem 2.2.2] we have that π is a homeomorphism. Hence $o(a/AB) \subseteq_* o(a/A) \iff \pi^{-1}[o(a/AB)] \subseteq_* \pi^{-1}[o(a/A)] \iff G_{AB}/G_{Aa} \subseteq_* G_A/G_{Aa}$. We finish using Proposition 1.3. ■

Corollary 1.13 *In every compact structure o-independence, nm-independence and m-independence coincide.*

In fact, for nm-independence we can strengthen Theorem 1.12.

Theorem 1.14 *Let (X, G) be a Polish structure such that G acts continuously on a Hausdorff space X. Let $a, A, B \subseteq X$ be finite. Assume that $o(a/A)$ is non-meager in its relative topology. Then $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_A B \iff o(a/AB) \subseteq_{nm} o(a/A)$.*

Proof. Assume for simplicity that $A = \emptyset$. Let $\pi := \pi_\emptyset : G \to o(a)$, i.e. $\pi(g) = ga$. By assumption, π is continuous. We need to show

$$(*) \qquad \pi^{-1}[o(a/B)] \subseteq_{nm} G \iff o(a/B) \subseteq_{nm} o(a).$$

Before the proof of $(*)$ we need to recall some facts from general topology. We work in a Hausdorff space Y. There is a classical result [1, Theorem 25.3] that the class of all sets with the Baire property is a σ-algebra and is closed under the (Souslin) operation $\mathcal{A}$. On the other hand, an easy modification of the proof of [7, Theorem 25.7] gives us that if $f : P \to Y$ is continuous and P is Polish, then $f[P] \in \mathcal{A}(CLO(Y))$ where $\mathcal{A}(CLO(Y))$ is the family of all sets obtained by application of the operation $\mathcal{A}$ to any family of closed subsets of Y. So we conclude that $f[P]$ has the Baire property.

Now we return to the proof of $(*)$.

Claim 1 If $M \subseteq_m o(a)$, then $\pi^{-1}[M] \subseteq_m G$.

Proof. We will be done when we show that $D \subseteq_{nwd} o(a)$ implies $\pi^{-1}[D] \subseteq_{nwd} G$. We can assume that D is closed. Suppose for a contradiction that $\pi^{-1}[D]$ is not nowhere dense in G. Since it is closed, it has a nonempty interior. So there are g_i, $i \in \omega$, such that $\bigcup_{i\in\omega} g_i\pi^{-1}[D] = G$. Then $\bigcup_{i\in\omega} g_i D = o(a)$. Since $o(a)$ is non-meager in itself, there is $i \in \omega$ such that $g_i D \subseteq_{nm} o(a)$, and so $D \subseteq_{nm} o(a)$, a contradiction. □

Claim 2 If N has the Baire property as a subset of $o(a)$ and $N \subseteq_{nm} o(a)$, then $\pi^{-1}[N] \subseteq_{nm} G$.

Proof. We have $N = U \triangle M$ where U is a nonempty, open and M a meager subset of $o(a)$. So $\pi^{-1}[N] = \pi^{-1}[U] \triangle \pi^{-1}[M]$. We also have that $\pi^{-1}[U] \neq \emptyset$ is open, and by Claim1, $\pi^{-1}[M]$ is meager in G. □

By Claim 1, we get $(\Rightarrow)$ in $(*)$. To show $(\Leftarrow)$, notice that $o(a/B) = \pi[G_B]$ and G_B is Polish, so by the remarks right below statement $(*)$, we have that $o(a/B)$ has the Baire property in $o(a)$. We finish using Claim 2. ■

In Section 3 (Remark 3.5) we will see that the pseudo-arc considered with the group of all homeomorphisms is an example of a small Polish structure where o-independence and nm-independence differ. Below we give simpler examples of Polish structures in which these two notions differ, but these structures are not small.

Example 1 (i) Let $X = (S^1)^\omega$ and $G = (Homeo(S^1))^\omega$. We consider the Polish structure (X, G) where G acts naturally on X on the appropriate coordinates. Then in (X, G) o-independence is different from nm-independence. Moreover, (X, G) satisfies neither the existence of o-independent nor nm-independent extensions. Thus, it is not small.

(ii) Let $X = (S^2)^\omega$ and $G = (Homeo(S^2))^\omega$. Then we get the same conclusions as in (i) except that now (X, G) satisfies the existence of nm-independent extensions.

Proof. (i) Take any $\overline{x} = \langle x_0, x_1, \ldots \rangle \in X$ and $\overline{y} = \langle y_0, y_1, \ldots \rangle \in X$ such that $x_i \neq y_i$ for all $i \in \omega$. Then $o(\overline{x}) = X$ and $o(\overline{x}/\overline{y}) = S^1 \setminus \{y_0\} \times S^1 \setminus \{y_1\} \times \ldots$. So $o(\overline{x}/\overline{y})$ is non-meager and not open in $o(\overline{x})$. Since $o(\overline{x}) = X$ is Polish, by Theorem 1.12, we get that $\overline{x} \overset{nm}{\downarrow} \overline{y}$ and $\overline{x} \overset{o}{\not\downarrow} \overline{y}$. This also shows that (X, G) does not satisfy the existence of o-independent extensions.

In order to prove the lack of nm-independent extensions, take any $\overline{x} = \langle x_0, x_1, \ldots \rangle \in X$, $\overline{y} = \langle y_0, y_1, \ldots \rangle \in X$ and $\overline{z} = \langle z_0, z_1, \ldots \rangle \in X$ such that for every $i \in \omega$ the elements x_i, y_i and z_i are pairwise distinct. Consider any $\overline{t} = \langle t_0, t_1, \ldots \rangle \in X$. Then $o(\overline{t}/\overline{xyz})$ is a product of singletons and interiors of closed arcs contained in S^1, so it is meager in $o(\overline{t}) = X$. By Theorem 1.12, we get $\overline{t} \overset{nm}{\not\downarrow} \overline{xyz}$.
(ii) is left to the reader. ■

2 Basic model theory

In this section we introduce counterparts of some basic notions from stability theory and we investigate their properties. We also study them in a more topological setting of G-spaces.

In compact [profinite] structures definable sets were defined as the sets which are closed and invariant over finite subsets. Since for a Polish structure (X, G) we do not have any topology on X, we need another definition of definable sets. Moreover, as in model theory we would like to have (imaginary) names for definable sets for which forking calculus works in the same way as for the real elements. Below we propose definitions of definable sets and imaginary elements that fulfill these requirements.

From now on we assume that (X, G) is a Polish structure. For $Y \subseteq X^n$, we define $Stab(Y) := \{g \in G : g[Y] = Y\}$. We say that Y is invariant [over a finite set A] if $Stab(Y) = G$ [$Stab(Y) = G_A$, respectively].

Definition 2.1 *The imaginary extension, denoted by X^{eq}, is the union of all sets of the form X^n/E with E ranging over all invariant equivalence relations such that for all $a \in X^n$, $Stab([a]_E) <_c G$. The sets X^n/E will be called the sorts of X^{eq}.*

Remark 2.2 *Let E be an invariant equivalence relation on X^n whose classes have closed stabilizers in G. Then G induces a group of permutations of X^n/E, denoted by $G \restriction X^n/E$, which is Polish, and $(X^n/E, G \restriction X^n/E)$ is a Polish structure.*

As in model theory, $(X^{eq})^{eq} = X^{eq}$ which means that if E is an invariant equivalence relation on a product of sorts $X^{n_1}/E_1 \times \ldots \times X^{n_k}/E_k$ whose classes have closed stabilizers in G, then the set of E-classes can be identified with the sort $X^{n_1} \times \ldots \times X^{n_k}/E'$ where

$$E'(x_1, \ldots, x_k; y_1, \ldots, y_k) \iff E(x_1/E_1, \ldots, x_k/E_k; y_1/E_1, \ldots, y_k/E_k).$$

Definition 2.3 *A subset D of X (or more generally of any sort of X^{eq}) is said to be definable over a finite subset A of X^{eq} if D is invariant over A and $Stab(D) <_c G$. We say that D is definable if it is definable over some A. We say that $d \in X^{eq}$ is a name for D if for every $f \in G$ we have $f[D] = D \iff f(d) = d$.*

Proposition 2.4 *Each set definable in (X, G) [or in X^{eq}] has a name in X^{eq}.*

Proof. The proof is similar to the proof of Proposition 1.9 in [13]. Suppose D is a-definable for some finite $a \in X^n$. We define an equivalence relation E on X^n by:

$$E(a_1, a_2) \iff [a_1 = a_2 \vee (a_1, a_2) \in S(a, a)]$$

where $S = \{(f, g) \in G \times G : f[D] = g[D]\}$. It is easy to check that E is invariant, every class of E has a closed stabilizer and a/E is a name for D. ■

Working in X^{eq}, we can define Acl^{eq} in the same way as in X. Then the results of Section 1 (including Theorems 1.5 and 1.10) are true in X^{eq} (the only exception is the fact that the Polish structure (X, G) considered in Example 1(ii) does not satisfy the existence of nm-independent extensions in X^{eq}). However, in the case of a compact [profinite] structure (X, G) both the family of definable sets and X^{eq} computed according to the definitions given in this paper are larger than those computed according to the definitions from [21, 22, 13]. For example, any invariant subset of X is definable in our sense, but it is not necessarily closed; also, if $Y \subseteq X$ is invariant, but not closed, then the equivalence relation E on X with classes Y and $X \setminus Y$ is invariant and $X/E \subseteq X^{eq}$ in our sense, but not in the sense of [21, 22, 13].

We could give more restrictive definitions of definable sets. For example, we could say that $D \subseteq X^n$ is definable if for every $a \in X^n$ the set $\{g \in G : ga \in D\}$ is closed. Then the family of definable sets would be closed under finite unions and intersections, but not necessarily under complements (as in compact structures). In the case of compact structures, this notion is also more general than Newelski's notion of definability, but it agrees with it on every orbit, i.e. D is definable in our sense iff for every $a \in X^n$, $o(a) \cap D$ is closed.

Recall some classical definitions. In the whole paper we assume that compact spaces are Hausdorff by definition.

Definition 2.5 *Let G be a Polish group.*
(i) A (topological) G-space is a Polish structure (X, G) such that X is a topological space and the action of G on X is continuous.
(ii) A Polish [compact] G-space is a G-space (X, G) where X is a Polish [compact] space.

In such topological situations it is natural to consider the same notion of definability as in profinite structures. To distinguish between this notion and our previous notion of definability, we will follow Wagner's terminology [27].

Definition 2.6 *Suppose (X, G) is a topological G-space. We say that $D \subseteq X^n$ is A-closed for a finite $A \subseteq X^{eq}$ if it is closed and invariant over A. Let E be an invariant equivalence relation on X^n whose classes have closed stabilizers in G, and equip $X^n/E \subseteq X^{eq}$ with the quotient topology. Then A-closed subsets of X^n/E are defined as above. We say that D is $*$-closed if it is A-closed for some finite A.*

It would be nice to know that if (X, G) is topologically nice, then so is (X^{eq}, G). However, in general, taking quotients does not preserve good topological properties like being Hausdorff or Polish. The following considerations tell us that everything looks nice for compact G-spaces, which we will be using in Section 4.

Proposition 2.7 *Let (X, G) be a compact G-space and E be a $\emptyset$-closed equivalence relation on X^n. Then X^n/E is compact, and $(X^n/E, G/G_{X^n/E})$ is a compact G-space where $G_{X^n/E} = \{g \in G : g[a]_E = [a]_E \text{ for every } a \in X^n\}$.*

Proof. We can assume $n = 1$. The fact that X/E is compact is a standard result in general topology. Let us give a few details. Only the fact that X/E is Hausdorff requires an explanation. Since E is closed and X is compact, the quotient map from X to X/E is closed. Since every compact space is normal and normality is preserved by closed functions, X/E is Hausdorff.

We also have that $G_{X/E} = \bigcap_{x \in X} G_{[x]_E}$, which implies that $G_{X/E} <_c G$ as each $[x]_E$ is a closed subset of X. So $G/G_{X/E}$ is a Polish group.

It remains to show that the action of $G/G_{X/E}$ on X/E is continuous, i.e. the function $\Phi : G/G_{X/E} \times X/E \to X/E$ given by $\Phi(gG_{X/E}, [x]_E) = [gx]_E$ is continuous. It is continuous if we equip $G/G_{X/E}$ with the compact-open topology because X/E is compact and it is easy to see that $G/G_{X/E}$ acts on X/E by homeomorphisms. So it is enough to show that the quotient topology on $G/G_{X/E}$ is stronger or equal to the compact-open topology. Take any set $\mathcal{U}$ of the compact-open subbasis on $G/G_{X/E}$, i.e. $\mathcal{U} = \{\overline{g} \in G/G_{X/E} : \overline{g}[K] \subseteq U\}$ where K is a compact and U is an open subset of X/E. Let $\pi_G : G \to G/G_{X/E}$ and $\pi_X : X \to X/E$ be the quotient maps. Then $K_1 := \pi_X^{-1}[K]$ is a compact and $U_1 := \pi_X^{-1}[U]$ is an open subset of X. We also have that $\pi_G^{-1}[\mathcal{U}] = \{g \in G : g[K_1] \subseteq U_1\}$. Thus, $\pi_G^{-1}[\mathcal{U}]$ is open in the compact-open topology on G, and hence in the original topology on G because X is compact and the action of G on X is continuous. So $\mathcal{U}$ is open in the quotient topology on $G/G_{X/E}$. ■

The above proposition leads to the following definition.

Definition 2.8 *Let (X, G) be a compact G-space. We define X^{teq} (topological imaginary extension) as the disjoint union of the spaces X^n/E with E ranging over all $\emptyset$-closed equivalence relations on X^n. The spaces X^n/E will be called topological sorts of X^{teq}.*

By Proposition 2.7, each topological sort X/E together with the group $G/G_{X/E}$ is a compact G-space. If E is A-closed for some finite set A, then replacing G by G_A, X/E can also be treated as a topological sort. We say that D is A-closed in X^{teq} if it is A-closed in a sort of X^{teq}. Quotients that occur in natural situations are often topological sorts, e.g. if we divide a compact G-group (Definition 4.1) by a $*$-closed subgroup.

Proposition 2.9 *If (X, G) is a small compact G-space, then X is second countable and hence metrizable. Thus, X is a Polish, compact space.*

Proof. We need to recall some topological terminology. If Y is a topological space, then a family $\mathcal{N}$ of subsets of Y is said to be a net of Y if for every $y \in Y$ and its open neighborhood U there is $M \in \mathcal{N}$ with $y \in M \subseteq U$. The smallest possible cardinality of a net of Y is called the net weight of Y and is denoted by $nw(Y)$. Of course, for every space Y we have $nw(Y) \leq w(Y)$. For us the following easy fact is essential [3, Twierdzenie 3.1.19]: If Y is compact, then $nw(Y) = w(Y)$.

By this fact, in order to prove that X is second countable, it is enough to show that $nw(X) \leq \aleph_0$. Fix a countable basis $\mathcal{U}$ of G. Let $\{a_i : i \in I\}$ be a countable set of representatives of all orbits, and $\pi^i : G \to o(a_i)$, $i \in I$, be defined by $\pi^i(g) = ga_i$. It is easy to see that the countable family $\{\pi^i[U] : i \in I, U \in \mathcal{U}\}$ is a net of X.

By [3, Theorem 4.2.8], we get that X is metrizable. ■

Let us formulate one more topological remark.

Remark 2.10 *(i) Let (X, G) be a Polish structure and D an A-definable [or only invariant over A] subset of X^{eq}. Then $(D, G_A/G_{AD})$ is a Polish structure. Moreover, if (X, G) is small, then $(D, G_A/G_{AD})$ is also small, and for tuples and subsets of D, nm-independence computed in (X, G_A) coincides with nm-independence computed in $(D, G_A/G_{AD})$.*
(ii) Let (X, G) be a topological G-space. If D is an A-closed subset of X^n [or X^{teq}, if X is compact], then $(D, G_A/G_{AD})$ is a topological G_A/G_{AD}-space.

Proof. (i) Only the fact that the two notions of nm-independence coincide requires an explanation. Let $\mathop{\smile\!\!\!\!\!\mid}\limits^{1}$ and $\mathop{\smile\!\!\!\!\!\mid}\limits^{2}$ be the notions of nm-independence computed in (X, G_A) and $(D, G_A/G_{AD})$, respectively. Let a be a finite tuple and A, B finite subsets of D.

Assume $a \mathop{\smile\!\!\!\!\!\mid}\limits^{1}_A B$. We need to show $a \mathop{\smile\!\!\!\!\!\mid}\limits^{2}_A B$. We have $G_{AB}G_{Aa} \subseteq_{nm} G_A$. Of course, we also know that G_{AD} is a closed, normal subgroup of G_A. So arguing as in the proof of Proposition 1.3, we get $(G_{AB}/G_{AD})(G_{Aa}/G_{AD}) = (G_{AB}G_{Aa})/G_{AD} \subseteq_{nm} G_A/G_{AD}$, which means that $a \mathop{\smile\!\!\!\!\!\mid}\limits^{2}_A B$.

In order to show that $a \mathop{\smile\!\!\!\!\!\mid}\limits^{2}_A B$ implies $a \mathop{\smile\!\!\!\!\!\mid}\limits^{1}_A B$, we also proceed as in the proof of Proposition 1.3.
(ii) is very easy. ■

We come back to the general situation when (X, G) is a Polish structure, but from now on we assume that (X, G) is small. In fact, more generally, one could only assume that (X, G) satisfies the existence of nm-independent extensions and, working in X^{eq}, one should also assume that the existence of nm-independent extensions holds in X^{eq}.

As usual, having a notion of independence satisfying the properties listed in Theorems 1.5 and 1.10, one can define a rank, which has some nice properties.

Definition 2.11 *The rank $\mathcal{NM}$ is the unique function from the collection of orbits over finite sets to the ordinals together with ∞, satisfying*

$$\mathcal{NM}(a/A) \geq \alpha + 1 \quad \text{iff} \quad \text{there is a finite set } B \supseteq A \text{ such that } a \not\downarrow^{nm}_A B \text{ and } \mathcal{NM}(a/B) \geq \alpha.$$

The results formulated below follow from a standard forking calculation (e.g. see [26, Lemma 5.1.4 and Theorem 5.1.6]).

Proposition 2.12 *Let a, b, A be any finite tuples (subsets) of X or X^{eq}.*

(1) $a \downarrow^{nm}_A b$ implies $\mathcal{N}M(a/Ab) = \mathcal{N}M(a/A)$. The converse holds whenever $\mathcal{N}M(a/A) < \infty$.

(2) $\mathcal{N}M(a/bA) + \mathcal{N}M(b/A) \leq \mathcal{N}M(ab/A) \leq \mathcal{N}M(a/bA) \oplus \mathcal{N}M(b/A)$.

(3) Suppose $\mathcal{N}M(a/Ab) < \infty$ and $\mathcal{N}M(a/A) \geq \mathcal{N}M(a/Ab) \oplus \alpha$. Then $\mathcal{N}M(b/A) \geq \mathcal{N}M(b/Aa) + \alpha$.

(4) Suppose $\mathcal{N}M(a/Ab) < \infty$ and $\mathcal{N}M(a/A) \geq \mathcal{N}M(a/Ab) + \omega^\alpha n$. Then $\mathcal{N}M(b/A) \geq \mathcal{N}M(b/Aa) + \omega^\alpha n$.

(5) If $a \downarrow^{nm}_A b$, then $\mathcal{N}M(ab/A) = \mathcal{N}M(a/bA) \oplus \mathcal{N}M(b/A)$.

As in stable or simple theories, the (in-)equalities in (2)-(5) will be called Lascar inequalities.

Using existence of nm-independent extensions in X^{eq} and Proposition 2.12(1), an easy induction yield the following remark.

Remark 2.13 *Let $a, A \subseteq X$ be finite. Then $\mathcal{N}M$-rank of $o(a/A)$ computed in X is the same as $\mathcal{N}M$-rank of $o(a/A)$ computed in X^{eq}.*

Definition 2.14 *(X, G) is nm-stable if every 1-orbit has ordinal $\mathcal{N}M$-rank.*

Remark 2.15 *(X, G) is nm-stable iff there is no infinite sequence $A_0 \subseteq A_1 \subseteq \ldots$ of finite subsets of X and $a \in X$ such that $a \not\downarrow^{nm}_{A_i} A_{i+1}$ for every $i \in \omega$.*

By Lascar inequalities and Remark 2.13 we easily get:

Remark 2.16 *(X, G) is nm-stable iff each n-orbit, $n \geq 1$, has ordinal $\mathcal{N}M$-rank iff each 1-orbit in X^{eq} has ordinal $\mathcal{N}M$-rank.*

Proposition 2.17 *TFAE:*

(1) (X, G) is nm-stable.

(2) There is no $a \in X$ and finite sets $A_0 \subseteq A_1 \subseteq \ldots \subseteq X$ such that $G_{A_{i+1}}G_{A_ia}$ is meager in G_{A_i} for every $i \in \omega$.

(3) For every $a \in X$ and finite sets $A_0 \subseteq A_1 \subseteq \ldots X$ there is $n \in \omega$ such that $G_{A_{n+i+1}}G_{A_na} \subseteq_{nm} G_{A_{n+i}}G_{A_na}$ for every $i \in \omega$.

Proof. (1) $\Leftrightarrow$ (2) is obvious by Proposition 1.3 and Remark 2.15.
(1) $\Rightarrow$ (3). By Proposition 1.3, Remark 2.15 and transitivity, we can find $n \in \omega$ such that $G_{A_{n+i+1}}G_{A_{n+i}a} \subseteq_{nm} G_{A_{n+i}}$ for every $i \in \omega$. So by Corollary 1.9(ii) and induction, we easily get that $G_{A_{n+i+1}}G_{A_na} \subseteq_{nm} G_{A_{n+i}}G_{A_na}$ for every $i \in \omega$.
(3) $\Rightarrow$ (1). Take any $a \in X$ and finite sets $A_0 \subseteq A_1 \subseteq \ldots \subseteq X$. By (3), there is $n \in \omega$ such that $G_{A_{n+i+1}}G_{A_na} \subseteq_{nm} G_{A_{n+i}}G_{A_na}$ for every $i \in \omega$. Hence we easily get that $G_{A_{n+i}}G_{A_na} \subseteq_{nm} G_{A_n}$ for every $i \in \omega$. So $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_{A_n} A_{n+i+1}$, and hence $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_{A_{n+i}} A_{n+i+1}$, for every $i \in \omega$. We are done by Remark 2.15. ■

One can adopt the proofs of [20, Proposition 1.1, Corollary 1.2] to conclude the following

Proposition 2.18 *Assume (X,G) is nm-stable. Let $a \in X^{eq}$, A be a finite subset of X and $\alpha \in Ord$. Then TFAE:*

(1) $\mathcal{N}M(a/A) \geq \alpha + 1$.

(2) There is a finite set $B \subseteq o(a/A)$ such that $a \not\mathop{\smile\!\!\!\!\!\!|}^{nm}_{A} B$ and $\mathcal{N}M(a/AB) \geq \alpha$.

Using this together with Remark 2.10, we get

Corollary 2.19 *Assume (X,G) is nm-stable. Let D be definable over A, and $a, B \subseteq D$ finite. Then $\mathcal{N}M(a/AB)$ computed in (X,G) is equal to $\mathcal{N}M(a/B)$ computed in $(D, G_A/D_{AD})$.*

As it was mentioned in the introduction I hope that *nm*-independence, $\mathcal{NM}$-rank and maybe some other model theoretic notions may be useful in descriptive set theory or topology. It is worth mentioning here that by Theorem 1.14, if orbits are non-megear in their relative topologies, then $\mathcal{NM}$-rank and *nm*-stability can be expressed in terms of X instead of G, and $\mathcal{NM}$-rank measures a 'topological size' of orbits. More precisely, Theorem 1.14 gives us the following remark.

Remark 2.20 *Suppose that X is a Hausdorff space and G acts continuously on X. Assume that all orbits over finite sets are non-meager in their relative topologies. Then*

(1) $\mathcal{NM}(a/A) \geq \alpha + 1$ iff there is a finite set $B \supseteq A$ such that $o(a/B)$ is meager in $o(a/A)$ and $\mathcal{NM}(a/B) \geq \alpha$.

(2) (X,G) is nm-stable iff there is no infinite sequence $A_0 \subseteq A_1 \subseteq \ldots$ of finite subsets of X and $a \in X$ such that $o(a/A_{i+1})$ is meager in $o(a/A_i)$ for every $i \in \omega$.

By Corollary 1.13, we see that in compact structures $\mathcal{NM}$-rank and nm-stability coincide with $\mathcal{M}$-rank and m-stability, respectively.

As in model theory and in profinite structures one can define a natural pregeometry on an orbit of $\mathcal{NM}$-rank 1. To introduce this definition, first we need to prove several remarks.

Remark 2.21 *For any finite $a, A \subseteq X^{eq}$ we have that $\mathcal{NM}(a/A) = 0$ iff $a \in Acl^{eq}(A)$.*

Proof. ($\Rightarrow$). Assume $\mathcal{NM}(a/A) = 0$. Then $a \mathop{\perp\!\!\!\perp}^{nm}_A B$ for every finite $B \subseteq X^{eq}$. By Theorem 1.5(3), we get $a \in Acl^{eq}(A)$.
($\Leftarrow$). Assume $a \in Acl^{eq}(A)$. By Theorem 1.5(3), we get that $a \mathop{\perp\!\!\!\perp}^{nm}_A B$ for every finite $B \subseteq X^{eq}$. Hence $\mathcal{NM}(a/A) = 0$. ■

Corollary 2.22 *If (X, G) is a G-space and $o(a/A)$ is Polish, then $\mathcal{NM}(a/A) = 0$ iff $o(a/A)$ is countable with the discrete topology.*

For a finite set $A \subseteq X^{eq}$ we define the operator Acl^{eq}_A by $Acl^{eq}_A(B) = Acl^{eq}(A \cup B)$.

Remark 2.23 *Assume $\mathcal{NM}(a/A) = 1$ and B is a finite subset of X^{eq}. Then $a \in Acl^{eq}_A(B)$ iff $a \mathop{\not\perp\!\!\!\perp}^{nm}_A B$.*

Proof. ($\Rightarrow$). Assume $a \mathop{\not\perp\!\!\!\perp}^{nm}_A B$. Then $\mathcal{NM}(a/AB) < \mathcal{NM}(a/A) = 1$. Hence $\mathcal{NM}(a/AB) = 0$. By Remark 2.21, we get $a \in Acl^{eq}_A(B)$.
($\Leftarrow$). Assume $a \in Acl^{eq}_A(B)$. By Remark 2.21, $\mathcal{NM}(a/AB) = 0 < \mathcal{NM}(a/A)$. Hence by Proposition 2.12(1), $a \mathop{\not\perp\!\!\!\perp}^{nm}_A B$. ■

Proposition 2.24 *Assume $\mathcal{NM}(a/A) = 1$. Then $(o(a/A), Acl^{eq}_A)$ is a pregeometry.*

Proof. The proof is the same as in model theory. Let us only check the exchange property. Take any $B \subseteq o(a/A)$ and $b \in o(a/A)$. Consider any $a' \in Acl^{eq}_A(Bb) \setminus Acl^{eq}_A(B)$. Wlog B is finite. By Remark 2.23, $a' \mathop{\not\perp\!\!\!\perp}^{nm}_A ABb$ and $a' \mathop{\perp\!\!\!\perp}^{nm}_A AB$. Hence by Theorem 1.5, $b \mathop{\not\perp\!\!\!\perp}^{nm}_{AB} a'$. So $b \mathop{\not\perp\!\!\!\perp}^{nm}_A Ba'$, which implies that $b \in Acl_A(Ba')$. ■

Definition 2.25 *If D is definable over A in X^{eq}, we define $\mathcal{N}M(D) = sup\{\mathcal{N}M(d/A) : d \in D\}$.*

Let D be definable over A. As above one can show that if $\mathcal{NM}(D) := 1$, then (D, Acl^{eq}) is a pregeometry.

3 Examples

In this section we give several examples of small Polish structures. To begin with, notice that all small profinite structures are such examples. This class contains for instance all abelian profinite groups of finite exponent presented as the inverse limit of a system indexed by the natural numbers and considered with the standard structural group [11, Theorem 1.9]. Thus, all such groups with the full group of topological automorphisms are also small. Below we give examples which are not profinite.

Example 2 (i) For every $n \geq 1$ the Polish structure $(S^n, Homeo(S^n))$ where S^n is the n-dimensional sphere and $Homeo(S^n)$ is the group of all homeomorphisms of S^n with the compact-open topology is small of $\mathcal{NM}$-rank 1.
(ii) For every $n \geq 1$ the Polish structure $((S^1)^n, Homeo((S^1)^n))$ where $(S^1)^n$ is the n-dimensional torus is small of $\mathcal{NM}$-rank 1.

Example 3 The Polish structure $(I^\omega, Homeo(I^\omega))$ where I^ω is the Hilbert cube is small of $\mathcal{NM}$-rank 1.

Proof. By [18, Section 6.1, Exercise 2] we know that the action is n-transitive for every $n \geq 1$. ■

Example 4 (i) Let $X = (S^1)^\omega$ and G be the group of all homeomorphisms of X respecting the inverse system consisting of initial subproducts $S^1 \longleftarrow S^1 \times S^1 \longleftarrow \dots$. Then (X, G) is a small, non-nm-stable Polish structure.
(ii) The Polish structure $((S^1)^\omega, Homeo((S^1)^\omega)$ is small of $\mathcal{N}M$-rank 1.

Proof. (i) It is easy to see that G is a closed subgroup of $Homeo((S^1)^\omega)$ so (X, G) is a Polish structure. The following claim is essential, and we leave it as a non-difficult topological exercise.

Claim Let $\overline{x} = \langle x_i \rangle_{i \in \omega} \in X$ and A be a finite subset of X. If $A \neq \emptyset$, choose an element $\overline{a} = \langle a_i \rangle_{i \in \omega} \in A$ with the longest common beginning with $\overline{x}$. If $\overline{x} \neq \overline{a}$, let n be the smallest natural number such that $x_n \neq a_n$. Then we have the following possibilities.

(1) If $A = \emptyset$, then $o(\overline{x}/A) = X$.

(2) If $A \neq \emptyset$ and $\overline{x} \notin A$, then $o(\overline{x}/A) = \{x_0\} \times \dots \times \{x_{n-1}\} \times U \times S^1 \times S^1 \times \dots$ for some open $U \subseteq S^1$.

(3) If $\overline{x} \in A$, then $o(\overline{x}/A) = \{\overline{x}\}$.

Smallness is a simple consequence of the Claim once we use the fact that on S^1 there is no uncountable family of open, pairwise disjoint sets. We also see that 1-orbits are non-meager in their relative topologies. Take any $\overline{x} = \langle x_i \rangle_{i \in \omega} \in X$ and $\overline{x^j} = \langle x_i^j \rangle_{i \in \omega} \in X$, $j \in \omega$, such that $x_{j+1}^j \neq x_{j+1}$ and $x_i^j = x_i$ for $i \leq j$. By the Claim, we see that for every $n \in \omega$ we have $o(\overline{x}/\{\overline{x^j} : j \leq n\})$ is meager in

$o(\overline{x}/\{\overline{x^j} : j \leq n+1\})$. Thus, in virtue of Remark 2.20, (X, G) is not nm-stable.
(ii) follows from the fact that the action is n-transitive for every $n \geq 1$. Smallness also follows immediately from (i). We give a brief sketch of the proof of n-transitivity. Take any finite set $A \subseteq X$ and $\overline{x} = \langle x_i \rangle_{i \in \omega} \in X \setminus A$. We need to show that $o(\overline{x}/A) = X \setminus A$. Consider any $\overline{y} = \langle y_i \rangle_{i \in \omega} \in X \setminus A$. There is $n \in \omega$ such that $\langle x_0, \ldots, x_n \rangle, \langle y_0, \ldots, y_n \rangle \notin A \restriction n+1$. Then there is a homeomorphism h of $(S^1)^{n+1}$ fixing $A \restriction n+1$ and mapping $\langle x_0, \ldots, x_n \rangle$ to $\langle y_0, \ldots, y_n \rangle$. As in the proof of the Claim, h can be 'extended' to a homeomorphism of $(S^1)^\omega$ fixing A and mapping $\overline{x}$ to $\overline{y}$. ■

Example 5 The Polish structure $(P, Homeo(P))$ where P is the pseudo-arc is small and not nm-stable.

Before the proof we recall some notions and results about continua. Recall that a continuum is a nonempty, compact, connected metric space. The pseudo-arc P is the unique nondegenerate (i.e. with more than one point), hereditarily indecomposable, chainable continuum (hereditary indecomposablity means that no subcontinuum of P can be written as the union of its two proper subcontinua; for the definition of a chainable continuum the reader is referred to [19, Chapter 12]). By hereditary idecomposability, we get that for every $A \subseteq P$ the intersection of all subcontinua of P containing A is the smallest subcontinuum of P containing A.

Let C be any nondegenerate continuum (e.g. P). We say that C is irreducible between subsets A and B if there is no proper subcontinuum of C containing A and B. For $p \in C$ we define the composant of p, denoted by $\kappa(p)$, as the set of the points $x \in C$ for which there is a proper subcontinuum A of C such that $p, x \in A$.

For every $p \in C$, $\kappa(p)$ is the union of countably many proper subcontinua of C containing p [19, Proposition 11.14] and $\kappa(p)$ is dense and connected [19, Exercise 5.20]. Moreover, by [19, Exercise 6.19] C is indecomposable iff every proper subcontinuum of C is nowhere dense in C. Hence

($*$) If C is indecomposable, then $\kappa(p)$ is meager and dense in C for every $p \in C$.

If C is indecomposable, we define an equivalence relation E on C by:

$$E_C(x, y) \iff y \in \kappa(x).$$

By ($*$) and the Baire category theorem, E_C has uncountably many classes (they are just composants). We will need the following two facts, which can be found in [16, Theorem 2, Theorem 6].

Fact 3.1 *The pseudo-arc is homeomorphic to each of its nondegenerate subcontinua.*

Fact 3.2 *Suppose $H_{1,1}, H_{1,2}, \ldots, H_{1,n}$ are proper subcontinua of the pseudo-arc P and that P is irreducible between each pair of them. Suppose T is a homeomorphism of $H_{1,1} \cup H_{1,2} \cup \ldots \cup H_{1,n}$ onto $H_{2,1} \cup H_{2,2} \cup \ldots \cup H_{2,n}$ where $H_{2,1}, H_{2,2}, \ldots, H_{2,n}$ are proper subcontinua of P such that P is irreducible between each pair of them. Then T can be extended to a homeomorphism of P onto P.*

Now we can prove the following Lemma which immediately implies that the Polish structure $(P, Homeo(P))$ is small.

Lemma 3.3 *For every $n \geq 1$, there are only finitely many orbits on P^n under the action of $Homeo(P)$.*

Proof. The proof is by induction on n. The case $n = 1$ follows from the fact that the pseudo-arc is homogeneous.

Assume that Lemma 3.3 is true for the tuples of length $< n$ where $n \geq 2$. Suppose for a contradiction that there are infinitely many n-tuples $t_0 = (t_0^0, \dots, t_0^{n-1}), t_1 = (t_1^0, \dots, t_1^{n-1}), \dots$ lying in different orbits under $Homeo(P)$. Then there is an infinite subsequence $t_{i_0}, t_{i_1}, \dots$ such that for every $j, k \in \omega$, the tuples t_{i_j} and t_{i_k} are isomorphic, via f_{jk}, with respect to the relation E_P. Wlog $i_j = j$ for every $j \in \omega$. Then there is $m \leq n-1$ such that for every $i \in \omega$, $\{t_i^0, \dots, t_i^{n-1}\}$ can be partitioned into classes modulo E_P, say $A_i^0, \dots, A_i^m$, so that $f_{jk}[A_j^l] = A_k^l$ for every $j, k \in \omega$ and $l \leq m$. Let $B_i^0, \dots, B_i^m$ be the smallest proper subcontinua of P containing $A_i^0, \dots, A_i^m$, respectively. Then $B_i^0, \dots, B_i^m$ are pairwise disjoint proper subcontinua of P such that P is irreducible between each pair of them.

Case 1 $m > 1$. By Fact 3.1, for every $i \in \omega$ we can find homeomorphisms $f_i^l : B_i^l \to B_0^l$, $l = 0, \dots, m$. By the inductive hypothesis, there are $i \neq j$ for which there exist homeomorphisms $g_l : B_0^l \to B_0^l$, $l = 0, \dots, m$, such that

$$T := (f_j^0)^{-1} g_0 f_i^0 \cup \dots \cup (f_j^m)^{-1} g_m f_i^m : B_i^0 \cup \dots \cup B_i^m \to B_j^0 \cup \dots \cup B_j^m$$

is a homeomorphism extending f_{ij}. Since both $\{B_i^0, \dots, B_i^m\}$ and $\{B_j^0, \dots, B_j^m\}$ are collections of proper subcontinua of P such that P is irreducible between each pair of them, by Fact 3.2, T can be extendend to an element of $Homeo(P)$. Since $T(t_i) = t_j$, we get a contradiction.

Case 2 $m = 1$. Then B_i^0, $i \in \omega$, are proper subcontinua of P, so by Fact 3.1, we can find homeomorphisms $f_i : B_i^0 \to B_0^0$. Let $t_i' = f_i(t_i)$. If we show that there exist $i \neq j$ and a homeomorphism $f : B_0^0 \to B_0^0$ with $f(t_i') = t_j'$, then $T := f_j^{-1} f f_i$ is a homeomorphism from B_i^0 onto B_j^0 such that $T(t_i) = t_j$, and we finish using Fact 3.2. By the minimality of $B_i^0, i \in \omega$, we see that each tuple t_i', $i \in \omega$, meets at least two composants computed within B_0^0. So applying the argument preceding Case1 for B_0^0 and $t_0', t_1', \dots$, we come to the situation described in Case 1, and the proof is completed. ■

Lemma 3.4 $(P, Homeo(P))$ *is not nm-stable.*

Proof. Take any $p \in P$. By $(*)$, we can find $p_0 \in \kappa(p)$ such that $p_0 \neq p$. Now choose the smallest proper subcontinuum P_0 of P containing p and p_0. Then P_0 is also the pseudo-arc and it is nowhere dense in P. We see that P_0 is definable over $\{p, p_0\}$. Let q_0 be a name of P_0. By Fact 3.2, $o(p/q_0) = P_0$, which is nowhere dense in $o(p) = P$. Since P is Polish, Theorem 1.12 implies that $p \not\perp^{nm} q_0$.

Let us repeat this step within P_0. By $(*)$, we can find $p_1 \in \kappa_{P_0}(p)$ such that $p_1 \neq p$, where κ_{P_0} is κ computed within P_0. Now choose the smallest proper subcontinuum P_1 of P_0 containing p and p_1. Then P_1 is also the pseudo-arc and it is nowhere dense in P_0. We see that P_1 is definable over $\{p, p_0, p_1\}$. Let q_1 be a name of P_1. By Fact 3.2, $o(p/q_0q_1) = P_1$, which is nowhere dense in $o(p/q_0) = P_0$. Since P_0 is Polish, Theorem 1.12 implies that $p \mathop{\not\perp}^{mm}_{q_0} q_1$.

We repeat this procedure and obtain an infinite sequence of imaginaries $q_0, q_1, \dots$ such that $p \mathop{\not\perp}^{mm}_{q_{<i}} q_{\leq i}$ for all $i \in \omega$. By Remarks 2.15 and 2.16, the proof is completed. ■

Remark 3.5 *For any $p_0 \in P$, the orbits over p_0 are $\{p_0\}$, $\kappa(p_0) \setminus \{p_0\}$ and $P \setminus \kappa(p_0)$.*

Proof. By Fact 3.2, we get that if $p \in P \setminus \kappa(p_0)$, then $o(p/p_0) = P \setminus \kappa(p_0)$. Now assume $p \in \kappa(p_0) \setminus \{p_0\}$. Of course, $o(p/p_0) \subseteq \kappa(p_0) \setminus \{p_0\}$. To show the opposite inclusion, consider any $q \in \kappa(p_0) \setminus \{p_0\}$. Let B_p and B_q be the smallest subcontinua containing $\{p_0, p\}$ and $\{p_0, q\}$, respectively. By Fact 3.1 and homogeneity of B_q, there is a homeomorphism $f : B_p \to B_q$ fixing p_0. So in virtue of Fact 3.2, f can be extended to a homeomorphism from P onto P, which will be still denoted by f. By the minimality of B_p and B_q, we see that $f(p)$ and q are not in the composant of p_0 computed within B_q. So the first paragraph of the proof gives us the existence of a homeomorphism $h : B_q \to B_q$ fixing p_0 and mapping $f(p)$ to q. By Fact 3.2, h can be extended to a homeomorphism from P onto P, which will be still denoted by h. Then $h \circ f : P \to P$ is a homeomorphism fixing p_0 and mapping p to q. ■

It is not clear how to repeat the proof of Lemma 3.4 without using imaginaries. Keeping the notation from this proof, by Remark 3.5 and $(*)$, we have that $o(p/p_0) = \kappa(p_0) \setminus \{p_0\}$ is meager in $o(p) = P$ and P is Polish, so by Theorem 1.12, $p \mathop{\not\perp}^{mm} p_0$. However, by $(*)$ and Effros' theorem (or directly from the comments preceding $(*)$), $o(p/p_0) = \kappa(p_0) \setminus \{p_0\}$ is meager in its relative topology, so starting from this point we cannot work just with orbits, but we should look at their preimages in $Homeo(P)$, which is rather complicated.

Remark 3.6 *(i) $(P, Homeo(P))$ does not satisfy the existence of o-independent extensions.*
(ii) In $(P, Homeo(P))$ the relations $\mathop{\perp}^{o}$ and $\mathop{\perp}^{mm}$ are different.

Proof. (i) Since P is homogeneous, by Theorem 1.12, it is enough to show that there is $p \in P$ such that every orbit over p is not open in P. Take any $p \in P$. Using Remark 3.5, we have that the only orbits over p are $\{p\}$, $\kappa(p) \setminus \{p\}$ and $P \setminus \kappa(p)$, so we are done by $(*)$.
(ii) It follows from (i), Lemma 3.3 and Theorem 1.10. We can also see it directly. Take any $p \in P$ and $q \notin \kappa(p)$. By Remark 3.5, $o(q/p) = P \setminus \kappa(p)$. So by $(*)$, we get that $o(q/p)$ is non-meager in $o(q) = P$, and hence, by Theorem 1.12, $q \mathop{\perp}^{mm} p$.

On the other hand, by $(*)$ we see that $P \setminus \kappa(p)$ is not open in P, and hence, by Theorem 1.12, $q \mathop{\not\perp}^{o} p$. ■

In fact, any natural, simple example of a compact (connected) metric space X with the full group of homeomorphisms seems to be a small Polish structure (e.g. S^n, $(S^1)^n$, I^ω, $(S^1)^\omega$ are small). Also, a much more complicated continuum, the pseudo-arc, turned out to be small. One could think that it is always the case (at least if X is connected). However, the main result of [9] says that a certain continuum, called the pseudo-circle, with the full group of homeomorphisms has uncountably many 1-orbits, so it is not small.

In the next section we will give several examples of small Polish groups regarded as Polish structures, i.e. small Polish structures (H, G) such that H is a Polish group and G acts continuously as a group of automorphisms of H.

We end up with a remark, which follows from Lascar inequalities, and yields examples of small Polish structures of arbitrary finite $\mathcal{NM}$-rank.

Remark 3.7 *If (X, G) is a small Polish structure of $\mathcal{NM}$-rank 1, then for every natural number $n \geq 1$, (X^n, G) is a small Polish structure of $\mathcal{NM}$-rank n.*

4 Small compact G-groups

In this paper all topological groups are Hausdorff by definition. We will be using the following terminology.

Definition 4.1 *Let G be a Polish group.*
(i) A Polish group structure is a Polish structure (H, G) such that H is a group and G acts as a group of automorphisms of H.
(ii) A (topological) G-group is a Polish group structure (H, G) such that H is a topological group and the action of G on H is continuous.
(iii) A Polish [compact] G-group is a topological G-group (H, G) where H is a Polish [compact] group.

Definition 4.2 *(i) We say that a group H is definable in a Polish structure (X, G) [or in X^{eq}] if H as well as the group operation on H are definable in (X, G) [X^{eq}].*
(ii) We say that a group H is $*$-closed in a G-space (X, G) [or in X^{teq}, if X is compact] if H as well as the group operation on H are $*$-closed.

In this section we mostly study small compact G-groups. We will give some examples and prove structural results about such groups.

Let us first discuss a notion of a generic orbit in the general context of small Polish group structures. So let (H, G) be a small Polish group structure and $a, A \subseteq H$ finite.

Definition 4.3 *We say that the orbit $o(a/A)$ is left nm-generic (or that a is left nm-generic over A) if for all $b \in H$ with $a \mathop{\smile\!\!\!\!\!\!\mid}^{nm}_A b$, one has that $b \cdot a \mathop{\smile\!\!\!\!\!\!\mid}^{nm} A, b$. We say that it is right nm-generic if, for b as above, we have $a \cdot b \mathop{\smile\!\!\!\!\!\!\mid}^{nm} A, b$. An orbit is nm-*generic *if it is both right and left nm-generic.*

All the facts about generic types in simple and rosy groups which use only formal properties of the underlying independence relation go through in our situation. In particular, Lemma 1.12, Remark 1.20 and Propositions 1.24, 1.26, 1.27(1) of [4] are true in our context, after replacing $*$ by nm, U^* by $\mathcal{NM}$, and assuming that H/L is contained in a sort of H^{eq} (whenever the set H/L of cosets modulo the definable subgroup L occurs) which is always the case when (H,G) is a G-group and L is a $*$-closed subgroup of H. For example, we have: if a is left (right) nm-generic over A, then $a \mathop{\smile\!\!\!\!\!\!\vert}^{nm} A$; being left (right) nm-generic is preserved under taking restrictions and nm-independent extensions; left nm-generic coincides with right nm-generic and so with nm-generic; in the nm-stable situation, being nm-generic means being of maximal $\mathcal{NM}$-rank. However, in order to get the existence of generics [thorn generics] in simple [rosy] groups, one uses suitable stratified local ranks, which are not present in our situation. In fact, even in the rosy context, if we consider an abstract independence relation, it is not clear whether a generic type with respect to this relation exists (see [4, Question 1.19]). In our context the corresponding question has the following form.

Question 4.4 *Let (G,H) be a small Polish group structure. Does H possess a generic orbit?*

The answer to the above question is positive in a rather big class of structures.

Proposition 4.5 *Suppose (H,G) is a small G-group such that H is not meager in itself (e.g. H is Polish or compact, or more generally Baire). Then at least one nm-generic orbit exists, and an orbit is nm-generic iff it is non-meager in H.*

Proof. By smallness and the fact that H is not meager, for any finite $A \subseteq H$ there is an orbit $o = o(h/A)$ which is non-meager in H. Let us show that o is nm-generic. Consider any $h' \mathop{\smile\!\!\!\!\!\!\vert}^{nm}_A h$. We claim that $o(h/A,h') \subseteq_{nm} H$. Otherwise there are closed subsets D_i, $i \in \omega$, of H with empty interiors such that $o(h/A,h') \subseteq \bigcup_{i\in\omega} D_i$. Then $\pi_A^{-1}[o(h/A,h')] \subseteq \bigcup_{i\in\omega} \pi_A^{-1}[D_i]$ where $\pi_A : G_A \to o(h/A)$ is given by $\pi_A(g) = gh$. Since $\pi_A^{-1}[o(h/A,h')] \subseteq_{nm} G_A$ and $\pi_A^{-1}[D_i]$, $i \in \omega$, are closed, there is k such that $\pi_A^{-1}[D_k]$ has nonempty interior in G_A. So countably many left translates of $\pi_A^{-1}[D_k]$ by some elements $g_0, g_1, \ldots \in G_A$ cover G_A. Hence the sets $g_i D_k$, $i \in \omega$, cover $o(h/A)$. Thus $o(h/A)$ is meager in H, a contradiction. From $o(h/A,h') \subseteq_{nm} H$ we get

$$o(h/A,h')h' \subseteq_{nm} H \Rightarrow o(hh'/A,h') \subseteq_{nm} H \Rightarrow o(hh'/A,h') \subseteq_{nm} o(hh').$$

By Theorem 1.14, we conclude that $hh' \mathop{\smile\!\!\!\!\!\!\vert}^{nm} A, h'$.

We have proved that every orbit which is non-meager in H is nm-generic, and so an nm-generic orbit exists. Now suppose that $o(k/A)$ is nm-generic. We need to show that $o(k/A) \subseteq_{nm} H$. Take $h \in H$ such that $o(h/A) \subseteq_{nm} H$ and $k \mathop{\smile\!\!\!\!\!\!\vert}^{nm}_A h$. Then $hk \mathop{\smile\!\!\!\!\!\!\vert}^{nm}_A h$ so $h \mathop{\smile\!\!\!\!\!\!\vert}^{nm}_A k^{-1}h^{-1}$. By the first paragraph of the proof, we get $o(h/A,k^{-1}h^{-1}) \subseteq_{nm} H$. So $o(k^{-1}/A,k^{-1}h^{-1}) \subseteq_{nm} H$, which implies $o(k^{-1}/A) \subseteq_{nm} H$, and so $o(k/A) \subseteq_{nm} H$. ■

The above discussion on generics extends easily to the situation when H is a $*$-closed subgroup of a G-group $\widetilde{H}$ (the difference is that here parameters from the supergroup $\widetilde{H}$ are allowed).

The above proposition together with Lascar inequalities for groups [4, Proposition 1.27(1)] easily imply the following corollary, whose direct proof is given below for the reader's convenience. In fact, only point (i) of this corollary will be very useful later. Notice that point (ii) does not require the existence of nm-generics.

Corollary 4.6 *(i) Let (H,G) be a small G-group whose all $*$-closed subgroups are non-meager in their relative topologies (e.g. (H,G) is a Polish or compact G-group). If $H_1 <_{nwd} H_2$ are $*$-closed subgroups of H and $\mathcal{N}M(H_2) < \infty$, then $\mathcal{N}M(H_1) < \mathcal{N}M(H_2)$.*
(ii) Let (H,G) be a small G-group. If $H_1 <_o H_2$ are $*$-closed subgroups of H, then $\mathcal{N}M(H_1) = \mathcal{N}M(H_2)$.

Proof. (i) Assume for simplicity that H_1 and H_2 are $\emptyset$-closed. The equivalence relation on H given by $xEy \iff xH_1 = yH_1$ is $\emptyset$-closed, so H/H_1 is a sort of H^{eq}. Since $H_1 <_{nwd} H_2$ and H_2 is non-meager in its relative topology, $[H_2 : H_1] > \aleph_0$. By smallness, $Acl^{eq}(\emptyset) \cap H/H_1$ is countable, so there is $h_2 \in H_2$ such that $h_2H_1 \notin Acl^{eq}(\emptyset)$. By Proposition 4.5, we can choose h_1 nm-generic in H_1 over h_2. Since $h_2h_1 \in h_2H_1$, we have $h_2H_1 \in dcl(h_2h_1)$. So $h_2H_1 \not\mathop{\smile\!\!\!\!\!\!|}^{nm} h_2h_1$. Thus $\mathcal{N}M(H_2) \geq \mathcal{N}M(h_2h_1) > \mathcal{N}M(h_2h_1/h_2H_1) \geq \mathcal{N}M(h_2h_1/h_2) = \mathcal{N}M(h_1/h_2) = \mathcal{N}M(H_1)$.
(ii) Assume for simplicity that H_1 and H_2 are $\emptyset$-closed. First we show that $[H_2 : H_1] \leq \omega$. Suppose it is not the case. Let $K = H_2/H_1$ and $G_0 = G/G_K$. Since $H_1 <_o H_2$, the quotient topology on H_2/H_1 is discrete. It is easy to check that (K, G_0) is a small G_0-space. Since K is uncountable, there is an uncountable orbit $o(a)$ in K. As the surjection $\pi : G_0 \to o(a)$ given by $\pi(g) = ga$ is continuous, the preimages of singletons form a partition of the Polish group G_0 into uncountably many open sets, a contradiction.

We need to show $\mathcal{N}M(H_2) \leq \mathcal{N}M(H_1)$ (the opposite inequality is obvious). It is enough to prove that for any $h_2 \in H_2$ there is $h_1 \in H_1$ with $\mathcal{N}M(h_2) \leq \mathcal{N}M(h_1)$. Take any $h_2 \in H_2$. Since H_2/H_1 is countable, it is contained in $Acl^{eq}(\emptyset)$. So $h_2 \mathop{\smile\!\!\!\!\!\!|}^{nm} h_2H_1$. Choose $a \in o(h_2/h_2H_1)$ so that $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_{h_2H_1} h_2$. Then $h_2 \mathop{\smile\!\!\!\!\!\!|}^{nm} a$ and $a^{-1}h_2 \in H_1$. Thus, $\mathcal{N}M(h_2) = \mathcal{N}M(h_2/a) = \mathcal{N}M(a^{-1}h_2/a) \leq \mathcal{N}M(a^{-1}h_2)$, and so $h_1 := a^{-1}h_2$ does the job. ■

Proposition 4.7 *(i) Let (H,G) be a small Polish G-group. Then any finitely generated subgroup of H is countable and does not have limit points in H.*
(ii) Let (H,G) be a small compact G-group. Then H is locally finite.

Proof. Let $S \subseteq H$ be finite.
(i) Then $\overline{\langle S\rangle} \subseteq dcl(S)$. So by smallness, $\overline{\langle S\rangle} < H$ is countable. If $\langle S\rangle$ had a limit point, $\overline{\langle S\rangle}$ would be perfect and hence uncountable, a contradiction.
(ii) By compactness, if $\langle S\rangle$ was infinite, then it would have a limit point, and we

would get a contradiction as in (i). ■

The next example is an example of a small Polish (but non-compact) G-group. It shows that point (i) of the above remark cannot be strengthen to get local finiteness of H. This is one of the reasons why our proofs of structural results for small compact G-groups in the further part of this section do not go through for Polish G-groups.

Example 6 Let us consider the discrete topology on $\mathbb{Q}$ and the product topology on $\mathbb{Q}^\omega$. We consider the additive group structure on $\mathbb{Q}$. Let $Aut^0(\mathbb{Q}^\omega)$ [$Aut^0(\mathbb{Q}^n)$, respectively] be the group of all automorphisms of $\mathbb{Q}^\omega$ [$\mathbb{Q}^n$, respectively] respecting the inverse system $\mathbb{Q} \longleftarrow \mathbb{Q} \times \mathbb{Q} \longleftarrow \dots$. Then $Aut^0(\mathbb{Q}^\omega)$ can be considered as the inverse limit of the system consisting of $Aut^0(\mathbb{Q}^n)$, $n \in \omega$, where on $Aut^0(\mathbb{Q}^n)$ we have the pointwise convergence topology. Then $(\mathbb{Q}^\omega, Aut^0(\mathbb{Q}^\omega))$ is a small Polish $Aut^0(\mathbb{Q}^\omega)$-group of $\mathcal{NM}$-rank 1, and $\mathbb{Q}^\omega$ is torsion-free, 0-dimensional and not compact.

Proof. We leave to the reader checking that $(\mathbb{Q}^\omega, Aut^0(\mathbb{Q}^\omega))$ is a Polish $Aut^0(\mathbb{Q}^\omega)$-group. We will show that it is small of $\mathcal{NM}$-rank 1.

The following claim is obvious.

Claim If $f \in Aut^0(\mathbb{Q}^n)$ and $\{\eta_1, \dots, \eta_n\}$ is a basis of $\mathbb{Q}^n$ over $\mathbb{Q}$, then for all $\epsilon_0, \epsilon_1, \dots, \epsilon_n, \epsilon_0', \epsilon_1', \dots, \epsilon_n' \in \mathbb{Q}$ such that $\epsilon_0, \epsilon_0' \neq 0$, there is $\overline{f} \in Aut^0(\mathbb{Q}^{n+1})$ such that $\overline{f}(0 ^\frown \epsilon_0) = 0 ^\frown \epsilon_0'$ and $\overline{f}(\eta_i ^\frown \epsilon_i) = f(\eta_i) ^\frown \epsilon_i'$, $i = 1, \dots, n$.

Now consider any finite set $A \subseteq \mathbb{Q}^\omega$ and $a \in \mathbb{Q}^\omega$. Then either $a \in Lin(A)$ or there is the largest natural number n such that $a{\restriction}n \in Lin(A){\restriction}n$. So by the Claim we get that either $a \in Lin(A)$, and then $o(a/A) = \{a\}$, or $a \notin Lin(A)$, and then $o(a/A) = \{\eta \in \mathbb{Q}^\omega : \eta{\restriction}n = a{\restriction}n \wedge \eta{\restriction}(n+1) \notin Lin(A){\restriction}(n+1)\}$ where n is the largest n such that $a{\restriction}n \in Lin(A){\restriction}n$. So each orbit over A is either a singleton from $Lin(A)$ or an open set. Hence there are countably many orbits over A, so $(\mathbb{Q}^\omega, Aut^0(\mathbb{Q}^\omega))$ is small; we also see that $\mathcal{NM}(\mathbb{Q}^\omega) = 1$. ■

Example 7 (i) If H is a compact metric group, then $Aut(H)$ is Polish and $(H, Aut(H))$ is a compact $Aut(H)$-group where $Aut(H)$ is the group of all topological automorphisms of H equipped with the compact-open topology.
(ii) If (H, G) is a small compact G-group, then $(H, Aut(H))$ is a small compact $Aut(H)$-group.

Proof. In order to show (i), it is enough to check that $Aut(H)$ is a closed subgroup of $Homeo(H)$, which is an easy exercise.
(ii) follows from (i) and Proposition 2.9. ■

From now on we assume that **(H, G) is a small compact G-group**. The following theorem is proved by Hewitt and Ross [5, Theorem 28.20].

Fact 4.8 *Every compact torsion group is profinite.*

This fact together with Proposition 4.7 imply

Corollary 4.9 *H is profinite.*

Notice that in spite of the fact that H is profinite we are still in a much more general situation than small profinite groups in the sense of Newelski. This is because G is a Polish group, which is not necessarily compact. In particular, we consider here the case when H is a profinite group being the inverse limit of a countable system and G is the full group of topological automorphisms of H (then G is Polish but often not compact), which is more natural from the point of view of group theory and topology than considering only automorphisms preserving a distinguished inverse system.

The next two remarks will be useful later.

Remark 4.10 *For every open subgroup U of H, $Stab(U)$ is a clopen subgroup of G and so $[G : Stab(U)] \leq \omega$.*

Proof. We see that U is clopen and hence compact. Since G is a group of automorphisms of H acting continuously on H, the topology on G must be stronger or equal to the compact-open topology. Thus, since

$$Stab(U) = \{f \in G : f[U] \subseteq U\} \cap \{f \in G : f^{-1}[U] \subseteq U\},$$

we get that $Stab(U)$ is an open subgroup of G, and so it is also closed. Since G is Polish, we get $[G : Stab(U)] \leq \omega$. ■

Remark 4.11 *If G_0 is a closed subgroup of G of countable index, then (H, G_0) is small, and $\mathop{\smile\!\!\!\!\!\!|}^{nm}$ as well as $\mathcal{NM}$-rank computed in (H, G) are the same as in (H, G_0).*

Proof. If (H, G_0) is not small, there are uncountably many n-tuples a_i, $i \in I$, in different orbits under G_0. Since (H, G) is small, there are $i_0 \in I_0 \subseteq I$ with $|I_0| > \aleph_0$ such that for every $i \in I_0$ there is $g \in G$ such that $ga_{i_0} = a_i$. As $[G : G_0] \leq \aleph_0$, we can find an uncountable set $I_1 \subseteq I_0$ so that for all distinct $i, j \in I_1$ there is $g \in G_0$ with $ga_i = a_j$, a contradiction. So we have proved that (H, G_0) is small.

Now suppose that $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_A B$ in the sense of (H, G_0). Then $G_{0AB}G_{0Aa} \subseteq_{nm} G_{0A}$. Since G_{0A} is a clopen subgroup of G_A and $G_{0AB}G_{0Aa} \subseteq G_{AB}G_{Aa}$, we easily get $G_{AB}G_{Aa} \subseteq_{nm} G_A$, i.e. $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_A B$ in the sense of (H, G_0).

Finally assume that $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_A B$ in the sense of (H, G). Then $G_{AB}G_{Aa} \subseteq_{nm} G_A$. But $G_{AB}G_{Aa} = \bigcup_{i \in I} \bigcup_{j \in J} l_i G_{0AB} G_{0Aa} r_j$ where the countable sets $\{l_i : i \in I\}$ and $\{r_j : j \in J\}$ are the sets of representatives of left cosets of G_{AB} modulo G_{0AB} and right cosets of G_{Aa} modulo G_{0Aa}, respectively. So we easily conclude that $G_{0AB}G_{0Aa} \subseteq_{nm} G_{0A}$, i.e. $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_A B$ in the sense of (H, G_0).

We have proved that $\mathop{\smile\!\!\!\!\!\!|}^{nm}$ is the same in both structures, and so must be $\mathcal{NM}$-rank. ■

Proposition 4.12 *(i) H is the inverse limit of a countable system of finite groups. Thus, it has a countable basis consisting of clopen sets.*
(ii) If $G = Aut(H)$ is the group of all topological automorphisms of H, then the topology on G must coincide with the compact-open topology.
(iii) If G is equipped with the compact-open topology, then G has a countable basis of open neighborhoods of id consisting of open subgroups. Thus, G is isomorphic to a closed subgroup of S_∞.

Proof. (i) By smallness and Proposition 2.9, H is second countable. As H is also profinite, we get that it has a countable basis of open neighborhoods of e consisting of clopen, normal subgroups H_i, $i \in \omega$, with $H_j < H_i$ whenever $i < j$. So H is the inverse limit of the countable system consisting of the finite quotients H/H_i, $i \in \omega$. Hence H has a countable basis consisting of clopen sets U_i, $i \in \omega$.
(ii) By smallness and Proposition 2.9, H is metrizable (without smallness we would have to assume metrizability). By Example 7, we get that $Aut(H)$ equipped with the compact-open topology is a Polish group. We also have that the given Polish topology on $G = Aut(H)$ is stronger or equal to the compact-open topology. So, by [2, Theorem 1.2.6], both topologies on G coincide.
(iii) By (i), H has a countable basis consisting of clopen sets U_i, $i \in \omega$. As it was noticed in the proof of Remark 4.10, we have that $Stab(U_{i_1} \cup \ldots \cup U_{i_n})$ is an open subgroup of G for any $i_1, \ldots, i_n \in \omega$. So it is enough to show that every set of the compact-open subbasis on G containing id contains a subgroup of the form $Stab(U_{i_1} \cup \ldots \cup U_{i_n})$.

Take any element $\mathcal{U}$ of the compact-open subbasis on G containing id, i.e. $\mathcal{U} = \{f \in G : f[C] \subseteq U\}$ where $C \subseteq H$ is compact, $U \subseteq H$ is open, and $id \in \mathcal{U}$. Then $C \subseteq U$. So, by compactness of C, $C \subseteq U_{i_1} \cup \ldots \cup U_{i_n} \subseteq U$ for some $i_1, \ldots, i_n \in \omega$. Therefore, $Stab(U_{i_1} \cup \ldots \cup U_{i_n}) \subseteq \mathcal{U}$.

Now the fact that G is isomorphic to a closed subgroup of S_∞ follows from [2, Theorem 1.5.1]. ■

Recall now the main conjecture concerning small profinite groups.

Conjecture 4.13 *Each small profinite group is abelian-by-finite.*

In [27] Wagner proved the conjecture assuming additionally m-stability.

Fact 4.14 *Each small, m-stable profinite group is abelian-by-finite.*

In fact, the following three intermediate conjectures are still open.

Conjecture 4.15 *Suppose (H, G) is a small profinite group. Then:*
(A) H is solvable-by-finite,
(B) if H is solvable(-by-finite), then H is nilpotent-by-finite,
(C) if H is nilpotent(-by-finite), then H is abelian-by-finite.

Remark 4.16 *If $S < H$ is abelian [solvable, nilpotent], then $\overline{S}$ is abelian [solvable, nilpotent].*

Below we give examples showing that all Conjectures (A), (B) and (C) are false in the more general context of small compact G-groups. Then we prove that, for small compact G-groups, in the nm-stable case Conjecture (A) is true, and in the finite $\mathcal{N}M$-rank case Conjecture (B) is also true.

Example A Let S be any finite group which is not solvable. Define an action of S_∞ on S^ω by

$$g\langle s_0, s_1, \ldots\rangle = \langle s_{g(0)}, s_{g(1)}, \ldots\rangle.$$

Then:
(i) S^ω is not solvable-by-finite.
(ii) (S^ω, S_∞) is a small compact S_∞-group.
(iii) (S^ω, S_∞) is not nm-stable.

Proof. (i) Let A be a solvable subgroup of S^ω. Then the projections of A on every coordinate are solvable and hence proper subgroups of S. So $[S^\omega : A]$ is infinite.
(ii) The fact that (S^ω, S_∞) is a compact S_∞-group is left to the reader. Now we will prove smallness. First, notice that for every $\langle s_i\rangle \in S^\omega$ we have

$$Stab(\langle s_i\rangle) = \bigcap_{s\in S}\{g \in S_\infty : g\{i \in \omega : s_i = s\} = \{i \in \omega : s_i = s\}\}.$$

So for any $\langle s_i^1\rangle, \ldots, \langle s_i^n\rangle \in S^\omega$ we get

$$Stab(\langle s_i^1\rangle, \ldots, \langle s_i^n\rangle) = \{g \in S_\infty : (\forall B \in \mathcal{B})(gB = B)\}$$

where $\mathcal{B}$ is the set of all atoms of the finite Boolean algebra generated by the sets $\{i \in \omega : s_i^j = s\}$, $s \in S$, $j = 1, \ldots, n$.

Now we look at 1-orbits over $\{\langle s_i^1\rangle, \ldots, \langle s_i^n\rangle\}$. We get that two elements $\langle x_i\rangle$ and $\langle y_i\rangle$ of S^ω lie in the same orbit over $\{\langle s_i^1\rangle, \ldots, \langle s_i^n\rangle\}$ iff for every $s \in S$ and $B \in \mathcal{B}$ the cardinality of $\{i \in B : x_i = s\}$ equals the cardinality of $\{i \in B : y_i = s\}$. Since S and $\mathcal{B}$ are finite and for every $B \in \mathcal{B}$ there are only countably many possibilities for the cardinality of a subset of B, we get that there are countably many 1-orbits over $\{\langle s_i^1\rangle, \ldots, \langle s_i^n\rangle\}$.
(iii) Let $I_0 \subseteq I_1 \subseteq \ldots \subseteq \omega$ be such that $I_{n+1} \setminus I_n$ is infinite for every $n \in \omega$. Choose any $s \notin Z(S)$. Define $x_n = \langle s_i^n\rangle \in S^\omega$ by $s_i^n = s$ if $i \in I_n$, and $s_i^n = e$ if $i \notin I_n$. Then the projection of $C(x_n)$ on the i-th coordinate is equal to $C(s) \subsetneq S$ if $i \in I_n$, and S if $i \notin I_n$. Therefore, $C(x_n)$, $n \in \omega$, form a descending sequence of $*$-closed subgroups of S^ω such that $C(x_{n+1}) <_{nwd} C(x_n)$ for every n. So by Corollary 4.6, (S^ω, S_∞) is not nm-stable. ■

Example B Let S be any finite solvable group which is not nilpotent. Define the action of S_∞ on S^ω in the same way as in Example A. Then (S^ω, S_∞) is a small compact S_∞-group which is solvable, not nilpotent-by-finite and not nm-stable.

Example C Let S be any finite nilpotent group which is not abelian. Define the action of S_∞ on S^ω in the same way as in Example A. Then (S^ω, S_∞) is a small compact S_∞-group which is nilpotent, not abelian-by-finite and not nm-stable.

As we will see in the next remark, the above examples also show that some basic properties of small profinite groups are not true for small compact G-groups, e.g. invariant subgroups are not necessarily closed.

Remark 4.17 *In Examples A, B and C we have that* $|Acl(\emptyset)| = \aleph_0$ *and* $Acl(\emptyset) < S^\omega$ *is* $\emptyset$*-invariant but not closed. In fact,* $Acl(\emptyset)$ *is dense in* S^ω*.*

Proof. It is easy to see that $Acl(\emptyset)$ consists of those $\langle s_i \rangle \in S^\omega$ for which there exists $s \in S$ such that $\{i \in \omega : s_i = s\}$ is cofinite. So $Acl(\emptyset)$ is dense in S^ω and of cardinality $\aleph_0$. ■

Newelski noticed [21, Remark 4.3] that if (H, G) is a small profinite group where $H = \prod_{n<\omega} H_n$ with H_n's being finite groups is regarded as the inverse limit of all initial subproducts, then almost all H_n's are abelian. Examples A, B and C show that the same result is not true if (H, G) is a small compact G-group. Easy modifications of the proof of [21, Remark 4.3] and the proof of (iii) in Example A yield:

Remark 4.18 *(i) If* $H = \prod_{i \in I} H_i$ *where* H_i*'s are finite groups and* (H, G) *is a small profinite group (but the distinguished inverse system does not necessarily consist of initial subproducts), then almost all* H_i*'s are abelian.*
(ii) If $H = \prod_{i \in I} H_i$ *where* H_i*'s are finite groups and* (H, G) *is a small, nm-stable, compact* G*-group, then almost all* H_i*'s are abelian.*

The above remark shows that Conjecture 4.13 holds for small, nm-stable, compact G-groups which are products of finite groups.

Now we will prove Conjecture (A) for small, nm-stable, compact G-groups. Our proof is completely different from the one for small, m-stable profinite groups, and it relies on some classical results about profinite groups. In fact, Wagner's proof that each small, m-stable profinite group is solvable-by-finite is a part of the inductive (on $\mathcal{M}$-rank) proof that each such group is abelian-by-finite [27, Proposition 17], and it uses the inductive hypothesis about abelianity of subgroups of smaller $\mathcal{M}$-rank. There are also some other steps in Wagner's proof of solvability which are not clear in our context, e.g. in our situation invariant subgroups are not necessarily closed.

Theorem 4.19 *If* (H, G) *is a small, nm-stable, compact* G*-group, then* H *is solvable-by-finite.*

Before the proof we need to recall some classical facts on profinite groups. There are two main tools that will be used in the proof of Theorem 4.19: Wilson's theorem on the structure of compact torsion groups [28, Theorem 1] and fundamental facts about Frattini subgroups [25, Section 2.8].

Fact 4.20 (Wilson) *Let H be a profinite torsion group. Then H has a finite series*

$$\{e\} = H_n < H_{n-1} < \ldots < H_0 = H$$

of closed, topologically characteristic subgroups such that each group H_i/H_{i+1} is either a pro-p group, for some prime p, or a direct product of isomorphic copies of a finite simple group.

Definition 4.21 *The Frattini subgroup $\Phi(H)$ of a profinite group H is the intersection of all maximal open subgroups of H. Then $\Phi^n(H)$, $n \in \omega$, are defined recursively by $\Phi^0(H) = H$ and $\Phi^{k+1}(H) = \Phi(\Phi^k(H))$, $k \in \omega$.*

Fact 4.22 *(i) If H is a profinite group, then $\Phi(H)$ is a proper, closed and topologically characteristic subgroup of H.*
(ii) If H is a pro-p group, then $\Phi(H) = \overline{H^p[H,H]}$.
(iii) If H is a pro-p group, then H is finitely generated iff $\Phi(H) <_o H$.

Now we can prove Theorem 4.19.

Proof of Theorem 4.19. By Wilson's theorem, there is a normal series

$$\{e\} = H_n < H_{n-1} < \ldots < H_0 = H$$

of $\emptyset$-closed subgroups such that each group H_i/H_{i+1} is either a pro-p group, for some prime p, or a direct product of isomorphic copies of a finite simple group. By Remarks 2.10 and 4.18, we get that each group H_i/H_{i+1} is either a pro-p group or a finite group. Replacing H by a clopen subgroup and using Remarks 4.10, 4.11 and 2.10, we can assume that each quotient H_i/H_{i+1} is a pro-p group. So wlog H is a pro-p group.

Of course, we can assume that H is infinite, and then by Proposition 4.7, we have that H is not finitely generated. So by Fact 4.22, $\Phi(H)$ is a $\emptyset$-closed, nowhere dense subgroup of H. We can continue this procedure (taking iterated Frattini subgroups), and by Corollary 4.6, we get that $\Phi^n(H)$ is finite for some $n \in \omega$. So there is m such that $\Phi^m(H) = \{e\}$. Since by Fact 4.22, $\Phi^i(H)/\Phi^{i+1}(H)$ are abelian, H is solvable.■

A modification of the proof of Theorem 4.19 yields the following strengthening of this theorem.

Corollary 4.23 *If (H,G) is a small, nm-stable, compact G-group, then H has a $*$-closed, solvable subgroup of finite index.*

Proof. We have to modify the first paragraph of the proof of Theorem 4.19. As before we have a normal series

$$\{e\} = H_n < H_{n-1} < \ldots < H_0 = H$$

of $\emptyset$-closed subgroups such that each group H_i/H_{i+1} is either a pro-p group or a finite group. We need to show that there is a $*$-closed subgroup $\widetilde{H}$ of finite index in H

such that for every $i \leq n-1$, $\widetilde{H} \cap H_i/\widetilde{H} \cap H_{i+1}$ is either a pro-p group or an abelian group; then we will be done by the second paragraph of the proof of Theorem 4.19.

To obtain $\widetilde{H}$, we will build a sequence $\widetilde{H}_0 > \ldots > \widetilde{H}_{n-1}$ of $*$-closed, normal, open subgroups of H such that $\widetilde{H}_j \cap H_i/\widetilde{H}_j \cap H_{i+1}$ is either a pro-p group or an abelian group for all $0 \leq i \leq j \leq n-1$. Then $\widetilde{H} := \widetilde{H}_{n-1}$ will be as required.

The first step of the construction is obvious: if H_0/H_1 is a pro-p group, we put $\widetilde{H}_0 := H_0$; otherwise H_0/H_1 is finite so we can put $\widetilde{H}_0 := H_1$.

For the induction step, suppose $\widetilde{H}_0, \ldots, \widetilde{H}_j$ have been constructed for some $j < n-1$. If $K := \widetilde{H}_j \cap H_{j+1}/\widetilde{H}_j \cap H_{j+2}$ is a pro-p group, we put $\widetilde{H}_{j+1} = \widetilde{H}_j$. Otherwise let $\widetilde{K} = C_{\widetilde{H}_j/\widetilde{H}_j \cap H_{j+2}}(K)$. Since $K \lhd \widetilde{H}_j/\widetilde{H}_j \cap H_{j+2}$ is finite, we have that for every $k \in K$, $C_{\widetilde{H}_j/\widetilde{H}_j \cap H_{j+2}}(k)$ is a clopen, $*$-closed subgroup of $\widetilde{H}_j/\widetilde{H}_j \cap H_{j+2}$, and hence $\widetilde{K}$ is a clopen, $*$-closed subgroup of $\widetilde{H}_j/\widetilde{H}_j \cap H_{j+2}$. Finally we define $\widetilde{H}_{j+1}$ as the preimage of $\widetilde{K}$ by the natural quotient map. ■

Next we will prove Conjecture (B) for small compact G-groups of finite $\mathcal{N}M$-rank. Unlike in the proof of Conjecture (A), this proof is a modification of the proof of Conjecture (B) for small, m-stable profinite groups [21, Theorem 3.3]. More precisely, the scheme of the proof is the same as in [21], but there are some extra difficulties arising from the fact that in our case: a group generated by an invariant set is not necessarily closed and generated in finitely many steps, we do not know if an increasing sequence of A-closed subgroups has to stabilize after finitely many steps, $Acl(\emptyset)$ is not necessarily finite, etc..

In the following theorem and lemmas we could skip the assumption that the group is solvable-by-finite because we know it by Theorem 4.19. But we do not do it in order to emphasize that Theorem 4.24 does not rely on Wilson's Theorem.

Theorem 4.24 *If (H,G) is a small compact G-group of finite $\mathcal{N}M$-rank, and H is solvable-by-finite, then H is nilpotent-by-finite.*

As in [21], first we need to prove some lemmas. The proof of the next lemma is based on the proof of [21, Lemma 3.4], but it is relatively more complicated in our case.

Lemma 4.25 *Suppose D, H are infinite, abelian groups $*$-closed in X^{teq} for some small compact G-space (X,G), and $\mathcal{N}M(H) < \infty$. Assume that D acts on H as a group of automorphisms, and the action is $*$-closed. Then some open subgroup of D fixes an infinite subgroup of H.*

Proof. Wlog everything is invariant over $\emptyset$. If Da is finite for some $a \in H$, then there is a clopen subgroup D_0 of D such that $a \in Fix(D_0)$. By Remark 4.10, we easily get that $Stab_G(D_0)$ is a clopen subgroup of countable index in G. If $Fix(D_0)$ is infinite, we are done. So assume $Fix(D_0)$ is finite. We also have that $Stab_G(D_0) \subseteq Stab_G(Fix(D_0))$. So $G \cdot Fix(D_0)$ is countable, hence $Fix(D_0) \subseteq Acl(\emptyset)$. Thus $a \in Acl(\emptyset)$.

We have proved that $H_0 := \{a \in H : Da \text{ is finite}\}$ is contained in $Acl^{eq}(\emptyset)$, and so it is countable.

Choose $a \in H$ so that Da is infinite and $H_1 := \overline{\langle Da \rangle} < H$ is of minimal possible $\mathcal{N}M$-rank. Then every D-ivariant, $*$-closed subgroup of H_1 is either finite or open in H_1.

Let R be the ring of endomorphisms of H_1 generated by D. Then R is commutative and every element of R is determined by its value on a. Moreover, one can check that the action of G_a on D extends to an action by automorphisms on R given by $g(d_1 + \ldots + d_n) = gd_1 + \ldots + gd_n$ for $g \in G_a$ and $d_1, \ldots, d_n \in D$, and this extended action preserves the action of R on H_1. This easily implies that R is locally finite. Indeed, take any $r_1, \ldots, r_n \in R$, and let $R_0 = \langle r_1, \ldots, r_n \rangle < R$. It is easy to check that $R_0 a \subseteq dcl(a, r_1 a, \ldots, r_n a)$. Hence $R_0 a$ must be finite (otherwise $\overline{R_0 a}$ would be an uncountable subset of $dcl(a, r_1 a, \ldots, r_n a)$, a contradiction). Since every element $r_0 \in R_0$ is determined by $r_0(a)$, R_0 is finite.

Now let $I = \{r \in R : r[H_1] \text{ is finite}\}$. Then I is a prime ideal of R because for every $r \in R$, $r[H_1]$ is a D-invariant, $*$-closed subgroup of H_1, and hence it is either finite or open (so of finite index) in H_1.

We conclude that R/I is a locally finite integral domain, and hence it is countable. Also, $\widetilde{D} := D/Stab_D(a)$ can be treated as a subgroup of the multiplicative group of R. Therefore, there are $d_i \in \widetilde{D}$, $i \in \omega$, such that

$$\widetilde{D} \subseteq \bigcup_{i \in \omega} d_i + I.$$

Put $D_i = \widetilde{D} \cap (d_i + I)$. Then $\widetilde{D} = \bigcup_{i \in \omega} D_i$. If $r \in I$, then $r[H_1]$ is finite and D-invariant, and so $r[H_1] < H_0$. Hence for every $i \in \omega$ and $b \in H_1$ we have $(D_i - d_i)b \subseteq Ib \subseteq H_0$. Thus, we easily get $D_i = \{d \in \widetilde{D} : (d - d_i)H_1 \subseteq H_0\}$.

For $h \in H_0$ and $i \in \omega$ put $K_{h,i} = \{(d, b) \in \widetilde{D} \times H_1 : (d - d_i)b = h\}$. We see that $\widetilde{D} \times H_1 = \bigcup_{h \in H_0} \bigcup_{i \in \omega} K_{h,i}$. Since H_0 is countable and $\widetilde{D} \times H_1$ is compact, there are $h \in H_0$ and $i \in \omega$ such that $K_{h,i} \subseteq_{nm} \widetilde{D} \times H_1$. By continuity of the action of $\widetilde{D}$ on H_1, we also have that $K_{h,i}$ is closed in $\widetilde{D} \times H_1$. Thus, there are $U \subseteq_o \widetilde{D}$ and $V \subseteq_o H_1$ such that $U \times V \subseteq K_{h,i}$.

Let $H_2 = \{b \in H_1 : (U - d_i)b = \{0\}\} < H_1$. Then H_2 is clopen in H_1 because $V - V \subseteq H_2$. For every $b \in H_2$ we have $Ub = d_i b$, so $d_i^{-1}Ub = b$. Hence $d_i^{-1}U \subseteq S_{H_2} := \{d \in \widetilde{D} : (\forall b \in H_2)(db = b)\} < \widetilde{D}$. Since $U \subseteq_o \widetilde{D}$, we get that S_{H_2} is an open subgroup of $\widetilde{D}$ fixing H_2. ■

The proof of the next lemma is very similar to the proof of [21, Lemma 3.5]. The only serious difficulty comes from the fact that we do not have a basis of open neighborhoods of e consisting of $\emptyset$-closed subgroups. Our previous remarks take care of that.

Lemma 4.26 *Assume that (H, G) is an infinite, small compact G-group of finite $\mathcal{N}M$-rank. If H is solvable-by-finite, then some open subgroup of H has an infinite center.*

Proof. By Remark 4.16, there is a clopen, solvable subgroup S of H. By Remarks 4.10 and 4.11, we can replace (H, G) by $(H, Stab(S))$. Then by Remark 2.10, we can replace (H, G) by $(S, Stab(S)/Stab(S)_S)$ where $Stab(S)_S = \{g \in Stab(S) : g(s) = s\}$. So wlog H is solvable.

Replacing H by a clopen subgroup (and changing G as above), we can assume that H has an infinite, invariant, normal, abelian subgroup. Taking its closure, we get an infinite, normal, abelian, $*$-closed subgroup of H. Let H_0 be a subgroup of H with all these properties and minimal possible $\mathcal{N}M$-rank, except for the fact that being $*$-closed is with respect to (H, G_0) where G_0 is a clopen subgroup of G (notice that by Remark 4.11, $\mathcal{N}M$-rank does not depend on the choice of G_0). Wlog H_0 is $\emptyset$-closed in (H, G).

Let $\{L_i : i \in I\}$ be a basis of clopen, normal subgroups of H. Put $H_i = H_0 \cap L_i$. It is enough to prove the following

Claim There exists $i \in I$ such that $[H : C_H(H_i)] < \omega$.

Proof. Suppose the Claim is false. First notice that it is enough to find $i_0, i_1, \ldots \in I$ such that $C_H(H_{i_j})$ is a subgroup of infinite index in $C_H(H_{i_{j+1}})$ for each $j \in \omega$. Indeed, since G acts continuously on H_0, by Remark 4.10, $G_j := Stab(H_{i_0}) \cap \ldots \cap Stab(H_{i_j})$ is a clopen subgroup of G. Hence, by Remarks 4.11 and 4.6, in (H, G_j) we have $\mathcal{N}M(C_H(H_{i_j})) > \mathcal{N}M(C_H(H_{i_{j-1}})) > \ldots > \mathcal{N}M(C_H(H_{i_0}))$, and so $\mathcal{N}M(H) > j$. Once again by Remark 4.11, $\mathcal{N}M(H) > j$ holds in (H, G) as well. So $\mathcal{N}M(H) \geq \omega$, a contradiction.

Suppose that $i_0, \ldots, i_n$ have been chosen. Then $D := H/C_H(H_{i_n})$ is an infinite group $\emptyset$-closed in $(H, G_n)^{teq}$. Then D acts on H_{i_n} by conjugation. Since D is solvable, there is a clopen, normal subgroup $\widetilde{D}$ of D with an infinite, abelian, invariant under the action of H by conjugation subgroup D_1 which is $\emptyset$-closed in $(H, Stab_{G_n}(\widetilde{D}))^{teq}$. By Lemma 4.25, $D_i := D_1 \cap (L_i/C_H(H_{i_n}))$ stabilizes an infinite subgroup $H^* = Fix(D_i)$ of H_{i_n} for some $i \in I$. Since D_1, L_i, H_{i_n} are invariant under the action of H by conjugation, H^* is normal in H. Moreover, H^* is $\emptyset$-closed in $(H, Stab_{G_n}(\widetilde{D}) \cap Stab_{G_n}(L_i))$. By Remark 4.10, $Stab_{G_n}(\widetilde{D}) \cap Stab_{G_n}(L_i)$ is a clopen subgroup of G, so by the choice of H_0, we get that H^* is open in H_0. We also see that $[C_H(H^*) : C_H(H_{i_n})] \geq |D_i| \geq \omega$. Now we can choose any $i_{n+1} \in I$ with $H_{i_{n+1}} < H^*$ in order to get $[C_H(H_{i_{n+1}}) : C_H(H_{i_n})] \geq \omega$. ■

Proof of Theorem 4.24. Suppose H is not nilpotent-by-finite. Then by Lemma 4.26, we can construct infinite sequences $H = H_0 > H_1 > \ldots > L_1 > L_0 = \{e\}$ such that H_i's are clopen, normal subgroups of H, each L_i is a normal subgroup of H and $*$-closed in $(H, Stab(H_0) \cap \ldots \cap Stab(H_i))$, and $L_{i+1}/L_i = Z(H_{i+1}/L_i)$ is infinite.

Then the same argument as in the first paragraph of the proof of the Claim in the proof of Lemma 4.26 shows that $\mathcal{N}M(H) \geq \omega$, a contradiction. ■

By Theorems 4.19 and 4.24, we get the following main result of this section.

Corollary 4.27 *If (H, G) is a small compact G-group of finite $\mathcal{N}M$-rank, then H is nilpotent-by-finite.*

By some standard tricks, Corollary 4.27 implies the following

Corollary 4.28 *If (H, G) is a small compact G-group of finite $\mathcal{N}M$-rank, then H has a $*$-closed, nilpotent subgroup of finite index.*

Proof. By Corollary 4.27 and Remark 4.16, there is a clopen, normal, nilpotent subgroup $\widetilde{H}$ of H. The rest of the proof is almost the same as in [24, Theorem 3.17] or [14, Remark 3.3(ii)]. Namely, by induction on the nilpotency class, we show that $\widetilde{H}$ has a nilpotent supergroup which is $*$-closed. Notice that in spite of the fact that we do not have icc on centralizers, it is true that for any $S \subseteq H$ such that $[H : C(S)] < \omega$, $C(S)$ is an intersection of finitely many centralizers of singletons, thus it is $*$-closed, and so is $Z(C(S))$. This is enough to repeat the proof of [14, Remark 3.3(ii)]. ■

Notice that unlike in [27], in our proofs we did not need to know that every infinite, small, *nm*-stable compact G-group has an infinite, $*$-closed, abelian subgroup. Anyway the next proposition tells us that this property holds in our case. We prove it modifying slightly the proof of [21, Proposition 3.1].

Proposition 4.29 *Every infinite, small, nm-stable compact G-group has an infinite, $*$-closed, abelian subgroup.*

Proof. Replacing H by an infinite, $*$-closed subgroup of minimal possible $\mathcal{N}M$-rank, we can assume that H does not have infinite, $*$-closed subgroups of smaller $\mathcal{N}M$-rank. We will show that then H has an abelian, $*$-closed subgroup of finite index.

Since H is locally finite, it has an infinite abelian subgroup [15, Corollary 2.5], say A. Taking the closure of A, we can assume that A is uncountable. So there is $a \in A \setminus Acl(\emptyset)$ with $C(a)$ infinite and hence open in H. Let d be a name for $C(a)$. By Remark 4.10, we see that $d \in Acl^{eq}(\emptyset)$. So $o(a/d)$ is infinite. Put $V = \overline{\langle o(a/d) \rangle}$. We see that V is an infinite, $*$-closed subgroup of H, and hence it is open. Moreover, since $a \in Z(C(a))$, we get $o(a/d) \subseteq Z(C(a))$ so $V < Z(C(a))$. Thus, V is an abelian, $*$-closed subgroup of finite index in H. ■

Corollary 4.30 *Each small, compact G-group of $\mathcal{N}M$-rank 1 is abelian-by-finite.*

Let us finish this section with several questions. We have shown that each small, *nm*-stable [of finite $\mathcal{N}M$-rank] compact G-group is solvable-by-finite [nilpotent-by-finite, respectively]. It seems that the following should be true.

Conjecture 4.31 *Each small, nm-stable compact G-group is abelian-by-finite.*

The next question is open even for small profinite groups. Notice that Examples A, B and C are not counterexamples for this question.

Question 4.32 *Does every small compact G-group contain an infinite, abelian, $*$-closed subgroup?*

The structure of small, nm-stable Polish G-groups is almost completely unknown. Example 6 yields an abelian, Polish G-group of $\mathcal{NM}$-rank 1 which is torsion free and not profinite. Yet, it is still 0-dimensional. It would be interesting to find a non-0-dimensional, small Polish G-group. A very interesting question is whether Conjectures (A), (B) and (C) (with 'by-finite' replaced with 'by-countable') are true for small, nm-stable [of finite $\mathcal{NM}$-rank] Polish G-groups. In particular, one can ask the following general

Question 4.33 *Is it true that every small, nm-stable Polish G-group is abelian-by-countable?*

Even the following question is open.

Question 4.34 *Is every small, Polish G-group of $\mathcal{NM}$-rank 1 abelian-by-countable?*

5 Final comments and questions

In model theory there are results, known as group configuration theorems, which say that under some general geometric assumptions one can find a definable group (e.g. [23, Chapters 5, 7]). Such theorems were also proved for small profinite structures [22, Theorem 1.7, Theorem 3.3] or, more generally, for compact structures satisfying the existence of m-independent extensions [13, Theorem 3.15]. Considerations concerning the existence of a definable (in some sense) group structure are the best example illustrating how stability-theoretic ideas may lead to new aspects in the analysis of classical topological objects. In the context of small Polish structures it is not clear how to prove such kind of results. We can ask here the following general questions.

Question 5.1 *Suppose (X, G) is a small Polish structure. When is there a function $\cdot : X \times X \to X$ definable (or invariant) over a finite subset A of X such that $(X, \cdot)$ is a group?*

Question 5.2 *Suppose (X, G) is a small Polish structure. When is there an infinite set $Y \subseteq X$ and a function $\cdot : Y \times Y \to Y$, both definable (or invariant) over a finite subset A of X, such that $(Y, \cdot)$ is a group?*

The group configuration theorem for small profinite structures [22, Theorem 1.7] yields a partial answer to Question 4.2. Namely, if we additionally assume that (X, G) is profinite and m-normal, then in every non-trivial orbit of $\mathcal{M}$-rank 1 there is an open definable group.

So one of the possible ways of further research is to prove for small Polish structures counterparts of some advanced results from geometric stability theory, e.g. a variant of the group configuration theorem.

There are also certain open questions about the existence of small profinite structures satisfying some additional assumptions. I think it would be interesting to find counterexamples for such questions in the bigger class of small Polish structures (e.g. Examples A, B and C in the previous section are counterexamples for Conjectures (A), (B) and (C), respectively). An interesting problem of this kind is

Conjecture 5.3 ($\mathcal{NM}$-gap conjecture) *Let (X,G) be a small Polish structure. Then for every orbit o over a finite subset A of X one has $\mathcal{NM}(o) \in \omega \cup \{\infty\}$.*

This conjecture is open in the class of small profinite structures; it was proved only for small m-stable profinite groups [27, Theorem 18]. In the context of small Polish structures it is open even for small nm-stable Polish [compact] G-groups.

At the end I would like to formulate several purely descriptive set theoretic (or topological) facts and questions which came up naturally during my considerations on Polish structures.

The following fact is Corollary 2.6.8 of [2].

Fact 5.4 *The Polish $Homeo(I^\omega)$-space $(I^\omega, Homeo(I^\omega))$ is universal for Borel G-spaces, i.e. every Borel G-space (X,G) can be embedded into $(I^\omega, Homeo(I^\omega))$ in the sense that there is a topological isomorphism $\psi : G \to \psi[G] <_c Homeo(I^\omega)$ and a Borel embedding $\phi : X \to I^\omega$ such that $\phi(gx) = \psi(g)\phi(x)$ for every $g \in G$.*

We know that this universal Borel G-space is a small Polish structure [Example 3]. The following question is a counterpart of the above fact for groups.

Question 5.5 *Does there exist a Polish G-group (U,G) which is universal for Borel H-groups (i.e. every Borel H-group (G_0,H) can be embedded into (U,G), preserving the group structure on G_0)? If yes, is (U,G) a small Polish G-group?*

From now on we fix a Polish group G. The following fact comes from [6, Theorem 0.3] and [2, Corollary 2.6.8].

Fact 5.6 *There is a Polish G-space which is universal for Polish [Borel] G-spaces.*

Question 5.7 *Does there exist a Polish G-group which is universal for Polish [Borel] G-groups?*

It is obvious that any universal Polish G-space (or G-group, if it exists) for a fixed G is not small. However, we can ask the following question (or its variants for Polish or Borel G-spaces or G-groups).

Question 5.8 *Does there exist a small Polish structure [group structure] which is universal for small Polish structures [group structures] of the form (X,G)?*

At the very end we would like to formulate a question concerning small profinite structures. The following is a well-known fact [17].

Fact 5.9 *If G is a locally compact group and (X, G) is a normal topological G-space, then there is a topological group $H \supseteq X$ such that the topology on X is inherited from H and the action of G on X can be extended to an action on H so that (H, G) is a topological G-group.*

Question 5.10 *Is it true that for every small profinite structure (X, G) one can find a profinite group $H \supseteq X$ such that X is closed in H, its topology is inherited from H and the action of G on X can be extended to an action on H so that (H, G) is a small profinite group?*

It is not difficult to construct H so that (H, G) is not necessarily small. Namely, if

$$X = \varprojlim X_i,$$

we can define H as the inverse limit of the system consisting of the linear spaces spanned freely by X_i over the two element field F_2.

Proposition 5.11 *If Question 5.10 has the positive answer, then $\mathcal{M}$-gap conjecture is true for small, m-stable profinite structures.*

Proof. Suppose for a contradiction that (X, G) is a small, m-stable profinite structure with an orbit o over some finite set A such that $\mathcal{M}(o) \in Ord \setminus \omega$; hence wlog $\mathcal{M}(o) = \omega$. By the assumption, we have that X is $\emptyset$-closed in a small profinite group (H, G).

By [21, Proposition 2.3], the subgroup $\langle X \rangle$ generated by X is generated in finitely many steps. This easily implies that $(\langle X \rangle, G)$ is a small, m-stable profinite group. Hence the $\mathcal{M}$-rank of o computed within $(\langle X \rangle, G)$ also equals ω, and we get a contradiction with the fact that $\mathcal{M}$-gap conjecture holds for small, m-stable profinite groups [27, Theorem 18]. ■

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