# Some model theory of Polish structures

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**Abstract**

We define a topological notion of independence in Polish structures and we prove that it has some nice properties. Using this notion, we prove counterparts of some basic results from geometric stability theory in the context of small Polish structures. Examples of small Polish structures are also given.

## 0 Introduction

We propose a new, model theoretic, approach to study classical descriptive set theoretic objects, like Polish $G$-spaces or Borel $G$-spaces. More generally, we are going to study Polish structures which are defined as follows.

**Definition 0.1** *A Polish structure is a pair $(X, G)$ where $G$ is a Polish group acting faithfully on a set $X$ so that the stabilizers of all singletons are closed subgroups of $G$. We say that $(X, G)$ is small if for every $n \in \omega$, there are only countably many orbits on $X^n$ under the action of $G$.*

Notice that the assumption that $G$ acts faithfully on $X$ is purely cosmetic as we can always divide $G$ by the maximal subgroup acting trivially on $X$.

Profinite structures ($X$ and $G$ are profinite and the action is continuous) introduced by Newelski in [13, 14] and then considered also by Wagner [17] and by myself [4, 5, 6, 8], and, more generally, compact structures [7] ($X$ is a compact metric space, $G$ is a compact group and the action is continuous) are particular cases of Polish structures. More generally, Polish $G$-spaces ($X$ is Polish and the action is continuous) and Borel $G$-spaces ($X$ is a standard Borel space and the action is Borel-measurable) are also examples of Polish structures.

In Section 1 we introduce the notion of $nm$-independence and we prove that it has some nice properties, similarly as forking independence in stable theories. We show that $nm$-independence generalizes the notion of $m$-independence introduced by Newelski for profinite structures. However, the proof that $nm$-independence is transitive is rather complicated and it uses some descriptive set theory, whereas transitivity of $m$-independence follows immediately from the definition. Similarly as in the case of profinite structures, in order to get the existence of $nm$-independent extensions we need to assume smallness. In fact, we could just assume the existence of $nm$-independent extensions (as in [7]) but we prefer to assume smallness since it is more natural and easier to check in concrete examples.

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In Section 2 we consider counterparts of some basic notions from geometric stability theory and we prove fundamental results about them.

In Section 3 we give examples of small Polish structures. We end up with a list of open questions.

The general goal of this paper, similarly as in [7] where I was investigating compact structures, is to make an attempt to apply stability theory ideas to various classical mathematical objects. A similar motivation appears in [13, 14] where Newelski was considering small profinite structures. However, Polish structures (particularly Polish $G$-spaces) seem to be more interesting than profinite structures from the point of view of descriptive set theory or topology. Moreover, in my opinion Polish structures yield a more adequate generalization of profinite structures than compact structures. This is because each small compact structure is profinite. So instead of smallness we should assume here the existence of $m$-independent extensions; but I do not know any interesting examples of such compact structures which are not profinite. In contrast, we have several natural examples of small Polish structures which are not profinite, and it seems to me that it should be not very difficult to find another ones.

Having all the notions and basic results established in this paper, the natural next step would be to try and prove the counterparts of some deep results from stability theory, e.g. a group configuration theorem. Such results were proved by Newelski for small profinite structures [14]. In [7] I noticed that most of them can be generalized to the case of compact [profinite] structures satisfying the existence of $m$-independent extensions. In small Polish structures the situation is more complicated and it is even not clear how to formulate the appropriate conjectures.

There are also certain open questions about the existence of small profinite structures with some model theoretic properties (e.g. $\mathcal{M}$-gap conjecture [13, 14, 17]). I think it would be interesting to find counterexamples for them in the wider context of small Polish structures.

The notions introduced in this paper (e.g. $nm$-independence, $\mathcal{N}M$-rank) may also turn out to be new tools to deal with purely descriptive set theoretic or topological problems.

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# 1 Independence relation

In this section we define a notion of independence, which we call $nm$-independence (read non-meager independence), and we study its properties. We also prove that it coincides with Newelski's $m$-independence in compact structures.

If $(X, G)$ is a Polish structure and $A \subseteq X$, then by $G_A$ we denote the pointwise stabilizer of $A$. For $a \in X^n$ we define $o(a/A) = \{f(a) : f \in G_A\}$ (the orbit of $a$ over $A$).

Let us recall the definition of $m$-independence.

**Definition 1.1** *Let $(X, G)$ be a compact structure, $a$ be a finite tuple and $A, B$ finite subsets of $X$. We say that $a$ is m-independent from $B$ over $A$ (written $a \overset{m}{\perp}_A B$) if $o(a/AB)$ is open in $o(a/A)$. We say that $a$ is m-dependent on $B$ over $A$ (written $a \overset{m}{\not\perp}_A B$) if $o(a/AB)$ is nowhere dense in $o(a/A)$.*

We cannot use the above definition for a Polish structure $(X, G)$ as there is no topology on $X$. Even if we assumed that $(X,G)$ is a Polish $G$-space, orbits could be weird, e.g. meager in their relative topologies, and then $m$-independence would not have nice properties in this context.

The idea to omit the above obstacle is to define a relation of independence in terms of the Polish group $G$.

**Definition 1.2** *Let $(X,G)$ be a Polish structure, $a$ be a finite tuple and $A, B$ finite subsets of $X$. Let $\pi_A : G_A \to o(a/A)$ be defined by $\pi_A(g) = ga$. We say that $a$ is nm-independent from $B$ over $A$ (written $a \mathop{\smile\!\!\!\!\!\!\mid}^{nm}_A B$) if $\pi_A^{-1}[o(a/AB)]$ is non-meager in $\pi_A^{-1}[o(a/A)]$. Otherwise, we say that $a$ is nm-dependent on $B$ over $A$ (written $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{nm}_A B$).*

One can also define $o$-independence just replacing the word 'non-meager' by 'open' in the above definition. Some of the results will work for both notions of independence. However, the proof of the existence of $nm$-independent extensions in small Polish structures does not work for $o$-independence. In Section 3 we will show that the pseudo-arc is an example of a small Polish structure without the existence of $o$-independent extensions. That is why $nm$-independence is a more appropriate notion of independence.

Notice that $nm$ and $o$ in Definition 1.2 come from topological properties 'non-meager' and 'open', whereas $m$ in Definition 1.1 comes from the word 'multiplicity'.

**Notation** If $T$ is a topological space and $U, V \subseteq T$, then $U \subseteq_{nm} V$ means that $U$ is a non-meager subset of $V$ and $U \subseteq_o V$ means that $U$ is an open subset of $V$. When we write $* = nm$, it means that $*$ stands for non-meager; similarly, $* = o$ means that $*$ stands for open.

**Proposition 1.3** *Let $(X,G)$ be a Polish structure, $a$ be a finite tuple and $A, B$ finite subsets of $X$. Assume $* = nm$ or $* = o$. Then TFAE:*

*(1) $a \mathop{\smile\!\!\!\!\!\!\mid}^{*}_B A$,*

*(2) $G_{AB}G_{Aa} \subseteq_* G_A$,*

*(3) $G_{AB}/G_{Aa} \subseteq_* G_A/G_{Aa}$.*

*Proof.* (1) $\leftrightarrow$ (2). An easy computation shows that $\pi_A^{-1}[o(a/AC)] = G_{AC}G_{Aa}$ for any $C \subseteq X$. Hence $\pi_A^{-1}[o(a/AB)] = G_{AB}G_{Aa}$ and $\pi_A^{-1}[o(a/A)] = G_A$. Now the desired equivalence follows directly from the definition of $*$-independence.
(2) $\to$ (3). Let $\pi : G_A \to G_A/G_{Aa}$ be the quotient map. Then $\pi$ is continuous and open.

If $* = o$, the implication follows from the fact that $\pi$ is open. So consider the case $* = nm$. Suppose $G_{AB}/G_{Aa}$ is a meager subset of $G_A/G_{Aa}$, i.e. $G_{AB}/G_{Aa}$ is covered by a countable union $\bigcup_{i\in\omega} D_i$ of closed and nowhere dense subsets of $G_A/G_{Aa}$. Then $G_{AB}G_{Aa} = \pi^{-1}[G_{AB}/G_{Aa}] \subseteq \bigcup_{i\in\omega} \pi^{-1}[D_i]$. Since $\pi$ is continuous and open, we get that $\pi^{-1}[D_i]$, $i \in \omega$, are closed and nowhere dense. So $G_{AB}G_{Aa}$ is a meager subset of $G_A$.
(3) $\to$ (2). Since $\pi$ is continuous and $G_{AB}$ is Polish, we see that $G_{AB}/G_{Aa}$ is analytic so it has the Baire property [3, Theorem 21.6], i.e. $G_{AB}/G_{Aa} = D \triangle U$ where $D$ is meager and $U$ is open in $G_A/G_{Aa}$. Assume now that $G_{AB}/G_{Aa} \subseteq_{nm} G_A/G_{Aa}$. Then $U$ is nonempty. We have $G_{AB}G_{Aa} = \pi^{-1}[D] \triangle \pi^{-1}[U]$. Since $\pi$ is continuous and open, we get that $\pi^{-1}[D]$ is meager and $\pi^{-1}[U] \neq \emptyset$ is open in $G_A$. Hence $G_{AB}G_{Aa} \subseteq_{nm} G_A$. ■

**Definition 1.4** *Let $(X, G)$ be a Polish structure and $A$ be a finite subset of $X$. We define the algebraic closure of $A$ (written $Acl(A)$) as the set of all elements of $X$ with countable orbits over $A$. The strong algebraic closure of $A$ (denoted by $acl(A)$) is the set of all elements of $X$ with finite orbits over $A$. If $A$ is infinite, we define $Acl(A) = \bigcup\{Acl(A_0) : A_0 \subseteq A \text{ is finite}\}$ and $acl(A) = \bigcup\{acl(A_0) : A_0 \subseteq A \text{ is finite}\}$.*

In Polish structures *Acl* plays a similar role as *acl* in compact [profinite] structures.

**Theorem 1.5** *In any Polish structure $(X, G)$ $*$-independence, where $* = nm$ or $* = o$, has the following properties.*

*(1) (Symmetry) For every finite $a, b, C \subseteq X$ we have that $a \perp^{*}_{C} b$ iff $b \perp^{*}_{C} a$.*

*(2) (Transitivity) For every finite $A \subseteq B \subseteq C \subseteq X$ and $a \subseteq X$ we have that $a \perp^{*}_{B} C \wedge a \perp^{*}_{A} B$ iff $a \perp^{*}_{A} C$.*

*(3) For every finite $A \subseteq X$ we have that $a \in Acl(A)$ iff for all finite $B \subseteq X$, $a \perp^{*}_{A} B$ .*

*Proof.* Only (2) requires a serious argument, but let us first check (1) and (3) for completeness.
(1) $a \perp^{*}_{A} b \iff G_{Ab}G_{Aa} \subseteq_{*} G_A \iff G_{Aa}G_{Ab} \subseteq_{*} G_A \iff b \perp^{*}_{A} a$.
(3) ($\Rightarrow$). $a \in Acl(A)$ means that $o(a/A)$ is countable. Hence $[G_A : G_{Aa}] \leq \omega$. So $G_{Aa}$ is non-meager in $G_A$ by the Baire category theorem. Since $G_{Aa}$ is a closed subgroup of $G_A$, we get $G_{Aa} \subseteq_o G_A \Longrightarrow G_{AB}G_{Aa} \subseteq_o G_A \Longrightarrow a \perp^{o}_{A} B \Longrightarrow a \perp^{nm}_{A} B$ for any finite $B \subseteq X$.
($\Leftarrow$). Suppose that $a \perp^{*}_{A} B$ for every finite $B \subseteq X$. Then $a \perp^{*}_{A} Aa$. Hence $G_{Aa} \subseteq_{*} G_A \Longrightarrow G_{Aa} \subseteq_o G_A \Longrightarrow [G_A : G_{Aa}] \leq \omega \Longrightarrow a \in Acl(A)$.

In order to prove transitivity we need first to prove some purely descriptive set theoretic lemmas, which seem to be interesting on their own rights.

**Lemma 1.6** *Suppose that $H_1$ and $H_2$ are closed subgroups of a Polish group $H$. Let $H_3 = H_1 \cap H_2$. Then $H_1H_2$ is a Borel subset of $H$. Moreover, for every $A_1 = A_1H_3$ a Borel subset of $H_1$ and $A_2 = H_3A_2$ a Borel subset of $H_2$ we have that $A_1A_2$ is a Borel subset of $H_1H_2$.*

*Proof.* Define $f : H_1 \times H_3 \times H_2 \to H$ by $f(h_1, h_3, h_2) = h_1h_3h_2$. By [3, Theorem 12.17] we can choose a Borel set $S_1$ meeting every coset from $H_1/H_3$ in exactly one point and a Borel set $S_2$ meeting every coset from $H_3 \backslash H_2$ in exactly one point.

Let $f_0$ be the function $f$ restricted to $S_1 \times H_3 \times S_2$. Notice that $f_0$ is an injection. Indeed, if $s_1, s_1' \in S_1$, $s_2, s_2' \in S_2$, $h, h' \in H_3$ and $s_1hs_2 = s_1'h's_2'$, then $s_1'^{-1}s_1 = h's_2's_2^{-1}h^{-1} \in H_1 \cap H_2 = H_3$. Hence $s_1 = s_1'$. Similarly $s_2 = s_2'$. So $h = h'$.

Since $S_1 \times H_3 \times S_2$ is a Borel subset of the Polish space $H_1 \times H_3 \times H_2$, we get that $f_0[S_1 \times H_3 \times S_2] = S_1H_3S_2 = H_1H_2$ is a Borel subset of $H$.

To prove the second part of the lemma, first notice that $A_1' := A_1 \cap S_1$ and $A_2' := A_2 \cap S_2$ are Borel subsets of $H_1$ and $H_2$, respectively. Hence $f_0[A_1' \times H_3 \times A_2']$ is a Borel subset of $H$. On the other hand, $f_0[A_1' \times H_3 \times A_2'] = A_1'H_3A_2' = A_1H_3A_2 = A_1A_2$. Hence $A_1A_2$ is a Borel subset of $H$, which implies that it is a Borel subset of $H_1H_2$. ■

**Lemma 1.7** *Suppose that $H_1$ and $H_2$ are closed subgroups of a Polish group $H$ such that $H_1H_2$ is non-meager in its relative topology. Let $H_3 = H_1 \cap H_2$. Then for every $A_1 = A_1H_3 \subseteq_o H_1$ and $H_2 = H_3A_2 \subseteq_o H_2$ we have $A_1A_2 \subseteq_o H_1H_2$.*

*Proof.* Let us define a function $\psi : H_1H_2 \to H_1/H_3 \times H_3\backslash H_2$ by

$$\psi(h_1h_2) = (h_1H_3, H_3h_2)$$

where $h_1 \in H_1$ and $h_2 \in H_2$.

We check that $\psi$ is well-defined. Suppose $h_1h_2 = h_1'h_2'$ where $h_1, h_1' \in H_1$ and $h_2, h_2' \in H_2$. Then $h_1'^{-1}h_1 = h_2'h_2^{-1} \in H_1 \cap H_2 = H_3$. Hence $h_1H_3 = h_1'H_3$ and $H_3h_2' = H_3h_2$.

To finish the proof it is enough to show that $\psi$ is continuous. The following Claim follows from Lemma 1.6.

**Claim 1** $\psi$ is Baire measurable.

By Claim 1 and [3, Theorem 8.38] there is a set $C \subseteq H_1H_2$ which is a countable intersection of dense open subsets of $H_1H_2$ such that $\psi \restriction C$ is continuous.

**Claim 2** $\psi$ is continuous.

*Proof of Claim 2.* Suppose for a contradiction that there are $h_1^i \in H_1$ and $h_2^i \in H_2$, $i \in \omega$, such that

1. $h_1^ih_2^i \longrightarrow h_1h_2$,
2. $(h_1^iH_3, H_3h_2^i) \not\longrightarrow (h_1H_3, H_3h_2)$.

Notice that (1) and (2) hold for $hh_1^i, h_2^ig, hh_1, h_2g$ instead of $h_1^i, h_2^i, h_1, h_2$ for arbitrary $h \in H_1$ and $g \in H_2$. Hence to get a contradiction it is enough to find $h \in H_1$ and $g \in H_2$ such that $hh_1h_2g \in C$ and $hh_1^ih_2^ig \in C$ for all $i \in \omega$. So we will be done if we prove the following:

**Subclaim** $(\forall h_1h_2 \in H_1H_2)(\forall^*(h,g) \in H_1 \times H_2)(hh_1h_2g \in C)$.

*Proof of Subclaim.* Since $C^* := \{h_1h_2 \in H_1H_2 : (\forall^*(h,g) \in H_1 \times H_2)(hh_1h_2g \in C)\}$ is invariant under multiplication by $H_1$ on the left and by $H_2$ on the right, it is enough to show that

$$(*) \quad (\exists h_1h_2 \in H_1H_2)(\forall^*(h,g) \in H_1 \times H_2)(hh_1h_2g \in C).$$

Since $C$ is comeager in $H_1H_2$, we have

$$(\forall(h,g) \in H_1 \times H_2)(\forall^* h_1h_2 \in H_1H_2)(hh_1h_2g \in C).$$

So by the Kuratowski-Ulam theorem [3, Theorem 8.41] we get

$$(\forall^* h_1h_2 \in H_1H_2)(\forall^*(h,g) \in H_1 \times H_2)(hh_1h_2g \in C).$$

Since $H_1H_2$ is non-meager in its relative topology, we get $(*)$ and we are done. ■

**Lemma 1.8** *Suppose that $H_1$ and $H_2$ are closed subgroups of a Polish group $H$ such that $H_1H_2$ is non-meager in its relative topology. Let $H_3 = H_1 \cap H_2$. Then for every analytic set $A_1 = A_1H_3 \subseteq_{nm} H_1$ and $A_2 = H_3A_2 \subseteq_{nm} H_2$ we have $A_1A_2 \subseteq_{nm} H_1H_2$.*

*Proof.* In the same way as in the proof of Proposition 1.3 we get that $A_1/H_3 \subseteq_{nm} H_1/H_3$ and $H_3\backslash A_2 \subseteq_{nm} H_3\backslash H_2$. Moreover, since $A_1$ and $A_2$ are analytic, $A_1/H_3$ and $H_3\backslash A_2$ have the Baire property [3, Proposition 14.4, Theorem 21.6]. So $A_1/H_3 = D_1 \triangle U_1$ and $H_3\backslash A_2 = D_2 \triangle U_2$ where $D_1, D_2$ are meager and $U_1, U_2$ are nonempty and open in $H_1/H_3$ and $H_3\backslash H_2$, respectively. So there are $h_i, g_i$, $i \in \omega$, such that $R_1 := \bigcup_{i\in\omega} h_i A_1/H_3$ and $R_2 := \bigcup_{i\in\omega} H_3\backslash A_2 g_i$ are comeager in $H_1/H_3$ and $H_3\backslash H_2$, respectively. So there are $U_i, V_i$, $i \in \omega$, open and dense in $H_1/H_3$ and $H_3\backslash H_2$, respectively, such that $\bigcap_{i\in\omega} U_i \subseteq R_1$ and $\bigcap_{i\in\omega} V_i \subseteq R_2$.

Let $\pi_1 : H_1 \to H_1/H_3$ and $\pi_2 : H_2 \to H_3\backslash H_2$ be the quotient maps. Let $U_i' = \pi_1^{-1}[U_i]$ and $V_i' = \pi_2^{-1}[V_i]$. Then

$(*)$ $U_i'$ is open and dense in $H_1$ and $V_i'$ is open and dense in $H_2$ for all $i \in \omega$,

$(**)$ $U_i' = U_i' H_3$ and $V_i' = H_3 V_i'$ for all $i \in \omega$,

$(***)$ $\bigcap_{i\in\omega} U_i' \subseteq \pi_1^{-1}[R_1] = \bigcup_{i\in\omega} h_i A_1$ and $\bigcap_{i\in\omega} V_i' \subseteq \pi_2^{-1}[R_2] = \bigcup_{i\in\omega} A_2 g_i$.

**Claim** $(\bigcap_{i\in\omega} U_i')(\bigcap_{i\in\omega} V_i') = \bigcap_{i\in\omega} U_i' V_i'$.

*Proof of Claim.* The inclusion $\subseteq$ is obvious. So let us prove $\supseteq$. Consider any $a \in \bigcap_{i\in\omega} U_i' V_i'$. Then for every $i \in \omega$ we can choose $u_i \in U_i'$ and $v_i \in V_i'$ so that $a = u_i v_i$. For every $i, j \in \omega$ we have $u_i v_i = u_j v_j \Longrightarrow u_j^{-1} u_i = v_j v_i^{-1} \in H_1 \cap H_2 = H_3$. So $u_i \in u_j H_3$ and $v_i \in H_3 v_j$. Hence by $(**)$ we get $u_0 \in \bigcap_{i\in\omega} U_i'$ and $v_0 \in \bigcap_{i\in\omega} V_i'$. So $a = u_0 v_0 \in (\bigcap_{i\in\omega} U_i')(\bigcap_{i\in\omega} V_i')$. □

By $(***)$ and Claim we get

$$\bigcup_{i,j\in\omega} h_i A_1 A_2 g_j = (\bigcup_{i\in\omega} h_i A_1)(\bigcup_{i\in\omega} A_2 g_i) \supseteq (\bigcap_{i\in\omega} U_i')(\bigcap_{i\in\omega} V_i') = \bigcap_{i\in\omega} U_i' V_i'.$$

By Lemma 1.7, $(*)$ and $(**)$, $U_i' V_i' \subseteq_o H_1 H_2$. By $(*)$, $U_i' V_i'$ is dense in $H_1 H_2$. Hence $\bigcap_{i\in\omega} U_i' V_i'$ is comeager, and thus non-meager, in $H_1 H_2$. Therefore $\bigcup_{i,j\in\omega} h_i A_1 A_2 g_j$ and so $A_1 A_2$ are non-meager in $H_1 H_2$. ■

To prove (2) in Theorem 1.5 we will need the following corollary of Lemma 1.7 and 1.8.

**Corollary 1.9** *Suppose that $H_1$ and $H_2$ are closed subgroups of a Polish group $H$ such that $H_1 H_2$ is non-meager in its relative topology. Then*
*(i) for every $A_1 \subseteq_o H_1$ we have $A_1 H_2 \subseteq_o H_1 H_2$,*
*(ii) for every analytic set $A_1 \subseteq_{nm} H_1$ we have $A_1 H_2 \subseteq_{nm} H_1 H_2$.*

*Proof.* Apply Lemma 1.7 and 1.8 for $A_2 = H_2$. ■

Now we are returning to the proof of Theorem 1.5.

*Proof of (2) in Theorem 1.5.* First consider the case $* = o$. We need to prove that

$$G_C G_{Ba} \subseteq_o G_B \wedge G_B G_{Aa} \subseteq_o G_A \iff G_C G_{Aa} \subseteq_o G_A.$$

$(\Rightarrow)$. Assume $G_C G_{Ba} \subseteq_o G_B \wedge G_B G_{Aa} \subseteq_o G_A$. Define $H := G_A, H_1 := G_B, H_2 := G_{Aa}, A_1 := G_C G_{Ba}$. Then $H_1 H_2 = G_B G_{Aa} \subseteq_o G_A$, so it is non-meager in its relative

topology. Moreover, $A_1 \subseteq_o H_1$. Hence by Corollary 1.9(i) we get $G_C G_{Aa} = G_C G_{Ba} G_{Aa} = A_1 H_2 \subseteq_o H_1 H_2 = G_B G_{Aa} \subseteq_o G_A$. So $G_C G_{Aa} \subseteq_o G_A$.
($\Leftarrow$). Assume $G_C G_{Aa} \subseteq_o G_A$. Then of course $G_B G_{Aa} \subseteq_o G_A$. On the other hand, taking the intersection with $G_B$, we get that $G_C G_{Ba} \subseteq_o G_B$.

Now consider the case $* = nm$. We need to prove that

$$G_C G_{Ba} \subseteq_{nm} G_B \wedge G_B G_{Aa} \subseteq_{nm} G_A \iff G_C G_{Aa} \subseteq_{nm} G_A.$$

($\Rightarrow$). The proof is similar as in the case $* = o$. We need only to check the last implication, namely $G_C G_{Aa} \subseteq_{nm} G_B G_{Aa} \subseteq_{nm} G_A$ implies $G_C G_{Aa} \subseteq_{nm} G_A$.

Suppose for a contradiction that there are closed and nowhere dense subsets $D_i$, $i \in \omega$, of $G_A$ such that $G_C G_{Aa} \subseteq \bigcup_{i \in \omega} D_i$. By the assumption that $G_C G_{Aa} \subseteq_{nm} G_B G_{Aa}$, there is $i \in \omega$ such that $D_i \cap G_B G_{Aa}$ has a non-empty interior in $G_B G_{Aa}$. So countably many translates (by elements of $G_B$ on the left and by elements of $G_{Aa}$ on the right) of $D_i$ cover $G_B G_{Aa}$. Since $D_i$ is nowhere dense in $G_A$, we get that $G_B G_{Aa}$ is meager in $G_A$. This is a contradiction.
($\Leftarrow$). Assume $G_C G_{Aa} \subseteq_{nm} G_A$. Then of course $G_B G_{Aa} \subseteq_{nm} G_A$. It remains to prove that $G_C G_{Ba} \subseteq_{nm} G_B$.

By Proposition 1.3 we get $G_C/G_{Aa} \subseteq_{nm} G_A/G_{Aa}$. Moreover, $G_C/G_{Aa}$ is Polish, so it is a $G_\delta$ subset of $G_A/G_{Aa}$. So $G_C/G_{Aa} = D \triangle U$ where $D$ is meager and analytic and $U \neq \emptyset$ is open in $G_A/G_{Aa}$. Let $\pi : G_A \to G_A/G_{Aa}$ be the quotient map. Define $D' = \pi^{-1}[D]$ and $U' = \pi^{-1}[U]$. Then we have that $G_C G_{Aa} = D' \triangle U'$, $D'$ is meager and analytic [3, Proposition 14.4] and $U' \neq \emptyset$ is open in $G_A$. Moreover, $D' G_{Aa} = D'$, $U' G_{Aa} = U'$ and $G_C \subseteq G_B$. Hence $U' \cap G_B \neq \emptyset$.

Suppose for a contradiction that $G_C G_{Ba}$ is meager in $G_B$. Since $G_C G_{Ba} = (G_C G_{Aa}) \cap G_B = (D' \triangle U') \cap G_B = (D' \cap G_B) \triangle (U' \cap G_B)$ and $U' \cap G_B \neq \emptyset$ is open in $G_B$, we get that $D' \cap G_B \subseteq_{nm} G_B$. We also know that $D' \cap G_B$ is analytic. Moreover, $G_B G_{Aa} \subseteq_{nm} G_A$ so it is non-meager in its relative topology. Hence by Corollary 1.9(ii) we get $(D' \cap G_B) G_{Aa} \subseteq_{nm} G_B G_{Aa} \subseteq_{nm} G_A$. So, in the same way as in the proof of ($\Rightarrow$), we get $(D' \cap G_B) G_{Aa} \subseteq_{nm} G_A$. Hence $D' G_{Aa} \subseteq_{nm} G_A$, which means that $D' \subseteq_{nm} G_A$, a contradiction. ■

In order to get the existence of $nm$-independent extensions we assume smallness.

**Theorem 1.10 (Existence of $nm$-independent extensions)** *Let $(X, G)$ be a small Polish structure. Then for all finite $a \subseteq X$ and $A \subseteq B \subseteq X$, there is $b \in o(a/A)$ such that $b \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$.*

Before the proof let us show the following remark.

**Remark 1.11** *A Polish structure $(X, G)$ satisfies the existence of $*$-independent extensions, where $* = nm$ or $* = o$, iff for all finite $a \subseteq X$ and $A \subseteq B \subseteq X$ there exists $f \in G_A$ such that $G_B f G_{Aa} \subseteq_* G_A$.*

*Proof.* The existence of $*$-independent extensions is equivalent to the fact that for every finite $a \subseteq X$ and $A \subseteq B \subseteq X$ there is $b \in o(a/A)$ such that $G_B G_{Ab} \subseteq G_A$, which in turn is equivalent to the conclusion of the remark. ■

*Proof of Theorem 1.10.* Consider $a, A, B$ as in the theorem. Let $\{a_i : i \in \omega\}$ be a set of representatives of all orbits over $B$ contained in $o(a/A)$. Then $o(a/A) = \bigcup_{i \in \omega} o(a_i/B)$.

Take $f_i \in G_A$, $i \in \omega$, such that $f_i(a) = a_i$.

**Claim** $G_A = \bigcup_{i\in\omega} G_B f_i G_{Aa}$.

*Proof of Claim.* Consider any $f \in G_A$. Then $f(a) \in o(a_i/B)$ for some $i \in \omega$. Hence there is $g \in G_B$ such that $g(f(a)) = f_i(a)$. Then $f_i^{-1} g f \in G_{Aa}$, so $f \in g^{-1} f_i G_{Aa} \subseteq G_B f_i G_{Aa}$. Hence $G_A \subseteq \bigcup_{i\in\omega} G_B f_i G_{Aa}$. The opposite inclusion is obvious. □

By Claim and the Baire category theorem there is $i \in \omega$ such that $G_B f_i G_{Aa} \subseteq_{nm} G_A$. So the proof is completed by Remark 1.11. ■

We see that the above application of the Baire category theorem works only for $nm$-independence. In Section 3 (Remark 3.5) we will see that the pseudo-arc is an example of a small Polish structure without the existence of $o$-independence extensions.

One justification for our definition of $nm$-independence is the fact that it satisfies all the fundamental properties (Theorem 1.5 and 1.10) necessary to develop a counterpart of basic geometric stability theory. Another justification is given by the next corollary, which shows that in compact [profinite] structures $nm$-independence coincides with $m$-independence.

**Theorem 1.12** *Let $(X, G)$ be a Polish structure such that $G$ acts continuously on a separable metrizable space $X$. Let $a, A, B \subseteq X$ be finite. Assume that $o(a/A)$ is non-meager in its relative topology (e.g. it is Polish). Then $a \mathop{\smile\!\!\!\!\!|}^{*}_{A} B \iff o(a/AB) \subseteq_* o(a/A)$ where $* = o$ or $* = nm$.*

*Proof.* Let $\pi : G_A/G_{Aa} \to o(a/A)$ be defined by $\pi(gG_{Aa}) = ga$. Since $o(a/A)$ is non-meager in its relative topology, by Effros' theorem [1, Theorem 2.2.2] we have that $\pi$ is a homeomorphism. Hence $o(a/AB) \subseteq_* o(a/A) \iff \pi^{-1}[o(a/AB)] \subseteq_* \pi^{-1}[o(a/A)] \iff G_{AB}/G_{Aa} \subseteq_* G_A/G_{Aa}$. We finish using Proposition 1.3. ■

**Corollary 1.13** *In every compact structure $o$-independence, $nm$-independence and $m$-independence coincide.*

In Section 3 (Remark 3.5) we will see that the pseudo-arc considered with the group of all homeomorphisms is an example of a small Polish structure where $o$-independence and $nm$-independence differ. Below we give a simpler example of a Polish structure in which these two notions differ, but this structure is not small.

**Example 1** Let $X = (S^1)^\omega$ and $G = (Homeo(S^1))^\omega$. We consider the Polish structure $(X, G)$ where $G$ acts naturally on $X$ on the appropriate coordinates. Then in $(X, G)$ $o$-independence is different from $nm$-independence. Moreover, $(X, G)$ is not small, it does not have the existence of $o$-independent extensions but it satisfies the existence of $nm$-independent extensions.

*Proof.* Take any $\overline{x} = \langle x_0, x_1, \ldots\rangle \in X$ and $\overline{y} = \langle y_0, y_1, \ldots\rangle \in X$ such that $x_i \neq y_i$ for all $i \in \omega$. Then $o(\overline{x}) = X$ and $o(\overline{x}/\overline{y}) = S^1 \setminus \{y_0\} \times S^1 \setminus \{y_1\} \times \ldots$. So $o(\overline{x}/\overline{y})$ is non-meager and not open in $o(\overline{x})$. Since $o(\overline{x}) = X$ is Polish, by Theorem 1.12 we get that $\overline{x} \mathop{\smile\!\!\!\!\!|}^{nm} \overline{y}$ and $\overline{x} \not\mathop{\smile\!\!\!\!\!|}^{o} \overline{y}$. The 'moreover' part is left to the reader. ■

## 2 Basic model theory

In this section we introduce counterparts of some basic notions from stability theory and we investigate their properties.

In compact [profinite] structures definable sets were defined as the sets which are closed and invariant over finite subsets. Since for a Polish structure $(X, G)$ we do not have any topology on $X$, we need another definition of definable sets. Moreover, as in model theory we would like to have (imaginary) names for definable sets for which forking calculus works in the same way as for the real elements. So we propose the following definition of definable sets and imaginary elements.

From now on we assume that $(X, G)$ is a Polish structure.

**Definition 2.1** *The imaginary extension, denoted by $X^{eq}$, is the union of all sets of the form $X^n/E$ with $E$ ranging over all invariant equivalence relations such that for all $a \in X^n$, $Stab([a]_E) <_c G$. The sets $X^n/E$ will be called the sorts of $X^{eq}$.*

**Remark 2.2** *Let $E$ be an invariant equivalence relation on $X^n$ whose classes have closed stabilizers in $G$. Then $G$ induces a group of permutations of $X^n/E$, denoted by $G \restriction X^n/E$, which is Polish, and $(X^n/E, G \restriction X^n/E)$ is a Polish structure.*

As in model theory, $(X^{eq})^{eq} = X^{eq}$ which means that if $E$ is an invariant equivalence relation on a product of sorts $X^{n_1}/E_1 \times \ldots \times X^{n_k}/E_k$ whose classes have closed stabilizers in G, then the set of $E$-classes can be identified with the sort $X^{n_1} \times \ldots \times X^{n_k}/E'$ where

$$E'(x_1, \ldots, x_k; y_1, \ldots, y_k) \iff E(x_1/E_1, \ldots, x_k/E_k; y_1/E_1, \ldots, y_k/E_k).$$

**Definition 2.3** *A subset $D$ of $X$ (or more generally of any sort of $X^{eq}$) is said to be definable over a finite subset $A$ of $X^{eq}$ if $D$ is invariant over $A$ and $Stab(D) <_c G$. We say that $d \in X^{eq}$ is a name for $D$ if for every $f \in G$ we have $f[D] = D \iff f(d) = d$.*

**Proposition 2.4** *Each set definable in $(X, G)$ [or in $X^{eq}$] has a name in $X^{eq}$.*

*Proof.* The proof is similar to the proof of Proposition 1.9 in [7]. Suppose $D$ is $a$-definable for some finite $a \in X^n$. We define an equivalence relation $E$ on $X^n$ by:

$$E(a_1, a_2) \iff [a_1 = a_2 \vee (a_1, a_2) \in S(a, a)]$$

where $S = \{(f, g) \in G \times G : f[D] = g[D]\}$. It is easy to check that $E$ is invariant, every class of $E$ has a closed stabilizer and $a/E$ is a name for $D$. ■

Working in $X^{eq}$, we can define $Acl^{eq}$ in the same way as in $X$. Then the results of Section 1 (including Theorem 1.5 and 1.10) are true in $X^{eq}$ (the only exception is the fact that the Polish structure $(X, G)$ considered in Example 1 does not satisfy the existence of $nm$-independent extensions in $X^{eq}$). However, in the case of a compact [profinite] structure $(X, G)$ both the family of definable sets and $X^{eq}$ computed according to the definitions given in this paper are larger than those computed according to the definitions from [13, 14, 7].

As usual, having a notion of independence satisfying the properties listed in Theorem 1.5 and 1.10, one can define a rank, which has some nice properties.

**Definition 2.5** *The rank $\mathcal{NM}$ is the unique function from the collection of orbits over finite sets to the ordinals together with $\infty$, satisfying*

$$\mathcal{NM}(a/A) \geq \alpha + 1 \quad \textit{iff} \quad \textit{there is a finite set } B \supseteq A \textit{ such that } a \not\mathop{\smile}\limits^{nm}_A B \textit{ and } \mathcal{NM}(a/B) \geq \alpha.$$

From now on we assume that $(X, G)$ is small. In fact, more generally, one could only assume that $(X, G)$ satisfies the existence of $nm$-independent extensions and, if we work in $X^{eq}$, one should also assume that the existence of $nm$-independent extensions holds in $X^{eq}$.

The results formulated below follow from a standard forking calculation (e.g. see [16, Lemma 5.1.4 and Theorem 5.1.6]). In the next proposition $a, b, A$ are finite tuples (subsets) of $X$ or $X^{eq}$.

**Proposition 2.6** *(1) $a \mathop{\smile}\limits^{m}_A b$ iff $\mathcal{N}M(a/Ab) = \mathcal{N}M(a/A)$.*
*(2) $\mathcal{N}M(a/bA) + \mathcal{N}M(b/A) \leq \mathcal{N}M(ab/A) \leq \mathcal{N}M(a/bA) \oplus \mathcal{N}M(b/A)$.*
*(3) Suppose $\mathcal{N}M(a/Ab) < \infty$ and $\mathcal{N}M(a/A) \geq \mathcal{N}M(a/Ab) \oplus \alpha$. Then $\mathcal{N}M(b/A) \geq \mathcal{N}M(b/Aa) + \alpha$.*
*(4) Suppose $\mathcal{N}M(a/Ab) < \infty$ and $\mathcal{N}M(a/A) \geq \mathcal{N}M(a/Ab) + \omega^\alpha n$. Then $\mathcal{N}M(b/A) \geq \mathcal{N}M(b/Aa) + \omega^\alpha n$.*
*(5) If $a \mathop{\smile}\limits^{m}_A b$, then $\mathcal{N}M(ab/A) = \mathcal{N}M(a/bA) \oplus \mathcal{N}M(b/A)$.*

As in stable or simple theories, the inequalities in the point (2) will be called Lascar inequalities.

An easy induction and Proposition 2.6(1) yield the following remark.

**Remark 2.7** *Let $a, A \subseteq X$ be finite. Then $\mathcal{N}M$-rank of $o(a/A)$ computed in $X$ is the same as $\mathcal{N}M$-rank of $o(a/A)$ computed in $X^{eq}$.*

**Definition 2.8** $(X, G)$ *is nm-stable if every 1-orbit has ordinal $\mathcal{N}M$-rank.*

**Remark 2.9** $(X, G)$ *is nm-stable iff there is no infinite sequence $A_0 \subseteq A_1 \subseteq \ldots$ of finite subsets of $X$ and $a \in X$ such that $a \not\mathop{\smile}\limits^{nm}_{A_i} A_{i+1}$ for every $i \in \omega$.*

By Lascar inequalities and Remark 2.7 we easily get:

**Remark 2.10** $(X, G)$ *is nm-stable iff each $n$-orbit, $n \geq 1$, has ordinal $\mathcal{N}M$-rank iff each $n$-orbit, $n \geq 1$, in $X^{eq}$ has ordinal $\mathcal{N}M$-rank.*

**Proposition 2.11** *TFAE:*

*(1) $(X, G)$ is nm-stable.*

*(2) There is no $a \in X$ and finite sets $A_0 \subseteq A_1 \subseteq \ldots \subseteq X$ such that $G_{A_{i+1}} G_{A_i a}$ is meager in $G_{A_i}$ for every $i \in \omega$.*

*(3) For every $a \in X$ and finite sets $A_0 \subseteq A_1 \subseteq \ldots X$ there is $n \in \omega$ such that $G_{A_{n+i+1}} G_{A_n a} \subseteq_{nm} G_{A_{n+i}} G_{A_n a}$ for every $i \in \omega$.*

*Proof.* (1) ⇔ (2) is obvious by Proposition 1.3 and Remark 2.9.
(1) ⇒ (3). By Proposition 1.3, Remark 2.9 and transitivity we can find $n \in \omega$ such that $G_{A_{n+i+1}} G_{A_{n+i}a} \subseteq_{nm} G_{A_{n+i}}$ for every $i \in \omega$. So by Corollary 1.9(ii) and induction we easily get that $G_{A_{n+i+1}} G_{A_n a} \subseteq_{nm} G_{A_{n+i}} G_{A_n a}$ for every $i \in \omega$.
(3) ⇒ (1). Take any $a \in X$ and finite sets $A_0 \subseteq A_1 \subseteq \ldots \subseteq X$. By (3) there is $n \in \omega$ such that $G_{A_{n+i+1}} G_{A_n a} \subseteq_{nm} G_{A_{n+i}} G_{A_n a}$ for every $i \in \omega$. Hence we easily get that $G_{A_{n+i}} G_{A_n a} \subseteq_{nm} G_{A_n}$ for every $i \in \omega$. So $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_{A_n} A_{n+i+1}$, and hence $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_{A_{n+i}} A_{n+i+1}$, for every $i \in \omega$. We are done by Remark 2.9. ■

As it was mentioned in the introduction I hope that $nm$-independence, $\mathcal{NM}$-rank and maybe some other model theoretic notions may be useful in descriptive set theory or topology. It is worth mentioning here that by Theorem 1.12, if orbits are non-megear in their relative topologies, then $\mathcal{NM}$-rank and $nm$-stability can be expressed in terms of $X$ instead of $G$, and $\mathcal{NM}$-rank measures a 'topological size' of orbits. More precisely, Theorem 1.12 gives us the following remark.

**Remark 2.12** *Suppose that $X$ is a separable metrizable space and $G$ acts continuously on $X$. Assume that all orbits over finite sets are non-meager in their relative topologies. Then*

*(1) $\mathcal{NM}(a/A) \geq \alpha + 1$ iff there is a finite set $B \supseteq A$ such that $o(a/B)$ is meager in $o(a/A)$ and $\mathcal{NM}(a/B) \geq \alpha$.*

*(2) $(X, G)$ is nm-stable iff there is no infinite sequence $A_0 \subseteq A_1 \subseteq \ldots$ of finite subsets of $X$ and $a \in X$ such that $o(a/A_{i+1})$ is meager in $o(a/A_i)$ for every $i \in \omega$.*

By Remark 2.12 we see that in compact structures $\mathcal{NM}$-rank and $nm$-stability coincide with $\mathcal{M}$-rank and $m$-stability, respectively.

Similarly as in model theory and in profinite structures one can define a natural pregeometry on an orbit of $\mathcal{NM}$-rank 1. To introduce this definition first we need to prove several remarks.

**Remark 2.13** *For any finite $a, A \subseteq X^{eq}$ we have that $\mathcal{NM}(a/A) = 0$ iff $a \in Acl^{eq}(A)$.*

*Proof.* (⇒). Assume $\mathcal{NM}(a/A) = 0$. Then $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_{A} B$ for every finite $B \subseteq X^{eq}$. By Theorem 1.5(3) we get $a \in Acl^{eq}(A)$.
(⇐). Assume $a \in Acl^{eq}(A)$. By Theorem 1.5(3) we get that $a \mathop{\smile\!\!\!\!\!\!|}^{nm}_{A} B$ for every finite $B \subseteq X^{eq}$. Hence $\mathcal{NM}(a/A) = 0$. ■

**Corollary 2.14** *If $o(a/A)$ is Polish, then $\mathcal{NM}(a/A) = 0$ iff $o(a/A)$ is countable with the discrete topology.*

For a finite set $A \subseteq X^{eq}$ we define the operator $Acl^{eq}_A$ by $Acl^{eq}_A(B) = Acl^{eq}(A \cup B)$.

**Remark 2.15** *Assume $\mathcal{NM}(a/A) = 1$ and $B$ is a finite subset of $X^{eq}$. Then $a \in Acl^{eq}_A(B)$ iff $a \not\mathop{\smile\!\!\!\!\!\!|}^{nm}_{A} B$.*

*Proof.* (⇒). Assume $a \not\mathop{\smile\!\!\!\!\!\!|}^{nm}_{A} B$. Then $\mathcal{NM}(a/AB) < \mathcal{NM}(a/A) = 1$. Hence $\mathcal{NM}(a/AB) = 0$. By Remark 2.13 we get $a \in Acl^{eq}_A(B)$.
(⇐). Assume $a \in Acl^{eq}_A(B)$. By Remark 2.13, $\mathcal{NM}(a/AB) = 0 < \mathcal{NM}(a/A)$. Hence by Proposition 2.6(1), $a \not\mathop{\smile\!\!\!\!\!\!|}^{nm}_{A} B$. ■

**Proposition 2.16** *Assume $\mathcal{NM}(a/A) = 1$. Then $(o(a/A), Acl_A^{eq})$ is a pregeometry.*

*Proof.* The proof is the same as in model theory. Let us only check the exchange property. Take any $B \subseteq o(a/A)$ and $b \in o(a/A)$. Consider any $a' \in Acl_A^{eq}(Bb) \setminus Acl_A^{eq}(B)$. Wlog $B$ is finite. By Remark 2.15, $a' \not\downarrow^{nm}_A ABb$ and $a' \downarrow^{nm}_A AB$. Hence by Theorem 1.5, $b \not\downarrow^{nm}_{AB} a'$. So $b \not\downarrow^{nm}_A Ba'$, which implies that $b \in Acl_A(Ba')$. ■

Similarly, one can show that if $\mathcal{NM}(X) := sup\{\mathcal{NM}(a) : a \in X\} = 1$, then $(X, Acl)$ is a pregeometry.

# 3 Examples

In this section we give several examples of small Polish structures. To begin with, notice that all small profinite structures are such examples. This class contains for instance all abelian profinite groups of finite exponent presented as the inverse limit of a system indexed by the natural numbers and considered with the standard structural group [5, Theorem 1.9]. Below we give examples which are not profinite.

**Example 2** For every $n \geq 1$ the Polish structure $(S^n, Homeo(S^n))$ where $S^n$ is the $n$-dimensional sphere and $Homeo(S^n)$ is the group of all homeomorphisms of $S^n$ with the compact-open topology is small of $\mathcal{NM}$-rank 1. Similarly, $(S^n, Diff(S^n))$ where $Diff(S^n)$ is the group of all diffeomorphisms of $S^n$ is also small of $\mathcal{NM}$-rank 1.

**Example 3** The Polish structure $(I^\omega, Homeo(I^\omega))$ where $I^\omega$ is the Hilbert cube is small of $\mathcal{NM}$-rank 1.

*Proof.* By [11, Section 6.1, Exercise 2] we know that the action is $n$-transitive for every $n \geq 1$. ■

**Example 4** The Polish structure $(P, Homeo(P))$ where $P$ is the pseudo-arc is small and not $nm$-stable.

Before the proof we recall some notions and results about continua. Recall that the pseudo-arc $P$ is the unique hereditarily indecomposable chainable continuum. By hereditary idecomposability we get that for every $A \subseteq P$ the intersection of all subcontinua of $P$ containing $A$ is the smallest subcontinuum of $P$ containing $A$.

Let $C$ be any nondegenerate continuum (e.g. $P$). We say that $C$ is irreducible between subsets $A$ and $B$ if there is no proper subcontinuum of $C$ containing $A$ and $B$. For $p \in C$ we define the composant of $p$, denoted by $\kappa(p)$, as the set of the points $x \in C$ for which there is a proper subcontinuum $A$ of $C$ such that $p, x \in A$.

For every $p \in C$, $\kappa(p)$ is the union of countably many proper subcontinua of $C$ containing $p$ [12, Proposition 11.14] and $\kappa(p)$ is dense and connected [12, Exercise 5.20]. Moreover, by [12, Exercise 6.19] $C$ is indecomposable iff every proper subcontinuum of $C$ is nowhere dense in $C$. Hence

(∗) If $C$ is indecomposable, then $\kappa(p)$ is meager and dense in $C$ for every $p \in C$.

If $C$ is indecomposable, we define an equivalence relation $E$ on $C$ by:

$$E_C(x, y) \iff y \in \kappa(x).$$

By $(*)$ and the Baire category theorem $E_C$ has uncountably many classes (they are just composants). We will need the following two facts, which can be found in [9, Theorem 2, Theorem 6].

**Fact 3.1** *The pseudo-arc is homeomorphic to each of its nondegenerate subcontinua.*

**Fact 3.2** *Suppose $H_{1,1}, H_{1,2}, \ldots, H_{1,n}$ are proper subcontinua of the pseudo-arc $P$ and that $P$ is irreducible between each pair of them. Suppose $T$ is a homeomorphism of $H_{1,1} \cup H_{1,2} \cup \ldots \cup H_{1,n}$ onto $H_{2,1} \cup H_{2,2} \cup \ldots \cup H_{2,n}$ where $H_{2,1}, H_{2,2}, \ldots, H_{2,n}$ are proper subcontinua of $P$ such that $P$ is irreducible between each pair of them. Then $T$ can be extended to a homeomorphism of $P$ onto $P$.*

Now we can prove the following Lemma which immediately implies that the Polish structure $(P, Homeo(P))$ is small.

**Lemma 3.3** *For every $n \geq 1$, there are only finitely many orbits on $P^n$ under the action of $Homeo(P)$.*

*Proof.* The proof is by induction on $n$. The case $n = 1$ follows from the fact that the pseudo-arc is homogeneous.

Assume that Lemma 3.3 is true for the tuples of length $< n$. Suppose for a contradiction that there are infinitely many $n$-tuples $t_0 = (t_0^0, \ldots, t_0^{n-1}), t_1 = (t_1^0, \ldots, t_1^{n-1}), \ldots$ lying in different orbits under $Homeo(P)$. Then there is an infinite subsequence $t_{i_0}, t_{i_1}, \ldots$ such that for every $j, k \in \omega$, the tuples $t_{i_j}$ and $t_{i_k}$ are isomorphic, via $f_{jk}$, with respect to the relation $E_P$. Wlog $i_j = j$ for every $j \in \omega$. Then there is $m \leq n-1$ such that for every $i \in \omega$, $\{t_i^0, \ldots, t_i^{n-1}\}$ can be partitioned into classes modulo $E_P$, say $A_i^0, \ldots, A_i^m$, so that $f_{jk}[A_j^l] = A_k^l$ for every $j, k \in \omega$ and $l \leq m$. Let $B_i^0, \ldots, B_i^m$ be the smallest proper subcontinua of $P$ containing $A_i^0, \ldots, A_i^m$, respectively. Then $B_i^0, \ldots, B_i^m$ are pairwise disjoint proper subcontinua of $P$ such that $P$ is irreducible between each pair of them.

**Case 1** $m > 1$. By Fact 3.1 for every $i \in \omega$ we can find homeomorphisms $f_i^l : B_i^l \to B_0^l$, $l = 0, \ldots, m$. By the inductive hypothesis and Fact 3.1 there are $i \neq j$ for which there exist homeomorphisms $g_l : B_0^l \to B_0^l$, $l = 0, \ldots, m$, such that

$$T := (f_j^0)^{-1} g_0 f_i^0 \cup \ldots \cup (f_j^m)^{-1} g_m f_i^m : B_i^0 \cup \ldots \cup B_i^m \to B_j^0 \cup \ldots \cup B_j^m$$

is a homeomorphism extending $f_{ij}$. Since both $\{B_i^0, \ldots, B_i^m\}$ and $\{B_j^0, \ldots, B_j^m\}$ are collections of proper subcontinua of $P$ such that $P$ is irreducible between each pair of them, by Fact 3.2, $T$ can be extendend to an element of $Homeo(P)$. Since $T(t_i) = t_j$, we get a contradiction.

**Case 2** $m = 1$. Then $B_i^0$, $i \in \omega$, are proper subcontinua of $P$, so by Fact 3.1 we can find homeomorphisms $f_i : B_i^0 \to B_0^0$. Let $t_i' = f_i(t_i)$. If we show that there exist $i \neq j$ and a homeomorphism $f : B_0^0 \to B_0^0$ with $f(t_i') = t_j'$, then $T := f_j^{-1} f f_i$ is a homeomorphism from $B_i^0$ onto $B_j^0$ such that $T(t_i) = t_j$, and we finish using Fact 3.2. By the minimality of $B_i^0, i \in \omega$, working with $B_0^0$ and $t_0', t_1', \ldots$, we are in the situation described in Case 1, so the proof is completed. ■

**Lemma 3.4** $(P, Homeo(P))$ *is not nm-stable.*

*Proof.* Take any $p \in P$. By $(*)$ we can find $p_0 \in \kappa(p)$ such that $p_0 \neq p$. Now choose the smallest proper subcontinuum $P_0$ of $P$ containing $p$ and $p_0$. Then $P_0$ is also the pseudo-arc and it is nowhere dense in $P$. We see that $P_0$ is definable over $\{p, p_0\}$. Let $q_0$ be a name of $P_0$. By Fact 3.2, $o(p/q_0) = P_0$, which is nowhere dense in $o(p) = P$. Since $P$ is Polish, Theorem 1.12 implies that $p \mathop{\not\perp}^{nm} q_0$.

Let us repeat this step within $P_0$. By $(*)$ we can find $p_1 \in \kappa_{P_0}(p)$ such that $p_1 \neq p$, where $\kappa_{P_0}$ is $\kappa$ computed within $P_0$. Now choose the smallest proper subcontinuum $P_1$ of $P_0$ containing $p$ and $p_1$. Then $P_1$ is also the pseudo-arc and it is nowhere dense in $P_0$. We see that $P_1$ is definable over $\{p, p_0, p_1\}$. Let $q_1$ be a name of $P_1$. By Fact 3.2, $o(p/q_0q_1) = P_1$, which is nowhere dense in $o(p/q_0) = P_0$. Since $P_0$ is Polish, Theorem 1.12 implies that $p \mathop{\not\perp}^{nm}_{q_0} q_1$.

We repeat this procedure and obtain an infinite sequence of imaginaries $q_0, q_1, \ldots$ such that $p \mathop{\not\perp}^{nm}_{q_{<i}} q_{\leq i}$ for all $i \in \omega$. By Remark 2.9 and 2.10 the proof is completed. ■

It is not clear how to repeat the above proof without using imaginaries. More precisely, by Fact 3.1, 3.2 and $(*)$ we have that $o(p/p_0) = \kappa(p)$ is meager in $o(p) = P$ and $P$ is Polish, so by Theorem 1.12, $p \mathop{\not\perp}^{nm} p_0$. However, by $(*)$ and Effros' theorem, $o(p/p_0) = \kappa(p)$ is meager in its relative topology, so starting from this point we cannot work just with orbits, but we should look at their preimages in $Homeo(S)$, which is rather complicated.

**Remark 3.5** *(i)* $(P, Homeo(P))$ *does not satisfy the existence of o-independent extensions.*
*(ii) In* $(P, Homeo(P))$ *the relations* $\perp^{o}$ *and* $\perp^{nm}$ *are different.*

*Proof.* (i) Since $P$ is homogeneous, by Theorem 1.12 it is enough to show that there is $p \in P$ such that every orbit over $p$ is not open in $P$. Take any $p \in P$. Using Fact 3.1 and 3.2 we easily get that the only orbits over $p$ are $\kappa(p)$ and $P \setminus \kappa(p)$, so we are done by $(*)$.
(ii) It follows from (i), Lemma 3.3 and Theorem 1.10. We can also see it directly. Take any $p \in P$ and $q \notin \kappa(p)$. By Fact 3.2, $o(q/p) = P \setminus \kappa(p)$. So by $(*)$ we get that $o(q/p)$ is non-meager in $P$, and hence, by Theorem 1.12, $q \perp^{nm} p$.

On the other hand, by $(*)$ we see that $P \setminus \kappa(p)$ is not open in $P$, and hence, by Theorem 1.12, $q \mathop{\not\perp}^{o} p$. ■

**Definition 3.6** *(i) A Polish group structure is a Polish structure* $(H, G)$ *such that* $H$ *is a group and* $G$ *acts as a group of automorphisms of* $H$*.*
*(ii) A Polish group regarded as a Polish structure is a Polish structure* $(H, G)$ *such that* $H$ *is a Polish group and* $G$ *acts continuously as a group of automorphisms of* $H$*.*

Of course every Polish group regarded as a Polish structure is a Polish group structure. The next example is an example of a small Polish group regarded as a Polish structure.

**Example 5** Let us consider the discrete topology on $\mathbb{Q}$ and the product topology on $\mathbb{Q}^\omega$. We consider the additive group structure on $\mathbb{Q}$. Let $Aut^0(\mathbb{Q}^\omega)$ be the group of all automorphisms of $\mathbb{Q}^\omega$ respecting the inverse system $\mathbb{Q} \longleftarrow \mathbb{Q} \times \mathbb{Q} \longleftarrow \ldots$. Then $Aut^0(\mathbb{Q}^\omega)$ can be considered as the inverse limit of the system consisting of $Aut^0(\mathbb{Q}^n)$, $n \in \omega$, where on $Aut^0(\mathbb{Q}^n)$ we have the pointwise convergence topology. Then $(\mathbb{Q}^\omega, Aut^0(\mathbb{Q}^\omega))$ is a small of $\mathcal{NM}$-rank 1 Polish group regarded as a Polish structure.

*Proof.* We leave to the reader checking that $(\mathbb{Q}^\omega, Aut^0(\mathbb{Q}^\omega))$ is a Polish group regarded as a Polish structure. Now we will show that it is small of $\mathcal{NM}$-rank 1.

The following claim is obvious.

**Claim** If $f \in Aut^0(\mathbb{Q}^n)$ and $\{\eta_1, \ldots, \eta_n\}$ is a basis of $\mathbb{Q}^n$ over $\mathbb{Q}$, then for all $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \epsilon'_0, \epsilon'_1, \ldots, \epsilon'_n \in \mathbb{Q}$ such that $\epsilon_0, \epsilon'_0 \neq 0$, there is $\overline{f} \in Aut^0(\mathbb{Q}^{n+1})$ such that $\overline{f}(0^\frown\epsilon_0) = 0^\frown\epsilon'_0$ and $\overline{f}(\eta_i^\frown\epsilon_i) = f(\eta_i)^\frown\epsilon'_i$, $i = 1, \ldots, n$.

Now consider any finite set $A \subseteq \mathbb{Q}^\omega$ and $a \in \mathbb{Q}^\omega$. Then either $a \in Lin(A)$ or there is the largest natural number $n$ such that $a{\restriction}n \in Lin(A){\restriction}n$. So by Claim we get that either $a \in Lin(A)$, and then $o(a/A) = \{a\}$, or $a \notin Lin(A)$, and then $o(a/A) = \{\eta \in \mathbb{Q}^\omega : \eta{\restriction}n = a{\restriction}n \wedge \eta{\restriction}(n+1) \notin Lin(A){\restriction}(n+1)\}$ where $n$ is the largest $n$ such that $a{\restriction}n \in Lin(A){\restriction}n$. So each orbit over $A$ is either a singleton from $Lin(A)$ or an open set. Hence there are countably many orbits over $A$, so $(\mathbb{Q}^\omega, Aut^0(\mathbb{Q}^\omega))$ is small; we also see that $\mathcal{NM}(\mathbb{Q}^\omega) = 1$. ■

We end up with a remark which follows from Lascar inequalities and yields examples of small Polish structures of arbitrary finite $\mathcal{NM}$-rank.

**Remark 3.7** *If $(X, G)$ is a small Polish structure of $\mathcal{NM}$-rank 1, then for every natural number $n \geq 1$, $(X^n, G)$ is a small Polish structure of $\mathcal{NM}$-rank $n$.*

# 4 Final comments and questions

In model theory there are results, known as group configuration theorems, which say that under some general geometric assumptions one can find a definable group (e.g. [15, Chapter 5, 7]). Such theorems were also proved for small profinite structures [14, Theorem 1.7, Theorem 3.3] or, more generally, for compact structures satisfying the existence of $m$-independent extensions [7, Theorem 3.15]. Considerations concerning the existence of a definable (in some sense) group structure are the best example illustrating how stability theory ideas may lead to new aspects in the analysis of classical topological objects. In the context of small Polish structures it is not clear how to prove such kind of results. We can ask here the following general questions.

**Question 4.1** *Suppose $(X, G)$ is a small Polish structure. When is there a function $\cdot : X \times X \to X$ definable (or invariant) over a finite subset $A$ of $X$ such that $(X, \cdot)$ is a group?*

**Question 4.2** *Suppose $(X, G)$ is a small Polish structure. When is there an infinite set $Y \subseteq X$ and a function $\cdot : Y \times Y \to Y$, both definable (or invariant) over a finite subset $A$ of $X$, such that $(Y, \cdot)$ is a group?*

The group configuration theorem for small profinite structures [14, Theorem 1.7] yields a partial answer to Question 4.2. Namely, if we additionally assume that $(X, G)$ is profinite and $m$-normal, then in every non-trivial orbit of $\mathcal{M}$-rank 1 there is an open definable group.

So one of the possible ways of further research is to prove for small Polish structures counterparts of some advanced results from geometric stability theory, e.g. a variant of the group configuration theorem.

There are also certain open questions about the existence of small profinite structures satisfying some additional assumptions. I think it would be interesting to find counterexamples for these questions in the bigger class of small Polish structures. Let us formulate two such questions.

**Conjecture 4.3 ($\mathcal{NM}$-gap conjecture)** *Let $(X, G)$ be a small Polish structure. Then for every orbit $o$ over a finite subset $A$ of $X$ one has $\mathcal{NM}(o) \in \omega \cup \{\infty\}$.*

This conjecture is open in the class of small profinite structures; it was proved only for small $m$-stable profinite groups [17, Theorem 18]. In the context of small Polish structures it is open even for small $nm$-stable Polish groups.

**Question 4.4** *Is it true that every small Polish group regarded as a Polish structure or, more generally, every small Polish group structure, is abelian-by-countable?*

In the case of small profinite structures the appropriate question asks if every small profinite group is abelian-by-finite. This question is open in general but it was answered positively for small $m$-stable profinite groups [17, Theorem 1]. Question 4.4 is open even for small $nm$-stable Polish groups.

Let us notice here that all the structural results about small profinite groups were based on the fact that such groups are locally finite [13, Proposition 2.4], which is not true for small Polish groups [Example 5]. So an interesting general question is what we can prove about small Polish groups regarded as Polish structures.

At the end I would like to formulate several purely descriptive set theoretic (or topological) facts and questions which came up naturally during my considerations on Polish structures.

The following fact is Corollary 2.6.8 of [1].

**Fact 4.5** *The Polish $Homeo(I^\omega)$-space $(I^\omega, Homeo(I^\omega))$ is universal for Borel $G$-spaces, i.e. every Borel $G$-space $(X, G)$ can be embedded into $(I^\omega, Homeo(I^\omega))$ in the sense that there is a topological isomorphism $\psi : G \to \psi[G] <_c Homeo(I^\omega)$ and a Borel embedding $\phi : X \to I^\omega$ such that $\phi(gx) = \psi(g)\phi(x)$ for every $g \in G$.*

We know that this universal Borel $G$-space is a small Polish structure [Example 3]. The following question is a counterpart of the above fact for groups.

**Question 4.6** *Does there exist a Polish $G$-group $(U, G)$ which is universal for Borel $H$-groups (i.e. every Borel $H$-group $(G_0, H)$ can be embedded into $(U, G)$, preserving the group structure on $G_0$)? If yes, is $(U, G)$ a small Polish group regarded as a Polish structure?*

From now on we fix a Polish group $G$. The following fact comes from [2, Theorem 0.3] and [1, Corollary 2.6.8].

**Fact 4.7** *There is a Polish $G$-space which is universal for Polish [Borel] $G$-spaces.*

**Question 4.8** *Does there exist a Polish $G$-group which is universal for Polish [Borel] $G$-groups?*

It is obvious that any universal Polish $G$-space (or $G$-group, if it exists) for a fixed $G$ is not small. However, we can ask the following question (or its variants for Polish structures which are additionally Polish or Borel $G$-spaces or $G$-groups).

**Question 4.9** *Does there exist a small Polish structure [group structure] which is universal for small Polish structures [group structures] of the form $(X, G)$?*

At the very end we would like to formulate a question concerning small profinite structures. The following is a well-known fact [10].

**Fact 4.10** *If $G$ is a locally compact group and $(X, G)$ is a topological $G$-space, then there is a topological group $H \supseteq X$ such that the topology on $X$ is inherited from $H$ and the action of $G$ on $X$ can be extended to an action on $H$ so that $(H, G)$ is a topological $G$-group.*

**Question 4.11** *Is it true that for every small profinite structure $(X, G)$ one can find a profinite group $H \supseteq X$ such that $X$ is closed in $H$, its topology is inherited from $H$ and the action of $G$ on $X$ can be extended to an action on $H$ so that $(H, G)$ is a small profinite group?*

It is not difficult to construct $H$ so that $(H, G)$ is not necessarily small. Namely, if

$$X = \varprojlim X_i,$$

we can define $H$ as the inverse limit of the system consisting of the linear spaces spanned freely by $X_i$ over the two element field $F_2$.

**Proposition 4.12** *If Question 4.11 has the positive answer, then $\mathcal{M}$-gap conjecture is true for small $m$-stable profinite structures.*

*Proof.* Suppose for a contradiction that $(X, G)$ is a small $m$-stable profinite structure with an orbit $o$ over some finite set $A$ such that $\mathcal{M}(o) \in Ord \setminus \omega$; hence wlog $\mathcal{M}(o) = \omega$. By the assumption we have that $X$ is $\emptyset$-definable (i.e. closed and invariant) in a small profinite group $(H, G)$.

By [13, Proposition 2.3] the subgroup $\langle X \rangle$ generated by $X$ is generated in finitely many steps. This easily implies that $(\langle X \rangle, G)$ is a small $m$-stable profinite group. Hence the $\mathcal{M}$-rank of $o$ computed within $(\langle X \rangle, G)$ also equals $\omega$, and we get a contradiction with the fact that $\mathcal{M}$-gap conjecture holds for small $m$-stable profinite groups [17, Theorem 18]. ■

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