A NOTE ON SUBVARIETIES OF POWERS OF OT-MANIFOLDS

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Abstract. It is shown that the space of finite-to-finite holomorphic correspondences on an OT-manifold is discrete. When the OT-manifold has no proper infinite complex-analytic subsets, it then follows by known model-theoretic results that its cartesian powers have no interesting complex-analytic families of subvarieties. The methods of proof, which are similar to [Moosa, Moraru, and Toma "An essentially saturated surface not of Kähler-type", *Bull. of the LMS*, 40(5):845–854, 2008], require studying finite unramified covers of OT-manifolds.

1. introduction

This note is concerned with complex-analytic families of subvarieties in cartesian powers of the compact complex manifolds introduced by Oeljeklaus and the second author in [7], here referred to as OT-manifolds. These manifolds are higher dimensional analogues of Inoue surfaces of type S_M. In [4], we, along with Ruxandra Moraru, showed that if X is an Inoue surface of type S_M then X^n contains no infinite complex-analytic families of subvarieties, except for the obvious ones such as $\big(\{a\}\times V : a\in X^m\big)$ where V is a fixed subvariety of X^{n-m}. Using model-theoretic techniques we were able to reduce the problem to considering only the case of $n=2$. That case amounted to showing that the set of finite-to-finite holomorphic correspondences on X, viewed as subvarieties of X^2, is discrete. Here we extend this result to OT-manifolds in general. Actually, it is useful to consider the following higher arity version of correspondences: for any compact complex manifold X, let $\mathrm{Corr}_n(X)$ denote the set of irreducible complex-analytic $S\subset X^n$ such that the co-ordinate projections $\mathrm{pr}_i : S\to X$ are surjective and finite for all $i=1,\dots,n$. So $\mathrm{Corr}_2(X)$ is the set of finite-to-finite holomorphic correspondences.[1]

Theorem 1. *If X is an* OT-*manifold then* $\mathrm{Corr}_n(X)$ *is discrete for all* $n>0$.

The proof, which we will give in Section 3, follows to some extent what was done for Inoue surfaces of type S_M in [4]. But this approach leads naturally to the consideration of finite unramified coverings of OT-manifolds, and the latter are not formally instances of the original construction in [7]. However, we show in Section 2 that a mild generalisation of that construction leads to a class of manifolds which is

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[1]It may be worth pointing out that the elements of $\mathrm{Corr}_n(X)$ are simply components of intersections of pull-backs of finite-to-finite holomorphic correspondences. That is, for $n>1$, if $S\in\mathrm{Corr}_n(X)$ and $\pi_i : X^n\to X^2$ is the co-ordinate projection $(x_1,\dots,x_n)\mapsto(x_1,x_i)$, for $i=2,\dots,n$, then each $\pi_i(S)\subset X^2$ is a correspondence and S is an irreducible component of $\bigcap_{i=2}^{n}\pi_i^{-1}(\pi_i(S))$. This is an easy dimension calculation, see [5, Lemma 3.2].

closed under finite unramified coverings. We call these manifolds also OT-manifolds and the theorem is valid for this larger class.

The theorem is particularly significant when X has no proper positive dimensional subvarieties, because of the following fact coming from model theory.

Fact 2. *Suppose X is a compact complex manifold that is not an algebraic curve, is not a complex torus, and has no proper infinite complex-analytic subsets. Then every irreducible complex-analytic subset of a cartesian power of X is a cartesian product of points and elements of $\mathrm{Corr}_n(X)$ for various $n > 0$.*

Proof. This is Proposition 5.1 of [9] together with Lemma 3.3(b) of [5]. □

That OT-manifolds without proper positive dimensional subvarieties are ubiquitous in all dimensions follows from work of Ornea and Verbitsky [8] showing that we get examples whenever X is the OT-manifold corresponding to a number field that has precisely two complex embeddings which are not real.

Putting together the Theorem and the Fact, we conclude:

Corollary 3. *Suppose X is an* OT*-manifold that has no proper infinite complex-analytic subsets. Then, for all $n > 0$, X^n has no infinite complex-analytic families of subvarieties that project onto each co-ordinate.*

Remark 4. The model theorist should note that for X to have no proper infinite complex-analytic subsets is exactly *strong minimality* of X as a first-order structure in the language of complex-analytic sets. Strongly minimal OT-manifolds are of *trivial* acl-*geometry* by the manifestation of the Zilber trichotomy in this context. By [5, Proposition 3.5], the discreteness of $\mathrm{Corr}_2(X)$ implies that strongly minimal OT-manifolds are *essentially saturated* in the sense of [3]. In particular, we obtain in every dimension examples of essentially saturated manifolds that are not of Kähler-type. This was the original motivation for both [4] and the current note.

2. Finite covers of OT-manifolds

We will quickly review the original construction of OT-manifolds from [7] and then describe how to generalise it.

Fix a number field K admitting $n = s + 2t$ distinct embeddings into $\mathbb{C}$, which we will denote by $\sigma_1, \ldots, \sigma_n$ where $\sigma_1, \ldots, \sigma_s$ are real and each σ_{s+i} is complex conjugate to σ_{s+i+t}. Assume that s and t are positive. By Dirichlet's Theorem the multiplicative group of units $\mathcal{O}_K^*$ of the ring of integers $\mathcal{O}_K$ of K has rank $s+t-1$. The subgroup

$$\mathcal{O}_K^{*,+} := \{a \in \mathcal{O}_K^* : \sigma_i(a) > 0 \text{ for all } 1 \leq i \leq s\}$$

of "positive" units is free abelian of finite index in $\mathcal{O}_K^*$. Let U be a rank s subgroup of $\mathcal{O}_K^{*,+}$ that is admissible for K in the sense of [7]. With respect to the natural action of U on the additive group $\mathcal{O}_K$, consider the semidirect product $\Gamma = U \ltimes \mathcal{O}_K$. Let $m = s + t$ and consider the action of Γ on $\mathbb{C}^m$ given by,

$$(a, x)(z_1, \ldots, z_m) := \big(\sigma_1(ax) + \sigma_1(a)z_1, \ldots, \sigma_m(ax) + \sigma_m(a)z_m\big).$$

As $U < \mathcal{O}_K^{*,+}$, this action leaves $\mathbb{H}^s \times \mathbb{C}^t$ invariant, and the admissibility condition is equivalent to the action being proper and discontinuous. The original OT-manifold, denoted by $X(K, U)$, is the quotient of $\mathbb{H}^s \times \mathbb{C}^t$ by this action. In the sequel we will denote these manifolds by $X(\mathcal{O}_K, U)$ in order to distinguish them from their generalisations.

The above construction is generalised by replacing the role of $\mathcal{O}_K$ in Γ by any rank n additive subgroup $M \leq \mathcal{O}_K$ that is stable under the action of U. We say then that U *is admissible for* M. Taking $\Gamma = U \ltimes M$, we again get a proper and discontinuous action on $\mathbb{H}^s \times \mathbb{C}^t$, and the quotient is denoted by $X(M, U)$. We will continue to call these compact complex manifolds OT-*manifolds*. To avoid confusing them with the previous construction we will occasionally say that they are of type $X(M, U)$ (otherwise of type $X(\mathcal{O}_K, U)$). Note that the possibility of generalising the original construction by replacing $\mathcal{O}_K$ with an order of K is already mentioned in [7]. However only the $\mathbb{Z}$-submodule structure of M and the stability under the U-action are necessary to make the construction work.

The universal cover of $X(M, U)$ is $\mathbb{H}^s \times \mathbb{C}^t$ and the fundamental group is $U \ltimes M$. As the latter is of finite index in $U \ltimes \mathcal{O}_K$, we see that $X(M, U)$ is a finite unramified covering of $X(\mathcal{O}_K, U)$. In fact, all finite unramified covers are of this form:

Lemma 5. *The class of* OT-*manifolds of type* $X(M, U)$ *is closed under finite unramified coverings.*

Proof. Given $X(M, U)$, such a covering would correspond to a finite index subgroup $\Gamma_1 \leq U \ltimes M$. Taking U_1 to be the image of Γ_1 in U, and setting $M_1 := \Gamma_1 \cap M$, it is not hard to check that U_1 is admissible for M_1 and that the covering is nothing other than $X(M_1, U_1)$. □

Much of the theory of OT-manifolds developed in [7] goes through in this more general setting. In particular,

Lemma 6. *If* $X = X(M, U)$ *is an* OT-*manifold then* $H^0(X, T_X) = 0$.

Proof. For OT-manifolds of type $X(\mathcal{O}_K, U)$ this is Proposition 2.5 of [7]. Imitating that argument, it suffices to prove for M a rank n additive subgroup of $\mathcal{O}_K$, that the image of M in $\mathbb{R}^s$ under $(\sigma_1, \ldots, \sigma_s)$ is dense. But this is the case because M has finite index in $\mathcal{O}_K$ and the latter does have dense image (see the proof of Lemma 2.4 of [7]). □

The following remarks serve as further evidence that the above extension of the definition of OT-manifolds is natural.

Remark 7. *Any* OT-*manifold of type* $X(M, U)$ *admits a finite unramified cover of type* $X(\mathcal{O}_K, U)$.

Indeed, since M is of maximal rank in $\mathcal{O}_K$, there exists a positive integer l such that $l\mathcal{O}_K \subset M$. Thus $X(l\mathcal{O}_K, U)$ is a finite unramified cover of $X(M, U)$. But the multiplication by l at the level of $\mathbb{H}^s \times \mathbb{C}^t$ conjugates the actions of $U \ltimes \mathcal{O}_K$ and of $U \ltimes l\mathcal{O}_K$ and thus induces an isomorphism between $X(\mathcal{O}_K, U)$ and $X(l\mathcal{O}_K, U)$.

Remark 8. *When* $s = t = 1$ *the class* OT-*manifolds of type* $X(M, U)$ *coincides with the class of Inoue surfaces of type* S_M *defined in* [2].

Indeed, if one starts with the manifold $X(M, U)$, then choosing a generator a of U with $\sigma_1(a) > 1$ and a base $(\alpha_1, \alpha_2, \alpha_3)$ of M over $\mathbb{Z}$ one obtains a matrix $A(a) \in GL(3, \mathbb{Z})$ which represents the action of a on M with respect to this basis. Applying the embedding σ_k to the relation $a(\alpha_1, \alpha_2, \alpha_3)^\top = A(a)(\alpha_1, \alpha_2, \alpha_3)^\top$ shows that $(\sigma_k(\alpha_1), \sigma_k(\alpha_2), \sigma_k(\alpha_3))^\top$ is an eigenvector of $A(a)$ associated to the eigenvalue

$\sigma_k(a)$. In particular this implies $A(a) \in SL(3,\mathbb{Z})$ since $\sigma_1(a) > 0$. At this point one sees that $X(M,U)$ coincides with the surface $S_{A(a)}$ as defined in [2].

Conversely, starting with any matrix $A \in SL(3,\mathbb{Z})$, with one real eigenvalue larger than 1 and two complex non-real eigenvalues, we denote by K the splitting field of the characteristic polynomial χ_A of A over $\mathbb{Q}$. Then there exists an element $a \in \mathcal{O}_K^{*,+}$ such that the eigenvalues of A (i.e the roots of χ_A) are precisely $\sigma_1(a), \sigma_2(a), \sigma_3(a)$. We find now an eigenvector $v \in \mathbb{Z}[\sigma_1(a)]^3$ associated to $\sigma_1(A)$ by solving the system $(A - \sigma_1(a)I_3)v^\top = 0$ over K. There exist now elements $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{O}_K$ such that $v = (\sigma_1(\alpha_1), \sigma_1(\alpha_2), \sigma_1(\alpha_3))$. Moreover $\alpha_1, \alpha_2, \alpha_3$ are linearly independent over $\mathbb{Q}$ since a linear relation would entail a linear relation between the components v_1, v_2, v_3 of v over $\mathbb{Q}$, which combined with the equations $(A - \sigma_1(a)I_3)v^\top = 0$ would show that $\sigma_1(a)$ is quadratic over $\mathbb{Q}$. Now choosing M to be the $\mathbb{Z}$-sumbodule of K generated by $\alpha_1, \alpha_2, \alpha_3$ and U the multiplicative group generated by a we get again $X(M,U) = S_{A(a)}$.

3. The Proof

As in the case of Inoue surfaces of type S_M studied in [4], we will make use of some deformation theory to prove the main theorem. But we will need a bit more than was used in [4]. We say that a holomorphic map $f : V \to W$ between compact complex manifolds is *rigid over* W if there are no nontrivial deformations of f that keep W fixed. More precisely: Whenever $\mathcal{V} \to D$ is a proper and flat holomorphic map of compact complex varieties with $V = \mathcal{V}_d$ for some $d \in D$, and $\mathcal{F} : \mathcal{V} \to D \times W$ is a holomorphic map over D with $\mathcal{F}_d = f$, then there is an open neighbourhood U of d in D and a diagram

$$\begin{tikzcd}
\mathcal{V}_U \arrow[dd, "\phi"'] \arrow[dr] \arrow[drr, "\mathcal{F}_U"] & & \\
 & U & U \times W \arrow[l] \\
U \times V \arrow[ur] \arrow[urr, "\mathrm{id}_U \times f"'] & &
\end{tikzcd}$$

where ϕ is a biholomorphism. In particular $\mathcal{F}_s(\mathcal{V}_s) = f(V)$ for all $s \in U$.

Fact 9 (Section 3.6 of [6])**.** *Suppuse $f : V \to W$ is a holomorphic map between compact complex manifolds such that*

- $H^0(V, f^*T_W) = 0$, *and*
- $f_* : H^1(V, T_V) \to H^1(V, f^*T_W)$ *is injective.*

Then f is rigid over W.

Lemma 10. *Suppose X and Y are compact complex manifolds, $H^0(Y, T_Y) = 0$, and $f : Y \to X^n$ is a holomorphic map such that $\mathrm{pr}_i \circ f : Y \to X$ is a finite unramified cover for each $i = 1, \ldots, n$. Then f is rigid over X^n.*

Proof. Note that here $\mathrm{pr}_i : X^n \to X$ is the projection onto the ith co-ordinate. Let $f_i := \mathrm{pr}_i \circ f : Y \to X$. As each f_i is unramified, we have that

$$f^*T_{X^n} = f^*\left(\bigoplus_{i=1}^n \mathrm{pr}_i^* T_X\right) = \bigoplus_{i=1}^n f_i^* T_X = \bigoplus_{i=1}^n T_Y$$

Hence, $H^0(Y, f^*T_{X^n}) = \bigoplus_{i=1}^{n} H^0(Y, T_Y) = 0$. On the other hand, the isomorphism $(f_1)_* : H^1(Y, T_Y) \to H^1(Y, f_1^* T_X)$ factors through $f_* : H^1(Y, T_Y) \to H^1(Y, f^*T_{X^n})$, and hence the latter is injective. So $f : Y \to X^n$ is rigid over X^n by Fact 9. □

We can now prove the main theorem.

Proof of Theorem 1. Suppose X is an OT-manifold of type $X(M, U)$. As in [4], in order to show that $\mathrm{Corr}_n(X)$ is discrete we let $S \in \mathrm{Corr}_n(X)$ be arbitrary, consider the irreducible component D of the Douady space of X^n in which S lives, and show that D is zero-dimensional. This suffices as it proves that each element of $\mathrm{Corr}_n(X)$ is isolated in the Douady space.

Let $Z \subset D \times X^n$ be the restriction of the universal family to D. By the flatness of $Z \to D$, for general $d \in D$, $Z_d \in \mathrm{Corr}_n(X)$ also. Let $\widetilde{Z} \to Z$ be a normalisation and denote by $f : \widetilde{Z} \to D \times X^n$ the composition of the normalisation with the inclusion of Z in $D \times X^n$. Then for general $d \in D$ we have that $f_d : \widetilde{Z}_d \to X^n$ is such that each projection $\mathrm{pr}_i \circ f_d : \widetilde{Z}_d \to X$ is a finite surjective map. In [1] it is shown that OT-manifolds of type $X(\mathcal{O}_K, U)$, and hence also OT-manifolds of type $X(M, U)$, have no divisors. So the purity of branch locus theorem (which applies as $\widetilde{Z}_d$ is normal and X is smooth) implies that $\mathrm{pr}_i \circ f_d$ is a finite unramified covering. In particular, $\widetilde{Z}_d$ is a generalised OT-manifold by Lemma 5, and so $H^0(\widetilde{Z}_d, T_{\widetilde{Z}_d}) = 0$ by Lemma 6. But moreover, by Lemma 10, f_d is rigid over X^n. It follows that for some open neighbourhood U of d in D, $f_U : \widetilde{Z}_U \to U \times X^n$ is biholomorphic over $U \times X^n$ with $\mathrm{id}_U \times f_d : U \times \widetilde{Z}_d \to U \times X^n$. In particular, for all $s \in U$, $Z_s = f_s(\widetilde{Z}_s) = f_d(\widetilde{Z}_d) = Z_d$. The universality of the Douady space now implies that $U = \{d\}$, so that in fact $D = \{d\}$, as desired. □

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