INDEPENDENCE PROPERTY AND HYPERBOLIC GROUPS

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Abstract. In continuation of [JOH04, OH07], we prove that existentially closed CSA-groups have the independence property. This is done by showing that there exist words having the independence property relatively to the class of torsion-free hyperbolic groups.

1. Introduction

A *CSA-group* is a group G in which every maximal abelian subgroup A is malnormal, i.e. satisfies $A \cap A^g = \{1\}$ for each element g of G which is not in A. The interest in these groups is motivated, on one hand, by the question of existence of new groups with well behaved first-order theories, and ultimately of the so-called *bad groups* of finite Morley rank which may be CSA-groups under certain circumstances [Jal01]. On the other hand, it is motivated by the fact that CSA-groups are usually more suitable than other groups for solving equations, an important step towards understanding their first-order theory.

The torsion of CSA-groups under consideration can be restricted as follows. If f is a function from the set of prime integers into $\mathbb{N} \sqcup \{\infty\}$, we call a *$CSA_f$-group* any CSA-group which contains no elementary abelian p-subgroup of rank $f(p)+1$ for any prime p such that $f(p)$ is finite. Once such a function f is fixed, it is easily seen that the first-order class of CSA_f-groups is inductive, and thus contains *existentially closed* CSA_f-groups. If a CSA-group contains an involution, then it must be abelian, a case of low interest from the group-theoretic point of view and from the model-theoretic point of view as well, since all abelian groups are known to be stable. Hence we prefer to work with CSA-groups without involutions, i.e. with CSA_f-groups where $f(2) = 0$. It is known, when $f(2) = 0$, that existentially closed CSA_f-groups are simple, in a strong sense that simplicity is provided by a first-order formula. Moreover, their maximal abelian subgroups are conjugate, divisible, and of Prüfer p-rank $f(p)$ for each prime p. We refer to [JOH04, Sect. 2, 5] for details.

Concerning the model theory of existentially closed CSA_f-groups, always assuming $f(2) = 0$, it has also been proved in [JOH04] that such groups are not ω-stable. This was done by counting the number of types, more precisely by showing the existence of $2^{\aleph_0}$ types over the empty set. The same method of counting types has later been used in [OH07] to prove that the first-order theory of existentially closed CSA_f-groups ($f(2) = 0$) is not superstable. We will show here by a much more elementary argument that their first-order theory is indeed far from being stable. We shall prove that they have the independence property using a standard small cancellation argument from combinatorial group theory.

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2. Relative independence property

The main result of the present paper leading to the independence property of existentially closed CSA-groups is the following.

Theorem 2.1. *There exists a group word $w(x, y)$ in two variables such that the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relatively to the class of torsion-free hyperbolic groups.*

For the moment we postpone the proof of Theorem 2.1, and rather give the definition used in this theorem and derive the corollary concerning the first-order theories of existentially closed CSA-groups which particularly interests us.

If $\phi(\overline{x}, \overline{y})$ is a formula in a given language $\mathcal{L}$, and $\mathcal{C}$ is any (not necessarily elementary) class of $\mathcal{L}$-structures, we say that ϕ has the *independence property relatively to* $\mathcal{C}$ if for any positive integer n, there exists a structure M_n in $\mathcal{C}$ with sequences of tuples $\overline{x}_1, \ldots, \overline{x}_i, \ldots, \overline{x}_n$, and $\overline{y}_1, \ldots, \overline{y}_\sigma, \ldots, \overline{y}_{2^n}$, where the indices σ vary over the set of subsets of $\{1, \ldots, n\}$, such that the following holds in M_n:

$$M_n \models \phi(\overline{x}_i, \overline{y}_\sigma) \quad \text{if and only if} \quad i \in \sigma.$$

When the class $\mathcal{C}$ consists of all models of a complete first-order theory T, this definition corresponds to the usual notion of independence property of the formula ϕ relatively to the first-order theory T, as defined in [She90]. As we are dealing with groups here, the language will be that of groups, and our formula will typically be a group equation in two variables, i.e. of the form $\ulcorner w(x, y) = 1 \urcorner$ with $w(x, y)$ a *group word* in two variables x and y (i.e. a word in $x^{\pm 1}$ and $y^{\pm 1}$).

Before the proof of Theorem 2.1, we derive the corollary concerning the model theory of existentially closed CSA-groups.

Corollary 2.2. *Assume $f(2) = 0$, and let G be an existentially closed CSA_f-group. Then the first-order theory of G has the independence property.*

For deriving Corollary 2.2 from Theorem 2.1, we will use the following fact.

Fact 2.3. *Assume $f(2) = 0$, and let G be an existentially closed CSA_f-group. Then any CSA_f-group embeds into a model of the first-order theory of G.*

Proof. This is explicitly seen in the proof of [JOH04, Corollary 8.2]. □

We will also use the following fact, according to which CSA-groups can be seen as a generalization of torsion-free hyperbolic groups.

Fact 2.4 ([MR96, Proposition 12])**.** *Every torsion-free hyperbolic group is CSA.*

We shall give here a short proof for reader's convenience.

Proof. By [CDP90, Corollaire 7.2] or [GdlH90, Théorème 38], the centralizer of any element of infinite order in a hyperbolic group is virtually cyclic.

Every torsion-free virtually cyclic group is cyclic. This can be shown as follows. First, it is easy to verify that in every torsion-free virtually cyclic group the center has finite index. (This is true even for virtually cyclic groups without involutions.) Second, there is a theorem, often attributed to Issai Schur, which states that if the index of the center of a group is finite, then the derived subgroup of the group is also finite. In particular, if in a torsion-free group the center has finite index, then the group is abelian. An elementary proof of this theorem of Schur may be found in [Ros62]: the idea is to show that in a group whose center has index n, the number of commutators is at most $(n-1)^2+1$, and that the $(n+1)$st power of each commutator is the product of n commutators. Hence every torsion-free virtually cyclic group is abelian. Last, torsion-free abelian virtually cyclic groups are cyclic. This follows from the classification of finitely generated abelian groups, or from the simpler fact that in a torsion-free abelian group, each two cyclic subgroups either intersect trivially, or are both contained in some cyclic subgroup.

Let G be an arbitrary torsion-free hyperbolic group. We already know that centralizers of non-trivial elements of G are infinite cyclic. In particular, the commutativity is an equivalence relation on $G \setminus \{1\}$. Consider an arbitrary maximal abelian subgroup A of G. Then A is infinite cyclic. Suppose A is not malnormal. Let a be a generator of A, and let b be an element of $G \setminus A$ such that $A \cap A^b \neq \{1\}$. Since commutativity is transitive and A is maximal, $A = A^b$. Therefore, $a^b \in \{a^{\pm 1}\}$, and hence $(a^b)^b = a$. Since a commutes with b^2, and b^2 commutes with b, we have that a commutes with b, and therefore $b \in A$, which gives a contradiction. Thus A is malnormal. □

Proof of Corollary 2.2. By Theorem 2.1, there exists a group word $w(x, y)$ in two variables x and y such that the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relatively to the class of torsion-free hyperbolic groups. This means that for any positive integer n, there exists a torsion-free hyperbolic group G_n with sequences of elements $x_1, \ldots, x_i, \ldots, x_n$, and $y_1, \ldots, y_\sigma, \ldots, y_{2^n}$, with $\sigma \subseteq \{1, \ldots, n\}$, such that

$$G_n \models w(x_i, y_\sigma) = 1 \quad \text{if and only if} \quad i \in \sigma.$$

By Fact 2.4, any torsion-free hyperbolic group G_n is a CSA-group, and even, as it is torsion-free, a CSA_f-group for an arbitrary function f.

Let now $\mathrm{Th}(G)$ denote the first-order theory of the existentially closed CSA_f-group G one wants to consider. By Fact 2.3, each group G_n embeds into a model of $\mathrm{Th}(G)$. In particular, as the truth of the formula $\ulcorner w(x, y) = 1 \urcorner$ is preserved under embeddings, $\mathrm{Th}(G)$ contains the formula

$$(\exists_{1 \le i \le n}\, x_i)\,(\exists_{\sigma \in 2^n}\, y_\sigma)\,\Big[\big(\bigwedge_{i \in \sigma} w(x_i, y_\sigma) = 1\big) \wedge \big(\bigwedge_{i \notin \sigma} w(x_i, y_\sigma) \neq 1\big)\Big].$$

Since this is true for any positive n, the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relatively to $\mathrm{Th}(G)$. Hence $\mathrm{Th}(G)$ has the independence property. □

3. Small-cancellation groups

In this section we consider groups together with their presentations by generators and defining relations. We take most definitions and results from [Ol′91], which is our main source of references for combinatorial treatment of group presentations. Some terms are also borrowed from [CCH81].

Recall that a *group presentation* is an ordered pair $\langle\, A \,\|\, \mathcal{R} \,\rangle$, also written as $\langle\, A \,\|\, R = 1; R \in \mathcal{R} \,\rangle$, where A is an arbitrary set, called the *alphabet*, and $\mathcal{R}$ is a set of *group words* over A, called *relators* or *defining relators*. Every relator from $\mathcal{R}$ is a word in the alphabet $A \sqcup A^{-1}$, composed of elements of A and their formal inverses. Relators can also be viewed as a simplified form of *terms* in the language of groups augmented with constants from A.

By abuse of notation, the same symbol $\langle\, A \,\|\, \mathcal{R} \,\rangle$ shall be used to denote a certain group given by this presentation, that is a group G where all constants from A are interpreted in such a way that G is generated by their interpretations, all the relations $\ulcorner R = 1 \urcorner$, $R \in \mathcal{R}$, hold in G, and all other relations of this form that hold in G are just their group-theoretic consequences. A presentation of a group G is any presentation $\langle\, A \,\|\, \mathcal{R} \,\rangle$ such that $G \cong \langle\, A \,\|\, \mathcal{R} \,\rangle$. Every group has a presentation (by the multiplication table, for example).

A presentation $\langle\, A \,\|\, \mathcal{R} \,\rangle$ is called *finite* if both sets A and $\mathcal{R}$ are finite. A group is *finitely presented* if it has a finite presentation.

Since *cyclic reduction* of relators, i.e. *cyclic shifts* and cancellation of pairs of adjacent mutually inverse letters, does not change the group given by the presentation, every group has a presentation with all relators *cyclically reduced*. Presentations, or sets of relators, in which all relators are cyclically reduced shall be called *cyclically reduced* themselves.

A cyclically reduced presentation $\langle\, A \,\|\, \mathcal{R} \,\rangle$, or a set of relators $\mathcal{R}$, shall be called *symmetrized* if $\mathcal{R}$ contains the *visual inverse* R^{-1} and every *cyclic shift* YX of every $R \equiv XY \in \mathcal{R}$. (The visual inverse of $ab^{-1}c$, for example, is $c^{-1}ba^{-1}$; XY denotes the concatenation of the words X and Y.) The *symmetrized closure*, or *symmetrization*, of $\mathcal{R}$ is the minimal symmetrized set of relators containing $\mathcal{R}$.

The opposite of "symmetrized" is "*concise*". Following [CCH81], we call a cyclically reduced presentation $\langle\, A \,\|\, \mathcal{R} \,\rangle$, or a set of relators $\mathcal{R}$, *concise* if no two distinct elements of $\mathcal{R}$ are cyclic shifts of each other or of each other's visual inverses.

In what follows, all group presentations shall be assumed cyclically reduced.

If $R_1 \equiv XY_1$ and $R_2 \equiv XY_2$ are two distinct words in a symmetrized set of relators $\mathcal{R}$, then X is called a *piece* (of R_1) relative to $\mathcal{R}$.

Definition 3.1. Let λ be a number in $[0, 1]$. Let $\langle\, A \,\|\, \mathcal{R} \,\rangle$ be a group presentation, and $\tilde{\mathcal{R}}$ be the symmetrization of $\mathcal{R}$. Then the set $\mathcal{R}$, or the presentation $\langle\, A \,\|\, \mathcal{R} \,\rangle$, is said to satisfy the *condition* $C'(\lambda)$ if for any $R \in \tilde{\mathcal{R}}$ and for any subword X of R which is a piece relative to $\tilde{\mathcal{R}}$, the following bound on the length of X holds: $|X| < \lambda|R|$.

The condition $C'(\lambda)$ with "small" λ is a standard example of so-called "small cancellation conditions". Generally, small cancellation conditions allow one to prove that large traces of relators remain in all their consequences.

The following fact about the condition $C'(\lambda)$ shows that it can be used for constructing hyperbolic groups.

Fact 3.2 ([GdlH90, Théorème 33])**.** *A finitely presented group which has a presentation satisfying* $C'(1/6)$ *is hyperbolic.*

As an intermediate step of our proof of Theorem 2.1, we are going to use *asphericity* of a group presentation. Various forms of asphericity and their mutual implications are studied in [CCH81]. What we will actually need is the asphericity defined in [Ol′91, §13], though it is defined there in a more general setting of *graded* presentations. To avoid confusion with other forms of asphericity, however, we will borrow the term *diagrammatic asphericity* from [CCH81]. (It follows from [Ol′91, Theorem 32.2] that the asphericity defined in [Ol′91, §13] in the case of a non-graded cyclically reduced presentation is indeed the same as diagrammatic asphericity in [CCH81].)

Taking small values of λ in the condition $C'(\lambda)$ results in asphericity. Indeed, the $C'(1/5)$ condition implies a related condition $C(6)$ [Ol′91, p. 127], which yields diagrammatic asphericity by [Ol′91, Theorem 13.3]. Hence we have the following fact.

Fact 3.3. *A presentation satisfying $C'(1/5)$ is diagrammatically aspherical.*

Asphericity of a group presentation implies many algebraic properties of the group. An important consequence is the following.

Fact 3.4 ([Ol′91, Theorem 13.4])**.** *If $\langle\, A \,\|\, \mathcal{R} \,\rangle$ is a concise aspherical group presentation, then the defining relations are independent, i.e. no relation in the set $\{\, \ulcorner R = 1 \urcorner \mid R \in \mathcal{R} \,\}$ is a consequence of the others.*

Following [CCH81] and [Mur07, Sect. 6], we say that a group presentation $\langle\, A \,\|\, \mathcal{R} \,\rangle$ is *singularly aspherical* if it is aspherical, concise, and no element of $\mathcal{R}$ can be decomposed as a concatenation of several copies of the same subword (i.e. does not represent a proper power in the free group $\langle\, A \,\|\, \varnothing \,\rangle$).

If $\langle\, A \,\|\, \mathcal{R} \,\rangle$ is a singularly aspherical presentation, then the relation module of $\langle\, A \,\|\, \mathcal{R} \,\rangle$ is a free G-module by [Ol′91, Corollary 32.1]. One can see then, using the fact that all odd-dimensional homology groups of nontrivial finite cyclic groups are nontrivial [Bro94], that this prevents the presence of torsion.

Fact 3.5 ([Mur07, Lemma 64])**.** *Every group with singularly aspherical presentation is torsion-free.*

Note that Fact 3.5 can also be deduced from [Hue79, Theorem 3], where the finite-order elements are classified in the more general case of *combinatorially aspherical* groups (defined in [Hue79, CCH81]).

We have now all the tools from combinatorial group theory needed to built families of finitely presented groups with the desired properties.

Proof of Theorem 2.1. Consider, for example, the following word $w(x, y)$ in two letters x and y:

$$w(x, y) = xy^7x^2y^6x^3y^5x^4y^4x^5y^3x^6y^2x^7y.$$

For an integer $n \geq 1$, let A be a set of $n + 2^n$ elements $a_1, \ldots, a_i, \ldots, a_n$, and $b_1, \ldots, b_\sigma, \ldots, b_{2^n}$, with the indices σ varying in the set of all subsets of $\{1, \ldots, n\}$. Define two sets of relators:

$$\mathcal{R} = \{\, w(a_i, b_\sigma) \mid i \in \sigma \,\}, \qquad \mathcal{S} = \{\, w(a_i, b_\sigma) \mid \,\}.$$

Denote by $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{S}}$ their respective symmetrizations. Clearly both $\mathcal{R}$ and $\mathcal{S}$ are concise.

Consider now the finitely presented group

$$G = \langle\, A \,\|\, \mathcal{R} \,\rangle = \langle\, A \,\|\, \tilde{\mathcal{R}} \,\rangle,$$

and denote by F the free group $\langle\, A \,\|\, \varnothing \,\rangle$, of rank $n + 2^n$. Then G is the quotient of F by the normal closure of $\mathcal{R}$, $\mathcal{R}$ being viewed as a subset of F.

One can check directly that any piece relative to $\tilde{\mathcal{S}}$ of any relator from $\tilde{\mathcal{S}}$ has at most two *syllables*, and deduce that the length of the piece is at most 8. As the length of every relator is 7×8, we obtain

$$|X| \leq \frac{1}{7}|R| < \frac{1}{6}|R|$$

for any $R \in \tilde{\mathcal{S}}$ and any piece X of R relative to $\tilde{\mathcal{S}}$. Hence the presentation $\langle\, A \,\|\, \mathcal{S} \,\rangle$ satisfies $C'(1/6)$. In particular, the subpresentation $\langle\, A \,\|\, \mathcal{R} \,\rangle$ satisfies $C'(1/6)$.

It follows now from Fact 3.2 that G is hyperbolic. By Fact 3.3, the presentation $\langle\, A \,\|\, \mathcal{S} \,\rangle$ is diagrammatically aspherical. As clearly no element of $\mathcal{S}$ is a proper power in the free group F, the presentation $\langle A \,\|\, \mathcal{S} \rangle$ is in fact singularly aspherical, and such is the subpresentation $\langle A \,\|\, \mathcal{R} \rangle$. Fact 3.5 implies now that G is torsion-free.

Finally, using Fact 3.4, we obtain that for all $i \in \{1, \dots, n\}$ and $\sigma \subseteq \{1, \dots, n\}$,

$$G \models w(a_i, b_\sigma) = 1 \quad \text{if and only if} \quad i \in \sigma.$$

As n can be any positive integer, the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relatively to the class of torsion-free hyperbolic groups. The proof of Theorem 2.1 is now complete. □

4. Concluding remarks

We conclude with a few remarks.

4.1. The group word $w(x, y)$ built in the proof of Theorem 2.1 is in two variables and without constants. Hence in Corollary 2.2 one finds directly a formula with the independence property having two single variables only (see [Poi85, Théorème 12.18] for general reductions of this kind), and without parameters.

4.2. As announced in [Sel07], a torsion-free hyperbolic group is stable, and hence cannot have the independence property. Hence one cannot imagine a version of Theorem 2.1 where the class of groups would consist of groups elementarily equivalent to a given torsion-free hyperbolic group, or more generally to a fixed *finite* set of torsion-free hyperbolic groups. Our proof provides however a countable set of torsion-free hyperbolic groups.

4.3. The proof of Theorem 2.1 actually provides a group word $w(x, y)$ such that the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relatively to the class of torsion-free finitely presented $C'(1/6)$-groups, a class significantly smaller than that of all torsion-free hyperbolic groups. Choosing w long enough, one can similarly produce formulas having the independence property relatively to the class of torsion-free finitely presented $C'(\lambda)$-groups with λ arbitrarily small. Indeed, if one denotes by P_n the "probability" that a cyclically reduced group word $w(x, y)$ of length $n \geq 1$ gives the independence property relatively to the class of torsion-free $C'(\lambda)$-groups (with the formula $\ulcorner w(x, y) = 1 \urcorner$), then one can check in a rather straightforward manner the rough estimate

$$P_n \geq 1 - \frac{n^2}{2^{\lambda n}} - \frac{4n}{2^{\lambda n}},$$

and hence P_n tends rapidly to 1 as n tends to the infinity. This estimate can be found by verifying that the "probability" of occurrence of a subword of a cyclic shift of w of length $\lceil \lambda n \rceil$ in two given distinct "positions" with respect to w is not greater than $1/2^{\lambda n}$, and that the "probability" that a cyclic shift of w contains a syllable of length $\lceil \lambda n \rceil$ is less than $4n/2^{\lambda n}$. For example, the probability that in a cyclically reduced group word $w(x, y)$ of length at least 6 the same word of length 4 occures as a subword starting from the first letter and also starting from third letter, is at most $1/(3 \cdot 3 \cdot 3 \cdot 2) = 1/54$; indeed, once all the letters of w except the first 4 are fixed, there is at most one way to complete w so as to obtain a word whose initial subword of length 4 occures again starting from the third letter, but there are at least $3 \cdot 3 \cdot 3 \cdot 2$ ways to complete it to obtain a cyclically reduced group word.

Roughly speaking, for any fixed $\lambda > 0$, by choosing an arbitrary but sufficiently long group word w, one will "most probably" obtain the formula $\ulcorner w(x, y) = 1 \urcorner$ with the independence property relatively to the class of torsion-free finitely presented $C'(\lambda)$-groups.

4.4. Another property of a formula witnessing the instability of a first-order theory, orthogonal in some sense to the independence property, is the *strict order property.* One may wonder whether the first-order theory $\mathrm{Th}(G)$ of an existentially closed CSA-group G has this property. We provide here some speculations on this question, again for formulas $\phi(x, y)$ of the form $\ulcorner w(x, y) = 1 \urcorner$ for some group word w, and we concentrate on weaker properties such as the *n-strong order property* SOP_n defined in [She96, Definition 2.5] for $n \geq 3$. We refer to [She96, Sect. 2] for a general discussion of these properties, and we just recall the implications

$$\begin{array}{lll} \text{strict order property} & \Longrightarrow & \cdots \\ & \Longrightarrow & \mathrm{SOP}_{n+1} \\ & \Longrightarrow & \mathrm{SOP}_n \\ & \Longrightarrow & \cdots \\ & \Longrightarrow & \mathrm{SOP}_3 \quad \Longrightarrow \text{non-simplicity} \end{array}$$

If w is a "long" and "arbitrarily chosen" cyclically reduced group word in two letters x and y (see above), then "most probably" the formula $\ulcorner w(x, y) = 1 \urcorner$ will not exemplify SOP_n. The reason is that essentially the same kind of arguments used in the proof of Theorem 2.1 provide torsion-free $C'(1/6)$-groups generated by elements $a_1, \ldots, a_n$ in which $w(a_i, a_j) = 1$ if and only if $j = i + 1$ modulo n, and thus the graph defined on an existentially closed CSA_f-group by the formula $\ulcorner w(x, y) = 1 \urcorner$ contains cycles of size n, contrary to one of the two requirements for the condition SOP_n (the other one being the existence in some model of an infinite chain in this graph). Hence one may be tempted to look at "short" words. As usual in our context of commutative transitive groups, the very short word $[x, y]$ witnessing the commutativity of x with y boils down to an equivalence relation, and hence is useless. Incidently, the formula $\phi(x, y)$ used in [SU06, Proposition 4.1] to prove that Group Theory "in general" has SOP_3 is

$$(xyx^{-1} = y^2) \wedge (x \neq y).$$

In our context of CSA-groups, it implies $y = 1$. Hence the absence of certain triangles (a_1, a_2, a_3) satisfying

$$\phi(a_1, a_2) \wedge \phi(a_2, a_3) \wedge \phi(a_3, a_1)$$

(provided by [Sta91, p. 493] in the context of arbitrary groups) is immediate in our case, but one cannot hope to find an infinite chain in the graph associated to $\phi(x, y)$. Hence this formula could not exemplify the SOP_3 in our context of CSA-groups. This means that a formula examplifying the SOP_n of an existentially closed CSA-group, if it exists, cannot involve only an "arbitrarily chosen long" equation, and it does not seem to involve only "short" equations.

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References

[Bro94] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[CCH81] I. M. Chiswell, D. J. Collins, and J. Huebschmann. Aspherical group presentations. *Math. Z.*, 178(1):1–36, 1981.

[CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], With an English summary.

[GdlH90] É. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*, volume 83 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[Hue79] J. Huebschmann. Cohomology theory of aspherical groups and of small cancellation groups. *J. Pure Appl. Algebra*, 14(2):137–143, 1979.

[Jal01] E. Jaligot. Full Frobenius groups of finite Morley rank and the Feit-Thompson theorem. *Bull. Symbolic Logic*, 7(3):315–328, 2001.

[JOH04] E. Jaligot and A. Ould Houcine. Existentially closed CSA-groups. *J. Algebra*, 280(2):772–796, 2004.

[MR96] A. G. Myasnikov and V. N. Remeslennikov. Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups. *Internat. J. Algebra Comput.*, 6(6):687–711, 1996.

[Mur07] A. Yu. Muranov. Finitely generated infinite simple groups of infinite commutator width. *Internat. J. Algebra Comput.*, 17(3):607–659, 2007. arXiv.org preprint: math.GR/0608688.

[OH07] A. Ould Houcine. On superstable CSA-groups. To appear in Annals of Pure and Applied Logic, http://math.univ-lyon1.fr/∼ould/, 2007.

[Ol′91] A. Yu. Ol′shanskiĭ. *Geometry of defining relations in groups*, volume 70 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the 1989 Russian original by Yu. A. Bakhturin.

[Poi85] B. Poizat. *Cours de théorie des modèles*. Bruno Poizat, Villeurbanne, 1985. Une introduction à la logique mathématique contemporaine. [An introduction to contemporary mathematical logic].

[Ros62] M. Rosenlicht. On a result of Baer. *Proc. Amer. Math. Soc.*, 13(1):99–101, 1962.

[Sel07] Z. Sela. Diophantine geometry over groups VIII: Stability. preprint: http://www.ma.huji.ac.il/∼zlil/, 2007.

[She90] S. Shelah. *Classification theory and the number of nonisomorphic models*. North-Holland Publishing Co., Amsterdam, second edition, 1990.

[She96] S. Shelah. Toward classifying unstable theories. *Ann. Pure Appl. Logic*, 80(3):229–255, 1996.

[Sta91] J. R. Stallings. Non-positively curved triangles of groups. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 491–503. World Sci. Publ., River Edge, NJ, 1991.

[SU06] S. Shelah and A. Usvyatsov. Banach spaces and groups—order properties and universal models. *Israel J. Math.*, 152:245–270, 2006.