Cell decomposition and dimension theory on p-optimal fields

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Abstract

We prove that for p-optimal fields (a very large subclass of p-minimal fields containing all the known examples) cell decomposition and cell preparation for definable functions follow from methods going back to Denef's paper [Den84]. We derive from it that the topological dimension introduced in [HM97] for P-minimal fields has even better properties in p-optimal fields, relatively to fundamental operations such as taking the boundary of a definable set, or the fibers of a definable function. As a consequence the topological dimension coincides with the topological rank, and every two infinite definable sets are isomorphic if and only if they have the same dimension.

1 Introduction

This paper is an attempt to continue the road opened by Haskell and MacPherson [HM97] toward a p-adic version of o-minimality. More general constructions have been given in [Sch01], [CL07], [CL12]. Our aim is not to encompass such a generality but rather to show how the methods introduced by Denef in his visionary paper [Den84] apply with striking efficiency to a very large subclass of p-minimal fields.

Recall that a **p-adically closed** field is a field K elementarily equivalent to a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers. Given a language $\mathcal{L}$ containing the language of rings, we say that an $\mathcal{L}$-structure $\mathcal{K}$ on K extending its ring structure is **p-minimal** if every definable subset of the affine line K is semi-algebraic. By "definable" we always mean definable in the language $\mathcal{L}$ with parameters from K. For sets and functions definable (with parameters as always) in the language of rings, we use the locution "semi-algebraic" instead. The $\mathcal{L}$-structure $\mathcal{K}$ is **P-minimal** (a.k.a. "strongly p-minimal") if every elementarily equivalent $\mathcal{L}$-structure is p-minimal. Abusing the notation we will always identify K and $\mathcal{K}$, and call K itself a p-minimal or P-minimal field.

Haskell and MacPherson introduced P-minimal fields in [HM97] and built a reasonably good dimension theory for definable sets over P-minimal fields. They left open several questions, such as the existence of a cell decomposition, and whether or not their dimension is a "dimension function" in the sense of [vdD89].

Mourgues proved in [Mou09] that a cell decomposition similar to the one of [Den84] holds for a P-minimal field K if and only if it has **definable Skolem**

functions (or "definable selection" in [Mou09]), that is if for every positive integers m, n and every definable subset S of K^{m+n} the coordinate projection of S onto K^m has a definable section. It is not known at the moment if every P-minimal field has definable Skolem functions.

As Cluckers noted in [Clu04], it was lacking in [Mou09] a preparation theorem for definable functions. He filled this lacuna for subanalytic functions, that is the functions which are definable in the analytic structure of K, initially introduced on $\mathbf{Z}_p$ by Denef and van den Dries in [DvdD88] (see definition 1.3 and further in [Clu04]). We call it the "classical analytic structure" in the remaining. Cluckers derived from his preparation theorem several important applications, for parametric integrals and classification of definable sets up to isomorphism.

In this article we extend these results to a large class of p-minimal fields, including all the P-minimal fields with definable Skolem functions studied in [Mou09]. Moreover we get for these fields, we call p-optimal, a dimension theory for definable sets similar to [HM97], with even better properties (see the paragraph "main results" below).

A **basic function** $f(x_1, \dots, x_m)$ is a polynomial function in x_m whose coefficients are global definable functions in $(x_1, \dots, x_{m-1})$. Thus unary basic functions are just ordinary polynomial functions in one variable with coefficients in K. A **basic set** $S \subseteq K^m$ of order N is a set of the form

$$S = \big\{x \in K^m : f(x) \in P_N\big\}$$

with f a basic function, $N \geq 1$ an integer and

$$P_N = \big\{x \in K : \exists y \in K,\ x = y^N\big\}.$$

We say that an $\mathcal{L}$-structure on a p-adically closed field K extending its ring structure is ***p*-optimal** if every definable subset of K^m (for every m) is a boolean combination of basic sets. When this happens we call K itself a ***p*-optimal field**.

***P*-minimality versus *p*-optimality.** Basic subsets of the affine line K are semi-algebraic, because unary basic functions are polynomial, hence every p-optimal field is p-minimal. On the other hand, the starting point of this paper is the easy observation that every P-minimal field with definable Skolem functions is p-optimal (proposition 2.3). Conversely we will prove that every p-optimal field has definable Skolem functions (theorem 5.3), hence a P-minimal field has definable Skolem functions if and only if it is p-optimal.

It is important to keep in mind that although p-optimality might seem at first glance a much stronger assumption than P-minimality, p-optimal fields are indeed p-minimal but are *not* supposed to be P-minimal. Moreover it is difficult to imagine any proof of p-minimality which does not involve in a way or another a quantifier elimination result similar to Macintyre's theorem 2.1. The condition defining p-optimality is actually very close to such kind of elimination. So close that it might be expectable that it can be proved simultaneously in most cases, if not all, without additional effort.

Examples. In the classical analytic structure, p-minimality was derived from the quantifier elimination theorem 1.1 in [DvdD88], the proof of which is based on the Weierstrass preparation and division theorem for analytic functions. Then P-minimality was proved later in [vdDHD99] by mean of a non-trivial improvement of this same Weierstrass division. But a detailed study of the original proof of theorem 1.1 in [DvdD88] shows that it directly proves (a strong form of) p-optimality for the classical analytic structure, without going through P-minimality.

The same holds true for the non-standard analytic structure on $\mathbf{Q}_p((t^{\mathbf{Q}}))$ studied in [Ble10], and for the expansions of $\mathbf{Q}_p$ with Weierstrass systems of [Mar08]: all of them are indeed examples of p-optimal fields (although the last ones would require a more precise examination).

Question 1.1 Does there exist a P-minimal (or at least p-minimal) field which is not p-optimal?

In the remaining of this paper K will always denote a p-optimal field. Topological notions such as continuity, interior, closure and so on refer to the topology of the p-valuation. We let $\overline{A}$ denote the topological closure of A.

Main results. This paper is threefold. Firstly, we present Denef's method and explain how it provides a cell decomposition (theorem 3.7) and a cell preparation result for definable functions (theorem 4.3) valid in every p-optimal field, and conversely (corollary 5.4). Secondly, we apply it in sections 5 and 6 in order to prove the existence of definable Skolem functions (theorem 5.3) and the following result of "continuity almost everywhere".

Theorem 1.2 *Every m-ary function f definable over a p-optimal field is continuous on a definable set which is dense and open in the domain of f.*

Finally we present in sections 7 and 8 a dimension theory for definable sets in p-optimal fields similar to, and inspired by, the analogous dimension theory for P-optimal fields in [HM97]. It not only keeps the nice properties of its ancestor, but has in addition several others of importance such as the following ones.

Theorem 1.3 *Let $f : A \subseteq K^m \to K^n$ be a definable map, and $B = f(A)$. Then for every positive integer d the set $B(d)$ defined by*

$$B(d) = \{b \in B : \dim f^{-1}(\{b\}) = d\}$$

is definable and $\dim f^{-1}(B(d)) = d + \dim B(d)$.

Theorem 1.4 *Two infinite subsets of K^m and K^n are isomorphic if and only if they have the same dimension.*

Theorem 1.5 *For every definable subset A of K^m,* $\dim \overline{A} \setminus A < \dim A$.

Combining theorems 1.2 and 1.5 we get[1]:

Corollary 1.6 *For every definable map $f : X \subseteq K^m \to K^n$, let $\mathcal{C}(f)$ denotes the set of points x in X such that f is continuous on a neighbourhood[2] of x in X. Then we have* $\dim X \setminus \mathcal{C}(f) < \dim X$.

[1] See footnote 3.

[2] Of course, as for "open" in theorem 1.2, "neighbourhood" refers here to the topology induced on X by K^m, whose open sets are the sets $U \cap X$ with U an open subset of K^m. In general the sets open in X are not open in K^m, unless X itself is open in K^m.

It follows immediately that every definable function is piecewise continuous. As a consequence, all the cells involved in the cell decomposition and cell preparation theorems can be chosen with continuous center and bounds (corollary 8.6.

We also derive from theorem 1.5 that our dimension coincides in P-optimal fields with the so-called "topological rank" (proposition 8.7).

Remark 1.7 All the results presented here are well known for the semi-algebraic and the classical analytic structure of p-adically closed fields, and new for p-optimal fields. Most of them are new also for the subclass of P-minimal fields with definable Skolem functions, and apparently yet unknown for general P-minimal fields[3]. The concept of p-optimal field itself seems to be new, although implicit in many papers on p-adic fields (specially [Den86], which was also a source of inspiration for us).

Acknowledgement. This paper is based on [Den84], with which the reader is expected to be familiar. Indeed we will constantly refer to Denef's proofs in sections 3 and 4. Apart of this it is essentially self-contained. However we borrowed ideas from papers of many other authors, specially Dreidre Haskell and Dugald Macpherson in [HM97], Lou van den Dries in [vdD84] and [vdD98], and Raf Cluckers in [Clu04].

Terminology and notation. For every a in K, $v(a)$ and $|a|$ denote the p-valuation of a and its norm. The norm is nothing but the valuation, but with a multiplicative notation so that $|0| = 0$, $|ab| = |a|.|b|$, $|a+b| \leq \max(|a|, |b|)$ and of course $|a| \leq |b|$ if and only if $v(a) \geq v(b)$. The valuation ring of v is R, and we fix some π in R such that πR is the maximal ideal of R.

For convenience we will sometimes add to K one more element ∞, with the property that $|x| < |\infty|$ for every x in K. We also denote ∞ any partial function with constant value ∞.

For every subset X of K we let $X^\times = X \setminus \{0\}$. Note the difference between $R^\times = R \setminus \{0\}$ and $R^* =$ the set of units in R.

Recall that K^0 is a one-point set. When a tuple $a = (x, t)$ is given in K^{m+1} it is understood that $x = (x_1, \ldots, x_m)$ and t is the last coordinate. We let $\widehat{a} = x$ denote the projection of a onto K^m. Similarly, the projection of a subset S of K^{m+1} onto K^m is denoted $\widehat{S}$. We let also:

$$\|x\| = \max\big(|x_1|, \ldots, |x_m|\big)$$

$$B(x, \rho) = \big\{y \in K^m : \|y - x\| < |\rho|\big\}$$

For every integer $e \geq 1$, $\mathbf{U}_e$ denotes the group of e-th roots of 1 in K.

Analogously to Landau's notation $\mathcal{O}(x^n)$ of calculus, we let $\mathcal{U}_{e,n}(x)$ denote *any* definable function in the multi-variable x with values in $\mathbf{U}_e.(1 + \pi^n R)$.

[3] I take this opportunity to mention that Remark 5.5 in [HM97] which outlines an argument proving corollary 1.6 in P-minimal fields is probably misleading. By mean of Lemma 7.1 in [Den84] it seems to prove it only piecewise: there is a finite partition of X in definable pieces A on each of which the restriction $f_{|A}$ of f satisfies $\dim A \setminus \mathcal{C}(f_{|A}) < \dim A$. But in the lack of theorem 1.5 one can not ensure that the pieces A of dimension $\dim X$ are open in X, hence that $\mathcal{C}(f)$ contains $\mathcal{C}(f_{|A})$. Thus corollary 1.6 does not follow from this argument, and Theorem 5.4 of [HM97] (which asserts that $\dim X \setminus \mathcal{C}(f) < m$) seems to be the best we can say at the moment for P-minimal fields.

So, given a family of functions f_i, g_i on the same domain X, we write that $f_i = \mathcal{U}_{e,n} g_i$ for every i, when there are definable functions $\omega_i : X \to R$ and $\chi_i : X \to \mathbf{U}_e$ such that for every x in X

$$f_i(x) = \chi_i(x)\big(1 + \pi^n \omega_i(x)\big)g_i(x).$$

$\mathcal{U}_{1,n}(x)$ is simply denoted $\mathcal{U}_n(x)$.

If $\tilde{K}$ is a finite extension of $\mathbf{Q}_p$ to which K is elementarily equivalent as a ring, and $\tilde{R}$ is the p-valuation ring of $\tilde{K}$, then the following set is semi-algebraic (see lemma 2.1, point 4, in [Den86])

$$\tilde{Q}_{N,M} = \{0\} \cup \bigcup_{k\in\mathbf{Z}} \pi^{kN}(1 + \pi^M \tilde{R}).$$

For every $M > v(N)$ we let $Q_{N,M}$ denote the semi-algebraic subset of K corresponding[4] by elementary equivalence to $\tilde{Q}_{N,M}$ in $\tilde{K}$. The condition $M > v(N)$ implies by Hensel's lemma that $1+\pi^M R$ is contained in $P_N^\times$. Note that $Q_{N,M}^\times$ is then a clopen subgroup of $P_N^\times$ with finite index. The next property also follows from Hensel's lemma (see for example lemma 1 and corollary 1 in [Clu01]).

Lemma 1.8 *The function $x \mapsto x^e$ is a group endomorphism of $Q_{N_0,M_0}^\times$. If $M_0 \geq 1 + v(e)$ this endomorphism is injective and its image is $Q_{eN_0,v(e)+M_0}^\times$.*

In particular $x \mapsto x^e$ defines a continuous bijection from $Q_{1,v(e)+1}$ to $Q_{e,2v(e)+1}$. We let $x \mapsto x^{\frac{1}{e}}$ denote the reverse continuous bijection. Note that it is defined on $Q_{N,M}$ for every N, M such that e divides N and $M > 2v(e)$. So $\mathcal{U}_n(x) = (\mathcal{U}_{e,n-v(e)}(x))^e$ whenever $n > 2v(e)$.

2 Basic sets

Recall the following celebrated result, stated for $\mathbf{Q}_p$ by Macintyre [Mac76]. The generalization to p-adically closed fields can be found in [PR84].

Theorem 2.1 (Macintyre) *Let F be a p-adically closed field. The semi-algebraic subsets of F^m are exactly the boolean combination of sets of the form*

$$\big\{x \in F^m : f(x) \in P_N\big\}$$

with f a polynomial function.

Remark 2.2 By the argument of lemma 2.1 in [Den84], the following sets are basic sets, for every basic functions f, g in m variables:

$$\big\{x \in F^m : f(x) = 0\big\}$$

$$\big\{x \in F^m : |g(x)| \leq |f(x)|\big\}$$

Moreover, since $P_N^\times$ is a subgroup of finite index in $F^\times$, the complement of a basic set in F^m is then a finite union of basic sets. Hence every (finite) boolean combination of basic sets is the union of intersections of finitely many basic sets. All of them can be taken the same larger order, because $P_{N'}^\times$ is a subgroup of $P_N^\times$ of finite index for every N' which is divisible by N.

[4]For a more intrinsic definition of $Q_{N,M}$ inside K, see [CL12].

Note also that if S is any definable subset of F^m then $S \times F$ is a basic subset of F^{m+1} by the above remark. Indeed the m-ary function $c_0(x)$ which equals 0 on S and 1 outside S determines on K^{m+1} a basic function $f(x,t) = c_0(x)$ of degree 0 in t, whose zero set it precisely $S \times F$.

Proposition 2.3 *Let F be a P-minimal field with m-ary definable Skolem functions, for some positive integer m. Then every definable subset of F^{m+1} is a boolean combination of basic sets.*

Thus every P-minimal field with definable Skolem functions (for every m) is p-optimal.

Proof: Let S be a definable subset of F^{m+1}, and S' the corresponding definable set in an elementary extension F' of F. For every x' in F'^m let $S'_{x'}$ denote the fiber of $\widehat{S}'$ over x':

$$S'_{x'} = \big\{t' \in F : (x', t') \in S'\big\}$$

For every x' in $\widehat{S}'$ the p-minimality of F' gives a tuple $z'_{x'}$ of coefficients of a description of $S'_{x'}$ as a boolean combination of basic sets. The model-theoretic compactness theorem then gives definable subsets $S_1, \dots, S_q$ of F^m and for every $i \leq q$ an $\mathcal{L}$–formula $\varphi_i(x,t,z)$ with $m+1+n_i$ free variables which is a boolean combination of formulas $f(x,t,z) \in P_N$ with $f \in \mathbf{Z}[x,t,z]$, such that for every x in S_i there is a list of coefficients z_x such that

$$S_x = \big\{t \in T : F \models \varphi(x,t,z_x)\big\}.$$

With other words, for every x in S_i

$$F \models \exists z \; \forall t \; \big((x,t) \in S \leftrightarrow \varphi_i(x,t,z)\big).$$

Our assumption then gives for each $i \leq q$ a definable Skolem function $\zeta_i : S_i \to F^{n_i}$ such that for every $x \in S_i$

$$F \models \forall t \; \big[(x,t) \in S \leftrightarrow \varphi_i\big(x,t,\zeta_i(x)\big)\big].$$

Let $B_i = \{(x,t) \in F^{m+1} : F \models \varphi_i(x,t,\zeta_i(x))\}$. By construction this is a boolean combination of basic subsets of F^{m+1}, hence so is $C_i = B_i \cap (S_i \times F)$. The conclusion follows, since S is the union of these C_i's.

■

It is worth mentioning that deriving p-optimality from P-minimality with Skolem functions is probably an unnecessary work-around in general (see the discussion in section 1 on P-minimality versus p-optimality).

3 Cell decomposition

Recall that, until the end of this paper, K always denotes a p-adically closed field endowed with a p-optimal structure. A **large** subgroup of $K^\times$ is a subgroup of finite index and[5] there is a positive integer n_0 such that $1 + \pi^{n_0} \subseteq G$. For instance, $P_N^\times$ and $Q_{N,M}^\times$ are large.

[5]Of course the second condition follows from the first when K is a finite extension of $\mathbf{Q}_p$ or if G is semi-algebraic.

The cells which usually appear in the literature on p-adic fields are non empty subsets of K^{m+1} of the form:

$$\{(x,t) \in X \times K : |\nu(x)| \square_1 |t - c(x)| \square_2 |\mu(x)| \text{ and } t - c(x) \in \lambda G\} \qquad (1)$$

where $X \subseteq K^m$ is a definable set, c, μ, ν are definable functions from X to K, $\square_1, \square_2$ are $\leq, <$ or no condition, $\lambda \in K$ and G is a large definable subgroup of $K^\times$. The interesting cases are when G is $K^\times$ (theorem 3.5), $P_N^\times$ (theorem 3.7) or $Q_{N,M}^\times$ (theorem 4.3).

In its simplest form, Denef's cell decomposition theorem asserts that every semi-algebraic subset of K^m is the disjoint union of finitely many cells. It will be convenient to fix a few more conditions on our cells, but most of all we want to pay attention on *how the functions defining the output cells depend on the input data.*

So we define **presented cells** in K^{m+1} as tuples $A = (c_A, \nu_A, \mu_A, \lambda_A, G_A)$ with c_A a definable function on a non-empty domain $X \subseteq K^m$ with values in K, ν_A and μ_A either definable functions on X with values in $K^\times$ or constant functions on X with values 0 or ∞, λ_A an element of K and G_A a large definable subgroup of $K^\times$ such that for every $x \in X$ there is $t \in K$ such that:

$$|\nu_A(x)| \leq |t - c_A(x)| \leq |\mu_A(x)| \text{ and } t - c_A(x) \in \lambda_A G_A \qquad (2)$$

We call it a **presented cell mod** G when $G_A = G$. Of course the set of tuples $(x,t) \in X \times K$ satisfying (2) is then a cell of K^{m+1} in the usual sense of (1). We call it the **underlying cellular set** of A. Abusing the notation we will also denote it A most often. The existence, for every $x \in X$, of t satisfying (2) simply means that X is exactly $\widehat{A}$. We call it the **basement** of A. The function c_A is called its **center**, μ_A and ν_A its **bounds**, G_A its **modulo**.

A is said to be of **type** 0 if $\lambda_A = 0$, and of type 1 otherwise. Contrary to its center, bounds, and modulo, the type of A only depends on its underlying set.

The word "cell" will usually refer to presented cells. However, for sake of simplicity, we will freely talk of disjoint cells, bounded cells, families of cells partitioning some set and so on, meaning that the underlying cellular sets of these (presented) cells have the corresponding properties. For instance, it is clear that every cellular set as in (1) is in that sense the disjoint union of finitely many (presented) cells mod G.

Lemma 3.1 (Denef) *Let S be a definable subset of K^{m+n}. Assume that there is an integer $\alpha \geq 1$ such that for every x in K^m the fiber*

$$S_x = \{y \in K^n : (x,y) \in S\}$$

has cardinality $\leq \alpha$. Then the coordinate projection of S on K^m has a definable section.

Proof: Identical to the proof of lemma 7.1 in [Den84].

■

Remark 3.2 The conclusion of the lemma remains true under the slightly weaker assumption that all the fibers S_x are finite. Indeed we are going to see that every definable subset S of K^{m+1} is the union of a finite family $\mathcal{A}$ of disjoint presented cells mod $P_N^\times$ for some N. As a consequence, if S_x is finite for every $x \in \widehat{S}$ then all the cells in $\mathcal{A}$ must be of type 0, hence the cardinality of S_x is bounded uniformly in x by the cardinality of $\mathcal{A}$.

Lemma 3.3 (Denef) *Let f be a basic function in $(x,t)=(x_1,\dots,x_m,t)$. Let $n\geq 1$ be a fixed integer. Then there exists a partition of K^{m+1} into sets A of the form*

$$A=\bigcap_{j\in S}\bigcap_{l\in S_j}\big\{(x,t)\in K^{m+1}: x\in C \text{ and } |t-c_j(x)|\square_{j,l}|a_{j,l}(x)|\big\}$$

where S and S_j are finite index sets, C is a definable subset of K^m, and c_j, $a_{j,l}$ are definable functions from K^m to K, such that for all (x,t) in A we have

$$f(x,t)=\mathcal{U}_n(x,t)h(x)\prod_{j\in S}\big(t-c_j(x)\big)^{e_j}$$

with $h:K^m\to K$ a definable function and $e_j\in\mathbf{N}$.

It is sufficient to check it for every n large enough so we can assume that:

$$1+\pi^nR\subseteq P_N\cap R^* \tag{3}$$

Thus $\mathcal{U}_n(x,t)$ in the conclusion could be replaced by $u(x,t)^N$ with u a definable function from A to R^*. This is indeed how this result is stated in lemma 7.2 of [Den84]. However it is the above equivalent (but slightly more precise) form which appears in Denef's proof, and which we retain in this paper.

Proof: Follow the proof of lemma 7.2 of [Den84], using the p-minimality assumption and basic functions in place of Macintyre's quantifier elimination and polynomial functions. Of course, lemma 7.1 used in Denef's proof has to be replaced with the analogous lemma 3.1.

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Remark 3.4 (co-algebraic functions) A remarkable by-product of Denef's proof is that the functions c_j and $a_{j,l}$ in the conclusion of lemma 3.3 belong to $\mathrm{coalg}(f)$, which we define now.

Given a basic function f in $m+1$ variables, we say that function $h:X\subseteq K^m\to K$ belongs to $\mathrm{coalg}(f)$ if there exists a finite partition of X in definable pieces H, on each of which the degree in t of $f(x,t)$ is constant, say e_H, and such that the following holds. If $e_H\leq 0$ then $h(x)$ is identically equal to 0 on H. Otherwise there is a family $(\xi_1,\dots,\xi_{r_H})$ of K-linearly independent elements in an algebraic closure of K and a family of definable functions $b_{i,j}:H\to K$ for $1\leq i\leq e_H$ and $1\leq j\leq r_H$, such that for every x in H

$$f(x,T)=a_{e_H}(x)\prod_{1\leq i\leq e_H}\Big(T-\sum_{1\leq j\leq r_H}b_{i,j}(x)\xi_j\Big)$$

and

$$h(x)=\sum_{1\leq i\leq e_H}\sum_{1\leq j\leq r_H}\alpha_{i,j}b_{i,j}(x)$$

With the $\alpha_{i,j}$'s in K. If $\mathcal{F}$ is any family of basic functions in $m+1$ variables we let $\mathrm{coalg}(\mathcal{F})$ denote the set of linear combinations of functions in $\mathrm{coalg}(f)$ for f in $\mathcal{F}$.

Theorem 3.5 (Denef) *Let $\mathcal{F}$ be a finite family of basic functions in $m+1$ variables. Let $n \geq 1$ be a fixed integer. Then there exists a finite partition of K^{m+1} into presented cells H mod $K^\times$ such that the center and bounds of H belong to* $\mathrm{coalg}(\mathcal{F}) \cup \{\infty\}$ *and for every (x,t) in H and every f in $\mathcal{F}$*

$$f(x,t) = \mathcal{U}_n(x,t) h_{f,H}(x) \big(t - c_H(x)\big)^{\alpha_{f,H}} \tag{4}$$

with $h_{f,H} : \widehat{H} \to K$ a definable function and $\alpha_{f,H} \in \mathbf{N}$.

Proof: Follow the proof of theorem 7.3 in [Den84], using once again the p-minimality assumption and basic functions in place of Macintyre's quantifier elimination and polynomial functions.

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Given two families $\mathcal{A}$, $\mathcal{B}$ of subsets of K^m, recall that $\mathcal{B}$ **refines** $\mathcal{A}$ if $\mathcal{B}$ is a partition of $\bigcup \mathcal{A}$ such that every A in $\mathcal{A}$ which meets some B in $\mathcal{B}$ contains it.

Corollary 3.6 (Denef) *Let $\mathcal{F}$ be a finite family of m-ary basic functions, $N \geq 1$ an integer and $\mathcal{A}$ a family of boolean combinations of subsets of K^m defined by $f(x) \in P_N$ with f in $\mathcal{F}$. Then there exists a finite family $\mathcal{H}$ of cells mod $P_N^\times$ with center and bounds in* $\mathrm{coalg}(\mathcal{F})$ *which refines $\mathcal{A}$.*

Proof: Theorem 3.5 applies to $\mathcal{F}$ with $n > v(N)$, so that $1 + \pi^n \subseteq P_N$. It gives a partition of K^m into presented cells B mod $K^\times$. Every such cell B is the disjoint union of finitely many presented cells H mod $P_N^\times$, whose centers and bounds are the restrictions to $\widehat{H}$ of the center and bounds of A (hence belong to $\mathrm{coalg}(\mathcal{F})$), on which $h_{f,B}(x)P_N^\times$ and $(t - c_B(x))P_N^\times$ are constant, simultaneously for every f in $\mathcal{F}$. Thus every A in $\mathcal{A}$ either contains H or is disjoint from H by (4) and our choice of n, which proves the result.

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The following simpler statement, which follows directly from corollary 3.6 by p-minimality, is sufficient in most cases.

Theorem 3.7 (Denef's cell decomposition) *For every finite family $\mathcal{A}$ of definable subsets of K^m there is for some N a finite family of presented cells mod $P_N^\times$ refining $\mathcal{A}$.*

Remark 3.8 Recall that $Q_{N,M}^\times$ is a subgroup of finite index in $P_N^\times$ (because $M > v(N)$ is assumed), and $P_N^\times$ is itself a subgroup of finite index in $K^\times$. It follows that every cell mod $K^\times$ or $P_N^\times$ is obviously the union of finitely many disjoint cells mod $Q_{N,M}^\times$. So the cells mod $Q_{N,M}^\times$ which are required in theorem 4.3 could have been used as well in all the results of the present section.

Boolean combination of cells. Denef derives theorem 3.5 (theorem 7.2 in [Den84]) from lemma 3.3 (lemma 7.1 in [Den84]) by proving that the intersection of a finite family $\mathcal{A}$ of presented cells mod $K^\times$ is the union of a finite family $\mathcal{H}$ of disjoint presented cells mod $K^\times$. Moreover, given an arbitrary integer $n \geq 1$, he builds the cells in $\mathcal{H}$ so that:

(I) The center and bounds of each H are linear combinations of the restrictions to $\widehat{H}$ of the center and bounds of the cells in $\mathcal{A}$ (except that μ_H can also be constantly equal to ∞, of course).

(II$_n$) For every $H \in \mathcal{H}$, every $A \in \mathcal{A}$ containing H and every $(x,t) \in H$

$$t - c_A(x) = \mathcal{U}_n(x,t)h(x)^\alpha \big(t - c_H(x)\big)^{1-\alpha}$$

with $h : \widehat{H} \to K$ a definable function and $\alpha \in \{0,1\}$ (both depending on H and A).

Let us quote that the same holds true for presented cells mod large groups.

Corollary 3.9 *Let $\mathcal{A}$ be a finite family of presented cells mod G, for some large definable subgroup G of $K^\times$. Then for every integer $n \geq 1$ there exists a finite family $\mathcal{H}$ of presented cells mod G refining $\mathcal{A}$ and satisfying the above conditions (I) and (II$_n$) for every H in $\mathcal{H}$ and every A in $\mathcal{A}$ containing H.*

Proof: It suffices to prove the result for n large enough, so that $1 + \pi^n R \subseteq G$. Let X be any atom of the finite boolean algebra generated by $\mathcal{A}$ among the subsets of K^m. It is the intersection of a finite family $\mathcal{B}$ of cells mod G whose centers are either 0 or c_A for some A in $\mathcal{A}$, and whose bounds are either 0, $\eta\nu_A$, $\eta\mu_A$ or ∞, for some A in $\mathcal{A}$ and some η in $K^\times$ (here we use that G has finite index in $K^\times$). Thus it suffices to prove that X is the union of a finite family $\mathcal{H}$ of disjoint cells mod G which satisfy conditions (I) and (II$_n$), with $\mathcal{B}$ in place of $\mathcal{A}$. By partitioning $\widehat{X}$ if necessary and refining $\mathcal{A}$ accordingly we can even assume that all the cells in $\mathcal{B}$ have the same basement.

For every B in $\mathcal{B}$ let $B^* = (c_B, \nu_B, \mu_B, \lambda_B, K^\times)$. Let $\mathcal{B}^*$ be the family of all these cells mod $K^\times$ as B ranges over $\mathcal{B}$. As already said, it appears in Denef's proof of theorem 3.5 that $\bigcap \mathcal{B}^*$ is the union of a finite family $\mathcal{C}^*$ of disjoint presented cells mod $K^\times$ which satisfy conditions (I) and (II$_n$) with $\mathcal{B}^*$, $\mathcal{C}^*$ in place of $\mathcal{A}$, $\mathcal{H}$. Each cell C^* in $\mathcal{C}^*$ splits into cells C mod G with the same center and bounds as C^*. For each B in $\mathcal{B}$, B^* contains C^* hence by (II$_n$) there is an exponent $\alpha_{C,B}$ in $\{0,1\}$ and a definable map $h_{C,B} : \widehat{C} \to K$ such that

$$t - c_B(x) = \mathcal{U}_n(x,t)h_{C,B}(x)^{\alpha_{C,B}} \big(t - c_C(x)\big)^{1-\alpha_{C,B}} \tag{5}$$

for every (x,t) in C^* hence *a fortiori* when $(x,t) \in C$. There is a finite partition $\mathcal{Y}$ of X in definable pieces on which $h_{C,B}(x)$ has constant residue class modulo G, simultaneously for every C, B as above. It only remains to check that every cell $H = C \cap (Y \times K)$ which meets X is contained in X, so that the family of all these cells, which obviously covers X, gives the conclusion.

In order to do so, it suffices to prove that for every B in $\mathcal{B}$, $t - c_B(x) \in \lambda_B.G$ for every (x,t) in C such that $x \in Y$. Pick any point (y,s) in $H \cap X$. By construction $s - c_B(y)$ belongs to $\lambda_B.G$ and one of the following happens. Either $\alpha_{C,B} = 0$ and by (5) applied to (y,s), $h_{C,B}(y)$ belongs to $\lambda_B.G$ (because $1 + \pi^n R$ is contained in G by assumption). Then $h_{C,B}(x)$ belongs to $\lambda_B.G$ as well because $y, x \in Y$, so $t - c_B(x)$ belongs to $\lambda_B.G$ by (5) applied to (x,t). Otherwise $\alpha_{C,B} = 1$ and by (5) applied to (y,s), $s - c_C(y)$ belongs to $\lambda_B.G$. It is then a common point of $\lambda_B.G$ and $\lambda_C.G$, hence λ_B and λ_C have the same residue class modulo G. But $t - c_C(x)$ belongs to $\lambda_C.G = \lambda_B.G$ hence $t - c_B(x)$ belongs to it as well by (5). In both cases we conclude that $t - c_B(x)$ belongs to $\lambda_B.G$, hence $(x,t) \in B$. So H is contained in every $B \in \mathcal{B}$, thus $H \subseteq \bigcap \mathcal{B} = X$.

■

4 Cell preparation for definable functions

Lemma 4.1 (Denef) *For every definable function $f : X \subseteq K^m \to K$ there is an integer $e \geq 1$ and a partition of X in finitely many definable sets A such that for every x in A*

$$\big|f(x)\big|^e = \left|\frac{p_A(x)}{q_A(x)}\right|$$

with p_A, q_A a pair of basic functions such that $q_A(x) \neq 0$ for every x in A.

Proof: The proof of Denef's theorem 6.3 in [Den84] applies word-for-word, with basic functions instead of polynomial functions, and the reference to Macintyre's theorem 2.1 replaced by our assumption that K is p-optimal.
■

Remark 4.2 Given an integer $n_0 \geq 1$, the set $1 + \pi^{n_0} R$ is a definable subgroup of R^* with finite index. Thus in lemma 4.1 we can always assume, refining the partition of X if necessary (but keeping the same integer e independently of n_0), that for every x in A

$$f(x)^e = \mathcal{U}_{n_0}(x)\frac{p_A(x)}{q_A(x)}$$

Moreover, since $P_N^\times$ has finite index in $P_{N_0}^\times$ whenever N_0 divides N, we can assume as well that A is a boolean combination of basic sets of the same power N, with N a multiple of any fixed N_0.

Theorem 4.3 (Cell preparation) *Let $(\theta_i : A_i \subseteq K^{m+1} \to K)_{i\in I}$ be a finite family of definable functions and $N_0 \geq 1$ an integer. Then there exists an integer $e \geq 1$ and, for every $n \in \mathbf{N}^\times$, a pair of integers M, N and a finite family $\mathcal{H}$ of presented cells mod $Q_{N,M}^\times$ such that $M > 2v(e)$, eN_0 divides N, $\mathcal{H}$ refines $(A_i)_{i\in I}$, and for every $(x,t) \in H$,*

$$\theta_i(x,t) = \mathcal{U}_{e,n}(x,t)h(x)\big[\lambda_H^{-1}\big(t - c_H(x)\big)\big]^{\frac{\alpha}{e}}$$

for every $i \in I$ and every $H \in \mathcal{H}$ contained in A_i, with $h : \widehat{H} \to K$ a definable function and $\alpha \in \mathbf{Z}$ (both depending on i and H)[6].

Proof: For each i let e_i be an integer, $\mathcal{A}_i$ a partition of A_i and $\mathcal{F}_i$ a family of basic functions, all given by lemma 4.1 applied to θ_i. By replacing each e_i with a common multiple we can assume that all of them are equal to some integer $e \geq 1$. Given an integer $n \geq 1$ we can refine the partition $\mathcal{A}_i$ as in remark 4.2 with $n_0 = n + 2v(e)$.

Let $\mathcal{A}$ be a finite family of definable sets refining $\bigcup_{i\in I} \mathcal{A}_i$. We can assume that each of them is a boolean combination of basic sets of the same power N, with N a multiple of eN_0. For every A in $\mathcal{A}$, every $i \in I$ such that A_i contains A and every (x,t) in A we have

$$\theta_i(x,t)^e = \mathcal{U}_{n_0}(x,t)\frac{p_{i,A}(x,t)}{q_{i,A}(x,t)} \tag{6}$$

[6] If H is of type 0 then it is understood that $\alpha = 0$ and we use the conventions that in this case $\lambda_H^{-1} = 0$ and $0^0 = 1$.

with $p_{i,A}$ and $q_{i,A}$ a pair of basic functions such that $q_{i,A}(x,t) \neq 0$ on A.

For each A in $\mathcal{A}$ let $\mathcal{F}_A$ be the set of basic functions involved in a description of A as a boolean combination of basic sets of power N. Theorem 3.5 applies to the family $\mathcal{F}$ of all the basic functions $p_{i,A}$, $q_{i,A}$ and the functions in $\mathcal{F}_A$, for every possible A's and i's. It gives a partition of K^{m+1} in finitely many presented cells B mod $K^\times$ such that for every f in $\mathcal{F}$ and every (x,t) in B

$$f(x,t) = \mathcal{U}_M(x,t) h_{f,B}(x) \big(t - c_B(x)\big)^{\beta_{f,B}} \tag{7}$$

with $M = n_0 + v(N)$, $h_{f,B} : \widehat{B} \to K$ a definable function and $\beta_{f,B}$ a positive integer.

Partitioning $\widehat{B}$ if necessary, we can assume that the cosets $h_{f,B}(x) Q_{N,M}^\times$ is constant on $\widehat{B}$. Since $M > v(N)$, $1 + \pi^M R$ is contained in $Q_{N,M}^\times$, so B itself can be partitioned into cells H mod $Q_{N,M}^\times$ such that $\widehat{H} = \widehat{B}$, $c_H = c_B$ and $f(x,t) Q_{N,M}^\times$ is constant on H by (7), for every f in $\mathcal{F}$. *A fortiori* $f(x,t) P_N^\times$ is constant on H for every f in $\mathcal{F}$ hence each A in $\mathcal{A}$ either contains H or is disjoint from H, for every A in $\mathcal{A}$. So the family $\mathcal{H}$ of all among these cells H which are contained in $\bigcup \mathcal{A}$ refines $\mathcal{A}$, hence refines $\{A_i : i \in I\}$ as well.

For every cell H in $\mathcal{H}$ there is a unique cell B as above containing H. For every $i \in I$ such that H is contained in A_i, the unique A in $\mathcal{A}$ containing B is also contained in A_i. By (7) applied to $f = p_{i,A}$ and to $f = q_{i,A}$, and by (6) we have for every $(x,t) \in H$

$$\theta_i(x,t)^e = \mathcal{U}_{n_0}(x,t) \frac{\mathcal{U}_M(x,t) h_{p_{i,A},B}(x) \big(t - c_B(x)\big)^{\beta_{p_{i,A},B}}}{\mathcal{U}_M(x,t) h_{q_{i,A},B}(x) \big(t - c_B(x)\big)^{\beta_{q_{i,A},B}}} \tag{8}$$

The $\mathcal{U}_{n_0}$ and $\mathcal{U}_M$ factors simplify in a single $\mathcal{U}_{n_0}$ since $M \geq n_0$. By construction $c_H = c_B$ and $\widehat{H} = \widehat{B}$. So, for every (x,t) in H we get

$$\theta_i(x,t)^e = \mathcal{U}_{n_0}(x,t) g(x) \big[\lambda_H^{-1} \big(t - c_H(x)\big)\big]^\alpha \tag{9}$$

with $g : \widehat{H} \to K$ a definable function and $\alpha \in \mathbf{Z}$ (both depending on i and H). In turn $\mathcal{U}_{n_0} = \mathcal{U}_{e,n_0 - v(e)}^e$ because $n_0 > 2v(e)$ (see lemma 1.8). The later can be replaced by $\mathcal{U}_n^e$ because $n_0 - v(e) = n + v(e) \geq n$. So (9) becomes

$$\theta_i(x,t)^e = \mathcal{U}_{e,n}(x,t)^e g(x) \Big(\big[\lambda_H^{-1} \big(t - c_B(x)\big)\big]^{\frac{\alpha}{e}} \Big)^e \tag{10}$$

This implies that g takes values in P_e, hence $g = h^e$ for some definable function $h : \widehat{H} \to K$, from which the conclusion follows.

■

Remark 4.4 From theorem 4.3 it follows by a straightforward induction that the norm of every definable function f on a p-optimal field is piecewise continuous.

5 Skolem functions

Lemma 5.1 *Let $(\theta_i : A \subseteq K^m \to K)_{i \in I}$ be a finite family of definable functions with the same domain. Then for every integer $n \geq 1$, there exists an integer e,* *a semi-algebraic set $\tilde{A} \subseteq K^m$ and a definable bijection $\varphi : \tilde{A} \to A$ such that for every $i \in I$ and every x in $\tilde{A}$*

$$\theta_i \circ \varphi(x) = \mathcal{U}_{e,n}(x)\tilde{\theta}_i(x)$$

with $\tilde{\theta}_i : \tilde{A} \subseteq K^m \to K$ a semi-algebraic functions.

Proof: The proof goes by induction on m. Let us assume that it has been proved for some $m \geq 0$ (it is trivial for $m = 0$) and that a finite family $(\theta_i)_{i \in I}$ of definable functions is given with domain $A \subseteq K^{m+1}$. If A is a disjoint union of sets B it suffices to prove the result for the restrictions of the θ_i's to B. So, for any given integer $n \geq 1$, theorem 4.3 with $N_0 = 1$ reduces to the case when A is a presented cell mod $Q_{N,M}^{\times}$ for some N, M such that for some $e_0 \geq 1$ dividing N, $M > 2v(e_0)$ and for every $i \in I$ and every (x, t) in A

$$\theta_i(x, t) = \mathcal{U}_{e_0,n}(x, t)h_i(x)\big[\lambda_A^{-1}\big(t - c_A(x)\big)\big]^{\frac{\alpha_i}{e_0}} \tag{11}$$

with $h_i : \widehat{A} \to K$ a definable function and $\alpha_i \in \mathbf{Z}$.

Let $e_1 \geq 1$ be an integer, $Y \subseteq K^m$ a semi-algebraic set, $\psi : Y \to \widehat{A}$ a definable bijection, $\tilde{f} : Y \to K$ a semi-algebraic function for each f in $\mathcal{F}$, all of this given by the induction hypothesis applied to $\mathcal{F} = \{\mu_A, \nu_A\} \cup \{h_i\}_{i \in I}$. Let $\tilde{A}$ be the set of $(y, s) \in Y \times K$ such that

$$|\tilde{\nu}_A(y)| \leq |s| \leq |\tilde{\mu}_A(x)| \text{ and } s \in \lambda_A Q_{N,M}^{\times}.$$

Then $\varphi : (y, s) \mapsto (\psi(y), c_A(\psi(y)) + s)$ defines a bijection from $\tilde{A}$ to A. For every $i \in I$ and every $(y, s) \in \tilde{A}$ we have

$$\theta_i \circ \varphi(y, s) = \mathcal{U}_{e_0,n}(y, s)\mathcal{U}_{e_1,n}(y, s)\tilde{h}_i(y)(\lambda_A^{-1}s)^{\frac{\alpha_i}{e_0}}$$

The first two factors can be replaced by $\mathcal{U}_{e,n}$ with e any common multiple of e_0 and e_1. Since $\tilde{\theta} : (y, s) \mapsto \tilde{h}_i(y)(\lambda_A^{-1}s)^{\frac{\alpha_i}{e_0}}$ is a semi-algebraic function on $\tilde{A}$ the conclusion follows.
■

Remark 5.2 In the above proof, given a presented cell A mod $Q_{N,M}^{\times}$ of K^{m+1} with basement X, a semi-algebraic set $Y \subseteq K^m$ and a definable bijection $\psi : Y \to X$, we have constructed a semi-algebraic set $S \subseteq K^{m+1}$ definable bijection $\varphi : S \to A$ such that

$$\widehat{\varphi(y, s)} = \psi(y) \tag{12}$$

for every (y, s) in S. Of course the same construction holds for a cell mod $P_N^{\times}$. It is known since [vdD84] that the projection of S onto Y has a semi-algebraic section, say ζ. Then (12) proves that $\varphi \circ \zeta \circ \psi^{-1}$ is a definable section of the projection of A onto $\widehat{A}$.

Theorem 5.3 *K has m-ary definable Skolem functions for every m.*

Proof: By a straightforward induction it suffices to prove that for every definable subset A of K^{m+1} the coordinate projection of A onto $\widehat{A}$ has a definable section. If A is a union of finitely many definable sets B and if a definable section $\sigma_B \widehat{B} \to B$ has been found for each projection of B onto $\widehat{B}$ we are done, because using lemma 3.1 we can choose definably for each x in $\widehat{A}$ one element among the finitely many $\sigma_B(x)$'s such that $x \in B$. Thus it suffices to prove the result when A is a cell mod $P_N^\times$ for some N, which has been done in remark 5.2. ■

Of course theorem 5.3 could as well be proved in a more constructive way by using the cell decomposition theorem 3.7), just as it is done for semi-algebraic sets in the appendix of [DvdD88].

Corollary 5.4 *The cell preparation theorem 4.3 for definable functions holds in an expansion $\mathcal{F}$ of a p-adically closed field F if and only if $\mathcal{F}$ is p-optimal.*

Proof: One direction has been proved. For the converse, the case of subsets of F^0 being trivial, let A be a definable subset of F^{m+1}. We have to prove that A is a boolean combination of basic sets. By p-optimality and theorem 3.7 we can assume that A is a presented cell $(c, \nu, \mu, \lambda, P_N^\times)$ for some N.

Let Y be a semi-algebraic subset of K^m and $\psi : Y \to \widehat{A}$ a definable bijection given by lemma 5.1 (which follows directly from theorem 4.3) applied to (c, ν, μ) with $n = 1$. Let $\widetilde{A}$ be the set of tuples (y, s) in $Y \times K$ such that

$$|\widetilde{\nu}(y)| \leq |s| \leq |\widetilde{\mu}(y)| \ \text{ and } \ s \in \lambda P_N^\times. \tag{13}$$

This is a semi-algebraic subset of F^{m+1}. By Macintyre's theorem 2.1 $\widetilde{A}$ is then defined by a boolean combination of conditions $f(y, s) \in P_{N'}$ with f a polynomial function and $N' \geq 1$ an integer. Since $|\nu \circ \psi| = |\widetilde{\nu}|$ and $|\mu \circ \psi| = |\widetilde{\mu}|$, for every (x, t) in $K^m \times K$ we have by (13):

$$(x, t) \in A \iff \big(\psi^{-1}(x), t - c(x)\big) \in \widetilde{A}.$$

So (x, t) belongs to A if and only it satisfies a boolean combination of conditions $f(\psi^{-1}(x), t - c(x)) \in P_{N'}$, and every such condition defines a basic set since f is polynomial. ■

6 Continuity

For every definable subsets $B \subseteq A$ of K^m let $\operatorname{Int}_A B$ denote the relative interior of B inside A, that is the largest set $U \cap A$ contained in B with U an open subset of K^m. We simply note $\operatorname{Int} B$ for $\operatorname{Int}_{K^m} B$.

Theorem 6.1 *Let $A_1, \dots, A_r \subseteq A$ be a family of definable subsets of K^m. If the union of the A_k's has non empty interior in A then at least one of them has non empty interior in A.*

The argument below is adapted from [HM97], Theorem 3.2.

Proof: It suffices to prove the result under the additional hypothesis that the A_k's are disjoint, and for $r = 2$. Assume that it has been done in K^m (it is obvious for $m = 0$) and let A_1, A_2 be two disjoint definable sets whose union has non empty interior inside a definable subset A_0 of K^{m+1}. So there is a box

$U = X \times Y$ with X (resp. Y) a non empty open subset of K^m (resp. K) such that $A_0 \cap U$ is non empty and contained in $A_1 \cup A_2$. Replacing each A_k by $A_k \cap U$ if necessary, we can assume that $A_0 \subseteq U$ hence $A_1 \cup A_2 = A$.

For $k = 0, 1, 2$ and every x in X, let $A_{k,x} = \{y \in K : (x, y) \in A_k\}$. This is a definable subset of K hence $A_{k,x} \setminus \mathrm{Int}(A_{k,x})$ is finite by p-minimality. So the coordinate projection of the definable set

$$B = \bigcup_{\substack{x\in X\\ k=1,2}} \{x\} \times \Big(A_{k,x} \setminus \mathrm{Int}(A_{k,x})\Big)$$

onto K^m has finite fibres B_x. By cell decomposition, the cardinality of these fibres is bounded by some integer N (see remark 2.2). Let $(y_i)_{0\le i\le N}$ be a sequence of distinct elements in the open set Y. For each $i \le N$ let

$$V_i = \{x \in \widehat{A}_0 : y_i \notin B_x\}.$$

For each x in $\widehat{A}_0$ at least one of the $N + 1$ elements y_i does not belong to B_x hence $\widehat{A}_0$ is covered by the V_i's. By induction hypothesis at least one of them, say V_0, has non empty interior Z_0 inside $\widehat{A}_0$. Then $y_0 \notin B_x$ for every x in Z_0, hence $Z_0 = W_1 \cup W_2$ where W_k is defined as

$$W_k = \big\{x \in Z_0 : y_0 \in \mathrm{Int}\, A_{k,x}\big\}.$$

By induction hypothesis it follows that at least one of the W_k's, for example W_1, has non empty interior Z_1 inside Z_0 hence inside $\widehat{A}_0$. For every x in Z_1 there is ρ_x in $K^\times$ such that the ball $B(y_0, \rho_x)$ is contained in $\mathrm{Int}(A_{1,x})$. Let $\rho : Z_1 \to K^\times$ be a definable Skolem function corresponding to this property. By remark 4.4 $|\rho|$ is piecewise continuous, that is Z_1 is the disjoint union of finitely many cells on which the restriction of $|\rho|$ is continuous. By induction hypothesis at least one of these cells has non empty interior inside Z_1 hence inside $\widehat{A}_0$. Fix an element a in this interior, and δ in $K^\times$ such that $B(a, \delta) \cap \widehat{A}_0$ is contained in Z_1. By continuity of $|\rho|$ there is $\delta' \in K^\times$ such that $|\rho|$ remains constant on $B(a, \delta') \cap \widehat{A}_0$. We can assume that $|\delta'[\le |\delta|$ so we have

$$A \cap \big[B\big(a, \delta'\big) \times B\big(y_0, \rho(a)\big)\big] \subseteq A_1.$$

Thus (a, y_0) belongs to $\mathrm{Int}_A A_1$, which is then non empty.

■

Remark 6.2 It follows that for every definable function $f : X \subseteq K^m \to K$ there is a set U dense and open in X on which $|f|$ is continuous. Indeed by remark 4.4 there is a finite partition $\mathcal{A}$ of X in definable pieces on which the restriction of $|f|$ is continuous, hence $|f|$ itself is continuous on $U = \bigcup\{\mathrm{Int}_X(A) : A \in \mathcal{A}\}$. This is a definable set open in X. As $A \setminus \mathrm{Int}_X(A)$ has empty interior in X, so does the union of these sets by theorem 6.1. That is $X \setminus U$ has empty interior in X, hence U is dense in X. If moreover f takes values in $K^\times$ it follows that $|f|$ is locally constant on U.

Corollary 6.3 *Let $B_1, \dots, B_r, B \subseteq A$ be definable sets in K^m.*

1. *If B is dense in A then $\mathrm{Int}_A(B)$ is dense in A.*

2. If $B_1, \dots, B_r$ are dense in A then so is their intersection.

Proof: For the first point consider $A \setminus B$, $B \setminus \mathrm{Int}_A(B)$, $\mathrm{Int}_A(B)$ and note that the two firsts of them have empty interior in A. It follows that their union has empty interior in A by theorem 6.1 hence their complement $\mathrm{Int}_A(B)$ is dense in A. For the second point apply theorem 6.1 to the complements of the B_k's.
■

Lemma 6.4 *Unary definable functions in K have at most finitely many discontinuities.*

Proof: Assume that a definable function $f : X \subseteq F \to F$ has infinitely many discontinuities. The set of points at which f is not continuous is definable, hence by p-minimality it contains a non-empty open set U. For every a in U there is $\varepsilon \neq 0$ such that

$$\forall \delta \in K^\times,\ \exists x \in U,\ |x - a| < |\delta| \text{ and } |f(x) - f(a)| \geq \varepsilon. \tag{14}$$

By theorem 5.3 there is a definable Skolem function $\varepsilon : U \to K^\times$ corresponding to this property. By remark 6.2) there exists ε_0 in $K^\times$ and a non-empty open subset V of U on which $|\varepsilon(x)| = |\varepsilon_0|$.

Note that $f(V)$ is infinite, otherwise V would contain on infinite subset on which f is constant, which will itself contain a non-empty open subset W by p-minimality. But then f will be continuous on W, contradicting the fact that $W \subseteq U$.

So $f(V)$ is infinite, hence it contains a ball B' which can be chosen with radius $\rho \neq 0$ such that $|\rho| \leq |\varepsilon_0|$. For the same reason $f^{-1}(B') \cap V$ contains a ball B of radius some $\delta \neq 0$. Pick some a in B. For every x in U such that $|x - a| < |\delta|$ we have $a, x \in B$ hence $f(a), f(x) \in B'$, so $|f(x) - f(a)| < |\varepsilon_0|$ (because B' has radius $|\rho| \leq |\varepsilon_0|$) which contradicts (14).
■

For every definable function $f : X \subseteq K^m \to K$ we let $\mathcal{C}(f)$ denote the set of elements x such that f is continuous on a neighbourhood[7] of x in X. This is a definable set, and the largest set open in X on which f is continuous.

Theorem 6.5 *For every definable function $f : X \subseteq K^m \to K$, $X \setminus \mathcal{C}(f)$ has empty interior inside X.*

Proof: We have to show that $\mathcal{C}(f)$ is dense in X. By the second point of corollary 6.3 we can prove it separately for each coordinate function of f hence we can assume that $n = 1$. In order to do so it suffices to prove, by induction on m, that there is a set U dense and open in X on which f is continuous. Assume that it has been done for some $m \geq 1$ (it is obvious for $m = 0$, and true for $m = 1$ by lemma 6.4). Let f be a definable function with domain $X \subseteq K^{m+1}$. For every $(a, b) \in X$ we let X^b (resp. X_a) be the set of elements x in K^m (resp. t in K) such that (x, b) (resp. (a, t)) belongs to X. Define $f^b : X^b \to K$ and $f_a : X_a \to K$ by $f^b(x) = f(x, b)$ and $f_a(t) = f(a, t)$ respectively, and let

$$Z_1 = \{(a, b) \in X : f^b \text{ is continuous on neighbourhood of } a \text{ in } X^b\}$$

[7] Of course "a neighbourhood of x in X" refers to the topology induced on X by K^m, that is a set $V = U \cap X$ with U an open subset of K^m containing x. In general V will not be open in K^m.

$$Z_2 = \big\{(a,b) \in X : f_a \text{ is continuous on neighbourhood of } b \text{ in } X_a\big\}.$$

Pick any $(a,b) \in X$ and any $\varepsilon \in K^\times$. By induction hypothesis $\mathcal{C}(f^b)$ is dense in X^b so there is a' in $\mathcal{C}(f^b)$ such that $\|a - a'\| < |\varepsilon|$. Then (a', b) belongs to Z_1 and to the ball of center (a,b) and radius ε. Thus Z_1 is dense in X, and so is Z_2 by a symmetric argument. By corollary 6.3 $Z_1 \cap Z_2$ has a dense interior Z in X. Thus, replacing f by its restriction to Z if necessary we can assume that $Z_1 = Z_2 = X$.

Under this assumption there is a definable Skolem function $\zeta_1 : X \times K^\times \to K^\times$ such that for every (a,b) in X and every non zero ε, for every x in K^m

$$\|x - a\| < |\zeta_1(a,b,\varepsilon)| \Rightarrow \Big(x \in X^b \text{ and } |f^b(x) - f^b(a)| < |\varepsilon|\Big). \tag{15}$$

Symmetrically there is a definable Skolem function $\zeta_2 : X \times K^\times \to K^\times$ such that for every (a,b) in X and every non zero ε, for every t in K

$$|t - b| < |\zeta_2(a,b,\varepsilon)| \Rightarrow \Big(t \in X_a \text{ and } |f_a(t) - f_a(b)| < |\varepsilon|\Big). \tag{16}$$

By remark 6.2 there is a definable set W dense and open in $X \times K^\times$ on which $|\zeta_1|$ and $|\zeta_2|$ are continuous. Let $\mathcal{C}$ be a partition of W into cells mod $P_N^\times$ for some N. Refining $\mathcal{C}$ if necessary we can assume that the center of each cell in $\mathcal{C}$ either is constantly equal to 0 or takes values only in $K^\times$. Using remark 4.4 we can even refine $\mathcal{C}$ so that the center and bounds of every cell in $\mathcal{C}$ have continuous norms. Moreover, as $\widehat{W}$ is dense and open in X, restricting f to $\widehat{W}$ if necessary we can assume that $\widehat{W} = X$.

Let $\mathcal{D} = \{\widehat{C} : C \in \mathcal{C}\}$. By refining $\mathcal{D}$ if necessary and then $\mathcal{C}$ accordingly, we can assume that $\mathcal{D}$ is a partition of X and that every D in $\mathcal{D}$ either is open in X or lacks interior in X. Now let $\mathcal{D}^\circ$ be the family of sets D in $\mathcal{D}$ which are open in X and Z_3 be their union. It is open in X, and its complement $X \setminus Z_3$ lacks interior by theorem 6.1 (because $X \setminus Z_3$ is the union of the sets D in $\mathcal{D} \setminus \mathcal{D}^\circ$ and each of them lacks interior in X). Thus it suffices to find for each D in $\mathcal{D}^\circ$ a definable set U_D dense and open in D on which f is continuous: the union of the U_D's will then be dense and open in Z_3 hence in X.

So pick any D in $\mathcal{D}^\circ$ and let $\mathcal{C}_D = \{C \in \mathcal{C} : \widehat{C} = D\}$. Let $\mathcal{C}_D^0$ be the family of cells C in $\mathcal{C}_D$ such that $c_C = \nu_C = 0$, and $\mathcal{C}_D^1 = \mathcal{C}_D \setminus \mathcal{C}_D^0$. Note that $\bigcup \mathcal{C}_D = W \cap (D \times K)$ is dense in $D \times K^\times$ because W is dense in $X \times K^\times$ and D is open in X. As every C in $\mathcal{C}_D^0$ is closed in $D \times K^\times$ (because $c_C = 0$ and the norm of the bounds of C are continuous) we have

$$D \times K^\times = \bigcup_{C \in \mathcal{C}_D} \overline{C} \cap (D \times K^\times) \subseteq \bigcup_{C \in \mathcal{C}_D^0} C \cup \bigcup_{C \in \mathcal{C}_D^1} \overline{C}. \tag{17}$$

We claim that there is a set U_D dense and open in D and a definable map $\rho_D : U_D \to K^\times$ with continuous norm such that the cell $W_D = (0, 0, \rho_D, 1, K^\times)$ is contained in W. Indeed, for every $z = (x,t)$ in D and every C in $\mathcal{C}_D$, $C_z = C \cap (\{z\} \times K)$ is closed except if C is of type 1 and $\nu_C = 0$, in which case its closure is $C_z \cup \{c_C(z)\}$. Thus $(z, 0)$ belongs to the closure of C_z if and only if C belongs to $\mathcal{C}_D^0$. So there is a definable Skolam function $\rho : D \to K^\times$ such that $\{z\} \times B(0, \rho(z))$ is disjoint from C_z for every C in $\mathcal{D}_D^1$ and every z in D. By remark 6.2 there is a definable set U_D dense and open in D on which $|\rho|$ is continuous. Let ρ_D be the restriction of ρ to U_D and W_D be the corresponding cell mod $K^\times$ as above. W_D is open (because $|\rho_D|$ is continuous), and disjoint from every C in $\mathcal{C}_D^1$ hence also disjoint from their closure. On the other hand W_D is contained in $D \times K^\times$ so by (17) it must be contained in the union of $\mathcal{C}_D^0$, hence in W.

It only remains to check that f is continuous on U_D. In order to do so, choose any (a,b) in U_D and fix any non zero ε. Without loss of generality we can assume that $|\varepsilon| < |\rho(a,b)|$. By continuity of $|\rho|$ there is a neighbourhood V of (a,b) in U_D on which $|\varepsilon| < |\rho(x,t)|$, so $V \times \{\varepsilon\}$ is contained in W. The continuity of $|\zeta_1|$ and $|\zeta_2|$ on W, hence on $V \times \{\varepsilon\}$, then gives a non zero δ such that $|\zeta_1(x,t,\varepsilon)|$ and $|\zeta_2(x,t,\varepsilon)|$ are constant on $B((a,b),\delta) \cap V$. Without loss of generality we can assume that $B((a,b),\delta) \cap U_D \subseteq V$ and

$$0 < |\delta| \leq \min\big(|\zeta_1(a,b,\varepsilon)|, |\zeta_2(a,b,\varepsilon)|\big).$$

For every (x,t) in $B((a,b),\delta) \cap U_D$ we have then

$$\|x-a\| < |\zeta_1(a,b,\varepsilon)| = |\zeta_1(x,t,\varepsilon)|$$

hence by (15), x belongs to X^t and

$$\big|f(x,t) - f(a,t)\big| = \big|f^t(x) - f^t(a)\big| < |\varepsilon|. \tag{18}$$

On the other hand we have $|t-b| < |\zeta_2(a,b,\varepsilon)|$ hence by (16), t belongs to X_a and

$$\big|f(a,t) - f(a,b)\big| = \big|f_a(t) - f_a(b)\big| < |\varepsilon|. \tag{19}$$

Combining (18) and (19) we get $|f(x,t) - f(a,b)| < |\varepsilon|$. This being true for every (x,t) in $B((a,b),\delta) \cap U_D$ and every non zero ε, it follows that f is continuous at (a,b).

■

7 Dimension

As in [HM97] for P-minimal fields we define over every p-optimal field K the **dimension** (the "topological dimension" in [HM97]) of a non-empty definable subset S of K^m, denoted $\dim S$, as the greatest integer d such that there exists a subset I of $\{1,\dots,m\}$ such that $\pi_I^m(S)$ has non-empty interior, where $\pi_I^m : F^m \to F^d$ is defined by

$$\pi_I^m : (x_i)_{1\leq i\leq m} \mapsto (x_{i_k})_{1\leq k\leq d}$$

with $i_1 < \cdots < i_d$ an enumeration of I. This projection will be denoted π_I when m is clear from the context. By convention $\dim \emptyset = -\infty$.

Note that every definable subset of K with dimension 0 has empty interior hence is finite by p-minimality, and conversely.

Proposition 7.1 *For every definable subsets $A_1,\dots,A_r$ of K^m*

$$\dim\big(A_1 \cup \cdots \cup A_r\big) = \max\big(\dim A_1,\dots,\dim A_r\big).$$

Proof: It suffices to prove it for $r = 2$. That $\dim A_1 \cup A_2 \geq \max(\dim A_1, \dim A_2)$ follows immediately from the definition. Conversely let $d = \dim A_1 \cup A_2$ and I be a subset of $\{1, \ldots, m\}$ with d elements such that $\pi_I(A_1 \cup A_2)$ has non empty interior in K^d. Since $\pi_I(A_1 \cup A_2)$ is the union of $\pi_I(A_1)$ and $\pi_I(A_2)$, by theorem 6.1 at least one of them must have non empty interior in K^d. Thus $\dim A_1 \geq d$ or $\dim A_2 \geq d$ and the conclusion follows.
■

Given a permutation σ of $\{1, \ldots, m\}$, for every $x = (x_1, \ldots, x_m)$ in K^m we let $x^\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(m)})$. Similarly we let $S^\sigma = \{x^\sigma : x \in S\}$ for every definable subset S of K^m. Note that $\pi_I(S^\sigma)$ has non empty interior if and only if $\pi_{\sigma(I)}(S)$ has non empty interior hence

$$\dim S^\sigma = \dim S \tag{20}$$

Following [vdD89], a dimension function on the definable sets of a first-order structure A is a function d with values in $\mathbf{N} \cup \{-\infty\}$ such that for every positive integer m and every definable sets S, S_1, S_2 of A^m we have:

(Dim 1) $d(S) = -\infty \Leftrightarrow S = \emptyset$, $d(\{a\}) = 0$ for each $a \in A$, $d(A) = 1$.

(Dim 2) $d(S_1 \cup S_2) = \max(d(S_1), d(S_2))$.

(Dim 3) $d(S^\sigma) = d(S)$ for each permutation σ of $\{1, \ldots, m\}$.

(Dim 4) For $k = 0, 1$ and every definable set $T \subseteq K^{m+1}$ the set $T(k)$ of x in $\widehat{T}$ such that $d(T_x) = k$, where T_x denotes the fiber of T over x, is definable and

$$d\big(\{(x,t) \in T : x \in T(k)\}\big) = \dim\big(T(k)\big) + k.$$

The aim of this section is to prove that in p-optimal fields dim is a dimension function. It satisfies (Dim 1) by definition, (Dim 2) by proposition 7.1, and (Dim 3) by (20). Moreover, if T is a definable subset of K^{m+1} then by p-minimality $x \in T(0)$ if and only if the points in T_x are isolated, hence $T(0)$ is indeed definable and so is $T(1) = \widehat{T} \setminus T(0)$. Thus we only have to prove the dimension formula of (Dim 4).

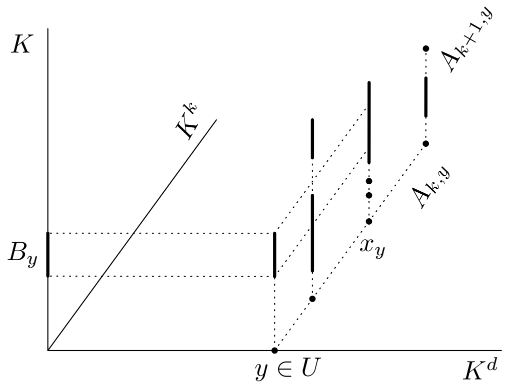
Lemma 7.2 *Let A be a definable subset of K^m, $d = \dim A$ and I a subset of $\{1, \ldots, m\}$ with d elements such that $\pi_I(A)$ has non empty interior. Let*

$$Z = \big\{y \in \pi_I(A) : \pi_I^{-1}(\{y\}) \cap A \text{ is infinite}\big\}.$$

Then Z has empty interior.

Proof: Let $i_1 < \cdots < i_d$ be an enumeration of I, and σ a permutation of $\{1, \ldots, m\}$ such that $\sigma(k) = i_k$ for every $k \leq d$. Replacing A by A^σ if necessary we can assume that $I = \{1, \ldots, d\}$.

For every $d \leq k \leq m$ let $A_k = \pi^m_{\{1,\ldots,k\}}(A)$. For every y in $A_d = \pi^m_I(A)$ let $A_{k,y}$ be the set of x in A_k such that $\pi^k_I(x) = y$. We let Z_k be the set of y in A_d such that $A_{k,y}$ is infinite. Clearly Z_d is empty, $Z_m = Z$ and $Z_k \subseteq Z_{k+1}$ for every k in between. Assume for a contradiction that Z has non empty interior. Let $k < m$ be then the greatest index such that Z_k lacks interior, and let $J = I \cup \{k+1\}$.



By theorem 6.1 $Z_{k+1} \setminus Z_k$ has non empty interior, so let U be a non empty open subset of K^d contained in it. For every y in U, $A_{k,y}$ is finite and $A_{k+1,y}$ is infinite, so there is an element x_y in $A_{k,y}$ and by p-minimality a ball B_y in K such that $\{x_y\} \times B_y$ is contained in $A_{k+1,y}$. *A fortiori* we have

$$\{y\} \times B_y \subseteq \pi_J^{d+1}(A_{k+1,y}) \subseteq \pi_J^{d+1}(A_{k+1}) = \pi_J^m(A).$$

By theorem 5.3 their is a pair of definable Skolem functions $c : U \to K$ and $\rho : U \to K^\times$ such that every B_y contains the ball of center $c(y)$ and radius $\rho(y)$. As U is open in K^d, by theorem 6.5 there is a non empty open set V in K^d contained in U such that c and ρ are continuous on V. We have

$$\big\{(y,t) \in V \times K : |t - c(y)| < |\rho(y)|\big\} \subseteq \pi_J^m(A).$$

This is a non empty open subset of K^{d+1} because c, d are continuous and V is non empty open in K^d. It follows that $\dim A \geq d+1$, a contradiction.
■

Lemma 7.3 *For every cell $A \subseteq K^{m+1}$, $\dim A = \dim \widehat{A} + \operatorname{tp} A$.*

Proof: Let A be a presented cell mod G in K^{m+1}, for some large definable subgroup of $K^\times$. Let d be its dimension.

We first prove that $\dim A \geq \dim \widehat{A} + \operatorname{tp} A$. Let $e = \dim \widehat{A}$ and $J \subseteq \{1, \dots, m\}$ an index set with e elements such that $\pi_J^m(\widehat{A})$ has non empty interior in K^e. Then $\pi_J^{m+1}(A) = \pi_J^m(\widehat{A})$ so $\dim A \geq \dim \widehat{A}$. If A is of type 0 we are done. Otherwise let $I = J \cup \{m+1\}$, let $\sigma : \pi_J^m(\widehat{A}) \to \widehat{A}$ be a definable section of π_I^m and let B be the presented cell in K^{d+1} defined by

$$B = (c_A \circ \sigma, \nu_A \circ \sigma, \mu_A \circ \sigma, \lambda_A, G).$$

This is a cell of type 1 contained in $\pi_I^{m+1}(A)$. Moreover $\widehat{B} = \pi_J^m(\widehat{A})$ has non empty interior in K^e. Thus by theorem 6.5 the center and bounds of B are continuous on a non-empty definable set V open in K^e. This continuity implies that $B \cap (V \times K)$ is open in K^{e+1}. As it is contained in $\pi_I^{m+1}(A)$ it follows that $\dim A \geq e+1$ so we are done.

For the reverse inequality, note first that $\dim \widehat{A} + 1 \geq \dim A$. Indeed, let I be a subset of $\{1, \dots, m+1\}$ with d elements such that $\pi_I^{m+1}(A)$ has non empty interior. If $m+1 \notin I$ then $\pi_I^m(\widehat{A}) = \pi_I^{m+1}(A)$ hence $\dim \widehat{A} \geq d$ and *a fortiori*

$\dim \widehat{A} + 1 \geq d$. Otherwise $m+1 \in I$ so $J = I \setminus \{m+1\}$ has $d-1$ elements. Then $\pi_J^m(\widehat{A})$ has non empty interior in K^{d-1} since

$$\pi_J^m(\widehat{A}) = \widehat{\pi_I^{m+1}(A)}.$$

So $\dim \widehat{A} \geq d-1$ hence $\dim \widehat{A} + 1 \geq d$.

We have proved that $\dim \widehat{A} + \operatorname{tp} A \leq \dim A \leq \dim \widehat{A} + 1$. If A is of type 1 the conclusion follows. Now assume that A is of type 0. It remains to prove that in this case $\dim A \leq \dim \widehat{A}$. So let I be a subset of $\{1, \ldots, m+1\}$ with d elements such that $\pi_I^{m+1}(A)$ has non empty interior. Assume for a contradiction that $\dim \widehat{A} < d$. This implies that $m+1$ belongs to I (otherwise $\pi_I^m(\widehat{A}) = \pi_I^{m+1}(A)$ has non empty interior hence $\dim \widehat{A} \geq d$). Let $J = I \setminus \{m+1\}$, $B = \pi_I^{m+1}(A)$ and $Y = \widehat{B} = \pi_J^m(\widehat{A})$.

By assumption B contains a non empty open subset U of K^d. Since J has $d-1$ elements and $\dim \widehat{A} = d-1$, lemma 7.2 implies that the set

$$Z = \big\{ y \in Y : \pi_J^{-1}(\{y\}) \cap \widehat{A} \text{ is infinite} \big\}$$

has empty interior. Now $\widehat{U}$ is a non empty open subset of K^{d-1} contained in Y, so it cannot be contained in Z. Pick any y in $\widehat{U} \setminus Z$. As $y \notin Z$ there are finitely many points $x_1, \ldots, x_N$ in $\widehat{A}$ such that $\pi_J^m(x_i) = y$. For every s in K such that (y,s) belongs to U there is (x,t) in A such that $\pi_I^{m+1}(x,t) = (y,s)$. But this implies that $\pi_J^m(x) = y$ and $s = t = c_A(x)$, so $s = c_A(x_i)$ for some i. Thus $U \cap (\{y\} \times K)$ is finite, contradicting that U is open in K^d.
■

Theorem 7.4 *The function* dim*, defined for definable sets over a p-optimal field, is a dimension function.*

Proof: Let T be a definable subset of K^{m+1}. By cell decomposition there is for some N a partition $\mathcal{A}$ of T into cells mod $P_N^\times$. Refining the basements of the cells in $\mathcal{A}$ if necessary, we can assume that $\{\widehat{A} : A \in \mathcal{A}\}$ is a partition of $\widehat{T}$ refining $\{T(0), T(1)\}$. For $k = 0, 1$ let $\mathcal{A}_k$ denote the cells in $\mathcal{A}$ such that $\widehat{A_k} \subseteq T(k)$. Then $\{(x,t) \in T : x \in T(k)\}$ is the union of the cells in $\mathcal{A}_k$ and $T(k)$ is the union of their basements, so by proposition 7.1 we have

$$\dim\{(x,t) \in T : x \in T(k)\} = \max_{A \in \mathcal{A}_k} \dim A \tag{21}$$

$$\dim T(k) = \max_{A \in \mathcal{A}_k} \dim \widehat{A} \tag{22}$$

For $k = 0$, every cell A in $\mathcal{A}_0$ is of type 0 hence $\dim A = \dim \widehat{A}$ by lemma 7.3. The conclusion follows from (21) and (22) in that case.

For $k = 1$, there is at least one cell B in $\mathcal{A}_1$ such that $\dim \widehat{B} = \dim T(1)$ by (22) and proposition 7.1. Since $\widehat{B} \subseteq T(1)$, $A \cap (\widehat{B} \times K)$ projects with infinite fiber onto $\widehat{B}$. Hence there is at least one cell C in $\mathcal{A}$ of type 1 such that $\widehat{C} = \widehat{B}$ (hence $C \in \mathcal{A}_1$). By lemma 7.3 we have

$$\dim C = \dim \widehat{C} + 1 = \dim T(1) + 1.$$

Now for every A in $\mathcal{A}_1$ we have by lemma 7.3 and (22)

$$\dim A = \dim \widehat{A} + \operatorname{tp} A \leq \dim T(1) + 1 = \dim C$$

So $\max_{A \in \mathcal{A}_k} \dim A = \dim C = \dim T(1) + 1$ as expected.
■

Corollary 7.5 *Let $f : A \subseteq K^m \to K^n$ be a definable map.*

1. $\dim f(A) \leq \dim A$ *and equality holds when f is injective.*
2. *For every positive integer d the set $S(d)$ defined by*
$$S(d) = \{y \in f(A) : \dim f^{-1}(\{y\}) = d\}$$
is definable and $\dim f^{-1}(S(d)) = d + \dim S(d)$.

Proof: This follows directly from theorem 7.4, as general properties of dimension functions (see corollary 1.5 in [vdD89]).
■

It is worth mentioning the following consequence of corollary 7.5, which will be needed in the next section.

Corollary 7.6 *Let $f : A \subseteq K^m \to K^n$ be a definable map, and B a definable subset of A. If* $\dim B \cap f^{-1}(\{y\}) < \dim f^{-1}(\{y\})$ *for every y in $f(A)$ then* $\dim B < \dim A$.

Proof: Let $S = f(A)$, g the restriction of f to B and $T = g(B)$. Let $S(d)$ (resp. $T(e)$) be as in corollary 7.5 for f (resp. g) and for every positive integers d, e. As $B = \bigcup_{e \leq n} g^{-1}(T(e))$ there is some $e \leq n$ such that $\dim B = \dim g^{-1}(T(e))$. By assumption $\dim g^{-1}(\{y\}) < \dim f^{-1}(\{y\})$ for every y in B hence $T(e)$ is contained in the union of $S(d)$ for $d > e$. It follows that $\dim T(e) \leq \dim S(d)$ for some $d > e$ by proposition 7.1, hence $e + \dim T(e) < d + \dim S(d)$. Thus by corollary 7.5, $\dim B = \dim g^{-1}(T(e)) < \dim f^{-1}(S(d)) \leq \dim A$.
■

Applications Theorem 4.3 and lemma 5.1 are exactly analogous to theorems 2.8 and 3.1 in [Clu04], except that we obtain a slightly more precise equality of functions mod $(1 + \pi^n R).\mathbf{U}_e$ instead of equality of their norm (which is the same as equality of functions mod R^*). Thus all the important consequences that are derived from these theorems in [Clu04] for the classical analytic structure remain valid in every p-optimal field.

For applications to parametric integrals, which require numerous specific definitions, we refer the reader to the proofs of theorems 4.2 and 4.4 in [Clu04]. For the classification of definable sets up to isomorphisms, we have the following.

Theorem 7.7 *There exists a definable bijection between two infinite definable sets $A \subseteq K^m$ and $B \subseteq K^n$ if and only if they have the same dimension.*

Proof: If there is a definable bijection (an "isomorphism") between A and B they have the same dimension by the first part of corollary 7.5. Conversely, if A and B have the same dimension d then by lemma 5.1 they are isomorphic to infinite semi-algebraic sets $\tilde{A}$ and $\tilde{B}$ respectively, both of which have dimension d by corollary 7.5. Then $\tilde{A}$ and $\tilde{B}$ are semi-algebraically isomorphic by the main result of [Clu01], hence A and B are isomorphic.
■

8 Boundary and topological rank

Theorem 8.1 *For every non empty definable subset A of K^m*

$$\dim \overline{A} \setminus A < \dim A.$$

As a consequence $\dim A = \dim \overline{A}$.

Remark 8.2 This result does not follow from proposition 2.23 of [vdD89] since p-optimal fields are *not* algebraically bounded in the restricted sense of [vdD89]. The classical analytic structure on $\mathbf{Q}_p$ is a counter-example (see [vdD89], page 191).

The following proof is borrowed from the o-minimal analogous theorem 1.8 in [vdD98].

Notation Given an index $i \in \{1, \ldots, m\}$ we let $\pi_i : K^m \to K$ denote the coordinate projection onto the i-th axis. For every $S \subseteq K^m$ and $z \in K$ we let

$$S_z^{(i)} = \big\{x \in K^{m-1} : (x_1, \ldots, x_{i-1}, z, x_i, \ldots, x_{m-1}) \in S\big\}.$$

Lemma 8.3 *For every definable subset A of K^m there are only finitely many z in K such that*

$$(\overline{A})_z^{(i)} \neq \overline{A_z^{(i)}}.$$

Proof: Permuting the axes if necessary we can assume that $i = 1$. For $z \in K$ and $S \subseteq K^n$ with $n \geq 1$ we simply write S_z for $S_z^{(1)}$.

Let Z be the set of elements z in K such that $(\overline{A})_z \neq \overline{A_z}$. Replacing A by $A \cap \pi_1^{-1}(Z)$ if necessary we can assume that $Z = \pi_1(A)$. Let C be the set of (z, x, ρ) in $Z \times K^{m-1} \times K$ such that

$$B(x, \rho) \cap (\overline{A})_z \neq \emptyset \ \text{ and } \ B(x, \rho) \cap A_z = \emptyset$$

As $\overline{A_z}$ is always contained in $(\overline{A})_z$, by construction $C_z \neq \emptyset$ for every z in Z. Moreover, if (x, ρ) belongs to C_z and we fix any element y in $B(x, \rho) \cap (\overline{A})_z$ then for every (x', ρ') in $K^{m-1} \times K$ such that

$$\|x' - y\| < |\rho'| < |\rho| \tag{23}$$

we have $y \in B(x', \rho') \subseteq B(y, \rho) = B(x, \rho)$ hence $(x', \rho') \in C_z$. The above condition (23) defines an open subset of K^m, hence $\dim C_z = m$. By corollary 7.5 it follows that

$$\dim C = m + \dim Z. \tag{24}$$

On the other hand, let us show that the projection $\Pi : C \to K^m$ which maps (z, x, ρ) to (x, ρ) has finite fibers. Assume the contrary. Then for some (x, ρ) in $K^{m-1} \times K^\times$ there is a ball B_0 contained in Z such that (z, x, ρ) belongs to C for every z in B_0. Then $B(x, \rho)$ is disjoint from A_z for every z in B_0, hence $B_0 \times B(x, \rho)$ is disjoint from $A \cap \pi_i^{-1}(B)$. It is disjoint as well from its closure since $B_0 \times B(x, \rho)$ is open. On the other hand $B(x, \rho)$ meets $(\overline{A})_z$ for every z in Z, and in particular it meets $(\overline{A})_z$ for some z in B_0. Then $B_0 \times B(x, \rho)$ meets $(\overline{A})_z$, which is contained in the closure of $A \cap \pi_1^{-1}(B_0)$, a contradiction.

So $\Pi : C \to K^m$ has finite fibers, which implies that $\dim C \leq m$. By (24) we conclude that $\dim Z = 0$ hence Z is finite.

■

We can turn now to theorem 8.1.

Proof: (of theorem 8.1). The result is obvious for $m \leq 1$ so we can assume, by induction, that $m \geq 2$ and it has been proved for $m-1$. For every definable set S we let $\operatorname{Fr} S = \overline{S} \setminus S$.

Let A be a definable subset of K^m. For each i in $\{1, \dots, m\}$ let

$$F_i = \left\{ z \in K : (\overline{A})_z^{(i)} \neq \overline{A_z^{(i)}} \right\}.$$

Note that for every z in K

$$(\operatorname{Fr} A)_z^{(i)} \setminus \operatorname{Fr}(A_z^{(i)}) = (\overline{A})_z^{(i)} \setminus \overline{A_z^{(i)}} \tag{25}$$

so $(\operatorname{Fr} A)_z^{(i)} = \operatorname{Fr}(A_z^{(i)})$ for every $z \in K \setminus F_i$.

Let $H_i = \pi_i^{-1}(F_i)$ and $H = \bigcap_{i=1}^m H_i = F_1 \times \cdots \times F_m$. Each F_i is finite by lemma 8.3 hence so is H. So $\dim \operatorname{Fr} A = \dim(\operatorname{Fr} A) \setminus H$ and moreover

$$(\operatorname{Fr} A) \setminus H = \bigcup_{i=1}^{m} (\operatorname{Fr} A) \setminus H_i.$$

Thus it suffices to prove that $\dim \operatorname{Fr} A \setminus H_i < \dim A$ for each i. By symmetry we can assume that $i = 1$ and remove the exponents $^{(i)}$ in order to ease the notation. By (25) we have then

$$(\operatorname{Fr} A) \setminus H_1 = \bigcup_{z \in K \setminus F_1} \{z\} \times (\operatorname{Fr} A)_z = \bigcup_{z \in K \setminus F_1} \{z\} \times \operatorname{Fr}(A_z). \tag{26}$$

By induction hypothesis $\dim \operatorname{Fr}(A_z) < \dim A_z$ for every z. Thus (26) implies that $\dim((\operatorname{Fr} A) \setminus H_1)_z < \dim A_z$ for every z. The conclusion follows by corollary 7.6.

■

Corollary 8.4 *Let $Y \subseteq X$ be a pair of definable subsets of K^m. If $\dim Y = \dim X$ then $\dim \operatorname{Int}_X Y = \dim X$.*

Proof: As $\operatorname{Int}_X Y = Y \setminus \overline{X \setminus Y}$ we have $Y \setminus \operatorname{Int}_X Y = Y \cap \overline{X \setminus Y}$ hence $Y \setminus \operatorname{Int}_X Y$ is contained in $\overline{X \setminus Y} \setminus (X \setminus Y)$. It follows from theorem 8.1 that

$$\dim \big(Y \setminus \operatorname{Int}_X Y\big) < \dim \big(X \setminus Y\big) \leq \dim X$$

hence $\dim \operatorname{Int}_X Y = \dim X$ by proposition 7.1.

■

Corollary 8.5 *For every definable function $f : X \subseteq K^m \to K^n$*

$$\dim X \setminus \mathcal{C}(f) < \dim X.$$

Proof: By theorem 6.5 $X \setminus \mathcal{C}(f)$ has empty interior in X, hence it has dimension $< \dim X$ by corollary 8.4.

■

Corollary 8.6 *Every definable function is piecewise continuous. As a consequence, the cells involved in the cell decomposition and cell preparation theorems can be chosen with continuous centers and bounds.*

Proof: Let $f : X \subseteq K^m \to K$ a definable function. If the restriction g of f to $A \setminus \mathcal{C}(f)$ is continuous we are done, otherwise repeat the argument for g. By corollary 8.5 the dimension of the domain decreases at each step, hence it must stop after at most m steps.

The last point follows by partitioning the cells given by the cell decomposition and cell preparation theorems *via* an appropriate refinement of their basements.

■

Topological rank For every subsets A, B of K^m we write $B \ll A$ when B is a subset of A with empty interior in A, that is:

$$B \ll A \iff B \subseteq A \subseteq \overline{A \setminus B}.$$

It is a strict partial order on the non empty sets. We call **topological rank** and denote $\operatorname{rk} A$ the corresponding rank on the non empty definable subsets of K^m. So $\operatorname{rk} A = 0$ if A is minimal for $\ll$, that is if every point of A is isolated; $\operatorname{rk} A \geq k + 1$ if $A \gg B$ for some non empty definable set B of rank $\geq k$. Of course $\operatorname{rk} A = k$ if $\operatorname{rk} A \geq k$ but $\operatorname{rk} A \not\geq k + 1$. By convention we let $\operatorname{rk} \emptyset = -\infty$.

Proposition 8.7 *For every definable set* $A \subseteq K^m$*,* $\dim A = \operatorname{rk} A$*.*

Proof: We have to prove that for every positive integers m, d and every definable subset A of K^m

$$\dim A \geq d \iff \operatorname{rk} A \geq d.$$

Assume that it has been proved for every (m, d) with $m \leq n$ (it is obvious in K^0) or $m = n + 1$ and $d \leq e$ (it is obvious for finite sets). Let A be a definable subset of K^{n+1}.

If $\operatorname{rk} A \geq e + 1$ then A contains a definable set B with rank e such that $\operatorname{Int}_A B = \emptyset$. Then $\dim B \geq e$ by induction hypothesis and $\dim A > \dim B$ by corollary 8.4, so $\dim A \geq e + 1$.

Conversely, if $\dim A \geq e+1$ then A contains a cell C of dimension $\geq e+1$. It can be chosen with continuous center and bounds by corollary 8.6. Let $X = \widehat{C}$, by lemma 7.3 $\dim X = \dim C - \operatorname{tp} C$. As X is contained in K^m the induction hypothesis gives $Y \ll Y$ with rank and dimension $\dim X - 1$. Then $D = C \cap (Y \times K)$ is a cell contained in C with the same type as C, hence by lemma 7.3 and the induction hypothesis

$$\dim D = \dim Y + \operatorname{tp} C = (\dim X - 1) + \operatorname{tp} C = \dim C - 1 \geq e$$

By induction hypothesis $\operatorname{rk} D \geq e$. But $Y \subseteq \overline{X \setminus Y}$ and the continuity of the center and bounds of C imply that $D \subseteq \overline{C \setminus D}$, hence $D \ll C$. *A fortiori* $D \ll A$ so $\operatorname{rk} A \geq e + 1$.

■

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