The universal covering homomorphism in o-minimal expansions of groups

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Abstract

Suppose that G is a definably connected, definable group in an o-minimal expansion of an ordered group. We show that the o-minimal universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ is a locally definable covering homomorphism and $\pi_1(G)$ is isomorphic to the o-minimal fundamental group $\pi(G)$ of G defined using locally definable covering homomorphisms.

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1 Introduction

Let $\mathcal{R}$ be an o-minimal expansion of an ordered group $(R, 0, +, <)$. The structure $\mathcal{R}$ will be fixed throughout and will be assumed to be $\aleph_1$-saturated. By definable we will mean definable in $\mathcal{R}$ with parameters.

In the paper [3] the first author introduced a notion of o-minimal fundamental group and o-minimal universal covering homomorphism for definable groups (or more generally for locally definable groups) in arbitrary o-minimal structures which we now recall.

First recall that a group $(G, \cdot)$ is a *locally definable group over* A, with $A \subseteq R$ and $|A| < \aleph_1$, if there is a countable collection $\{Z_i : i \in I\}$ of definable subsets of R^n, all definable over A, such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$ and (iii) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map over A into R^n.

Given two locally definable groups H and G over A, we say that H is a *locally definable subgroup of* G *over* A if H is a subgroup of G.

A homomorphism $\alpha : G \longrightarrow H$ between locally definable groups over A is called a *locally definable homomorphism over* A if for every definable subset $Z \subseteq G$ defined over A, the restriction $\alpha_{|Z}$ is a definable map over A.

In the terminology of [9], locally definable groups (respectively homomorphisms) are $\bigvee$-definable groups (respectively homomorphisms). Therefore, every locally definable group $G \subseteq R^n$ over A is equipped with a unique topology τ, called the *τ-topology*, such that: (i) (G, τ) is a topological group; (ii) every generic element of G has an open definable neighborhood $U \subseteq R^n$ such that $U \cap G$ is τ-open and the topology which $U \cap G$ inherits from τ agrees with the topology it inherits from R^n; (iii) locally definable homomorphisms between locally definable groups are continuous with respect to the τ topologies. Note also that when G is a definable group, then its τ-topology coincides with the its *t-topology* from [10].

Definition 1.1 A locally definable homomorphism $p : H \longrightarrow G$ over A between locally definable groups over A is called a *locally definable covering homomorphism* if p is surjective and there is a family $\{U_l : l \in L\}$ of τ-open definable subsets of G over A such that $G = \cup\{U_l : l \in L\}$ and, for each $l \in L$, $p^{-1}(U_l)$ is a disjoint union of τ-open definable subsets of H over A, each of which is mapped homeomorphically by p onto U_l.

We call $\{U_l : l \in L\}$ a *p-admissible family of definable τ-neighborhoods* over A.

We denote by $\mathrm{Cov}(G)$ the category whose objects are locally definable covering homomorphisms $p : H \longrightarrow G$ (over some A with $|A| < \aleph_1$) and whose morphisms are surjective locally definable homomorphisms $r : H \longrightarrow K$ (over some A with $|A| < \aleph_1$) such that $q \circ r = p$, where $q : K \longrightarrow G$ is a locally definable covering homomorphism (over some A with $|A| < \aleph_1$). Let $p : H \longrightarrow G$ and $q : K \longrightarrow G$ be locally definable covering homomorphisms. If $r : H \longrightarrow K$ is a morphism in $\mathrm{Cov}(G)$, then by [3] Theorem 3.6, $r : H \longrightarrow K$ is a locally definable covering homomorphism.

Definition 1.2 The category $\mathrm{Cov}(G)$ and its full subcategory $\mathrm{Cov}^0(G)$ with objects $h : H \longrightarrow G$ such that H is a definably connected locally definable group, form inverse systems ([3] Corollary 3.7 and Lemma 3.8). The inverse limit $\widetilde{p} : \widetilde{G} \longrightarrow G$ of the inverse system $\mathrm{Cov}^0(G)$ is called the *(o-minimal) universal covering homomorphism of G*.

The kernel of the universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ of G is called the *(o-minimal) fundamental group of G* and is denoted by $\pi(G)$.

Inverse limits of inverse systems of groups always exist in the category of groups ([11] Proposition 1.1.1), but in general we do not know if the o-minimal universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ is locally definable. The main result of this paper is that this is the case in o-minimal expansions of groups.

On the other hand, in the paper [5], the second author and S. Starchenko use definable t-continuous paths to define the o-minimal fundamental group $\pi_1(G)$ of a definably t-connected, definable group G following the classical case in [7] and the case in o-minimal expansions of fields treated by Berarducci and Otero in [1]. We will adapt that definition to the category of locally definable groups. As in [5] we will run the definition in parallel with respect to the τ-topology of a definably connected locally definable group G and the usual topology on an arbitrary definable subset X of R^n.

A *(τ-)path* $\alpha : [0, p] \longrightarrow X$ ($\alpha : [0, p] \longrightarrow G$) is a ($\tau$-)continuous definable map. A (τ-)path $\alpha : [0, p] \longrightarrow X$ ($\alpha : [0, p] \longrightarrow G$) is a *($\tau$-)loop* if $\alpha(0) = \alpha(p)$. A concatenation of two (τ-)paths $\gamma : [0, p] \longrightarrow X$ ($\gamma : [0, p] \longrightarrow G$) and $\delta : [0, q] \longrightarrow X$ ($\delta : [0, q] \longrightarrow G$) with $\gamma(p) = \delta(0)$ is a (τ-)path $\gamma \cdot \delta : [0, p + q] \longrightarrow X$ ($\gamma \cdot \delta : [0, p + q] \longrightarrow G$) with:

$$(\gamma \cdot \delta)(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, p] \\ \delta(t - p) & \text{if } t \in [p, p + q]. \end{cases}$$

Given two definable (τ-)continuous maps $f, g : Y \subseteq R^m \longrightarrow X$ ($f, g : Y \subseteq R^m \longrightarrow G$), we say that a definable (τ-)continuous map $F(t, s) : Y \times [0, q] \longrightarrow X$ ($F(t, s) : Y \times [0, q] \longrightarrow G$), is a *($\tau$-)homotopy between f and g* if $f = F_0$ and $g = F_q$, where $\forall s \in [0, q]$, $F_s := F(\cdot, s)$. In this situation we say that f and g are *(τ-)homotopic*, denoted $f \sim g$ ($f \sim_\tau g$).

Definition 1.3 Two (τ-)paths $\gamma : [0,p] \longrightarrow X$ ($\gamma : [0,p] \longrightarrow G$), $\delta : [0,q] \longrightarrow X$ ($\delta : [0,q] \longrightarrow G$), with $\gamma(0) = \delta(0)$ and $\gamma(p) = \delta(q)$, are called *(τ-)homotopic* if there is some $t_0 \in [0, \min\{p,q\}]$, and a (τ-)homotopy $F(t,s) : [0, \max\{p,q\}] \times [0,r] \longrightarrow X$ ($F(t,s) : [0, \max\{p,q\}] \times [0,r] \longrightarrow G$), for some $r > 0$ in R, between

$$\gamma_{|[0,t_0]} \cdot \mathbf{c} \cdot \gamma_{|[t_0,p]} \text{ and } \delta \text{ (if } p \leq q\text{), or}$$

$$\delta_{|[0,t_0]} \cdot \mathbf{d} \cdot \delta_{|[t_0,q]} \text{ and } \gamma \text{ (if } q \leq p\text{).}$$

where $\mathbf{c}(t) = \gamma(t_0)$ and $\mathbf{d}(t) = \delta(t_0)$ are the constant (τ-)paths with domain $[0, |p-q|]$.

If $\mathbb{L}(G)$ denotes the set of all τ-loops that start and end at the identity element e_G of G, the restriction of $\sim_\tau$ to $\mathbb{L}(G) \times \mathbb{L}(G)$ is an equivalence relation on $\mathbb{L}(G)$. We define

$$\pi_1(G) := \mathbb{L}(G)/\sim_\tau$$

and $[\gamma] :=$ the class of $\gamma \in \mathbb{L}(G)$. Note that $\pi_1(G)$ is indeed a group with group operation given by $[\gamma][\delta] = [\gamma \cdot \delta]$.

In a similar way we define the o-minimal fundamental group $\pi_1(X)$ of a definable set $X \subseteq R^n$.

Given the above two possible definitions of o-minimal fundamental groups it is natural to try to find out if they coincide. Our main result shows that this is the case:

Theorem 1.4 *Let $\mathcal{R}$ be an o-minimal expansion of a group and G a definably t-connected definable group. Then the o-minimal universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ is a locally definable covering homomorphism and $\pi_1(G)$ is isomorphic to $\pi(G)$.*

Theorem 1.4 will actually be proved for definably τ-connected locally definable groups. See Theorem 3.11 below. As a consequence of our work we obtain the following corollary which is proved at the end of the paper.

Corollary 1.5 *Let $\mathcal{R}$ be an o-minimal expansion of a group and G a definably t-connected definable group. Then $\pi_1(G)$ is a finitely generated abelian group. Moreover, if G is abelian, then there is $l \in \mathbb{N}$ such that $\pi_1(G) \simeq \mathbb{Z}^l$ and, for each $k \in \mathbb{N}$, the subgroup $G[k]$ of k-torsion points of G is given by $G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^l$.*

When G is a definably compact, abelian definable group, we conjecture that l above is the dimension of G. This is known to be the case when $\mathcal{R}$ is linear ([5]) or $\mathcal{R}$ is an o-minimal expansion of a real closed field ([4]). So the conjecture is open for $\mathcal{R}$ eventually linear but not linear.

2 Preliminary results

This section contains all the lemmas that come from other references and are used later in the paper. Thus we generalize the theory of [3] and [4] Section 2 to the category of locally definable covering *maps* of locally definable groups in $\mathcal{R}$. Since the arguments are similar we will omit the details.

Definition 2.1 A set Z is a *locally definable set over* A, where $A \subseteq R$ and $|A| < \aleph_1$, if there is a countable collection $\{Z_i : i \in I\}$ of definable subsets of R^n, all definable over A, such that: (i) $Z = \cup\{Z_i : i \in I\}$; (ii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$.

Given two locally definable sets X and Z over A, we say that X is a *locally definable subset of* Z *over* A if X is a subset of Z.

A map $\alpha : Z \longrightarrow X$ between locally definable sets over A is called a *locally definable map over* A if for every definable subset $V \subseteq Z$ defined over A, the restriction $\alpha_{|V}$ is a definable map over A.

By saturation, the set Z does not depend on the choice of the collection $\{Z_i : i \in I\}$. Furthermore, if $\alpha : Z \longrightarrow X$ is a locally definable map over A between locally definable sets over A and Y is a locally definable subset of X over A, then the following hold:

(1) $\alpha(Z)$ is a locally definable subset of X over A and $\alpha^{-1}(Y)$ is a locally definable subset of Z over A.

(2) If Y is such that $V \cap Y$ is definable for every definable subset V of X, then $W \cap \alpha^{-1}(Y)$ is definable for every definable subset W of Z. (Since $W \cap \alpha^{-1}(Y) = \alpha^{-1}_{|W}(\alpha(W) \cap Y))$).

Definition 2.2 Let G be a locally definable group over A and W a locally definable set over A. A locally definable map $w : W \longrightarrow G$ over A is called a *locally definable covering map* if w is surjective and there is a family $\{U_l : l \in L\}$ of τ-open definable subsets of G over A such that $G = \cup\{U_l : l \in L\}$ and, for each $l \in L$, the locally definable subset $w^{-1}(U_l)$ of W over A is a disjoint union of definable subsets of W over A, each of which is mapped bijectively by w onto U_l.

We call $\{U_l : l \in L\}$ a *w-admissible family of definable τ-neighborhoods* over A.

Given a locally definable covering map $w : W \longrightarrow G$ over A there is a topology on W, which we call the *w-topology*, generated by the definable sets of the form $w^{-1}(U) \cap V$, where U is a τ-open definable subset of G and V is one of the definable subsets of the disjoint union $w^{-1}(U_l)$ for some U_l in the w-admissible family of definable τ-neighborhoods.

Clearly, with respect to the w-topology on W (and the τ-topology on G), $w : W \longrightarrow G$ is continuous. Furthermore, $w : W \longrightarrow G$ is an open surjection. In fact, let V be a w-open definable subset of W over A and, for each $l \in L$, let $\{U_s^l : s \in S_l\}$ be the collection of w-open disjoint definable subsets of W over A such that $w^{-1}(U_l) = \cup\{U_s^l : s \in S_l\}$ and $w_{|U_s^l} : U_s^l \longrightarrow U_l$ is a definable homeomorphism over A for every $s \in S_l$. Since $|A| < \aleph_1$, by saturation, there is $\{W_1, \ldots, W_m\} \subseteq \{U_s^l : l \in L,\ s \in S_l\}$ such that $V \subseteq \cup\{W_i : i = 1, \ldots, m\}$. But then $V = \cup\{V \cap W_i : i = 1, \ldots, m\}$ and $w(V) = \cup\{w(V \cap W_i) : i = 1, \ldots, m\}$ is τ-open.

Lemma 2.3 *Let $w : W \longrightarrow G$ be a locally definable covering map and suppose that W is also a locally definable group. Then on W the w-topology coincides with the τ-topology.*

Proof. Let $a \in W$ be a generic point and U a definable w-open neighborhood of a in W. We may assume that $w_{|U} : U \longrightarrow w(U)$ is a definable homeomorphism. Since $w(a)$ is also generic, there exists a definable subset $V \subseteq w(U)$ containing $w(a)$ such that V is both τ-open in G and open in G with the induced topology on G from R^n. Thus $w^{-1}(V)$ is also both a w-neighborhood of a in W and in W with the induced topology on W from R^n. Hence, $w^{-1}(V)$ is a τ-neighborhood of a in W. By uniqueness of τ-topology, this implies that the w-topology and the τ-topology on W agree. $\square$

Let $w : W \longrightarrow G$ be a locally definable covering map (over some A with $|A| < \aleph_1$). Let X be a definable subset of W equipped with the induced w-topology from W. We will now introduce certain notions in parallel for X and W.

A *w-path* $\alpha : [0, p] \longrightarrow X$ ($\alpha : [0, p] \longrightarrow W$) is a w-continuous definable map. A w-path $\alpha : [0, p] \longrightarrow X$ ($\alpha : [0, p] \longrightarrow W$) is a *$w$-loop* if $\alpha(0) = \alpha(p)$. A concatenation of two w-paths $\gamma : [0, p] \longrightarrow X$ ($\gamma : [0, p] \longrightarrow W$) and $\delta : [0, q] \longrightarrow X$ ($\delta : [0, q] \longrightarrow W$) with $\gamma(p) = \delta(0)$ is a w-path $\gamma \cdot \delta : [0, p + q] \longrightarrow X$ ($\gamma \cdot \delta : [0, p + q] \longrightarrow W$) with:

$$(\gamma \cdot \delta)(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, p] \\ \\ \delta(t - p) & \text{if } t \in [p, p + q]. \end{cases}$$

Given two definable w-continuous maps $f, g : Y \subseteq R^m \longrightarrow X$ ($f, g : Y \subseteq R^m \longrightarrow W$), we say that a definable w-continuous map $F(t, s) : Y \times [0, q] \longrightarrow X$ ($F(t, s) : Y \times [0, q] \longrightarrow W$) is a *$w$-homotopy between f and g* if $f = F_0$ and $g = F_q$, where $\forall s \in [0, q]$, $F_s := F(\cdot, s)$. In this situation we say that f and g are *w-homotopic*, denoted $f \sim_w g$.

Definition 2.4 Two w-paths $\gamma : [0,p] \longrightarrow X$ ($\gamma : [0,p] \longrightarrow W$), $\delta : [0,q] \longrightarrow X$ ($\delta : [0,q] \longrightarrow W$), with $\gamma(0) = \delta(0)$ and $\gamma(p) = \delta(q)$, are called *w-homotopic* if there is some $t_0 \in [0, \min\{p,q\}]$, and a w-homotopy $F(t,s) : [0, \max\{p,q\}] \times [0,r] \longrightarrow X$ ($F(t,s) : [0, \max\{p,q\}] \times [0,r] \longrightarrow W$), for some $r > 0$ in R, between

$$\gamma_{|[0,t_0]} \cdot \mathbf{c} \cdot \gamma_{|[t_0,p]} \text{ and } \delta \text{ (if } p \leq q), \text{ or}$$

$$\delta_{|[0,t_0]} \cdot \mathbf{d} \cdot \delta_{|[t_0,q]} \text{ and } \gamma \text{ (if } q \leq p).$$

where $\mathbf{c}(t) = \gamma(t_0)$ and $\mathbf{d}(t) = \delta(t_0)$ are the constant w-paths with domain $[0, |p-q|]$.

If $\mathbb{L}(W)$ denotes the set of all w-loops that start and end at a fixed element e_W of W such that $w(e_W) = e_G$, the restriction of $\sim_w$ to $\mathbb{L}(W) \times \mathbb{L}(W)$ is an equivalence relation on $\mathbb{L}(W)$. We define

$$\pi_1(W) := \mathbb{L}(W)/ \sim_w$$

and $[\gamma] :=$ the class of $\gamma \in \mathbb{L}(W)$. Note that $\pi_1(W)$ is indeed a group with group operation given by $[\gamma][\delta] = [\gamma \cdot \delta]$. Also this group depends on the w-topology on W.

In a similar way we define the o-minimal fundamental group $\pi_1(X)$ of a definable subset $X \subseteq W$ with respect to the induced w-topology.

Clearly, any two constant w-loops at the same point $c \in W$ are w-homotopic. We will thus write ϵ_c for the constant w-loop at c without specifying its domain.

In view of Lemma 2.3, we obtain the above notions with w replaced by τ for definable subsets of a locally definable group equipped with the induced τ-topology.

Lemma 2.5 *Let $w : W \longrightarrow G$ and $v : V \longrightarrow H$ be locally definable covering maps. Then $(w,v) : W \times V \longrightarrow G \times H$ is a locally definable covering map and $\theta : \pi_1(W) \times \pi_1(V) \longrightarrow \pi_1(W \times V) : ([\gamma],[\delta]) \mapsto [(\gamma,\delta)]$ is a group isomorphism.*

Proof. The inverse of θ is $\pi_1(W \times V) \longrightarrow \pi_1(W) \times \pi_1(V) : [\rho] \mapsto ([q_1 \circ \rho], [q_2 \circ \rho])$ where q_1 and q_2 are the projections from $W \times V$ onto W and V, respectively. □

Let $w : W \longrightarrow G$ be a locally definable covering map (over some A with $|A| < \aleph_1$). Let Z be a definable set and let $f : Z \longrightarrow G$ be a definable continuous map (with respect to the τ-topology on G). A *lifting of* f is a continuous definable map $\widetilde{f} : Z \longrightarrow W$ (with respect to the w-topology on W) such that $p \circ \widetilde{f} = f$.

Lemma 2.6 *Let $w : W \longrightarrow G$ be a locally definable covering map, Z a definably connected definable set and $f : Z \longrightarrow G$ a definable continuous map. If $\widetilde{f_1}, \widetilde{f_2} : Z \longrightarrow W$ are two liftings of f, then $\widetilde{f_1} = \widetilde{f_2}$ provided there is a $z \in Z$ such that $\widetilde{f_1}(z) = \widetilde{f_2}(z)$.*

Proof. As in the proof of [3] Lemma 3.2, both sets $\{w \in Z : \widetilde{f_1}(w) = \widetilde{f_2}(w)\}$ and $\{w \in Z : \widetilde{f_1}(w) \neq \widetilde{f_2}(w)\}$ are definable and open, the first one is nonempty. □

Lemma 2.7 *Suppose that $w : W \longrightarrow G$ is a locally definable covering map. Then the following hold.*

(1) Let γ be a τ-path in G and $y \in W$. If $w(y) = \gamma(0)$, then there is a unique w-path $\widetilde{\gamma}$ in W, lifting γ, such that $\widetilde{\gamma}(0) = y$.

(2) Suppose that F is a τ-homotopy between the τ-paths γ and σ in G. Let $\widetilde{\gamma}$ be a w-path in W lifting γ. Then there is a unique definable lifting $\widetilde{F}$ of F, which is a w-homotopy between $\widetilde{\gamma}$ and $\widetilde{\sigma}$, where $\widetilde{\sigma}$ is a w-path in W lifting σ.

Proof. In our category, the path and the homotopy liftings can be proved as in [4] by observing that, by saturation, a definable subset of G is covered by finitely many open definable subsets of G. □

Notation: Referring to Lemma 2.7, if $\gamma : [0, q] \longrightarrow G$ is a τ-path in G and $y \in W$, we denote by $y * \gamma$ the final point $\widetilde{\gamma}(q)$ of the lifting $\widetilde{\gamma}$ of γ with initial point $\widetilde{\gamma}(0) = y$.

The following consequence of Lemma 2.7 is proved in exactly the same way as its definable analogue in [4] Corollary 2.9. Below, for $w : W \longrightarrow G$ a locally definable covering map, we say that W is *definably w-connected* if there is no proper locally definable subset of W which is both w-open and w-closed and whose intersection with any definable subset of W is definable. In view of Lemma 2.3, this notion generalizes the notion of definably connected in locally definable groups studied in [3].

Remark 2.8 Suppose that $w : W \longrightarrow G$ is a locally definable covering map and let $y \in W$ be such that $w(y) = e_G$. Suppose that W and G are definably w-connected and τ-connected respectively. Then we have a well defined homomorphism $w_* : \pi_1(W) \longrightarrow \pi_1(G) : [\gamma] \mapsto [w \circ \gamma]$ and the following hold.

(1) If σ is a τ-path in G from e_G to e_G, then $y = y * \sigma$ if and only if $[\sigma] \in w_*(\pi_1(W))$.

(2) If σ and σ' are two τ-paths in G from e_G to x, then $y * \sigma = y * \sigma'$ if and only if $[\sigma \cdot \sigma'^{-1}] \in w_*(\pi_1(W))$.

Let $w : W \longrightarrow G$ be a locally definable covering map. We say that W is *w-path connected* if for every $u, v \in W$ there is a w-path $\alpha : [0, q] \longrightarrow W$ such that $\alpha(0) = u$ and $\alpha(q) = v$.

Lemma 2.9 *Let $w : W \longrightarrow G$ be a locally definable covering map. Then W is definably w-connected if and only if W is w-path connected. In fact, for any definably w-connected definable subset X of W there is a uniformly definable family of w-paths in X connecting a given fixed point in X to any other point in X.*

Proof. Since $w : W \longrightarrow G$ is a locally definable covering map, it is enough to prove the result for locally definable groups. By the first part of the proof of [9] Lemma 2.13, there is a locally definable subset U of G such that $\dim(G \setminus U) < \dim G$, the intersection of any definable subset of G with U is a definable subset and the induced τ-topology on U coincides with the induced topology from R^n. So a definable subset B of U is τ-connected if and only if B is definably connected (in R^n). Thus the result follows from by [6] Chapter VI, Proposition 3.2 and its proof, saturation and [3] Lemma 3.5 (i.e., countably many translates of U cover G). $\square$

The next proposition is also a consequence of Lemma 2.7 and is proved in exactly the same way as its definable analogue in [4] Corollary 2.8 and Proposition 2.10.

Proposition 2.10 *Let $w : W \longrightarrow G$ be a locally definable covering map. Suppose that W and G are definably w-connected and τ-connected respectively. Then the following hold:*

(1) $w_* : \pi_1(W) \longrightarrow \pi_1(G)$ is an injective homomorphism;

(2) $\pi_1(G)/w_*(\pi_1(W)) \simeq \mathrm{Aut}(W/G)$ (the group of all locally definable w-homeomorphisms $\phi : W \longrightarrow W$ such that $w = w \circ \phi$).

Below we will also require the following generalization of Lemma 2.6:

Lemma 2.11 *Let $w : W \longrightarrow G$ and $v : V \longrightarrow H$ be locally definable covering maps and let $f, g : V \longrightarrow W$ be two continuous locally definable maps (with respect to the v and w topologies) such that $w \circ f = w \circ g$. If V is definably v-connected and $f(x) = g(x)$ for some $x \in V$, then $f = g$.*

Proof. This is as in [3] Lemma 3.2 once we show that $\{x \in V : f(x) = g(x)\}$, which is open and closed, is a locally definable subset whose intersection with any definable subset of V is a definable subset of V. If C, $D \subseteq V$ are definable, then $(V \times_W V) \cap (C \times D) = \{(x, y) \in C \times D : f_{|C}(x) = g_{|D}(y)\}$ is definable, and so $(V \times_W V) \cap E$ is definable for every definable subset E of $V \times V$. Similarly, $\Delta_V \cap E$ is definable for every definable subset E of $V \times V$. Hence, $(V \times_W V) \cap \Delta_V \cap E$ is definable for every definable subset E of $V \times V$. From this and the observation (2) on page 5 we get our result since $\{x \in V : f(x) = g(x)\} = i^{-1}((V \times_W V) \cap \Delta_V)$, where $i : V \longrightarrow \Delta_V : x \mapsto (x, x)$ is a locally definable map. □

Finally we include the following result ([3] Proposition 3.4) which will also be useful later:

Proposition 2.12 *Let $h : H \longrightarrow G$ be a locally definable covering homomorphism and suppose that H is definably τ-connected. Then*

$$\mathrm{Ker} h \simeq \mathrm{Aut}(H/G)$$

and $\mathrm{Aut}(H/G)$ *is abelian.*

3 The universal covering homomorphism

Here we will present the proof of our main result. We start however with a special case.

3.1 A special case of the main result

The main result of the paper [5], in the language of the theory of locally definable covering homomorphisms, is the following (compare with [5] Remark 6.14). For a related result see also [8].

Theorem 3.1 ([5]) *Suppose that $\mathcal{R}$ is an ordered vector space over an ordered division ring and G is a definably t-connected, definably compact, definable group of dimension n. Then there is a locally definable group V which is a subgroup of $(R^n, +)$ and a locally definable covering homomorphism $v : V \longrightarrow G$ such that $\pi_1(G) \simeq \mathrm{Ker} v \simeq \mathbb{Z}^n$.*

In [5] Remark 6.14 it is suggested that $v : V \longrightarrow G$ is in some sense the universal cover of G since we have $\pi_1(V) = 1$ ([5] Corollary 6.7). This claim can now be made more precise:

Theorem 3.2 *Suppose that $\mathcal{R}$ is an ordered vector space over an ordered division ring and G is a definably t-connected, definably compact, definable group of dimension n. Then the locally definable covering homomorphism $v : V \longrightarrow G$ is isomorphic to $\widetilde{p} : \widetilde{G} \longrightarrow G$ and $\pi_1(G) \simeq \pi(G) \simeq \mathbb{Z}^n$.*

Proof. Suppose that $q : K \longrightarrow V$ is a locally definable covering homomorphism. Then from Propositions 2.10 and 2.12 we obtain $\mathrm{Ker}q \simeq \mathrm{Aut}(K/V) \simeq \pi_1(V)/q_*(\pi_1(K)) = 1$ since $\pi_1(V) = 1$, by [5] Corollary 6.7. So $q : K \longrightarrow V$ is a locally definable isomorphism (since it is surjective). Consequently, by [3] Lemma 3.8, the set of all $h : H \longrightarrow G$ in $\mathrm{Cov}^0(G)$ which are locally definably isomorphic to $v : V \longrightarrow G$ is cofinal in $\mathrm{Cov}^0(G)$ and hence the inverse limit $\widetilde{p} : \widetilde{G} \longrightarrow G$ is isomorphic to $v : V \longrightarrow G$. By Propositions 2.10 and 2.12 we obtain $\pi(G) \simeq \mathrm{Ker}v \simeq \mathrm{Aut}(V/G) \simeq \pi_1(G)$ since $\pi_1(V) = 1$. Thus the result holds as required. □

3.2 The main result

Here we prove the main result of the paper. Before we proceed we need the following propositions.

Proposition 3.3 *Let G be a definably τ-connected locally definable group of dimension k. Then there is a countable collection $\{O_s : s \in S\}$ of τ-open definably τ-connected definable subsets of G with $G = \cup\{O_s : s \in S\}$ and, for each $s \in S$, O_s is definably homeomorphic to an open cell in R^k. In particular, for each $s \in S$, the o-minimal fundamental group $\pi_1(O_s)$ with respect to the induced τ-topology on O_s is trivial*

Proof. By the first part of the proof of [9] Lemma 2.13, there is a locally definable subset U of G such that $\dim(G \setminus U) < \dim G$, the intersection of any definable subset of G with U is a definable subset and the induced τ-topology on U coincides with the induced topology from R^n. Without loss of generality we can assume that U is a countable union of cells of dimension $k = \dim G$. Note that on each of these k-cells in U, the induced τ-topology coincides with the induced topology from R^n. By [3] Lemma 3.5 countably many translates of U cover G, so countably many τ-open definably τ-connected subsets of G which are definably τ-homeomorphic to k-cells in U cover G.

Let $\{O_s : s \in S\}$ be this collection. To finish, it is enough to show that if C is an open cell in R^k then $\pi_1(C) = 1$ (since definable homeomorphisms induce isomorphisms between the o-minimal fundamental groups).

We will show this by induction on the construction of cells. If C has dimension zero then this is obvious. Assume that $C = (a, b) \subseteq R \cup \{-\infty, +\infty\}$ is an open cell of dimension one and $\alpha : [0, q] \longrightarrow C$ is a definable loop at $c \in C$. Consider the continuous definable map $H : [0, q] \times [0, q] \longrightarrow C$ given by

$$H(t, x) := \alpha(\frac{t + x + |t - x|}{2}).$$

Then H is a definable homotopy between α and ϵ_c. So $[\alpha] = 1$ and $\pi_1(C) = 1$ as required.

Suppose that B is a cell, $\pi_1(B) = 1$ and $C = (f, g)_B$ with $f, g : B \longrightarrow R \cup \{-\infty, +\infty\}$ continuous definable maps such that $f < g$. Let $c = (b, a) \in C$ and let $\sigma : [0, q] \longrightarrow C$ be a definable loop at c. We can write $\sigma(t) = (\beta(t), \alpha(t))$ for some definable loop $\beta : [0, q] \longrightarrow B$ at b and $\alpha : [0, q] \longrightarrow R$ a definable loop at a. By assumption there is a definable homotopy $F : [0, q] \times [0, p] \longrightarrow B$ between β and ϵ_b and a definable homotopy $E : [0, q] \times [0, r] \longrightarrow R$ between α and ϵ_a. Let $H : [0, q] \times [0, \max\{r, p\}] \longrightarrow C$ be the definable map such that if $r \leq p$ then

$$H(t, x) = \begin{cases} (F(t, x), E(t, x)) & \text{if } x \leq r, \\ (F(t, x), E(t, r)) & \text{if } x \geq r, \end{cases}$$

and if $p \leq r$ then

$$H(t, x) = \begin{cases} (F(t, x), E(t, x)) & \text{if } x \leq p, \\ (F(t, p), E(t, x)) & \text{if } x \geq p. \end{cases}$$

Then H is a definable homotopy between σ and ϵ_c. So $[\sigma] = 1$ and $\pi_1(C) = 1$ as required. □

Proposition 3.4 *Let G be a definably τ-connected locally definable group. Then the o-minimal fundamental group $\pi_1(G)$ of G (with respect to the induced τ-topology) is countable. In fact, if G is definable, then $\pi_1(G)$ is finitely generated.*

Proof. Consider the countable cover $\{O_s : s \in S\}$ of G by τ-open definably τ-connected definable subsets given by Proposition 3.3. For each pair of distinct elements $s, t \in S$ such that $O_s \cap O_t \neq \emptyset$ and for each definably

τ-connected component C of this intersection choose a point $a_{s,t,C} \in C$. For each pair $(a_{s,t,C}, a_{s',t',D})$ of distinct points and $l \in \{s,t\} \cap \{s',t'\}$ let $\sigma^l_{(C,D),s,t,s',t'}$ be a τ-path in O_l from $a_{s,t,C}$ to $a_{s',t',D}$. Also, for each $a_{s,t,C}$ such that $e_G \in O_s$, let $\sigma^s_{(e_G,C),s,t}$ (respectively, $\sigma^s_{(C,e_G),s,t}$) be a τ-path in O_s from e_G to $a_{s,t,C}$ (respectively, from $a_{s,t,C}$ to e_G).

Let Σ be the countable collection of all τ-paths $\sigma^l_{(C,D),s,t,s',t'}$, $\sigma^s_{(e_G,C),s,t}$ and $\sigma^s_{(C,e_G),s,t}$ as above. The set Σ generates a free countable language Σ^* such that some of its words correspond in an obvious way to τ-paths in G. To finish the proof it is enough to show that any τ-loop in G is τ-homotopic to a τ-loop which is a concatenation of τ-paths in Σ and thus corresponds to a word in Σ^*.

Let λ be a τ-loop in G. Then by saturation and o-minimality there exists a minimal k for which we can choose points $0 = t(0) < t(1) < \cdots < t(k) < t(k+1) = q_\lambda$ such that for each $j = 0, \dots, k$, we have $\lambda([t(j), t(j+1)]) \subseteq O_{s(j)}$ for some $s(j) \in S$. Thus $\lambda = \lambda_0 \cdot \cdots \cdot \lambda_k$ where, for each j, $\lambda_j : [0, q_{\lambda_j}] \longrightarrow G$ is the τ-path with $q_{\lambda_j} = t(j+1) - t(j)$ and given by $\lambda_j(t) = \lambda(t + t(j))$. For $i = 0, \dots, k-1$, let C_i be the definably τ-connected component of $O_{s(i)} \cap O_{s(i+1)}$ containing $\lambda_i(q_{\lambda_i})$ and let ϵ_i be a τ-path in C_i from $a_{s(i),s(i+1),C_i}$ to $\lambda_i(q_{\lambda_i})$. Let σ_0 be the τ-path $\sigma^{s(0)}_{(e_G,C_0),s(0),s(1)}$ in $O_{s(0)}$ and let σ_k be the τ-path $\sigma^{s(k)}_{(C_{k-1},e_G),s(k-1),s(k)}$ in $O_{s(k)}$. Finally, for $i = 1, \dots, k-1$, let σ_i be the τ-path $\sigma^{s(i)}_{(C_{i-1},C_i),s(i-1),s(i),s(i),s(i+1)}$ in $O_{s(i)}$. Since by Proposition 3.3, $\pi_1(O_{s(j)}) = 1$ for all $j = 0, \dots, k$, we have that σ_0 is τ-homotopic to $\lambda_0 \cdot \epsilon_0^{-1}$, σ_k is τ-homotopic to $\epsilon_{k-1} \cdot \lambda_k$ and, for each $i = 1, \dots, k-1$, σ_i is τ-homotopic to $\epsilon_{i-1} \cdot \lambda_i \cdot \epsilon_i^{-1}$. Hence, λ is τ-homotopic to $\sigma_0 \cdot \sigma_1 \cdot \cdots \cdot \sigma_k \in \Sigma^*$ as required.

Assume now that G is definable. Let K be the simplicial complex of dimension one whose vertices are the end points of the τ-paths in Σ and whose edges are the images of the τ-paths in Σ. Clearly we have a homomorphism $\pi_1(|K|, e_G) \longrightarrow \pi_1(G)$ which sends an edge loop in K into the τ-loop it determines in G. This is well defined since if two edge loops are homotopic in $|K|$ then they are obviously τ-homotopic in G. The argument in the previous paragraph shows that the homomorphism $\pi_1(|K|, e_G) \longrightarrow \pi_1(G)$ is surjective. Now as explained in [2] Chapter 3, Subsection 3.5.3, the fundamental group of a (finite) simplicial complex is finitely generated. Hence $\pi_1(G)$ is also finitely generated. $\square$

For the rest of the section, fix G a definably τ-connected locally definable group.

We will construct now an "abstract universal covering map" $u : U \longrightarrow G$ from which we will obtain a locally definable covering map $v : V \longrightarrow G$ which will be a locally definable covering homomorphism once we put a suitable locally definable group structure on V. The later will then be shown to be isomorphic to $\widetilde{p}:\widetilde{G}\longrightarrow G$.

Given two τ-paths $\sigma:[0,q_\sigma]\longrightarrow G$ and $\lambda:[0,q_\lambda]\longrightarrow G$ in G, we put $\sigma\simeq\lambda$ if and only if $\sigma(0)=\lambda(0)=e_G$, $\sigma(q_\sigma)=\lambda(q_\lambda)$ and $[\sigma\cdot\lambda^{-1}]=1\in\pi_1(G)$. Here, $\lambda^{-1}:[0,q_{\lambda^{-1}}]\longrightarrow G$ is the τ-path such that $q_{\lambda^{-1}}=q_\lambda$ and $\lambda^{-1}(t)=\lambda(q_\lambda-t)$ for every t in $[0,q_{\lambda^{-1}}]$. The relation $\simeq$ is an equivalence relation and we denote the equivalence class of σ under $\simeq$ by $\langle\sigma\rangle$. For each $s\in S$, let $U_s=\{\langle\sigma\rangle:\sigma$ is a τ-path in G such that $\sigma(0)=e_G$ and $\sigma(q_\sigma)\in O_s\}$ and fix a τ-path $\sigma_s:[0,q_s]\longrightarrow G$ such that $\sigma(0)=e_G$ and $\sigma(q_s)\in O_s$.

Claim 3.5 *There is a well-defined bijection*

$$\phi_s:U_s\longrightarrow O_s\times\pi_1(G):\langle\lambda\rangle\mapsto(\lambda(q_\lambda),[\lambda\cdot\eta\cdot\sigma_s^{-1}]),$$

where $\eta:[0,q_\eta]\longrightarrow O_s$ *is a* τ*-path in* O_s *such that* $\eta(0)=\lambda(q_\lambda)$ *and* $\eta(q_\eta)=\sigma_s(q_s)$.

Proof. Clearly, ϕ_s is well-defined, i.e. it does not depend on the choice of η since $\pi_1(O_s)=1$ (Proposition 3.3) and for $\langle\lambda\rangle=\langle\lambda'\rangle$ we have $\lambda(q_\lambda)=\lambda(q_{\lambda'})$ and

$$\begin{array}{rcl}[\lambda\cdot\eta\cdot\sigma_s^{-1}] & = & [\lambda\cdot\lambda'^{-1}\cdot\lambda'\cdot\eta\cdot\sigma_s^{-1}]\\ & = & [\lambda\cdot\lambda'^{-1}][\lambda'\cdot\eta\cdot\sigma_s^{-1}]\\ & = & [\lambda'\cdot\eta\cdot\sigma_s^{-1}].\end{array}$$

Also, for $o\in O_s$ and $[\gamma]\in\pi_1(G)$ we have $\phi_s(\langle\lambda\rangle)=(o,[\gamma])$ for $\lambda=\gamma\cdot\sigma_s\cdot\eta^{-1}$, where $\eta:[0,q_\eta]\longrightarrow G$ is a τ-path in O_s such that $\eta(0)=o$ and $\eta(q_\eta)=\sigma_s(q_s)$. Thus ϕ_s is surjective. On the other hand, suppose that $\phi_s(\langle\lambda\rangle)=\phi_s(\langle\lambda'\rangle)$. Then $\lambda(q_\lambda)=\lambda'(q_{\lambda'})$ and $[\lambda\cdot\eta\cdot\sigma_s^{-1}]=[\lambda'\cdot\eta'\cdot\sigma_s^{-1}]$. But we also have

$$\begin{array}{rcl}[\lambda\cdot\eta\cdot\sigma_s^{-1}] & = & [\lambda\cdot\lambda'^{-1}][\lambda'\cdot\eta\cdot\sigma_s^{-1}]\\ & = & [\lambda\cdot\lambda'^{-1}][\lambda'\cdot\eta'\cdot\sigma_s^{-1}]\\ & = & [\lambda\cdot\lambda'^{-1}][\lambda\cdot\eta\cdot\sigma_s^{-1}]\end{array}$$

(the fact $\pi_1(O_s)=1$ (Proposition 3.3) implies that $\lambda'\cdot\eta\cdot\sigma_s^{-1}$ is τ-homotopic to $\lambda'\cdot\eta'\cdot\sigma_s^{-1}$). Thus we have $[\lambda\cdot\lambda'^{-1}]=1$, $\langle\lambda\rangle=\langle\lambda'\rangle$ and ϕ_s is injective. □

Set $U=\cup\{U_s:s\in S\}$ and let $u:U\longrightarrow G$ be the surjective map given by $u(\langle\lambda\rangle)=\lambda(q_\lambda)$. By Claim 3.5 and its proof we have, for each $s\in S$,

(•) $u^{-1}(O_s)$ is the disjoint union of the subsets $\phi_s^{-1}(O_s\times\{[\gamma]\})$ with $[\gamma]\in\pi_1(G)$;

($\bullet\bullet$) u restricted to $\phi_s^{-1}(O_s \times \{[\gamma]\})$ is a bijection onto O_s.

Claim 3.6 *If $s, t \in S$ are such that $O_s \cap O_t \neq \emptyset$ and C is a definably τ-connected component of $O_s \cap O_t$, then the restriction of the bijection*

$$\phi_t \circ \phi_s^{-1} : (O_s \cap O_t) \times \pi_1(G) \longrightarrow (O_s \cap O_t) \times \pi_1(G)$$

to $C \times \{[\gamma]\}$ is the same as $C \times \{[\gamma]\} \longrightarrow C \times \{[\gamma_C]\} : (o, [\gamma]) \mapsto (o, [\gamma_C])$ for some $[\gamma_C] \in \pi_1(G)$.

Proof. Let $o \in C$. By Claim 3.5 and its proof , $\phi_t \circ \phi_s^{-1}(o, [\gamma]) = (o, [\lambda \cdot \eta' \cdot \sigma_t^{-1}])$, where $\lambda = \gamma \cdot \sigma_s \cdot \eta^{-1}$ and $\eta : [0, q_\eta] \longrightarrow O_s$ and $\eta' : [0, q_{\eta'}] \longrightarrow O_t$ are τ-paths such that $\eta(0) = \eta'(0) = o$, $\eta(q_\eta) = \sigma_s(q_s)$ and $\eta'(q_{\eta'}) = \sigma_t(q_t)$. Thus to prove the claim it is enough to show that $[\gamma \cdot \sigma_s \cdot \eta^{-1} \cdot \eta' \cdot \sigma_t^{-1}] = [\gamma \cdot \sigma_s \cdot \theta^{-1} \cdot \theta' \cdot \sigma_t^{-1}]$ whenever $\theta : [0, q_\theta] \longrightarrow O_s$ and $\theta' : [0, q_{\theta'}] \longrightarrow O_t$ are τ-paths such that $\theta(0) = \theta'(0) \in C$, $\theta(q_\theta) = \sigma_s(q_s)$ and $\theta'(q_{\theta'}) = \sigma_t(q_t)$.

Since C is τ-path connected, let $\rho : [0, q_\rho] \longrightarrow C$ be a τ-path such that $\rho(0) = o$ and $\rho(q_\rho) = \theta(0) = \theta'(0)$. Now using the fact that $\pi_1(O_s) = \pi_1(O_t) = 1$ (Proposition 3.3) we see that $\rho \cdot \theta$ (respectively $\theta' \cdot \rho^{-1}$) is τ-homotopic to η (respectively η'^{-1}). Thus $\eta^{-1} \cdot \eta'$ is τ-homotopic to $\theta^{-1} \cdot \theta'$. From here we get $[\gamma \cdot \sigma_s \cdot \eta^{-1} \cdot \eta' \cdot \sigma_t^{-1}] = [\gamma \cdot \sigma_s \cdot \theta^{-1} \cdot \theta' \cdot \sigma_t^{-1}]$ as required. □

We will let $1 \in R$ be a fixed 0-definable positive element of R and denote the element $n \cdot 1$ of the group $(R, 0, +)$ by n. By Proposition 3.4, we will identify $\pi_1(G)$ with a subset of $\mathbb{N} \subseteq R$ and thus, assuming that $G \subseteq R^l$,

$$O_{(s,[\gamma])} := O_s \times \{[\gamma]\}$$

is a definable subset of R^{l+1} and $O := \cup\{O_{(s,[\gamma])} : (s, [\gamma]) \in S \times \pi_1(G)\}$ is a locally definable subset of R^{l+1}.

Let $\{(s_i, l_j) : (i, j) \in \mathbb{N} \times \mathbb{N}\}$ be an enumeration of $S \times \pi_1(G)$. Define inductively (on i) the sets $N_i, O'_{(s_i,l_j)}$ and $V_{(s_i,l_j)}$ in the following way:

$$N_0 = \emptyset \text{ and } O'_{(s_0,l_j)} = V_{(s_0,l_j)} = O_{(s_0,l_j)};$$

assuming that $N_i, O'_{(s_i,l_j)}$ and $V_{(s_i,l_j)}$ were already defined, put

$$N_{i+1} = \{n : n < i + 1 \text{ and } O_{s_{i+1}} \cap O_{s_n} \neq \emptyset\};$$

$O'_{(s_{i+1},l_j)} = O_{(s_{i+1},l_j)} \setminus \cup\{C \times \{l_j\} : C$ is a definably τ-connected component of $O_{s_{i+1}} \cap O_{s_n}$, $n \in N_{i+1}$ and $(\phi_{s_{i+1}} \circ \phi_{s_n}^{-1})_{|C \times \{l_C\}}(o, l_C) = (o, l_j)\}$;

$V_{(s_{i+1},l_j)} = O'_{(s_{i+1},l_j)} \cup \bigcup\{V^C_{(s_n,l_C)} : C$ is a definably τ-connected component of $O_{s_{i+1}} \cap O_{s_n}$, $n \in N_{i+1}$ and $(\phi_{s_{i+1}} \circ \phi_{s_n}^{-1})_{|C \times \{l_C\}}(o, l_C) = (o, l_j)\}$, where $V^C_{(s_n,l_C)} = \{x \in V_{(s_n,l_C)} : x = (o, l) \text{ with } o \in C\}$. [1]

[1]We wish to thank here Elias Baro (Universidad Autónoma de Madrid) for pointing out an imprecision on an early version of our inductive construction.

By Claim 3.6, the sets $V_{(s_i,l_j)}$ are well defined definable subsets of R^{l+1}.

Claim 3.7 *Let* $V = \cup\{V_{(s_i,l_j)} : (i,j) \in \mathbb{N}\times\mathbb{N}\}$*. Then* V *is a locally definable set and the surjective map* $v : V \longrightarrow G$ *given by the projection onto the first coordinate is a locally definable covering map, i.e., for each* i*, we have:*

(1) $v^{-1}(O_{s_i}) = \cup\{V_{(s_i,l_j)} : j \in \mathbb{N}\}$ *(disjoint union);*

(2) $v_{|V_{(s_i,l_j)}}$ *is a definable bijection onto* O_{s_i}*.*

Proof. This follows by induction on the definition of the definable sets $V_{(s_i,l_j)}$ together with Claim 3.6. □

Fix $s_{e_G} \in S$ such that $e_G \in O_{s_{e_G}}$ and assume without loss of generality that $\sigma_{s_{e_G}} = \epsilon_{e_G}$ (the trivial τ-loop at e_G, see page 7). Let $e_V = (e_G, [\epsilon_{e_G}]) \in V$.

Claim 3.8 *Let* $(o,[\gamma]) \in V$ *and suppose that* $\lambda : [0,q_\lambda] \longrightarrow G$ *is a* τ*-path such that* $\lambda(0) = e_G$*,* $\lambda(q_\lambda) = o$ *and* $\phi_s(\langle\lambda\rangle) = (o,[\gamma])$*. Then there exists a* v*-path* $\widetilde{\lambda} : [0,q_{\widetilde{\lambda}}] \longrightarrow V$ *in* V *such that* $\widetilde{\lambda}(0) = e_V$*,* $\widetilde{\lambda}(q_{\widetilde{\lambda}}) = (o,[\gamma])$ *and* $v\circ\widetilde{\lambda} = \lambda$*. In particular,* V *is* v*-path connected and the o-minimal fundamental group* $\pi_1(V)$ *of* V *with respect to the* v*-topology is trivial.*

Proof. By saturation and o-minimality there exists a minimal k for which we can choose points $0 = t(0) < t(1) < \cdots < t(k) < t(k+1) = q_\lambda$ such that for each $j = 0,\ldots,k$, we have $\lambda([t(j),t(j+1)]) \subseteq O_{s(j)}$ for some $s(j) \in S$.

We prove the result by induction on k. If $k = 0$, then $\lambda([0,q_\lambda]) \subseteq O_{s(0)}$ and $[\gamma] = [\epsilon_{e_G}]$, and we put $\widetilde{\lambda} := (v_{|V_{(s(0),[\epsilon_{e_G}])}})^{-1}\circ\lambda$. For the inductive step let $\eta := \lambda_{|[0,t(k)]}$ and $\delta : [0,q_\lambda - t(k)] \longrightarrow O_{s(k)} : t \mapsto \lambda(t+t(k))$. By the induction hypothesis, let $\widetilde{\eta} : [0,t(k)] \longrightarrow V$ be a v-path such that $\widetilde{\eta}(0) = e_V$, $\widetilde{\eta}(t(k)) = (\eta(t(k)),[\gamma'])$ and $v\circ\widetilde{\eta} = \eta$, where $\phi_{s(k-1)}(\langle\eta\rangle) = (\eta(t(k)),[\gamma'])$. Assume that $s(k)$ appear after $s(k-1)$ in the enumeration of S introduced before. The other case is treated symmetrically. If $\phi_{s(k)}(\langle\eta\rangle) = (\eta(t(k)),[\gamma''])$, then $(\eta(t(k)),[\gamma'])$ and $(\eta(t(k)),[\gamma''])$ are the same point in $V_{(s(k),[\gamma''])}$. Since $\lambda = \eta\cdot\delta$ and $\pi_1(O_{s(k)}) = 1$ (Proposition 3.3), we have $[\gamma] = [\gamma'']$. Thus, if $\widetilde{\delta} := (v_{|V_{(s(k),[\gamma''])}})^{-1}\circ\delta$, then $\widetilde{\eta}(t(k)) = \widetilde{\delta}(0)$, and $\widetilde{\lambda} := \widetilde{\eta}\cdot\widetilde{\delta}$ satisfies the claim. So, in particular, V is v-path connected.

By Lemma 2.7, any v-loop δ in V at e_V is the unique lifting $\widetilde{\lambda}$ of a τ-loop $\lambda = v\circ\delta$ in G at e_G as defined in the previous paragraph. So we see that $(e_G,[\epsilon_{e_G}]) = e_V = \widetilde{\lambda}(0)$ and $e_V = \widetilde{\lambda}(q_{\widetilde{\lambda}}) = (e_G,[\lambda])$. This implies that $[\lambda] = 1$ and so $v_*([\widetilde{\lambda}]) = [\lambda] = 1$. Therefore, since by Proposition 2.10 (i), $v_* : \pi_1(V) \longrightarrow \pi_1(G)$ is injective, it follows that $\pi_1(V) = 1$. □

Our next goal is to make the locally definable covering map $v : V \longrightarrow G$ into a locally definable covering homomorphism. For this we will need the following claim:

Claim 3.9 *Let $h : Y \longrightarrow X$ be either $v : V \longrightarrow G$ or $(v, v) : V \times V \longrightarrow G \times G$, and let e_Y be e_V or (e_V, e_V) respectively, and e_X be e_G or (e_G, e_G) respectively. Suppose that $g : X \longrightarrow G$ is a continuous locally definable map such that $g(e_X) = e_G$. Then there is a unique continuous locally definable map $\widetilde{g} : Y \longrightarrow V$ such that $\widetilde{g}(e_Y) = e_V$ and $v \circ \widetilde{g} = g \circ h$.*

Proof. The uniqueness of such a locally definable lifting $\widetilde{g}$ of $g \circ h$ follows from Lemma 2.11. To construct $\widetilde{g} : Y \longrightarrow V$ we will use the fact that $h : Y \longrightarrow X$ is a locally definable covering map, and by Lemma 2.5 and Claim 3.8, $\pi_1(V \times V) \simeq \pi_1(V) \times \pi_1(V) = 1$. We will also use the notation introduced right after Lemma 2.7.

Let $\{U_l : l \in L\}$ be either $\{O_s : s \in S\}$ or $\{O_s \times O_t : s, t \in S\}$. Let $f = g \circ h : Y \longrightarrow G$ and for each $l \in L$, let $\{V_i^l : i \in I_l\}$ be the definably h-connected components of $f^{-1}(U_l)$. For all $l \in L$, $i \in I_l$, choose $y_i^l \in V_i^l$ such that if $e_Y \in V_i^l$ then $e_Y = y_i^l$, and let η_i^l be an h-path in Y from e_Y to y_i^l. Since each V_i^l is definably h-connected, by Lemma 2.9 there is a uniformly definable family $\{\gamma_i^l(w) : w \in V_i^l\}$ of h-paths in V_i^l from y_i^l to w. For $w \in V_i^l$, let $\delta_i^l(w)$ be the h-path $\eta_i^l \cdot \gamma_i^l(w)$ from e_Y to w. Let $\sigma_i^l(w) = f \circ \delta_i^l(w)$ and put $\widetilde{f}(w) = e_Y * \sigma_i^l(w)$.

If $w \in V_i^l \cap V_j^k$ then we have another h-path $\delta_j^k(w)$ from e_Y to w obtained from V_j^k, and $f \circ (\delta_j^k(w) \cdot (\delta_i^l(w))^{-1}) = \sigma_j^k(w) \cdot (\sigma_i^l(w))^{-1}$ is a τ-path from e_G to e_G. By hypothesis, $[\sigma_j^k(w) \cdot (\sigma_i^l(w))^{-1}] \in f_*(\pi_1(Y)) = 1$ and by Remark 2.8 (2), $e_Y * \sigma_i^l(w) = e_Y * \sigma_j^k(w)$ and so $\widetilde{f}$ is well defined. Note that the same argument shows that $\widetilde{f}$ does not depend on the choice of the points $y_i^l \in V_i^l$ or of the h-paths η_i^l.

We now show that $\widetilde{f}$ is a locally definable map. For this it is enough to show that $\widetilde{f}_{|V_i^l}$ is a definable map since by saturation any definable subset of Y is contained in a finite union of V_i^l's. But for $w \in V_i^l$, we have $\widetilde{f}(w) = e_Y * \sigma_i^l(w)$ which is the endpoint of the lifting $\widetilde{\sigma_j^l(w)}$ of $\sigma_j^l(w)$ starting at e_Y. Since $\sigma_j^l(w) = (f \circ \eta_i^l) \cdot (f \circ \gamma_i^l(w))$, $\widetilde{f}(w)$ is the endpoint of the lifting $\widetilde{f \circ \gamma_i^l(w)}$ of $f \circ \gamma_i^l(w)$ starting at the endpoint $\widetilde{f \circ \eta_i^l}(q_{\eta_i^l})$ of the lifting $\widetilde{f \circ \eta_i^l}$ of $f \circ \eta_i^l$. Thus, if W_i^l is a v-open subset of $v^{-1}(O_l)$ such that $v_{|W_i^l} : W_i^l \longrightarrow O_l$ is a definable homeomorphism and $\widetilde{f \circ \eta_i^l}(q_{\eta_i^l}) \in W_i^l$, then $\widetilde{f}(w) = ((v_{|W_i^l})^{-1} \circ (f \circ \gamma_i^l(w)))(q_{\gamma_i^l(w)})$ where $q_{\gamma_i^l(w)}$ is the end point of the domain of $\gamma_i^l(w)$.

To finish we need to show that $\widetilde{g} := \widetilde{f}$ is continuous. For this we use $v \circ \widetilde{g} = g \circ h = f$ (which is immediate from the above characterization of $\widetilde{f}(w)$) and the fact that, as remarked after Definition 2.2, $v : V \longrightarrow G$ is an open mapping. $\square$

Let $\mu : G \times G \longrightarrow G$ and $\iota : G \longrightarrow G$ be the multiplication and the inverse in G. Let $\widetilde{\mu} : V \times V \longrightarrow V$ and $\widetilde{\iota} : V \longrightarrow V$ be the unique continuous locally definable maps given by Claim 3.9.

Claim 3.10 *$(V, \widetilde{\mu}, \widetilde{\iota}, e_V)$ is a locally definable group and $v : V \longrightarrow G$ is a locally definable covering homomorphism.*

Proof. We have that $\widetilde{\mu} \circ (\widetilde{\mu} \times \mathrm{id}_V)$ and $\widetilde{\mu} \circ (\mathrm{id}_V \times \widetilde{\mu})$ are the liftings of the same continuous locally definable map $\mu \circ (\mu \times \mathrm{id}_G) = \mu \circ (\mathrm{id}_G \times \mu)$ and they coincide at (e_V, e_V, e_V). Thus by Lemma 2.11, we have $\widetilde{\mu} \circ (\widetilde{\mu} \times \mathrm{id}_V) = \widetilde{\mu} \circ (\mathrm{id}_V \times \widetilde{\mu})$ and so $(V, \widetilde{\mu})$ is a locally definable semigroup. Similarly, we see that $\widetilde{\mu} \circ (\widetilde{\iota} \times \mathrm{id}_V) \circ \Delta_V = e_V = \widetilde{\mu} \circ (\mathrm{id}_V \times \widetilde{\iota}) \circ \Delta_V$ and $\widetilde{\mu} \circ i_1^V = \mathrm{id}_V = \widetilde{\mu} \circ i_2^V$ where $\Delta_V : V \longrightarrow V \times V$ is the diagonal map, $i_1^V : V \longrightarrow V \times V : v \mapsto (v, e_V)$ and $i_2^V : V \longrightarrow V \times V : v \mapsto (e_V, v)$. Thus $(V, \widetilde{\mu}, \widetilde{\iota}, e_V)$ is a locally definable group as required. Since $v \circ \widetilde{\mu} = \mu \circ (v, v)$ and $v \circ \widetilde{\iota} = \iota \circ v$, it follows that $v : V \longrightarrow G$ is a locally definable homomorphism which must be a locally definable covering homomorphism since it is also a locally definable covering map. $\square$

We are now ready to prove the main theorem of the paper (Theorem 1.4 in the introduction is a special case of this):

Theorem 3.11 *Let G be a definably τ-connected locally definable group. Then the o-minimal universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ is a locally definable covering homomorphism and $\pi_1(G)$ is isomorphic to $\pi(G)$.*

Proof. Suppose that $q : K \longrightarrow V$ is a locally definable covering homomorphism. Then from Propositions 2.10 and 2.12 we obtain $\mathrm{Ker} q \simeq \mathrm{Aut}(K/V) \simeq \pi_1(V)/q_*(\pi_1(K)) = 1$ since $\pi_1(V) = 1$, by Claim 3.8. So $q : K \longrightarrow V$ is a locally definable isomorphism (since it is surjective). Consequently, by [3] Lemma 3.8, the set of all $h : H \longrightarrow G$ in $\mathrm{Cov}^0(G)$ which are locally definably isomorphic to $v : V \longrightarrow G$ is cofinal in $\mathrm{Cov}^0(G)$ and hence the inverse limit $\widetilde{p} : \widetilde{G} \longrightarrow G$ is isomorphic to $v : V \longrightarrow G$. By Propositions 2.10 and 2.12 we obtain $\pi(G) \simeq \mathrm{Ker} v \simeq \mathrm{Aut}(V/G) \simeq \pi_1(G)$ since $\pi_1(V) = 1$. Thus the result holds as required. $\square$

Proof of Corollary 1.5: Let G be a definably t-connected definable group. By Proposition 3.4, $\pi_1(G)$ is finitely generated and, by the isomorphism $\pi_1(G) \simeq \pi(G)$ (Theorem 3.11) and [3] Proposition 3.11, $\pi_1(G)$ is abelian. If G is abelian, then by [12] the assumptions of [3] Theorem 3.15 hold for G. Therefore we have $\pi_1(G) \simeq \pi(G) \simeq \mathbb{Z}^l$ and $G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^l$ for some $l \in \mathbb{N}$ as required.

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