UNIFORM COMPANIONS FOR RELATIONAL EXPANSIONS OF LARGE DIFFERENTIAL FIELDS IN CHARACTERISTIC 0

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Abstract. We give uniform companions for theories of partial differential and large fields of characteristic 0 that are expanded by relations and constants which are independent from the derivations. This is done by generalising the techniques presented in [Tre05] and the notion of large fields. As applications we provide an alternative uniform companion for partial differential henselian valued fields as well giving model companions for theories of partial differential with several orderings and valuations. We also apply our results to provide model complete theories of partial differential valued fields enriched by a predicate for a subfield of the residue field.

1. Terminology, Notation and Conventions

Let $\mathcal{L}_{Ri} := \{+, -, \cdot, 0, 1\}$ and $\mathcal{L}$ an extension of $\mathcal{L}_{Ri}$ by constant and relation symbols. Let also $N \in \mathbb{N}$, $\partial_1, \dots, \partial_N$ be unary function symbols and $\bar{\partial} := (\partial_1, \dots, \partial_N)$.

We shall denote $\mathcal{L}_{val} := \mathcal{L}_{Ri}(|)$.

By an **partial differential field (in $N \in \mathbb{N}$ derivations)** we mean a differential field with $N \in \mathbb{N}$ commuting derivations.

Definition 1.1. By a **$\mathcal{L}$-field** we mean a $\mathcal{L}$-structure that expands a field. Similarly, a **differential $\mathcal{L}$-field** is a $\mathcal{L}(\bar{\partial})$-structure that expands a differential field.

For any given language $\mathcal{L}'$, a $\mathcal{L}'$-formula is a formula of the language of $\mathcal{L}$ and an $\exists$-$\mathcal{L}'$-formula is a formula which,in prenex normal form, has only one block of quantification which are existential. Also for any $\mathcal{L}'$-theory T, we denote by $Mod(T)$ the class of models of T. Given a structure $\mathcal{N}$ in some language $\mathcal{L}''$ extending $\mathcal{L}'$ we write $\mathcal{N} \restriction_{\mathcal{L}'}$ for the reduct of $\mathcal{N}$ to $\mathcal{L}'$. If $\mathcal{M}$ is substructure of $\mathcal{N}$ (in $\mathcal{L}''$), we write $\mathcal{M} \prec_{\mathcal{L}'} \mathcal{N}$ to mean that $\mathcal{M} \restriction_{\mathcal{L}'}$ embeds elementarily into $\mathcal{N} \restriction_{\mathcal{L}'}$.

1.1. **Algebraic Geometry Terminology.** The terminology from algebraic geometry shall be that used in [FJ08]. Therefore, for any field K, a K-variety shall be a separated reduced irreducible scheme of finite type over K. A K-variety V is **absolutely irreducible** if for each $V \times_K \overline{K}$ is a $\overline{K}$-variety where $\overline{K}$ is the algebraic closure of K. We denote by $\mathbb{A}^n$ the affine n-space ($n \in \mathbb{N}$) over the algebraic closure of K. Also for any K-variety V let V_{Reg} denote the regular of V. For moer information on K-varieties the reader is referred to [FJ08] although note that there K-varieties are also irreducible.

2. Introduction

In [Tre05] M. Tressl presented what he called an *uniform companion* of all model complete theories of large and partial differential (i.e. in several commuting derivations) fields in characteristic 0. Recall that a large field K is one such that every smooth K-curve with K-rational point has infinitely many K-rational points. An uniform companion can be defined in the following way

Definition 2.1. Let $\mathbb{M}$ be a class of fields in some language $\mathcal{L}$ extending $\mathcal{L}_{Ri}$ and let $\mathbb{M}_{Diff,N}$ be the class of all partial differential $\mathcal{L}(\bar{\partial})$-fields whose $\mathcal{L}$-reducts are in $\mathbb{M}$. We say that a theory U_N in a language extending $\mathcal{L}(\bar{\partial})$ is a **uniform companion** for $\mathbb{M}_{Diff,N}$ if for each model complete $\mathcal{L}$-theory T such that $Mod(T) \subseteq \mathbb{M}$ and each extension T_1 by definitions and constants of T that contains T we have

- If there T_1 a model companion of another theory T_0 in the language of T_1, then $T_1 \cup U_N$ a model companion of $T_0 \cup DF_N$.
- Moreover if T_1 a model completion of T_0, then $T_1 \cup U_N$ a model companion of $T_0 \cup DF_N$.
- If T_1 has quantifier elimination then so does $T_1 \cup U_N$

- If T is complete and $\mathcal{M} \models T$, then $T_1 \cup U_N \cup \mathrm{diag}(\mathcal{C})$ is complete, where $\mathcal{C}$ is the $\mathcal{L}(\bar{\partial})$-substructure generated by $\emptyset$ in $\mathcal{M}$ (so $\mathcal{C}$ is the $\mathcal{L}(\bar{\partial})$-substructure of $\mathcal{M}$ generated by $c^{\mathcal{M}}$, c a constant symbol of $\mathcal{L}$).

Then the theorem proved by Tressl in [Tre05] is the following

Theorem 2.2. [Tre05, Th. 7.2] *For each $N \in \mathbb{N}$, there exists a uniform companion UC_N in the language $\mathcal{L}_{Ri}(\bar{\partial}) := \mathcal{L}_{Ri}(\partial_1, \ldots, \partial_N)$ for the large and partial differential fields with N commuting derivations in charactersitic 0.*

This work was followed by N. Guzy who, in [Guz06], found an uniform companion for the class of partial differential henselian valued fields with N commuting derivations for each $N \in \mathbb{N}$ (where the valuation is independent from the derivations) in the language $\mathcal{L}_{val}(\partial_1, \ldots, \partial_N)$ where $|$ is a binary relation symbol. He and F. Point subsequently also found uniform methods for companioning theories of topological field structures with one derivative (known as ordinary differential fields) in [GP10] which satisfy a condition that generalises the henselian property for a more general class of definable field topologies.

One may notice that henselian valued fields are themselves large fields (cf. [Pop00, 1.1]). It can also be shown that the fields to which the results of [GP10] apply are also large. This leads to the question of whether there is general over-arcing method that would subsume all three of these cases using simply large fields.

This paper presents work in that direction. We show that the methods presented by Tressl in [Tre05] and the notion large fields can be generalised to expansions of large fields by relations and constants to obtain uniform companions for certain classes of such field structures. Henselian valued fields and the other examples (of t-theories) covered in [GP10] shall all fit into our framework. Therefore, the results here generalise both [Tre05] and [Guz06] as well generalising the applications mentioned in [GP10]. It maybe possible that the results presented here may generalise [GP10] for topological fields entirely but this is yet to be proved (or refuted for that matter).

The main idea in [Tre05] is to construct UC_N in such a way that for any model K, satisfiability certain $\exists$-$\mathcal{L}_{Ri}(\bar{\partial})$-sentences with parameters from K can be reduced to where it has a "regular algebraic realization" in K. These certain $\exists$-$\mathcal{L}_{Ri}$-sentences are in fact those that assert the existence of vanishing points of differential prime ideals of differential polynomials. More precisely, the axioms of UC_N state that for a model K the following holds:

($*$) Let $G := \{f_1, \ldots, f_m\} \subseteq K\{\bar{Y}\}$ ($\bar{Y} := (Y_1, \ldots, Y_n)$) be a characteristic set of a differential prime ideal $\mathfrak{p}$ and V_G the irreducible K-variety defined by the system

$$f_1 = \ldots = f_m = 0, H(f_1, \ldots, f_m) \neq 0$$

where the terms $\partial_1^{i_1} \ldots \partial_N^{i_N} Y_j$ ($1 \leq j \leq N, i_1, \ldots, i_N \in \mathbb{N}$) have been replaced by standard variables. If V_G has a regular K-rational point, then there exists a vanishing point of $\mathfrak{p}$ in K.

That UC_N is a uniform companion for large differential fields is deduced in [Tre05] from the following theorem.

Theorem 2.3. [Tre05, Theorem 6.2]

(I) Given two differential fields K and L of characteristic 0 who both model UC_N and that have a common differential subring A such that $K \equiv_{\exists,A,\mathcal{L}_{Ri}} L$. Then $K \equiv_{\exists,A,\mathcal{L}_{Ri}(\bar{\partial})} L$.

(II) If K is a differential field of characteristic 0 that is large as a pure field then there exists a differential field extension L of K that is an elementary extension in $\mathcal{L}_{Ri}$ and $L \models UC_N$

Part (I) of this theorem follows by reducing the satisfiability of quantifier free formulae $\mathcal{L}_{Ri}$ (respectively $\mathcal{L}_{Ri}(\bar{\partial})$) to those that define vanishing locus of (respectively, differential) prime ideal of (respectively, differential) polynomials. By Robinson's test for model completeness (and similar for quantifier elimination), 2.3 gives that UC_N transfers model completeness and quantifier elimination.

In the proof of part (II) the importance of the K-rational point of V_G being regular in ($*$) is seen. Indeed if K is also a large field then the K-rational points are the K-rational points of V_G be Zariski-dense. Hence the compactness theorem and the structure theorem [Tre05, Theorem 1] give us a differential field extension

L of K which is $\mathcal{L}_{Ri}$-elementary and $K\{\bar{Y}\}/\mathfrak{p} \subseteq L$. Then, since large fields are $\mathcal{L}_{Ri}$-axiomatisable, (II) is obtained by transfinite induction. (II) ensures the consistency of UC_N and transference of embeddings required for model companions and completions. Thus Theorem 2.3 gives Tressl's Main Theorem 2.2.

Similarly to [Tre05], the main idea in this paper will be to construct uniform companions for classes of differential $\mathcal{L}$-fields in which certain $\exists$-$\mathcal{L}(\bar{\partial})$-sentences with parameters are satisfied if they have a "regular algebraic" realisation. For such theories we shall prove a generalisation of Theorem 2.3. These certain $\exists$-$\mathcal{L}(\bar{\partial})$-sentences are ones that say that "there exists a vanishing point of a given differential prime ideal of differential polynomials in a given $\exists$-$\mathcal{L}$-definable set". More precisely, give a set $\mathfrak{B}$ of $\exists$-$\mathcal{L}$-formulae we define an $\mathcal{L}(\bar{\partial})$-theory $UC_N(\mathfrak{B})$ that says for any model $\mathcal{M} \models UC_N(\mathfrak{B})$ expanding some differential field K the following holds

> (†) Let $\lambda(x_1, \ldots, x_n) \in \mathfrak{B}$ and G and $\mathfrak{p}$ as in ($*$). If V_G has a regular K-rational point whose coordinate that correspond to $(Y_1, \ldots, Y_n)$ satisfy λ in $\mathcal{M}$ are Zariski-dense then $\mathfrak{p}$ has a vanishing point in $\mathcal{M}$ which satisfies λ.

To obtain an analogy of Theorem 2.3 we need to be able to reduce any $\exists$-$\mathcal{L}$-formula to one that defines a the (regular) vanishing points of a (respectively, differential) prime ideal that also satisfying some $\lambda \in \mathfrak{B}$. To do this we need to be able to reduce the new relations and their complements to sets defined by formulae from $\mathfrak{B}$. To do this we introduce the following notion of covers.

Definition 2.4. Let C be the set of constants in $\mathcal{L}$ and $\mathfrak{B}$ a set of $\mathcal{L}$-formulae. We say $\mathfrak{B}$ **induces a cover in a class $\mathbb{M}$ on differential $\mathcal{L}$-fields** if for any two $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{M}$ and any relation symbol $\mathcal{R}(\bar{x}) \in \mathcal{L}$ there exists a set $\mathfrak{A}_{\mathcal{R}}$ (resp. $\mathfrak{A}_{\neg\mathcal{R}}$) of tuples (ψ, φ) where ψ is a $\exists$-$\mathcal{L}_{Ri}(C)$-formula and $\varphi \in \mathfrak{B}$ such that

$$\mathcal{R}^{\mathcal{M}_i} = \bigcup_{(\psi,\varphi)\in\mathfrak{A}_{\mathcal{R}}} \exists\bar{y}(\psi(\mathcal{M}_i, \bar{y}) \cap \varphi(\mathcal{M}_i, \bar{y})) \text{ (resp. } (\mathcal{R}^{\mathcal{M}_i})^c = \bigcup_{(\psi,\varphi)\in\mathfrak{A}_{\neg\mathcal{R}}} \exists\bar{y}(\psi(\mathcal{M}_i, \bar{y}) \cap \varphi(\mathcal{M}_i, \bar{y})))$$

for $i = 1, 2$ (here $(\mathcal{R}^{\mathcal{M}_i})^c$ is the complement of $\mathcal{R}^{\mathcal{M}_i}$).

For example, in section 5.1 we show that certain sets of formulae that define basic open sets of valued fields and ordered fields induce covers for the class of these structures (cf. 5.1). This notion of covers gives us the following analogy of 2.3.

Theorem 2.5 (Theorem I). *Whenever $\mathcal{M}, \mathcal{N}$ are two differential $\mathcal{L}$-fields that model $UC_N(\mathfrak{B})$ and $\mathcal{A}$ a common $\mathcal{L}(\bar{\partial})$-substructure with universe A, if $\mathcal{M} \equiv_{\exists,A,\mathcal{L}} \mathcal{N}$ and $\mathfrak{B}$ induces a cover on $\{\mathcal{M}, \mathcal{N}\}$, then $\mathcal{M} \equiv_{\exists,A,\mathcal{L}(\bar{\partial})} \mathcal{N}$.*

The purpose of requiring the K-rational point to be regular in (†) is similar to that in [Tre05]. Indeed, as we shall see in sections 6.2, 1 and **??**, (cartesian products of) open sets of a henselian valued fields and pseudo real closed fields satisfy the following generalised notion of largeness.

Definition 2.6. Let K be a field. A subset $A \subseteq K^m$ ($m \in \mathbb{N}$) is **large over** K if for each absolutely irreducible affine K-variety $V \subseteq \mathbb{A}^n$ with $n \geq m$ and each natural number $r \in \mathbb{N}$, where $rm \leq n$, with a regular K-rational $\bar{a}$ such that $\bar{a} \in A^r \times K^{n-rm}$ then the set $V \cap (A^r \times K^{n-rm})$ is Zariski-dense in V. If $\mathcal{L}$ is a language extending $\mathcal{L}_{Ri}$, $\mathcal{M}$ an $\mathcal{L}$-structure extending K. If each formula λ of a set $\mathfrak{B}$ of $\mathcal{L}$-formulae defines a large set in $\mathcal{M}$ over K, then we shall say that $\mathfrak{B}$ **is large in** $\mathcal{M}$.

Thus by using compactness, [Tre02, Theorem 1] and transfinite induction in the same way that Tressl did we shall obtain the following analogy of Theorem 2.3.

Theorem 2.7 (Theorem II'). *Every differential $\mathcal{L}$-field $\mathcal{M}$ of characteristic 0 in which $\mathfrak{B}$ is large can be extended to a model $\mathcal{N}$ of $UC_N(\mathfrak{B})$ (expanding a partial differential field), such that $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ in the language $\mathcal{L}$.*

Putting 2.5 and 2.7 shall give us a uniform companion in the following sense.

Theorem 2.8 (Main Theorem). *Let $\mathcal{L}$ be a language extending $\mathcal{L}_{Ri}$ by relations and constants and let $\mathfrak{B}$ be a set of $\exists$-$\mathcal{L}$-formulae. Then $UC_N(\mathfrak{B})$ is an uniform companion for any class $\mathbb{M}$ of differential $\mathcal{L}$-fields in characteristic 0 in which $\mathfrak{B}$ induces a cover and is large in each $\mathcal{M} \in \mathbb{M}$*

In practice all of our sets $\mathfrak{B}$ of formulae that we shall consider can be thought of as being "generated" by smaller set $\mathfrak{S}$ of $\mathcal{L}$-formulae in the following sense: if for each $\varphi \in \mathfrak{S}$ we introduce a new predicate symbol

$\mathcal{P}_\varphi$ of arity equal to the number of free variables in φ which is interpreted as the set defined φ, then $\mathfrak{B}$ can be thought of as the is the set of positive conjunctions $\bigwedge_{i=1}^n \psi_i(\bar{x})$ $(n \in \mathbb{N})$ where for each $1 \leq i \leq n$ is either $x = x$ or a literal involving $\mathcal{P}_\varphi$ for some $\varphi \in \mathfrak{S}$. We shall denote the set generated by $\mathfrak{S}$ by $\langle\mathfrak{S}\rangle$. With this notation it is clear that $UC_N(\langle\emptyset\rangle) = UC_N$ and so indeed 2.8 does generalise 2.2.

We shall show that 2.8 generalises [Guz06], by showing that $UC_N(\langle 1|x, \neg 1|x\rangle)$ is a uniform companion for henselian valued fields in 8.2. This shall be because the set $\langle 1|x, \neg 1|x\rangle$ induces a cover for the class of henselian valued fields and since these define open sets they are also large in henselian valued fields.

We shall also apply our results to obtain model companions for theories of differential fields with several independent valuations and/or orderings (which are also independent from the derivations). Such theories were already handled in [GP10] in case of one derivations and case of differential fields with several orderings and one derivative was initially addressed by C. Rivière in [Riv06].

We shall give one more application of the main theorem which shall concern certain two-sorted valued field structure where the second sort has the residue field and a subfield. Such a structure we shall call a **valued field with a residual subfield**. In section 8.3 we use our main result is provide model companions for certain partial differential valued fields with a residual subfield whose valued field sort and field pair sort are models of model complete theories.

We shall only consider (differential) fields of characteristic 0 in this paper and, hence, all (differential) fields in this paper shall be assumed to be of characteristic 0.

We note that these are results that were obtained in the author's PhD thesis [Sol14]. However, the terminology from [Sol14] has been slightly modified here for better exposition

3. Preliminaries For Differential Algebra

In this section we recall notions from basic differential algebra; mainly we explain what a characteristic set is in the differential setup. Our main source here is Kolchin's book [Kol73] on differential algebra and algebraic groups.

Let R be a differential ring in N pairwise commuting derivations $\partial_1, ..., \partial_N$. Let $\bar{Y} := (Y_1, ..., Y_n)$ be a tuple of $n \in \mathbb{N}$ indeterminates over R and let

$$\mathcal{D} := \{\partial_1^{i_1} ... \partial_N^{i_N} \mid i_1, ..., i_N \in \mathbb{N}_0\}$$

be the free abelian monoid generated by $\{\partial_1, ..., \partial_N\}$, which we denote multiplicatively. For each $\Theta \in \mathcal{D}$ and $i \in \{1, ..., n\}$ let ΘY_i be an indeterminate, where $\Theta Y_i = Y_i$ if $\Theta = \partial_1^0 ... \partial_N^0$ by definition. Moreover let

$$\mathcal{D}\bar{Y} := \{\Theta Y_i \mid \Theta \in \mathcal{D}, 1 \leq i \leq n\}.$$

The differential polynomial ring over R in N derivations and n indeterminates is the polynomial ring $R\{\bar{Y}\} := R[y \mid y \in \mathcal{D}\bar{Y}]$ together with the uniquely determined derivations ∂_i such that $\partial_i(r\Theta Y_j) = (\partial_i r)\Theta Y_j + r(\partial_i \Theta) Y_j$ $(1 \leq i \leq N \ 1 \leq j \leq n, \ r \in R)$. So $R\{\bar{Y}\}$ is a differential ring extension of R and $R\{\bar{Y}\}$ is the free object generated by n elements over R in the category of differential rings with K commuting derivations. The set of all powers of variables from $\mathcal{D}\bar{Y}$ is denoted by

$$\mathcal{D}\bar{Y}^* := \{y^p \mid y \in \mathcal{D}\bar{Y}, p \in \mathbb{N}\}.$$

Definition 3.1. The **rank** on $\mathcal{D}\bar{Y}^*$ is the map $\mathrm{rk} : \mathcal{D}\bar{Y}^* \longrightarrow \mathbb{N}_0 \times \{1, ..., n\} \times \mathbb{N}_0^N \times \mathbb{N}$ defined by

$$\mathrm{rk}(\partial_1^{j_1} ... \partial_N^{j_N} Y_i)^p := (j_1 + ... + j_N, i, j_N, ..., j_1, p).$$

The set $O := \mathbb{N}_0 \times \{1, ..., n\} \times \mathbb{N}_0^K \times \mathbb{N}$ equipped with the lexicographic order (hence the first component is the dominating one) is well ordered.

Definition 3.2. We say a variable $y \in \mathcal{D}\bar{Y}$ **appears** in $f \in R\{\bar{Y}\}$ if y appears in f considered as an ordinary polynomial (hence Y_1 does not appear in $\partial_1 Y_1$). The **leader** u_f of $f \in R\{\bar{Y}\} \setminus R$ is the variable $y \in \mathcal{D}\bar{Y}$ of highest rank which appears in f. Moreover $u_f^* := u_f^{\deg_{u_f} f} \in \mathcal{D}\bar{Y}^*$ denotes the highest power of u_f in f. We extend the rank to polynomials $f \in R\{\bar{Y}\}$ by

$$\mathrm{rk}(f) := \mathrm{rk}(u_f^*) \in O.$$

Definition 3.3. If $g, f \in R\{\bar{Y}\}$, $g \notin R$, are differential polynomials, then f is called **weakly reduced** with respect to g if no proper derivative of u_g (i.e. any element $v \in \mathcal{D}\bar{Y}$ such that $v = \Theta u_g$ where $\Theta \in \mathcal{D}\backslash\{\partial_1^0 \dots \partial_N^0\}$) appears in f. f is called **reduced** with respect to g if f is weakly reduced with respect to g and if $\deg_{u_g} f < \deg_{u_g} g$.

The polynomial f is called (weakly) reduced with respect to a nonempty set $G \subseteq R\{\bar{Y}\}\backslash R$ if f is (weakly) reduced with respect to every $g \in G$.

A nonempty subset $G \subseteq R\{\bar{Y}\} \setminus R$ is called **autoreduced** if every $f \in G$ is reduced with respect to all $g \in G$, $g \neq f$. If G consists of a single element, then G is autoreduced.

It easy to see that $u_f \neq u_g$ — hence $\operatorname{rk} f \neq \operatorname{rk} g$ — if f, g are different polynomials from an autoreduced set. Moreover, by [Kol73], Chap. O, Section 17, Lemma 15(a), we have

Proposition 3.4. *Every autoreduced set is finite.* □

Let ∞ be an element bigger than every element in O and let $(O\cup\{\infty\})^{\mathbb{N}}$ be equipped with the lexicographic order. We define the rank of an autoreduced set G to be an element of $(O \cup \{\infty\})^{\mathbb{N}}$ as follows. Let $G = \{g_1, ..., g_l\}$ with $\operatorname{rk} g_1 < ... < \operatorname{rk} g_l$. Then

$$\operatorname{rk} G := (\operatorname{rk} g_1, ..., \operatorname{rk} g_l, \infty, \infty, ...).$$

Proposition 3.5. *There is no infinite sequence $G_1, G_2, ...$ of autoreduced sets with the property $\operatorname{rk} G_1 > \operatorname{rk} G_2 > ....$*

Proof. [Kol73], Chap. I, Section 10, Proposition 3. □

Definition 3.6. If $M \subseteq R\{\bar{Y}\}$ is a set not contained in R, then by Proposition 3.5 the set $\{\operatorname{rk} G \mid G \subseteq M \text{ is autoreduced}\}$ has a minimum. Every autoreduced subset G of M with this rank is called a **characteristic** set of M.

Proposition 3.7. *If G is a characteristic set of $M \subseteq R\{\bar{Y}\}$ and $f \in M \setminus R$, then f is not reduced with respect to G.*

Proof. If $f \in M \setminus R$ is reduced with respect to G, then the set $\{g \in G \mid \operatorname{rk} g < \operatorname{rk} f\} \cup \{f\}$ is an autoreduced subset of M of rank strictly lower than the rank of G, which is impossible. □

Throughout the rest of this chapter assume that R is a differential domain in N derivations containing $\mathbb{Z}$.

Definition 3.8. Let $f \in R\{\bar{Y}\} \setminus R$, $f = f_d u_f^d + ... + f_1 u_f + f_0$ with polynomials $f_d, ..., f_0 \in R[y \in \mathcal{D}\bar{Y} \mid y \neq u_f \text{ and } y \neq \Theta u_f \text{ forall } \Theta \in \mathcal{D}]$ and $f_d \neq 0$. The **initial** $I(f)$ of f is defined as

$$I(f) := f_d.$$

The **separant** $S(f)$ of f is defined as

$$S(f) := \frac{\partial}{\partial u_f} f = d \cdot f_d u_f^{d-1} + ... + f_1.$$

Moreover, for every subset $G = \{g_1, ..., g_l\}$ of $R\{\bar{Y}\} \setminus R$ we define

$$H(G) := \prod_{i=1}^{l} I(g_i) \cdot S(g_i) \text{ and } H_G := \{\prod_{i=1}^{l} I(g_i)^{n_i} S(g_i)^{m_i} \mid n_i, m_i \in \mathbb{N}_0\}.$$

Since R is a domain and $\mathbb{Z} \subseteq R$, the set H_G does not contain 0. Moreover, $S(g)$ and $I(g)$ are reduced with respect to G ($g \in G$), if G is an autoreduced set.

Theorem 3.9. *Let $G \subseteq R\{\bar{Y}\}$ be an autoreduced set and let $f \in R\{\bar{Y}\}$. Let $[G]$ denote the differential ideal generated by G in $R\{\bar{Y}\}$ and let (G) denote the ideal generated by G in $R\{\bar{Y}\}$. Then there is some $\tilde{f} \in R\{\bar{Y}\}$ which is reduced with respect to G and some $H \in H_G$ such that $H \cdot f \equiv \tilde{f} \mod [G]$. If f is weakly reduced with respect to G, then we can take H such that $H \cdot f \equiv \tilde{f} \mod (G)$.*

Proof. [Kol73], Chap. I, Section 9, Proposition 1. □

Notation. For any ring A, ideal $\mathfrak{a}$ of A and element $a \in A$, we denote

$$\mathfrak{a} : a := \{b \in A \mid a{\cdot}b \in \mathfrak{a}\}$$

and

$$\mathfrak{a} : a^\infty = \{b \in A \mid a^n{\cdot}b \in \mathfrak{a}\text{for some}n \in \mathbb{N}\}$$

Also, if A is a differential domain and G a subset of A then $[G]$ denotes differential ideal generated by G or, in other words, the smallest differential ideal containing G.

Corollary 3.10. *If G is a characteristic set of a differential prime ideal $\mathfrak{p}$ of $R\{\bar{Y}\}$ with $\mathfrak{p} \cap R = 0$, then*

$$\mathfrak{p} = \{f \in R\{\bar{Y}\} \mid H(G)^n{\cdot}f \in [G] \text{ for some } n \in \mathbb{N}_0\}.$$

Moreover if $f \in \mathfrak{p}$ is weakly reduced with respect to G, then $H(G)^n{\cdot}f \in (G)$ for some $n \in \mathbb{N}_0$.

Proof. From Theorem 3.9 and Proposition 3.7, since $H_G \cap \mathfrak{p} = \emptyset$. □

Finally we collect some facts, which will be used later on.

Proposition 3.11. *Let $K \subseteq L$ be an extension of fields of characteristic 0 and let K be equipped with N commuting derivations. Then there are N commuting derivations on L extending those on K.*

Proof. [Kol73], p. 90. □

Proposition 3.12. *Let K be a differential field of characteristic 0 and let Y be a set of differential indeterminates. Let $\mathfrak{a} \subseteq K\{\bar{Y}\}$ be a differential ideal. Then:*

(i) *Every prime ideal $\mathfrak{p}$ of $K\{\bar{Y}\}$, minimal with the property $\mathfrak{a} \subseteq \mathfrak{p}$, is a differential ideal.*
(ii) *Let $\mathfrak{a}$ be radical and differential. Then $\mathfrak{a}$ is finitely generated as a differential radical ideal. Moreover, if $K \subseteq L$ is an extension of differential fields, then the ideal generated by $\mathfrak{a}$ in $L\{\bar{Y}\}$ is differential and radical.*

Proof. For (i) see [Kol73], Chap. 1, Sect. 2, Exercise 3. Item (ii) can be found in [Kol73], Chap. 4, Sect. 3 and 4. □

Proposition 3.13. *Let $K \subseteq L$ be fields of characteristic 0 and let Y be a set of indeterminates. Let $G \subseteq K[\bar{Y}]$ be a set and let $H \subseteq K[\bar{Y}]$ be a multiplicatively closed set. Let $\mathfrak{a}$ be the ideal $(G)_{K[\bar{Y}]} : H$ of $K[\bar{Y}]$. Then the ideal $(G)_{L[\bar{Y}]} : H$ of $L[\bar{Y}]$ is generated by $\mathfrak{a}$ and $((G)_{L[\bar{Y}]} : H) \cap K[\bar{Y}] = \mathfrak{a}$.*

Proof. We omit the easy proof. □

Definition 3.14. An autoreduced set G is **coherent** if for each $g_1, g_2 \in G$, if v is a common derivative of u_{g_1} and u_{g_2}, say $v = \Theta_1 u_{g_1} = \Theta_2 u_{g_2}$ where $\Theta_1, \Theta_2 \in \mathcal{D}$, then $S(g_2)\Theta_1 g_1 - S(g_2)\Theta_2 g_2 \in (G_v) : H(G)^\infty$, where G_v denotes the set of all differential polynomials Θf with $f \in G, \Theta \in \mathcal{D}$, and Θu_f of lower rank than v.

Theorem 3.15 (The Rosenfeld Lemma)**.** *Let K be a differential field of characteristic 0 in N derivations and let A be the differential polynomial ring of K in $\bar{Y} := (Y_1, ..., Y_n)$. Let $G \subseteq A$ be an autoreduced set. Then the following are equivalent:*

1. *G is a characteristic set of $[G] : H(G)^\infty$ and $[G] : H(G)^\infty$ is prime.*
2. *(a) G is coherent and*
   (b) the ideal $(G)_A : H(G)^\infty$ of A is prime and does not contain nonzero elements of A reduced with respect to G.
3. *Let B denote the K-algebra $K[y \in \mathcal{D}\bar{Y} \mid y$ appears in g for some $g \in G]$.*
   (a) G is coherent and
   (b) the ideal $(G)_B : H(G)^\infty$ of B is prime and does not contain nonzero elements of B, reduced with respect to G.

Proof. 1⇔2 is [Kol73], IV, 9, Lemma 2, and 2⇔3 can be easily derived. □

Remark 3.16. Theorem 3.15 tells us that if $G \subseteq K\{\bar{Y}\}$ is a characteristic set of a differential prime ideal, then the prime ideal $(G)_B : H(G)^\infty$ defines an irreducible K-variety. We shall denote this variety by V_G. If $G := \{g_1, \ldots, g_m\}$ then V_G is precisely the K-variety defined by the system of (standard) polynomial equations

$$g_1 = \ldots = g_m = 0, H(G) \neq 0$$

Here we view as standard polynomials i.e. $g_1, \ldots, g_m, H(G) \in A$. The fact that V_G is irreducible follows from the following proposition

Proposition 3.17. *Let A be a domain, let $\mathfrak{a}$ be an ideal of A, let $h \in A$ and let z be an indeterminate over A. Then*

$$\mathfrak{a} : h^\infty := \{a \in A \mid h^n \cdot a \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\} = (\mathfrak{a}, z \cdot h - 1)_{A[z]} \cap A.$$

Moreover $h \notin \sqrt{\mathfrak{a}} \iff A \neq \mathfrak{a} : h^\infty$, and in this case h is a nonzero divisor of $A/(\mathfrak{a} : h^\infty)$. Also, mapping $\frac{1}{h/(\mathfrak{a}:h^\infty)}$ to $z/(\mathfrak{a}, z \cdot h - 1)$ induces an isomorphism between the localisation $(A/(\mathfrak{a} : h^\infty))_{h/(\mathfrak{a}:h^\infty)}$ and the factor ring $A[z]/(\mathfrak{a}, z \cdot h - 1)$. In particular, $\mathfrak{a} : h^\infty$ is prime if and only if $(\mathfrak{a}, z \cdot h - 1)_{A[z]}$ is prime, provided that $h \notin \sqrt{\mathfrak{a}}$.

Proof. We omit the easy proof. □

Proposition 3.18. *Let $K \subseteq L$ be differential fields of characteristic 0. If $G \subseteq K\{\bar{Y}\}$ is a characteristic set of $K\{\bar{Y}\}$, then G is a characteristic set of $L\{\bar{Y}\}$, too.*

Proof. Again, this is easy and left to the reader. □

4. Definition of $UC_N(\mathfrak{S})$

Let $\mathfrak{B}$ be a set of $\mathcal{L}$-formulae. The method by which we shall axiomatise the theory $UC_N(\mathfrak{B})$, as described in the introduction, in the language $\mathcal{L}(\bar{\partial})$ shall closely followed that presented in [Tre05] to axiomatise UC_N. There Tressl defines UC_N using what he calls *algebraically prepared systems*.Here we shall generalise these systems using $\mathfrak{B}$ in the following way-

Definition 4.1. A **$\mathfrak{B}$-algebraically prepared system** of a differential $\mathcal{L}$-field $\mathcal{M}$ a sequence $(f_1, \ldots, f_m, \lambda)$ of differential polynomials $f_1, \ldots, f_m \in K\{\bar{Y}\} \backslash K$, where K is the field $\mathcal{M}$ expands and $\bar{Y} := (Y_1, \ldots, Y_n)$, and a formula $\lambda \in \mathfrak{B}$ in n free variables such that

(1) $\{f_1, \ldots, f_m\}$ is a characteristic set, thus $\{f_1, \ldots, f_m\}$ is a reduced and coherent set and the ideal $(f_1, \ldots, f_m) : H(f_1, \ldots, f_m)^\infty$ of

$$A(f_1, \ldots, f_m) := \{y \in \mathcal{D}\bar{Y} | y \text{ y appears in some } f_1, \ldots, f_m\}$$

does not contain nonzero elements, reduced with respect to $f_1, \ldots, f_m$.

(2) V_G has a regular K-rational point $\bar{a}$ such that $(a_1, \ldots, a_n)$ (i.e. the coordinates corresponding to the $\bar{Y}$) satisfies λ in $\mathcal{M}$.

Note that algebraically prepared systems are simply $\mathfrak{S}$-algebraically prepared systems where $\mathfrak{S} = \{x = x\}$ or even $\mathfrak{S} = \langle \emptyset \rangle$.

We say that an $\mathfrak{B}$-algebraically prepared system $(f_1, \ldots, f_m, \lambda)$ of $\mathcal{M}$ is **solvable** in some differential $\mathcal{L}$-field extension $\mathcal{N}$ if there exists an n-tuple $\bar{a}$ in $\mathcal{N}$ such that $f_1(\bar{a}) = 0, \ldots, f_m(\bar{a}) = 0, H(f_1, \ldots, f_m)(\bar{a}) \neq 0$ and $\mathcal{N} \models \lambda(\bar{a})$. We call $\bar{a}$ a **solution** of $(f_1, \ldots, f_m, \lambda)$.

Remark 4.2. Keeping with the notation of 4.1, suppose that $(f_1, \ldots, f_m, \lambda)$ is a $\mathfrak{B}$-algebraically prepared system of $\mathcal{M}$. Then since the variety $V_{f_1,\ldots,f_m}$ has a regular K-rational point, then [Jar11, 5.1.1] tells us that $V_{\{f_1,\ldots,f_m\}}$ is absolutely irreducible. Thus, for any field extension $L \supseteq K$, $V_{f_1,\ldots,f_m}$ can be thought of as an irreducible L-variety. In particular, a solution of $(f_1, \ldots, f_m, \lambda)$ in an extension of $\mathcal{M}$ can be thought of as a point in $V_{f_1,\ldots,f_m}$.

We now construct $UC_N(\mathfrak{S})$ as the axiomatisation of the class of all differential $\mathcal{L}$-fields that solve all their own $\mathfrak{B}$-algebraically prepared systems. The following proposition, which is an analogue of [Tre05, Prop. 4.1], shall enable us to do this.

Proposition 4.3. *Let $n, m, N \in \mathbb{N}$. Let $\bar{Y} = (Y_1, \dots, Y_n)$ and let $f_1(\bar{t}, \bar{Y}), \dots, f_m(\bar{t}, \bar{Y}) \in \mathbb{Z}\{\bar{Y}\}[\bar{t}]$ be general polynomials in indeterminates $\{\Theta Y_j | \Theta \in \mathcal{D}, 1 \leq j \leq n\}$ and in the indeterminate coefficients $\bar{t} = (t_1, \dots, t_r)$. Let also $\lambda \in \mathfrak{B}$. Then there is a sentence φ in the language $\mathcal{L}(\bar{\partial})$ such that for all differential $\mathcal{L}$-field $\mathcal{M}$ of characteristic 0 in N commuting derivations we have*

$\mathcal{M} \models \varphi \iff$ *for all r-tuples $\bar{c}$ from $\mathcal{M}$, if $(f_1(\bar{c}, \bar{Y}), \dots, f_m(\bar{c}, \bar{Y}), \lambda))$ is an $\mathfrak{B}$-algebraically prepared system of $\mathcal{M}$ then it is solvable in $\mathcal{M}$.*

Thus, for all $N \in \mathbb{N}$, the class of all differential $\mathcal{L}$-fields of characteristic 0 in N commuting derivations, which solve all their $\mathfrak{B}$-algebraically prepared systems, is first order axiomatisable in the language $\mathcal{L}(\bar{\partial})$

Proof. We firstly we claim that there is a formula $\phi(\bar{v}), \bar{v} = (v_1, \dots, v_r)$, such that for all differential $\mathcal{L}$-fields $\mathcal{M}$ of characteristic 0 in N commuting derivations and $\bar{c} \in dom(\mathcal{M})^r$ we have

$$\mathcal{M} \models \phi(\bar{c}) \iff (f_1(\bar{c}, \bar{Y}), \dots, f_n(\bar{c}, \bar{Y}), \lambda)) \textit{ is an } \mathfrak{B}\textit{-algebraically prepared system of } \mathcal{M}.$$

In [Tre05, 4.3], Tressl shows that there exists some $\mathcal{L}_{Ri}$-formula $\phi(\bar{t})$ such that such that for any field K and any r-tuple $\bar{c}$ from K $K \models \phi(\bar{c})$ if and only if "$\{f_1(\bar{c}, \bar{Y}), \dots, f_n(\bar{c}, \bar{Y})\}$ is a characteristic set of differential prime ideal $\mathfrak{p}_{\bar{c}}$ of $K\{\bar{x}\}$". Also in [Tre05, 4.3], Tressl shows that there exists an $\mathcal{L}_{Ri}$-formula $\psi(\bar{x}, \bar{t})$, where $\bar{x}$ enumerates the variables ΘY_j that occur in some $f_i(\bar{t}, \bar{Y})$ (with $x_i = Y_i$ for $1 \leq i \leq N$), such that for any field K and all tuples $\bar{a}$, $\bar{c}$ from K of the same lengths as $\bar{x}$ and $\bar{t}$,

$$\mathcal{M} \models \psi(\bar{a}, \bar{c}) \textit{ if and only if } \bar{a} \textit{ is a regular } K\textit{-rational point of } V_G$$

Thus we may take $\nu(\bar{t})$ to be the formula $\phi(\bar{t}) \wedge \exists \bar{x}(\psi(\bar{x}, \bar{t}) \wedge \lambda(x_1, \dots, x_n))$ which proves the claim. But now the proof of the theorem is now completed by taking φ to be the sentence

$$\forall \bar{v}(\nu(\bar{v}) \rightarrow \exists \bar{y}(\bigwedge_{i=1}^{m} f_i(\bar{v}, \bar{y}) = 0 \wedge \lambda(\bar{y})))$$

□

Definition 4.4. Let $UC_N(\mathfrak{B})$ be the theory of differential $\mathcal{L}$-fields of characteristic 0 in N commuting derivations, which solves all its $\mathfrak{B}$-algebraically prepared systems.

5. Proof of (I')

5.1. **Covers.** Before we proceed to prove (I') we shall first develop the idea of covers that we presented in the introduction (definition 2.4). Firstly we shall generalise the notion.

Throughout this section let $\mathcal{L}$ an extension of $\mathcal{L}_{Ri}$ by relation and constant symbols only and let C be the set of constant symbols in $\mathcal{L}$. Let also $\mathfrak{B}$ be a set of $\mathcal{L}$-formulae.

Definition 5.1. Let $\mathbb{M}$ be a class of differential $\mathcal{L}$-fields and $\varphi(\bar{x})$ an $\mathcal{L}(\bar{\partial})$-formula. We say $\mathfrak{B}$ **induces a cover of** φ **in** $\mathbb{M}$ if for any two $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{M}$ there exists a set $\mathfrak{A}_\varphi$ of tuples $(\psi(\bar{x}, \bar{y}), \varphi(\bar{x}, \bar{y}))$ where ψ is a $\exists$-$\mathcal{L}_{Ri}(\bar{\partial}, C)$-formula and $\varphi \in \mathfrak{B}$ such that

$$\varphi\mathcal{M}_i = \textstyle\bigcup_{(\psi,\varphi) \in \mathfrak{A}_\mathcal{R}} \exists \bar{y}(\psi(\mathcal{M}_i, \bar{y}) \cap \varphi(\mathcal{M}_i, \bar{y})).$$

for $i = 1, 2$. We call the set $\mathfrak{A}_\varphi$ a **cover of** φ **in** $\mathcal{M}_1$ **and** $\mathcal{M}_2$ **(induced by** $\mathfrak{B}$**).**

We shall say that $\mathfrak{B}$ induces a cover on a $\mathcal{L}$-theory T of fields if it induces a cover on $Mod(T)$.

We immediately obtain the following results.

Proposition 5.2. *(1) For any class $\mathbb{M}$ of $\mathcal{L}$-fields the following are equivalent-*
 (a) $\mathfrak{B}$ induces a total cover of $\mathbb{M}$.
 (b) For every $\exists$-$\mathcal{L}(\bar{\partial})$-formula, $\mathfrak{B}$ induces a cover of φ in $\mathbb{M}$.
(2) If $\mathfrak{B}$ induces a cover on a model complete $\mathcal{L}$-theory T of fields then it also induces a cover on any extension by definitions of T.

Example 5.3. (1) Clearly $\langle\emptyset\rangle$ induces a cover on the class of fields in the language $\mathcal{L}_{Ri}$.
(2) Let $\mathcal{L}' := \mathcal{L} \backslash \mathcal{L}_{Ri} \cup \{=\}$, we can take the $\mathfrak{B}_\mathcal{L}$ is the set of $\mathcal{L}'$-literals. Then clearly $\mathfrak{B}$ induces a cover on the class of $\mathcal{L}$-fields.

(3) taking $\mathfrak{S}_{val}$ to be the set $\{1|x, \neg 1|x\}$ of $\mathcal{L}_{val}$ formulae, then $\mathfrak{B}_{val} := \langle \mathfrak{S}_{val} \rangle$ induces a cover on the class of valued fields in the language $\mathcal{L}_{val}$.
(4) Then (2) of 5.2 along with example (1) above tells us that $\langle \emptyset \rangle$ induces a cover the $\mathcal{L}_{<}$-theory RCF of the class of real closed fields.
(5) Similarly, $\langle \emptyset \rangle$ induces a cover of the theory pCF in the language $\mathcal{L}_{val}(P_n)_{n\in\mathbb{N}}$ (for each $n \in \mathbb{N}$ P_n is a unary predicate).
(6) Generalising (4), consider the theory OF_n of n-ordered fields (that is, a field with $n \in \mathbb{N}$ independent orderings) in the language $\mathcal{L}_{Ri}(<_1, \ldots, <_1)$. Then $\mathfrak{B}_{n-ord} := \langle 0 <_1 x, \ldots, 0 <_n x \rangle$ induces a cover on OF_n. Indeed, for each $1 \leq i \leq n$

$$OF_n \vdash \forall x_1, x_2(\neg x_1 <_i x_2 \leftrightarrow x_2 - x_1 = 0 \vee \exists y(x_2 - x_1 = y \wedge 0 <_i y))$$

Remark 5.4. Notice that in all the examples mentioned above all covers are finite. Indeed, all the examples that we shall consider throughout this paper shall be finite. However, the definition allows for infinite covers for the sake of proving 2.5 in more general terms.

5.2. **Proof of (I').** We shall prove an analogue of [Tre05, Th. 3.3] from which 2.5 will clearly be a corollary.

Notation. Give two $\mathcal{L}$-structures $\mathcal{M}_1, \mathcal{M}_2$ with universes M_1, M_2 respectively and $A \subseteq M_1 \cap M_2$, then the notation

$$\mathcal{M}_1 \Rrightarrow_{\exists,A,\mathcal{L}} \mathcal{M}_2$$

means that every $\exists$-$\mathcal{L}(\bar{\partial})$-formula with parameters form A which holds in $\mathcal{M}_1$ also holds in $\mathcal{M}_2$.

Theorem 5.5. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two differential $\mathcal{L}$-fields expanding partial differential fields L_1, L_2 (of characteristic 0 in N commuting derivations). Let $\mathfrak{B}$ be a set of $\exists$-$\mathcal{L}$-formulae that induces a cover $\{\mathcal{M}_1, \mathcal{M}_2\}$. Let A be a common differential subring of L_1, L_2 such that the elements of $C^{\mathcal{M}_1}, C^{\mathcal{M}_2} \subseteq Quot(A) =: K_0$, where C is the set of constants in $\mathcal{L}$. Let K_i be the algebraic closure of K_0 in L_i. Suppose*
(1) $\mathcal{M}_1 \equiv_{\exists,A,\mathcal{L}} \mathcal{M}_2$.
(2) $\mathcal{M}_2$ solves all $\mathfrak{B}$-algebraically prepared systems of $\mathcal{M}_2$ defined over K_2.
Then $\mathcal{M}_1 \Rrightarrow_{\exists,K_0,\mathcal{L}(\bar{\partial})} \mathcal{M}_2$.

Proof. Clearly, condition (1) implies $L_1 \equiv_{\exists,K_0,\mathcal{L}_{Ri}} L_2$ as pure fields. By standard arguments we get that K_1 and K_2 are isomorphic as fields over K_0. This isomorphism maps respective interpretations of constants to each other and respects the derivations (observe that K_i is a differential subfield of L_i). Hence we may assume that $K = K_1 = K_2$ is the algebraic closure of K_0 in L_i for each $i = 1, 2$ and $C^{\mathcal{M}_1} = C^{\mathcal{M}_2} \subseteq K$.

Let $\varphi(\bar{x}_1, \bar{v})$ be a quantifier free formula in the language $\mathcal{L}(\bar{\partial})$ where $\bar{x}_1$ and $\bar{v}$ are tuples of variables of lengths $n_1, q \in \mathbb{N}$ respectively. Suppose there is some $\bar{c} \in K_0^q$ such that $\mathcal{M}_1 \models \exists \bar{x}\varphi(\bar{x}, \bar{c})$. We have to show that $\mathcal{M}_2 \models \exists \bar{x}\varphi(\bar{x}, \bar{c})$.

Since $\mathfrak{B}$ is induces a cover on $\{\mathcal{M}_1, \mathcal{M}_2\}$, (1) of 5.2 tells us that there exists a cover $\mathfrak{A}_\varphi$ of φ in $\{\mathcal{M}_1, \mathcal{M}_2\}$ induced by $\mathfrak{B}$. So there exists some $(\phi, \lambda) \in \mathfrak{A}_{\varphi,\{\mathcal{M}_1,\mathcal{M}_2\}}$, where $\phi(\bar{x})$ is an $\exists$-$\mathcal{L}_{Ri}(\bar{\partial}, C)$-formula abd $\lambda \in \mathfrak{B}$ such that $\mathcal{M}_1 \models \exists \bar{x}_1\phi(\bar{x}_1, \bar{c}) wedge \lambda(x_1, \ldots, x_n)$. It then suffices to show that there exists $\bar{b} \in L_2^n$ such that $\mathcal{M}_2 \models \phi(\bar{b}, \bar{c}) \wedge \lambda(\bar{b})$. Since $C^{\mathcal{M}_1} = C^{\mathcal{M}_2} \subseteq K_0$ we can assume that ϕ is a $\mathcal{L}_{Ri}(\bar{\partial})$-formula by subsuming the all constants that occur in ϕ into $\bar{c}$. We can also assume without any loss of generality that ϕ of the form $\exists \bar{x}_2(\bigwedge_{i=1}^r p_i(\bar{x}_1, \bar{x}_2) = 0)$ where $p_1, \ldots, p_r \in K_0\{\bar{Y}\}$ and $\bar{Y} := (Y_1, \ldots, Y_n)$ is a tuple of differential indeterminates of over K_0 of length $n = n_1 + n_2$. Thus, writing $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and $\lambda(\bar{x})$ for $\lambda(\bar{x}_2)$, we get that there exists some $\bar{a} \in L_1^n$ such that $p_1(\bar{a}) = \ldots = p_r(\bar{a}) = 0$ and $\mathcal{M}_1 \models \lambda(\bar{a})$. Then it suffices to show that there exists $\bar{b} \in L_2^n$ such that $p_1(\bar{b}) = \ldots = p_r(\bar{b}) = 0$ and $\mathcal{M}_2 \models \lambda(\bar{b})$.

From here the proof follows analogously to the proof of Thereom 3.3 in [Tre05]. □

6. Large Sets

Throughout this section let K be a field.

In the introduction we defined the notion of large sets to be "over K" but as we now show that this is redundant since a set can only be large over the smallest field containing it. Given a subset $A \subseteq K^M$, the **smallest field containing** A is the smallest subfield $K_0 \subseteq K$ such that $A \subseteq K_0^M$.

Proposition 6.1. *Let $A \subseteq K^m$ for some $m \in \mathbb{N}$. If A is large over K then K is the smallest field containing A.*

Proof. Suppose that the smallest field containing A is not K. Then it must be some field $K_0 \subsetneq K$. Thus there exists some element $\alpha \in K \backslash K_0$. Consider the firstly the case when $A \subseteq K_0^n \subsetneq K^n$ where $n \geq 2$. Let $\bar{b} = (b_1, b_2, \ldots, b_n) \in A$ and consider the K-curve C defined by the equation $\alpha(b_1 - x_1) - (b_2 - x_2) = 0$ considered as a curve embedded into (the first two coordinates of) $\mathbb{A}^n$. Then $\bar{b}$ is a regular point of C. Since A is large over K, $C \cap A$ is Zariski-dense in C. In particular, there is some $\bar{c} = (c_1, c_2, \ldots, c_n) \in A$ such that $\bar{c} \in C \cap U$ where $U := \{\bar{x} \in \mathbb{A}^n | b_1 - x_1 \neq 0\}$. Then $\alpha = b_2 - c_2/(b_1 - c_1) \in K_0$ which is a contradiction.

In the case that $A \subseteq K_0$ we simply replace A by $A \times A$ and follow the same argument. □

Thus from here on we shall drop the "over K" and just simply say a set is large.

The definition of largeness for sets was in regards to *absolutely* irreducible affine K-varieties. However, by [Jar11, Lemma 4.1.1.] any K-variety with a regular K-rational point *is* absolutely irreducible. Therefore, we shall no longer assume that our varieties are absolutely irreducible unless otherwise stated.

We now note that the classical reduction down to curves for large fields also can be generalised for sets of relations over a field-

Proposition 6.2. *Let $B \subseteq K^m$ and $n \geq m$. The following are equivalent-*

(1) B is large.

(2) For every affine K-curve $C \subseteq \mathbb{A}^n$ for $n \geq m$, one has: If $\bar{a} \in C_{Reg}$, then $C \cap A$ is infinite.

Proof. Proof follows analogously to [Jar11, Lemma 5.3.1]. □

6.1. **First Order Largeness.** Now we consider large sets that are definable in a language $\mathcal{L}$ extending $\mathcal{L}_{Ri}$. Let $\mathfrak{B}$ be a set of $\mathcal{L}$-formulae. We now show that we can axiomtise the class of $\mathcal{L}$-fields in which $\mathfrak{B}$ defines large sets.

Proposition 6.3. *The class of $\mathcal{L}$-fields in which $\mathfrak{B}$ defines large sets is axiomatisable in the language $\mathcal{L}$.*

Proof. We show that there exists an axiom schema in $\mathcal{L}$ that will say if a model $\mathcal{M}$ expands a field K and $C \subseteq \mathbb{A}^n$ is a K-curve with a regular K-rational point that satisfies some $\lambda \in \mathfrak{B}$ then there infinitely many K-rational points satisfying λ in $\mathcal{M}$. Then condition (2) of 6.2 tells us such a schema axiomatises the class of $\mathcal{L}$-fields in which $\mathfrak{B}$ defines large sets.

Let $f_1, ..., f_m(\bar{U}, \bar{X}) \in \mathbb{Z}[\bar{U}, \bar{X}]$, with $\bar{X} = (X_1, \ldots, X_{n_1})$ and $\bar{U} = (U_1, \ldots, U_{n_2})$. Then there exists an $\mathcal{L}_{Ri}$-formula $\varphi_{(f_1,\ldots,f_m)}(\bar{x}, \bar{u})$ such that for every field and all $\bar{a} \in K^{n_1}$ and $\bar{c} \in K^{n_2}$

$K \models \varphi_{(f_1,\ldots,f_m)}(\bar{a}, \bar{c})$ if and only if the system $f_1(\bar{c}, \bar{X}) = 0, ..., f_m(\bar{c}, \bar{X}) = 0$ defines an K-curve and $\bar{a}$ is a regular (K-rational) point of that curve.

Now fix some $\lambda \in \mathfrak{B}$. Now let $\psi_{(f_1,\ldots,f_m,\lambda)}(\bar{u})$ denote the $\mathcal{L}$-formula $\exists \bar{x}(\varphi_{(f_1,\ldots,f_m)}(\bar{x}, \bar{u}) \wedge \lambda(\bar{x}))$. Then for any $\mathcal{L}$-structure $\mathcal{M}$ expanding a field K and $\bar{c} \in K^{n_2}$,

$\mathcal{M} \models \psi_{(f_1,\ldots,f_m,\lambda)}(\bar{c}))$, if and only if $f_1(\bar{c}, \bar{X}) = 0, ..., f_m(\bar{c}, \bar{X}) = 0$ defines an K-curve that has $\bar{c}$ as a regular K-rational point satisfying λ in $\mathcal{M}$.

Therefore by varying over all $r, m, n_1, n_2 \in \mathbb{N}$ and all $f_1, \ldots, f_r \in \mathbb{Z}[\bar{U}, \bar{X}]$, the set of all formulas

$$\forall \bar{u} \left[\psi_{(f_1,\ldots,f_m,\lambda)}(\bar{u}) \to \exists \bar{x}_1, ..., \bar{x}_r \; (\bigwedge_{i \neq j} \bar{x}_i \neq \bar{x}_j \wedge \bigwedge_{i=1}^{r} f_1(\bar{u}, \bar{x}_i) = 0 \wedge ... \wedge f_n(\bar{u}, \bar{x}_i) = 0 \wedge \bigwedge_{i=1}^{r} \lambda(\bar{x}_i)) \right]$$

gives the desired axiom schema. □

6.1.1. *Extensions of $\mathcal{L}$-fields in which $\mathfrak{B}$ defines large sets.* Recall that for a large field and K and an K-variety V with a regular K-rational point we have that K elementarily embeds into a field containing $K(V)$. This can be proved using the compactness theorem. Similarly, the compactness theorem gives us a generalisation of this property for large sets. Before we give this generalisation we establish some notation.

Notation. $\overline{K}$ is the algebraic closure of K. Let $n \in \mathbb{N}$ and $Z \subseteq \overline{K}^n$. Then we denote-

$$I_K(Z) := \{f \in K[X_1, \dots, X_n] | \forall \bar{a} \in Z(f(\bar{a}) = 0)\}$$

Given an ideal $\mathfrak{p} \subseteq K[X_1, \dots, X_n]$ then we shall denote $\bar{X} + \mathfrak{p} := (X_1 + \mathfrak{p}, \dots, X_n + \mathfrak{p})$.

Recall that given an affine K-variety $V \subseteq \mathbb{A}^n$ with $\mathfrak{p} := I_K(V) \subseteq K[\bar{X}]$ it is common notation in algebraic geometry to write $K[V] := K[\bar{X}]/\mathfrak{p}$ and when V is irreducible, $K(V) := \mathrm{Quot}(K[V])$. We use this notation here.

Proposition 6.4. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure expanding a field K, λ a set of $\mathcal{L}$-formulae in at most n free variables, and $V \subseteq \mathbb{A}^n$ an irreducible affine K-variety with a regular K-rational point satisfying λ in $\mathcal{M}$. Let also $I_K(V) = \mathfrak{p} \subseteq K[\bar{X}]$ where $\bar{X} = (X_1, \dots, X_n)$. Then the following are equivalent-*

(1) The set

$$V_\lambda(\mathcal{M}) := \{\bar{a} \in \overline{K}^n | \bar{a} \in V_K, \mathcal{M} \models \lambda(\bar{a})\}$$

is Zariski-dense in V

(2) There exists an elementary extension $\mathcal{M}(V) \succ \mathcal{M}$ such that THE field $\mathcal{M}(V)$ expands contains $K(V)$ and $\mathcal{M}(V) \models \lambda(\bar{X} + \mathfrak{p})$.

Proof. Easy proof using compactness. □

6.2. **Large Topologies.** The various applications of our main theorem shall be to topological fields. So now we consider fields topologies whose open sets are large.

Definition 6.5. We say that a topology τ of a field K is **large** if each open subset $U \subseteq K^n$ in topology induced on K^n by τ is large.

When τ is a field topology then the question of whether it is large can simplified in various ways down to fundamental system neighbourhoods of 0. We firstly define fundamental systems of neighbourhoods.

Definition 6.6. Let τ be a field topology on K. A fundamental system of neighbourhoods (of 0) $\mathscr{F}_\tau$ is a subset of τ such each element of $\mathscr{F}_\tau$ is an open neighbourhood of 0 and for any open neighbourhood U of 0 there exists some $B \in \mathscr{F}_\tau$ such that $B \subseteq U$.

Proposition 6.7. *Let τ be a field topology on a field K with $\mathscr{F}_\tau$ a fundamental system of neighbourhoods. Then the following are equivalent.*

(1) τ is large.

(2) For each affine K-curve C embedded in affine n-space for some $n \in \mathbb{N}$, with $\bar{0} \in C_{Reg}$, there are infinitely many K-rational points of C in every open neighbourhood B^n where $B \in \mathscr{F}_\tau$.

(3) For each plane K-curve $C \subseteq \mathbb{A}^2$ with $(0,0) \in C_{Reg}$ then there are infinitely many K-rational points of C in every open neighbourhood B^2 where $B \in \mathscr{F}_\tau$.

Now suppose there exists some subset $\mathscr{F}_0 \subseteq \mathscr{F}$ such that for each $B \in \mathscr{F}$ and each plane K-curve C with $(0,0) \in C_{Reg}$, there exists some $B_0 \subseteq \mathscr{F}_0$ and some K-isomorphism $\gamma : D \mapsto C$ where D is another plane K-curve regular at the origin such that $\gamma(B_0^2) \subseteq B^2$. Then the above conditions are also equivalent to the following.

(4) For each irreducible plane K-curve $C \subseteq \mathbb{A}^2$ with $(0,0) \in C_{Reg}$ then there are infinitely many K-rational points of C in B_0^2 for each $B_0 \in \mathscr{F}_0$.

Proof. (1) implies (2) is trivial. We now prove the converse. Firstly, since for any $n \in \mathbb{N}$, any translations of $\mathbb{A}^n$ is a isomorphism of varieties as wells an isomorphism of the topological space induced by τ on K^n (since τ is a field topology), we can reduce to considering varieties with $\bar{0}$ as a regular K-rational point. Secondly, use 6.2 to reduce down to K-curves. So now consider a K-curve $C \subseteq \mathbb{A}^n$ with $\bar{0} \in C_{Reg}$ and suppose that $U \subseteq K^n$ is neighbourhood of $\bar{0}$ in the product topology on K^n induced by τ. Without loss of generality we can assume that U is a basic open set i.e. we can write $U = U_1 \times \ldots \times U_n$ where for each $1 \leq i \leq n$ $U_i \subseteq K$ is a neighbourhood of 0. Then for each $1 \leq i \leq n$ there exists some $B_i \in \mathscr{F}_\tau$ such that $0 \in B_i \subseteq U_i$. Let $B := \bigcap_{i=1}^{n} B_i$. Thus $\bar{0} \in B^n \subseteq U$. From here the proof is straightforward.

(2) implies (3) is obvious. We now prove the converse. So suppose C is affine K-curve C embedded in affine n-space for some $n \in \mathbb{N}$, with $\bar{0}$ a regular K-rational point and suppose that $U \subseteq K^n$ is a open

neighbourhood of $\bar{0}$ in τ. By [Jar11, Lemma 5.1.2] there is a K-birational correspondence $\pi : C \dashrightarrow D$ to a plane curve D that is regular at $(0,0)$ that is defined in $\bar{0}$ and such that $\pi(\bar{0}) = (0,0)$. Thus, the inverse K-birational map $\rho : D \dashrightarrow C$ is defined at $(0,0)$ and moreover $\rho((0,0)) = \bar{0}$. Now notice that on its domain of definition ρ is defined by rational functions over K. Thus on its domain of definition, ρ is a continuous map in the (product topology induced by) topology τ. Furthermore, clearly the restrictions of the domain and codomain of ρ to K^n are open sets in the topology induced by τ. Thus $A = \rho^{-1}(U)$ is an open neighbourhood of $(0,0)$ in τ. Without loss of generality we can assume that A is a basic set. Therefore we may write $A = A_1 \times A_2$ where A_1 and A_2 are open neighbourhoods of 0 in τ. Then there exists sets $B_1, B_2 \in \mathscr{F}_\tau$ such that $B_i \subseteq A_i$ for each $i = 1,2$. Let $B = B_1 \cap B_2$. Then, by assumption, there are infinitely many K-rational points of D in B^2 and hence in A. These get mapped to infinitely many K-rational points of C in $\rho(A) \subseteq U$ which proves the claim.

If there exists some $\mathscr{F}_0 \subseteq \mathscr{F}$ with the property outlined above then it is clear that (3) is equivalent to (4). □

These reductions help us to obtain the following examples of large topologies.

Corollary 6.8. *(1) If K is a real closed field with ordering $<$, then the topology $\tau_<$ induced by $<$ on K is large.*
(2) If v is a henselian valuation of a field K then the topology τ_v that v induces on K is large.

Proof. (1) is more or less immediate from (2) of 6.7 and [JB98, Theorem 3.25].

We now prove (2). Write $\mathcal{O}_v$ for the valuation ring of (K, v). It is clear that the following is fundamental system of neighbourhood

$$\mathscr{F}_v := \{a\mathcal{O}_v | a \in K \backslash \{0\}\}$$

Thus it suffices to show that (4) of 6.7 is true for $\mathscr{F}_0 := \{\mathcal{O}_v\}$. However, this holds by [JD, Theorem 2.4]. □

7. Proof of (II') and Main Theorem

7.1. **Proof of (II').** The proof of 2.7 uses [Tre02, Theorem 1]. Thus we recall this Theorem before proceeding to the proof of 2.7.

Theorem 7.1. [Tre02, Theorem 1] *Let K be a differential field of characteristic 0 in $N \in \mathbb{N}$ commuting derivations and let $\mathfrak{p} \subseteq K\{\bar{Y}\}, \bar{Y} := (Y_1, \ldots, Y_n)$ be a differential prime ideal. Let $\varphi : K\{\bar{Y}\} \to K\{\bar{Y}\}/\mathfrak{p} =: R$ be the residue map and let G be a characteristic set of $\mathfrak{p}$. Let $H(G)$ be the product of all initials and separants of polynomials in G. Let $h := \varphi(H(G))$*

$$\Delta := \{y \in \mathcal{D}Y | y \text{ is not a proper derivative of any leader of an element } g \in G\}$$

$$\Delta_B := \{y \in \Delta | y \text{ appears in some } g \in G\}$$

$$B := \varphi(K[\Delta_B]) \text{ and } P := \varphi(K[\Delta \backslash \Delta_B])$$

then $h \in B_R, h \neq 0$ and

1. *B is a finitely generated K-algebra and P is K-isomorphic to a polynomial ring over K in at most countably many variables (the case that $K = P$ is not excluded).*
2. *$R_h = (B \cdot P)_h$ is a differentially finitely generated K-algebra.*
3. *The homomorphism $B \otimes P \to B \cdot P$ induced from multiplication is an isomorphism of K-algebras.*
4. *The restriction of φ to $K[\Delta \backslash \Delta_B]$ is injective.*

Theorem 7.2. *Let $\mathfrak{B}$ be a set $\mathcal{L}$-formulae. Suppose that $\mathcal{M}$ is a differential $\mathcal{L}$-field expanding a field K, in which the sets defined by $\mathfrak{B}$ are large. Then there is some differential $\mathcal{L}$-field extension $\mathcal{N}$ such that $\mathcal{M} \prec_{\mathcal{L}} \mathcal{N}$ and $\mathcal{N} \models UC_N(\mathfrak{B})$ i.e. it solves every $\mathfrak{B}$-algebraically prepared system over itself.*

Proof. Claim: *Every $\mathfrak{B}$-algebraically prepared system $(f_1, \ldots, f_m, \lambda)$ of $\mathcal{M}$ has a differential solution in some differential $\mathcal{L}$-field extension $\mathcal{N}$ which does not annihilate $H(f_1, \ldots, f_m)$ and such that $\mathcal{M} \prec_{\mathcal{L}} \mathcal{N}$.*

Let $(f_1, \ldots, f_m, \lambda)$ be a $\mathfrak{B}$-algebraically prepared systems of $\mathcal{M}$ with $f_1, \ldots, f_m \in K\{Y_1, \ldots, Y_n\}$. By condition 1 for algebraically prepared systems, $\{f_1, \ldots, f_m\}$ is a characteristic set. Moreover, by condition 2 and 3.15, $f_1, \ldots, f_m$ is the characteristic set of the differential prime ideal $\mathfrak{p} := [f_1, \ldots, f_m] : H(f_1, \ldots, f_m)^\infty \subseteq K\{\bar{Y}\}$. Define B and P as in 7.1 with respect to $\mathfrak{p}$ and let $A := K[\Lambda_B]$. By 3.15 $\tilde{\mathfrak{p}} := \mathfrak{p} \cap A$ is a prime ideal of A

and in fact the ideal of polynomials over K that vanish on the affine K-variety V_G. Thus $B = A/\tilde{\mathfrak{p}} = K[V_G]$. From condition (2) of the definition of $\mathfrak{B}$-algebraically prepared systems, we know that V_G has a regular K-rational point $\bar{a}$ whose first n coordinates satisfies $\lambda(\bar{x})$ in $\mathcal{M}$. So, by 6.4 there exists an elementary $\mathcal{L}$-extension $\mathcal{N}_0$ of $\mathcal{M}$ containing B_h and such that $\mathcal{N}_0 \models \lambda(X_1 + \tilde{\mathfrak{p}}, \ldots, X_n + \tilde{\mathfrak{p}})$ where $\bar{X}$ is an enumeration of the variables in Δ_B (with $X_i = Y_i$ for $1 \leq i \leq n$). Now we can choose $\mathcal{N}$ to be an elementary $\mathcal{L}$-extension of $\mathcal{N}_0$ such that $L := \mathcal{N} \restriction_{\mathcal{L}_{Ri}}$ is of infinite transcendence degree over $L_0 := \mathcal{N}_0 \restriction_{\mathcal{L}_{Ri}}$. In particular, L is of infinite transcendence degree over $\mathrm{Quot}(B)$. Thus, there is an K-embedding that fixes K of P into L by mapping each of the indeterminates of P to elements of L that are algebraically independent over L_0. This gives us an K-embedding of $A_h = B_h \otimes P$ into L. Notice that by (4) of the structure theorem for each $1 \leq i \leq n$, $X_i + \tilde{\mathfrak{p}}$ maps to $Y_i + \mathfrak{p}$ in this embedding. Then since $\mathcal{N}_0 \prec_{\mathcal{L}} \mathcal{N}$, $\mathcal{N} \models \lambda(Y_1 + \mathfrak{p}, \ldots, Y_n + \mathfrak{p})$. We now extend the derivations of A_h to L to give us a desired differential $\mathcal{L}$-field extension, $(\mathcal{N}, \partial_1, \ldots, \partial_N)$, of $\mathcal{M}$ that solves $(f_1, \ldots, f_m, \lambda)$. This proves the claim.

From the claim, we get the theorem by transfinite induction. This is possible because any $\mathfrak{B}$-algebraically prepared system of a $\mathcal{M}$ is an $\mathfrak{B}$-algebraically prepared system over $\mathcal{N}$ any differential $\mathcal{L}$-field extension $\mathcal{N}$ where $\mathcal{M} \prec_{\mathcal{L}} \mathcal{N}$. Moreover, by 6.3 and the fact $\mathcal{N} \models Th_{\mathcal{L}}(\mathcal{M})$, $\mathfrak{B}$ is also large in $\mathcal{N}$. So we can iterate the claim until all $\mathfrak{B}$-algebraically prepared systems constructed so far are solvable in the union of all the differential $\mathcal{L}$-fields. So the union will be a model of $UC_N(\mathfrak{B})$ and an elementary $\mathcal{L}$-extension of $\mathcal{M}$. $\square$

This also concludes the proof of 2.8.

7.2. **Proof of Main Theorem.**

Definition 7.3. Given an $\mathcal{L}$-theory T in which $\mathfrak{B}$ is large and $\mathfrak{B}$ and induces a covering then we shall abbreviate and just write that $\mathfrak{B}$ induces a **large cover** on T.

Theorem 7.4. *Let $\mathfrak{B}$ be a set of $\exists$-$\mathcal{L}$-formulae that induces a large cover a model complete $\mathcal{L}$-theory T of fields.*

Let T^* be a theory in a language $\mathcal{L}^* \supseteq \mathcal{L}$ such that T^* contains T and T^* is an extension by definitions of T.

Let $\mathcal{A}$ be an $\mathcal{L}^*(\bar{\partial})$-structure with universe A such that $\mathcal{A}$, viewed as an $\mathcal{L}(\bar{\partial})$-structure, is a substructure of a partial differential $\mathcal{L}$-field.

If $T^* \cup \mathrm{diag}(\mathcal{A} \restriction \mathcal{L}^*)$ is complete, then $T^* \cup UC_N(\mathfrak{B}) \cup \mathrm{diag}(\mathcal{A})$ is complete.

Proof. The proof follows analogously as to that of [Tre05, Theorem 7.1] except easier since the constatns are included in our language $\mathcal{L}$ already. $\square$

By gathering the consequences of 7.4, the next theorem actually proves the main theorem 2.8.

We now show that the theory $UC_N(\mathfrak{B})$ also preserves NIP

Proposition 7.5. *Let T be a model complete $\mathcal{L}$-theory and $\mathcal{M} \models T$. Suppose that $\mathcal{M}$ embeds into some $\mathcal{L}(\bar{\partial})$-structure $\mathcal{M}^*$ whose $\mathcal{L}$-reduct is also a model of T. If $\phi(x, \bar{y})$ is a quantifier free $\mathcal{L}(\bar{\partial})$-formula that has the independence property in $\mathcal{M}^*$ then the quantifier free $\mathcal{L}$-formula ϕ^* is has the independence property in $\mathcal{M}$.*

Proof. Assume that, in the $\mathcal{L}(\bar{\partial})$-quantifier free formula $\phi(x, \bar{y})$, with $\bar{y} := (y_1, \ldots, y_s)$, that x occurs with order in at most $d_x := (d_{x,1}, \ldots, d_{x,N})$ and for each y_j, $1 \leq j \leq s$, occurs with order at most $d_y := (d_{y,1}, \ldots, d_{y,N})$. Let $\bar{u} := (u_{(0,\ldots,0)}, \ldots, u_{d_x})$, $\bar{v} := (v_{1,(0,\ldots,0)}, \ldots, v_{1,d_y}, v_{2,(0,\ldots,0)}, \ldots, v_{s,d_y})$ and let $\theta(\bar{u}, \bar{v})$ be the following $\mathcal{L}(\bar{\partial})$-formula: $\bigwedge_{i_1 \leq d_{x,1} \ldots, i_N \leq d_{x,N}} \partial_1^{d_{x,1}} \ldots u_{i_1,\ldots,i_N} = \partial_N^{d_{x,N}}(u_{(0,\ldots,0)}) \wedge \bigwedge_{j=1}^{s} \bigwedge_{l_1 \leq d_{y,1},\ldots,l_N \leq d_{y,N}} v_{j,(l_1,\ldots,l_N)} = \partial_1^{d_{y,1}} \ldots \partial_N^{d_{y,N}}(v_{j,(0,\ldots,0)})$. Then the rest of the proof proceeds as in [GP10, Lemma 4.2]. $\square$

Corollary 7.6. *Let T be a complete and model complete $\mathcal{L}$-theory of fields and $\mathfrak{B}$ a set of $\exists$-$\mathcal{L}$-formulae that induces a large cover on T. If T has NIP then so does $T \cup UC_N(\mathfrak{B})$.*

8. Applications

8.1. **Henselian Valued Fields.**

We now give an alternative uniform companion for henselian valued fields to that given by N. Guzy [Guz06]. Recall that $\mathfrak{S}_{val} := \{1|x, \neg 1|x\}$ and $\mathfrak{B}_{val} := \langle \mathfrak{S}_{val} \rangle$.

Lemma 8.1. *$UC_N(\mathfrak{B}_{val})$ is an uniform companion for the class $\mathbb{M}_{val}$ of all valued fields whose valuation topology is large.*

Proof. By 2.8, it suffices to notice that for each $\mathcal{L}_{val}$-theory such that each model of T is a henselian valued field, (3) of 5.3 tells us that $\mathfrak{B}_{val}$ induces a cover of $\mathbb{M}_{val}$. Since the sets of $\mathfrak{B}_{val}$ define open sets in (the product topology of) any valued field, they are also define a set of large sets in any member of $\mathbb{M}_{val}$. □

Corollary 8.2. *$UC_N(\mathfrak{B}_{val})$ is an uniform companion for the class of henselian valued fields.*

Proof. By (2) of 6.8, the class of henselian valued fields is a subclass of $\mathbb{M}_{val}$ □

Remark 8.3 (Open Question)**.** Is each member of $\mathbb{M}_{val}$ a henselain valued field?

8.2. **Fields with Several Valuations and Orderings.** In this subsection we apply our results to theories of fields with several independent orderings and valuations hence generalising work in [GP10]. In his PhD thesis L. van den Dries proved that such theories have a model companion (cf. [vdD78, Chapter 3]). Given a theory T of fields with several independent orderings and valuations with a model companion $\bar{T}$ as presented van den Dries, Guzy and Point showed in [GP10] that $\bar{T}$ can be extended to be a model companion of $T \cup DF_1$. We now use 2.8 to generalise this i.e. we show that for each $N \in \mathbb{N}$, $\bar{T}$ can be extended to a model companion of $T \cup DF_N$.

Definition 8.4. We call any field with several valuations and orderings a **multi-local field**

Throughout this subsection we denote a multi-local field by $(K, A_1, \ldots, A_n)$ where for each $1 \leq i \leq n$ A_i is either a valuation ring of K or a positive cone for some ordering of K. By the notation H_i $(1 \ldots i \leq n)$ we mean the henselianization of K with respect to A_i if A_i is a valuation ring or the real closure of K with respect A_i if A_i is a positive cone of K. Let also τ_K denote the smallest topology that contains $\bigcup_{i=1}^{n} \tau_{A_i}$ where, for each $1 \leq i \leq n$, τ_{A_i} is the topologies induced by A_i on K.

Definition 8.5. [JRH84, Theorem 1.9] Let $(K, A_1, \ldots, A_n)$ be a multi-local field. We say that $(K, A_1, \ldots, A_n)$ has the **Approximation Property** if for each absolutely irreducible affine K-variety V, there are K-rational points $\bar{a}$ which are simultaneously (arbitrarily) close to any sequence of regular points $\bar{a}_1 \in H_1, \ldots, \bar{a}_n \in H_n$ of V.

Proposition 8.6. *For any multi-local field $(K, A_1, \ldots, A_n)$ with the Approximation Property τ_K is large.*

Proof. Recall from 6.8, the topology induced by (the extension of) A_i in H_i must be large. Since these topologies are Hausdorff, it is clear that if $(K, A_1, \ldots, A_n)$ has the Approximation Property then this must imply that τ_K is large □

Now suppose that for each $1 \leq i \leq n$ that T_i is one of the following theories.

1. The theory OF of ordered fields in the language $\mathcal{L}_{Ri}(<)$.
2. The theory VF of valued fields in the language $\mathcal{L}_{val}$.
3. The theory pCF of p-adically closed fields in the language $\mathcal{L}_{val}(\{\mathcal{P}_n\}_{n \in \mathbb{N}})$, where for each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a unary relation symbol interpreted in models of pCF as the set of n^{th} powers.
4. The theory πCF of π-adically closed fields in the language $\mathcal{L}_{val}(\{\mathcal{P}_n\}_{n \in \mathbb{N}}, \pi)$ where π is a constant symbol interpreted in models of πCF as some element such that $v(\pi) = 1$.

For each $1 \leq i \leq n$ let $\mathcal{L}_i$ denote the language of T_i and

$$\mathcal{L} := \mathcal{L}_{Ri} \cup \bigsqcup_{i=1}^{n} (\mathcal{L}_i \backslash \mathcal{L}_{Ri})$$

where $\bigsqcup$ stands for disjoint union. If $<, |$ or $\mathcal{P}_n$ $(n \in \mathbb{N})$ is a symbol of the language $\mathcal{L}_i$ for some $1 \leq i \leq n$ then we denote the corresponding symbol in $\mathcal{L}$ by $|_i, <_i$ or P_{in} respectively. Then the $\mathcal{L}$-theory $(T_1, \ldots, T_n)$ is defined as the theory whose models are $\mathcal{L}$-structures $\mathcal{M}$ such that $\mathcal{M} \upharpoonright_{\mathcal{L}_i} \models T_i$ for each $1 \leq i \leq n$. Theorem 1.12 in [vdD78, Chp. 3] tells us that $(T_1, \ldots, T_n)$ has a model companion in $\mathcal{L}$ which we shall denote by $\overline{(T_1, \ldots, T_n)}$. Furthermore, Theorem 4.1 in [JRH84] tells us that every model of $\overline{(T_1, \ldots, T_n)}$ has the Approximation Property. Hence, 8.6 gives us the following.

Corollary 8.7. *For every model $(K, A_1, \ldots, A_n) \models \overline{(T_1, \ldots, T_n)}$, τ_K is large.*

Remark 8.8. The works [Pre81] of A. Prestel and [Jar83] of M. Jarden show that if $T_i = OF$ for each $1 \leq i \leq n$, then the class of models of $\overline{(T_1, \ldots, T_n)}$ is precisely the class of PRC$_n$ fields with no proper algebraic extensions (called a **maximal PRC$_n$ field**). Thus Corollary 8.7 tells us that the topology on maximal PRC$_n$ field is large. It can in fact be shown that the topology on any PRC$_n$ field is large ([Sol14, Lemma 4.4.15.]). However, as we will not require this fact here we shall show it here it.

Now we turn to find a set of $\exists$-$\mathcal{L}$-formulae that define open sets in models in of $\overline{(T_1, \ldots, T_n)}$ (and so are large in $\overline{(T_1, \ldots, T_n)}$) and also induce covers in $\overline{(T_1, \ldots, T_n)}$.

Corollary 8.9. *For each $1 \leq i \leq n$ let $\mathfrak{S}_i$ be one of the following sets of $\exists$-$\mathcal{L}_i$-formulae formulae such*

(1) $\{0 <_i x\}$ if $T_i = OF$,
(2) $\{1|_i x, \neg 1|_i x\}$ if $T_i = VF$,
(3) $\{1|_i x, \neg 1|_i x\} \cup \{\mathcal{P}_{in}(x) \wedge x \neq 0\}_{n\in\mathbb{N}}$ if $T_i = pCF$ or $T_i = \pi CF$,

Then $\mathfrak{B} := \langle \bigcup_{i=1}^{n} \mathfrak{S}_i \rangle$ induces a large cover in $\overline{(T_1, \ldots, T_n)}$ (and thus $\overline{(T_1, \ldots, T_n)} \cup UC_N(\mathfrak{B})$ is the model companion of $(T_1, \ldots, T_n) \cup DF_N$).

Proof. From Examples 5.3 it is clear that $\mathfrak{B}$ induces a cover in $(T_1, \ldots, T_n)$ (and thus in $\overline{(T_1, \ldots, T_n)}$ also). We now show that for each $\mathcal{M} := (K, A_1, \ldots, A_n) \models \overline{(T_1, \ldots, T_n)}$ and each $\varphi(x) \in \mathfrak{B}$, $\varphi(\mathcal{M}) \subset K$ is an open set of τ_m. It is clear that it suffices to check that for each $1 \leq i \leq n$ and each $\varphi \in \mathfrak{S}_i$, $\varphi(\mathcal{M})$ is an open set. If for some $1 \leq i \leq n$ $T_i = pCF$ or $T_i = \pi CF$ and $\varphi(x)$ is the formula $\mathcal{P}_n(x) \wedge x \neq 0$ then $\varphi(\mathcal{M})$ is open subset of K because the set $\mathcal{P}_n^{\mathcal{M}} \cap K^\times$ is open in $K^\times$ (cf. [vdD78, Examples]) where $K^\times := K \backslash \{0\}$ and $K^\times \in \tau_K$ itself. For each other formulae $\varphi \in \bigcup_{i=1}^n \mathfrak{S}_i$, it is clear that $\varphi(\mathcal{M})$ is an open set. Thus, by 8.7 we know that $\mathfrak{B}$ is a large cover. We apply 2.8 to get that $\overline{(T_1, \ldots, T_n)} \cup UC_N(\mathfrak{B})$ is the model companion of $(T_1, \ldots, T_n) \cup DF_N$. ☐

Remark 8.10 (Open Question)**.** The theories considered in this subsection fall under a class of theories van den Dries defined called ***t*-theories** (cf. [vdD78, Chp. 3]). Could the method presented here be generalised to obtain model companion for t-theories extend by the theory of partial differential fields in N commuting derivations?

8.3. **Valued Fields with a Residual Subfield.** In this final section we apply our results to valued fields with a designated subfield of the residue field. By this we mean a two sorted structure $((K, v), (k_v, k))$ where

- (K, v) is a valued field in the language $\mathcal{L}_{val}$.
- k_v is the residue field of v and k is a subfield of k_v. So (k_v, k) is a structure in the language $\mathcal{L}_{pairs} := \mathcal{L}_{Ri}(\mathcal{R})$ where $\mathcal{R}$ is a unary predicate.
- We have the residue map $\mu_v : \mathcal{O}_v \to k_v$, where $\mathcal{O}_v$ is the valuation ring of v, between the two sorts.

Of course, to be formal μ_v has to be a map from K but this can easily be arranged by setting $\mu(a) = 0$ for all $a \in K \backslash \mathcal{O}_v$. However it shall be more convenient for us to view μ as the actual residue map i.e. with domain $\mathcal{O}_v$ so that is indeed what we shall do. We shall call such structures **valued fields with a residual subfield**. By a **partial differential valued field with a residual subfield (in $N \in \mathbb{N}$ commuting derivations)** we shall mean a structure of the form $((K, v, \partial_1, \ldots, \partial_n), (k_v, k))$ where-

- $((K, v), (k_v, k))$ is a valued field with a residual subfield.
- $(K, v, \partial_1, \ldots, \partial_N)$ is partial differential valued field where the derivations and the pair (k_v, k) are independent.

Then given a theory T of valued fields with a residual subfield, by $T \cup DF_N$ we shall mean the theory of partial differential valued fields with a residual subfield $((K, v, \partial_1, \ldots, \partial_n), (k_v, k))$ such that $((K, v), (k_v, k)) \models T$.

Remark 8.11. By replacing variables of the residue sort with a string $\mu(x)$, where x is a variable of the valued field sort, it is not difficult to see that in the theory of (respectively, partial differential) valued fields with a residual subfield, every formula in the language of (respectively, partial differential) valued fields with a residual subfield is equivalent to one in which no variables of the residue sort occur (bound or free).

We shall focus on theories of valued field with a residual subfield that are unions of a theory of valued fields and a theory of pairs if fields i.e. theories T such that there exists theories T_{val} of valued fields and T_{pairs} of pairs of fields with the property that:

$$(K, v, (k_v, k)) \models T \text{ if and only if } (K, v) \models T_{val} \text{ and } (k_v, k) \models T_{pairs}$$

In other words, theories where there is no assumed interaction between the valued field and the subfield k of the residue field. For such a theory T we can of course denote $T = T_{val} \cup T_{pairs}$

The advantage of considering such theories T is that the following Ax-Kochen-Eršov principle reduces questions of model completeness for T to a question of model completeness of the theories T_{val} and T_{pairs}.

Theorem 8.12. [BÓ0, Corollary 4.3] *Suppose v_1, v_2 are two henselian unramified valuations. Then*

(1) $((K_1, v_1), (k_{v_1}, k_1)) \subseteq ((K_2, v_2), (k_{v_2}, k_2))$ if and only if $(k_{v_1}, k_1) \subseteq (k_{v_2}, k_2)$ and $v_1(K_1^\times) \subseteq v_2(K_2^\times)$
(2) $((K_1, v_1), (k_{v_1}, k_1)) \prec ((K_2, v_2), (k_{v_2}, k_2))$ if and only if $(k_{v_1}, k_1) \prec (k_{v_2}, k_2)$ and $v_1(K_1^\times) \prec v_2(K_2^\times)$

Corollary 8.13. *For any two theories T_{val} of valued fields and T_{pairs} of pairs of fields, the theory $[T_{val}, T_{pairs}]$ is model complete if and only if T_{val} and T_{pairs} are both model complete.*

However, as far as the author is aware, there are no known model complete $\mathcal{L}_{pairs}$-theories of pairs of fields. Therefore, to obtain model complete theories of pairs of fields one needs to Morleyise a given $\mathcal{L}_{pairs}$-theory of pairs of fields to a some degree i.e. extend the $\mathcal{L}_{pairs}$-theories of pairs of fields by definitions somehow. We now list certain model complete extensions by definitions of theories of pairs of fields that we shall use later.

Example 8.14. In the following, let FL_{pairs} be the $\mathcal{L}_{pairs}$-theory of pairs of fields.

(1) ([Rob59, Theorem 3.6]) Let T_{dense} be the $\mathcal{L}_{pairs}$-theory of dense pairs i.e. pairs of real closed fields (k_1, k_2) such that $\emptyset \neq k_2 \subsetneq k_1$ and k_2 is dense in k_1 with respect to the interval topology. For each $n, d \in \mathbb{N}$ let $D_{n,d}$ be a new n-ary relation symbol and let T_{Rob} be the extension by definitions of T_{dense} to $\mathcal{L}_{Rob} := \mathcal{L}_{pairs}(D_{n,d})_{n,d \in \mathbb{N}}$ where each $D_{n,d}(x_1, \dots, x_n)$ is defined by the $\mathcal{L}_{pairs}$ formula that says (in the theory of pairs of fields) "$x_1, \dots, x_n$ are a zero of a non-zero polynomial of degree at most d with coefficients from the subfield". Then T_{Rob} is model complete.
(2) ([Lel90, Théorème 4.1.3]) The extension T_{Lel} of T_{dense} to the language $\mathcal{L}_{Lel} := \mathcal{L}_{pairs}((E_{n,m,f,g})_{n,m \in \mathbb{N}, f,g \in \mathbb{Z}[v_1,\dots,v_m,X_1,\dots,X_n]})$ where for each $n, m \in \mathbb{N}$ and $f, g \in \mathbb{Z}[X_1, \dots, X_n]$, $E_{n,m,f,g}$ is an m-ary relation defined by the $\mathcal{L}_{pairs}$-formula-
$$\exists x_1, \dots, x_n \left(\bigwedge_{i=1}^{n} \mathcal{R}(x_i) \wedge f(y_1, \dots, y_m, x_1, \dots, x_n) = 0 \wedge g(y_1, \dots, y_m, x_1, \dots, x_n) > 0 \right)$$
Then T_{Lel} has quantifier elimination.
(3) ([Del12, Théorème 1]) Let T_{ACLP} be the $\mathcal{L}_{pairs}$-theory of pairs algebraically closed fields. Now let T_{Del} be the extension of T_{ACLP} by definitions to the language $\mathcal{L}_{Del} = \mathcal{L}_{pairs}((l_n)_{n \in \mathbb{N}_{\geq 2}}, (f_{n,i})_{n \in \mathbb{N}_{\geq 2}, 1 \leq i \leq n})$ where
   (a) For each $n \in \mathbb{N}_{\geq 2}$, $l_n(x_1, \dots, x_n)$ is an n-ary relation defined by the $\mathcal{L}_{pairs}$ formula saying in FL_{pairs} "$x_1, \dots, x_n$ are linearly independent over the smaller field".
   (b) For each $n \in \mathbb{N}_{\geq 2}$ and $1 \leq i \leq n$ $f_{n,i}(y, x_1, \dots, x_n)$ an $(n+1)$-ary function symbol defined in T'_{ACLP} to be a function in the following way
$$z = f_{n,i}(y, x_1, \dots, x_n) \leftrightarrow$$
$$l_n(x_1, \dots, x_n) \wedge \exists z_1, \dots, z_n \left(z = z_i \wedge y = \sum_{i=1}^{n} x_i z_i \wedge \bigwedge_{i=1}^{n} \mathcal{R}(z_i) \right)$$
   (and one can set $f_{n,i}$ to be zero everywhere where the formula at the bottom is not defined)

   Then T_{Del} has quantifier elimination in $\mathcal{L}_{Del}$

Thus we now consider expansions $((K, v), \mathcal{K})$ of valued field with a residual subfield $((K, v), (k_v, k))$ where $\mathcal{K}$ is an expansion of (k_v, k). If the language of $\mathcal{K}$ is $\mathcal{L}'_{pairs}$ then we shall denote the (two sorted) language of $((K, v), \mathcal{K})$ by $(\mathcal{L}_{val}, \mathcal{L}'_{pairs})$. Furthermore, if T_{val} is an $\mathcal{L}_{val}$-theory and T'_{pairs} is an $\mathcal{L}'_{pairs}$-theory then we shall write $T_{val} \cup T'_{pairs}$ for the $(\mathcal{L}_{val}, \mathcal{L}'_{pairs})$ theory they determine in the same sense as above. For this case, 8.12 also gives us a generalisation of 8.13.

Proposition 8.15. *Let T_{val} be an $\mathcal{L}$-theory of henselian unramified valued fields, T_{pairs} be a theory of pairs of fields and T'_{pairs} be an extension of T_{pairs} by definitions to some language $\mathcal{L}'_{pairs}$. If T_{val} and T'_{pairs} are model complete then $T_{val} \cup T'_{pairs}$ is a model complete in the language $(\mathcal{L}_{val}, \mathcal{L}'_{pairs})$.*

Proof. Suppose that $((K_1, v_1), \mathcal{K}_1), ((K_2, v_2), \mathcal{K}_2) \models T_{val} \cup T'_{pairs}$ such that $((K_1, v_1), \mathcal{K}_1) \subseteq ((K_2, v_2), \mathcal{K}_2)$. Then for both $i = 1, 2$

- $(K_i, v_i) \models T_{val}$
- $\mathcal{K}_i \models T'_{pairs}$
- $\mathcal{K}_i$ is an expansion of $(K_{\mathcal{O}_i}, k_i) \models T_{pairs}$ for some subfield k_i of $k_{\mathcal{O}_i}$.

It is not difficult to see that also $(K_1, v_1) \subseteq (K_2, v_2)$ and $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Since, T_{val} and T'_{pairs} are model complete, $(K_1, v_1) \prec (K_2, v_2)$ and $\mathcal{K}_1 \prec \mathcal{K}_2$. Thus in particular $(k_{v_1}, k_1) \prec (k_{v_2}, k_2)$. Then by 8.12 we have that $((K_1, v_1), (k_{v_1}, k_1)) \prec ((K_2, v_2), (k_{v_2}, k_2))$. But then, since for each $i = 1, 2$, $((K_i, v_i), \mathcal{K}_i)$ is an extension by definitions of $((K_i, v_i), (k_{\mathcal{O}_i}, k_i))$ we have $((K_1, v_1), \mathcal{K}_1) \prec ((K_2, v_2), \mathcal{K}_2)$. □

Corollary 8.16. *The following theories of (expansions of) valued fields with a residual subfield are model complete-*

(1) $RCVF \cup T_{Rob}$ in the language $(\mathcal{L}_{val}, \mathcal{L}_{Rob})$
(2) $RCVF \cup T_{Lel}$ in the language $(\mathcal{L}_{val}, \mathcal{L}_{Lel})$
(3) $ACVF \cup T_{Del}$ in the language $(\mathcal{L}_{val}, \mathcal{L}_{Del})$

Remark 8.17. We consider both expansions T_{Rob} and T_{Lel} because though T_{Lel} has quantifier elimination, due to the choice of new relations in the extensions T_{Rob}, this theory is also of interest in many cases.

Remark 8.18. There a couple questions that arise from the above corollary:

(1) Are the theories listed above model companions of natural theories and if so, what are they?
(2) Considering $ACVF$ has quantifier elimination in $\mathcal{L}_{val}$ and T_{Del} has quantifier elimination in the language $\mathcal{L}_{Del}$, does $ACVF \cup T_{Del}$ have quantifier elimination in the language $(\mathcal{L}_{val}, \mathcal{L}_{Del})$?

Now we turn to finding model companions of $T \cup DF_N$ where T is one of the model complete theories in corollary 8.16. We shall do this by applying our results from section 7.2. However, since our results are regarding one-sorted structures we must first look at how to appropriately interpret the structures we have been considering as a one sorted structure. This is what we do now.

In fact, any valued field with a residual subfield $((K, v), (K_v, k))$ is bi-interpretable with the one-sorted structure (K, R) where R is, what is called, a *pseudovaluation* ring such that $K = \mathrm{Quot}(R)$. Pseudovaluation rings were first introduced in [JRH78] and can be defined as follows-

Definition 8.19. An integral domain R is called a **pseudovaluation domain** if there is a valuation v of the fraction field $K = \mathrm{Quot}(R)$ and a subfield k of the residue field k_v of v such that R is the pullback of the residue map $\mu : \mathcal{O}_v \to k_v$ (where $\mathcal{O}_v$ is the valuation ring of v) along the inclusion $i : k \hookrightarrow k_v$ i.e. $R = \mu^{-1}(k)$.

Call any structure of $(\mathrm{Quot}(R), R)$, where R is a pseudovalued domain, a **pseudovalued field**.

Clearly, for any valued field with a residual subfield $((K, v), (k_v, k))$, the pseudovaluation $R = \mu_v^{-1}(k)$ $((K, v), (k_v, k))$. On the other hand, $\mathcal{O}_v$ is definable in (K, R) and k is interpretable in (K, R) (cf. [B0́0] for details). This gives rise to a bi-interpretation between valued fields with a residual subfield and pseudovalued fields. What is more, this bi-interpretation preserves sub-structures. Thus, the following fact tells us that this bi-interpretation also preserves properties such as model completeness and quantifier elimination between theories of such field structures.

Theorem 8.20. [B0́0, Theorem 3.1] *Let R and S be two pseudovaluations associated to valuation rings $\mathcal{O}_v \subseteq Quot(R)$ and $\mathcal{O}_w \subseteq Quot(S)$ respectively. Let also k_R and k_S be the images of R and S respectively under their residue maps.*

(1) $R \equiv_{\mathcal{L}_{Ri}} S$ if and only if $((Quot(R), \mathcal{O}_v), (k_v, k_R)) \equiv ((Quot(R), \mathcal{O}_v), (k_v, k_R))$.
(2) Suppose $R \subseteq S$ is a local inclusion. Then $R \prec_{\mathcal{L}_{Ri}} S$ if and only if $((Quot(R), \mathcal{O}_v), (k_v, k_R)) \prec ((Quot(R), \mathcal{O}_v), (k_v, k_R))$.

Further details on this bi-interpretability can be found in [B0́0]. This bi-interpretation also gives us general framework for how to proceed to expansions $((K, v), \mathcal{K})$ where $\mathcal{K}$ is an expansion of (k_v, k): firstly, by replacing new functions by the graphs and new constants by unary predicates, we assume that $\mathcal{K}$ is a relational expansion of (k_v, k). Denote by Rel the set of new relations of $\mathcal{K}$. Then it is not hard to see that $((K, v), \mathcal{K})$ is bi-interpretable with the structure -

$$(K, R, \{\mu^{-1}(P)\}_{P \in Rel})$$

We shall call any such an expansion of a pseudovalued field a **residual expansion**. Notice that if $\mathcal{L}$ is the language of $\mathcal{K}$ then the language of $(K, R, \{\mu^{-1}(P)\}_{P \in Rel})$ is the language formed by replacing the function and constant symbols in $\mathcal{L}$ replaced by appropriate relation symbols. Denote this new language by $\mathcal{L}^\circ$. Once again, this bi-interpretation preserves substructures. Furthermore, 8.20 again tells us that this bi-interpretation preserves model completeness and quantifier elimination of theories of such expansions as well. Therefore, henceforth we consider the all the model complete theories of expansions of valued fields with a residual subfield presented in 8.16 as model complete theories of residual expansions of pseudovalued fields in their respective languages.

Now consider one of these theories T (as a theory residual expansions of pseudovalued fields) listed in 8.16 and suppose its language is $\mathcal{L}$. Then $\mathcal{L}$ is a relational extension of $\mathcal{L}_{Ri}$. Now consider the set $\mathfrak{B}_{\mathcal{L}}$ of $\mathcal{L}$-formulae defined in example (2) of 5.3 that induces a cover on T. Then notice that in every model of T the formulae of $\mathfrak{B}_{\mathcal{L}}$ are just unions of products of cosets of the maximal ideal of the valuation ring (in the valuation ring). Thus the formulae of $\mathfrak{B}_{\mathcal{L}}$ define open sets in the models of T with respect to the henselian topology. So by (2) of 6.8 we obtain the following.

Proposition 8.21. *If v is a henselian valuation then sets defined by the formulae of $\mathfrak{B}_{\mathcal{L}}$ in $\mathcal{M}$ are large.*

This now, along with **??** gives us the following corollary.

Corollary 8.22. *We have the following-*

(1) $ACVF \cup T_{Del} \cup UC_N(\mathfrak{B}_{\mathcal{L}^\circ_{Del}})$ is the model companion of $ACVF \cup T_{Del} \cup DF_N$ in the language $\mathcal{L}^\circ_{Del}(\bar{\partial})$.
(2) $RCVF \cup T_{Rob} \cup UC_N(\mathfrak{B}_{\mathcal{L}_{Rob^\circ}})$ is the model companion of $RCVF \cup T_{Rob} \cup DF_N$ in the language $\mathcal{L}^\circ_{Rob}(\bar{\partial})$.
(3) $RCVF \cup T_{Lel} \cup UC_N(\mathfrak{B}_{\mathcal{L}^\circ_{Lel}})$ is the model companion of $RCVF \cup T_{Lel} \cup DF_N$ in the language $\mathcal{L}^\circ_{Lel}(\bar{\partial})$.

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