A definable Borsuk-Ulam type theorem

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Abstract

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field R. A definably compact definable group G is a definable Borsuk-Ulam group if there exists an isovariant definable map $f : V \to W$ between representations of G, then $\dim V - \dim V^G \leq \dim W - \dim W^G$. We prove that if a finite group G satisfies the prime condition, then G is a definable Borsuk-Ulam group.

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1 . Introduction.

Let G be a topological group. A continuous map $f : X \to Y$ is a G *map* if $f(gx) = gf(x)$ for all $x \in X, g \in G$. A G map $f : X \to Y$ is *isovariant* if for any $x \in X$, $G_{f(x)} = G_x$.

Let C_k be the cyclic group of order k and $\mathbb{S}^n$ the n-dimensional unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the antipodal C_2 action. If a real closed field R is the field $\mathbb{R}$ of real numbers, then the Borsuk-Ulam theorem states that if there exists a continuous C_2 map from $\mathbb{S}^n$ to $\mathbb{S}^m$, then $n \leq m$. There are several equivalent statements of it and many related generalizations (e.g. [1], [8] [10], [11], [12], [13], [14], [15], [19]). The above theorem is generalized the case where spheres with free C_k actions and a definable version in an o-minimal expansion of a real closed field of it is known in [13].

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field R. Let G be a definably compact definable group. A group homomorphism from G to some $O_n(R)$ is a *representation* if it is definable, where $O_n(R)$ means the nth orthogonal group of R. A *representation space* of G is R^n with the orthogonal action induced from a representation of G. In this paper, we consider isovariant definable maps between representation spaces of definably compact definable groups as a definable generalization of related results of the Borsuk-Ulam theorem. Everything is considered in $\mathcal{N}$ and a definable map is assumed to be continuous unless otherwise stated.

A positive integer n *satisfies the prime condition* if n is expressed as $p_1^{r_1} \dots p_s^{r_s}$, each p_i is a prime and $r_i \geq 1$ for $1 \leq i \leq s$, then $\sum_{i=1}^{s} \frac{1}{p_i} \leq 1$. A finite simple group

G satisfies the prime condition if for any $g \in G$, the order $|g|$ of g satisfies the prime condition. A finite group G *satisfies the prime condition* if each composition factor of G satisfies the prime condition.

A definably compact definable group G is a *definable Borsuk-Ulam group* if there exists an isovariant definable map $f : V \to W$ between representations of G, then $\dim V - \dim V^G \leq \dim W - \dim W^G$.

Theorem 1.1. *If a finite group G satisfies the prime condition, then G is a definable Borsuk-Ulam group.*

Theorem 1.1 is a definable generalization of [20].

2 . Preliminaries.

General references on o-minimal structures are [2], [4]. See also [18], [3], [5], [9] for examples and constructions of them.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset R^n \times R^m$) is a definable set. A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A definable subset X of R^n is *definably compact* if for every definable map $f : (a, b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in X, where $(a, b)_R = \{x \in R | a \leq x < b\}, -\infty \leq a < b \leq \infty$. A definable subset X of R^n is definably compact if and only if X is closed and bounded ([17]). Note that if X is a definably compact definable set and $f : X \to Y$ is a definable map, then $f(X)$ is definably compact.

If $R = \mathbb{R}$, then for any definable subset X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general, a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$ is definably compact but not compact.

Note that every definable subgroup of a definable group is closed ([16]) and a closed subgroup of a definable group is not necessarily definable. For example $\mathbb{Z}$ is a closed subgroup of $\mathbb{R}$ but not a definable subgroup of $\mathbb{R}$.

Recall existence of definable quotient.

Theorem 2.1. *(Existence of definable quotient (e.g. 10. 2.18 [2])). Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.*

Let G, G' be definable groups. A group homomorphism $f : G \to G'$ is a *definable group homomorphism* if f is definable. A definable group homomorphism $h : G \to G'$ is a *definable group isomomorphism* if there exists a definable group homomorphism $k : G' \to G$ such that $h \circ k = id_{G'}, k \circ h = id_G$.

Let G be a finite group and X a representation space of G. The character $\chi_X : G \to R$ is defined by $\chi_X(g) =$ the trace of the orthogonal transformation of g. Note that $\chi_X(e) = \dim X$ and $\dim X^G = \frac{\sum_{g \in G} \chi_X(g)}{|G|}$, where e denotes the unit element of G and $|G|$ stands for the order of G. Thus $\dim X - \dim X^G = \chi_X(e) - \frac{\sum_{g \in G} \chi_X(g)}{|G|}$
$= \frac{1}{|G|}(\sum_{g \in G}(\chi_X(e) - \chi_X(g)))$.

There exist some examples which are continuous actions but not definable actions.

Example 2.2. *(1) Let g denote the generator of C_2. Corresponding g to the map $f_g : \mathbb{R}_{alg} \to \mathbb{R}_{alg}$,*
$f_g(x) = \begin{cases} x, & x < -\pi \\ -x, & -\pi < x < \pi \\ x, & x > \pi \end{cases}$, *we have a non-trivial continuous C_2 action of $\mathbb{R}_{alg}$. But this action is not a definable C_2 action.*

(2) Let g denote a generator of C_p with $p \geq 2$ and $D = \{(x, y) \in \mathbb{R}^2_{alg} | x^2 + y^2 < \pi^2\}$. Suppose that $F_p : \mathbb{R}^2_{alg} \to \mathbb{R}^2_{alg}$ denotes the $\frac{2\pi}{p}$ rotation centered the origin. Corresponding g to the map $f_g : \mathbb{R}^2_{alg} \to \mathbb{R}^2_{alg}$,
$f_g(x) = \begin{cases} x, & x \in \mathbb{R}^2_{alg} - D \\ F_p(x), & x \in D \end{cases}$, *we have a non-trivial continuous C_p action of $\mathbb{R}_{alg}$. But this action is not a definable C_p action.*

3 . Proof of Theorem 1.1

Proposition 3.1. *Let G be a definably compact definable group and H a definable normal subgroup of G. If G is a definable Borsuk-Ulam group, then G/H is a definably compact definable Borsuk-Ulam group.*

Proof. Since H is normal and by Theorem 2.1, G/H is a definable group. Since G is definably compact, so is G/H. Any representation of G/H can be pulled back to the group G via the projection $\pi : G \to G/H$ and every G/H isovariant definable map is seen to be G isovariant. □

Proposition 3.2. *If $1 \to H \to G \to K \to 1$ is an exact sequence of definably compact definable groups and H, K are definable Borsuk-Ulam groups, then G is a definable Borsuk-Ulam group.*

Proof. Let V, W be representation spaces of G and $f : V \to W$ a G isovariant definable map. Since f is H isovariant and H is a definable Borsuk-Ulam group, $\dim V - \dim V^H \leq \dim W - \dim W^H$. Moreover V^H, W^H are representation spaces of G/H because H is normal in G, and $f|V^H : V^H \to W^H$ is a G/H isovariant definable map. Since G/H is definably group isomorphic to K and K is a definable Borsuk-Ulam group, $\dim V^H - \dim(V^H)^K \leq \dim W^H - \dim(W^H)^K$. On the other hand, $(V^H)^K$ (resp. $(W^H)^K$) is definably isomorphic to V^G (resp. W^G). Thus $\dim V^H - \dim V^G \leq \dim W^H - \dim W^G$. Hence $\dim V - \dim V^G \leq \dim W - \dim W^G$. Therefore G is a definable Borsuk-Ulam group. □

The proof also proves the following inequality.

If H is a normal definable subgroup of G, then $\dim W - \dim W^G - (\dim V - \dim V^G) \geq \dim W - \dim W^H - (\dim V - \dim V^H)$. Moreover,
$\frac{1}{|G|}(\sum_{g \in G}(\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g))) \geq \frac{1}{|H|}(\sum_{g \in H}(\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g)))$.

Corollary 3.3. *If G is a definably compact definable group and the identity definable component G_0 and G/G_0 are definable Borsuk-Ulam groups, then G is a definable Borsuk-Ulam group.*

A *composition series* of a finite group G is a collection of subgroups $G_j, 0 \leq j \leq r-1$ such that $G_0 = \{e\}, G_r = G$ and G_j is a maximal normal subgroup of G_{j+1} for $0 \leq j \leq r-1$. Each quotient group G_{j+1}/G_j is a finite simple group and is called a *composition factor* of G. They are independent of the choice of the composition series.

Proposition 3.4. *(1) If any composition factor of a finite group G is a definable Borsuk-Ulam group, then G is a definable Borsuk-Ulam group.*

(2) If p is a prime, then C_p is a definable Borsuk-Ulam group.

(3) Any finite abelian group is a definable Borsuk-Ulam group.

Proof. (1) If $G = G_1$, then $G_1/G_0 \simeq G$ and G is a definable Borsuk-Ulam group. Assume that it is true for groups with n factors. Let $G = G_{n+1}$. Considering the sequence $1 \to G_n \to G_{n+1} \to G_{n+1}/G_n \to 1$, since G_n is a definable Borsuk-Ulam group and G_{n+1}/G_n is a composition factor and by Proposition 3.2, $G = G_{n+1}$ is a definable Borsuk-Ulam group.

(2) follows from [13].

(3) follows from (2) and Proposition 3.2. □

Let G be a definably compact definable group. We recall orbit types ([7]). We say that two homogeneous definable G sets are *equivalent* if they are definably G homeomorphic. Let (G/H) denote the equivalence class of G/H. The set of all equivalence classes of homogeneous definable G sets has a natural order defined as $(X) \geq (Y)$ if there exists a definable G map $X \to Y$. If $(X) = (G/H)$ and $(Y) = (G/K)$, then $(X) \geq (Y)$ if and only if H is conjugate to a definable subgroup of K. The reflexivity and the transitivity clearly hold and the anti-symmetry is true by the following lemma.

Lemma 3.5 ([7]). *Let G be a definably compact definable group, H a definable subgroup of G and $g \in G$. If $gHg^{-1} \subset H$, then $gHg^{-1} = H$.*

Theorem 3.6 (2.2 [6]). *Let G be a definably compact definable group. Then every definable G set has only finitely many orbit types.*

The unit circle S^1 of R^2 is defined as $S^1 = \{(x, y) \in R^2 | x^2 + y^2 = 1\}$. For a general real closed field R, S^1 is definably compact and definably connected but netiher compact nor connected. We say that $T^n = S^1 \times \cdots \times S^1$ (n-times) is the *n-dimensional torus*.

Proposition 3.7. *The n-dimensional torus T^n is a definable Borsuk-Ulam group.*

Proof. Using the exact sequence $1 \to T^{n-1} \to T^n \to S^1 \to 1$ and Proposition 3.2, it is enough to prove the case where $G = S^1$.

Let V, W be representations of S^1 and $f : V \to W$ an isovariant definable S^1 map. By Theorem 3.6, there exist only finitely many definable subgroups of S^1 that occur as isotropy subgroups in V or W, say $C_{n_1}, \ldots, C_{n_r}, S^1$ with $n_i < n_{i+1}$ for all i. Take a prime p such that $p > n_r$. Then considering f as a C_p isovariant definable map, $\dim V - \dim V^{C_p} \leq \dim W - \dim W^{C_p}$. Moreover $V^{C_p} = V^{S^1}, V^{C_p} = W^{S^1}$ and hence $\dim V - \dim V^{S^1} \leq \dim W - \dim W^{S^1}$. □

By a way similar to the proof of Lemma 13 [20].

Lemma 3.8. *Let C be a finite cyclic group and $|C|$ satisfies the prime condition and $f : V \to W$ an isovariant definable C map. Then $\sum_{gen\ C}(\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g)) \geq 0$, where g ranges all of generators of C.*

Proof of Theorem 1.1. Using Propositiomn 3.2, Lemma 3.8, by a way similar to the proof of Theorem 12 [20], we have theorem 1.1. □

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