Relative Manin-Mumford for semi-abelian surfaces

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Abstract.- We show that Ribet sections are the only obstruction to the validity of the relative Manin-Mumford conjecture for one dimensional families of semi-abelian surfaces. Applications include special cases of the Zilber-Pink conjecture for curves in a mixed Shimura variety of dimension four, as well as the study of polynomial Pell equations with non-separable discriminants.

Contents

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1 Introduction

1.1 The viewpoint of group schemes : relative Manin-Mumford

Let $\mathbb{Q}^{alg}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, let S be an irreducible algebraic curve over $\mathbb{Q}^{alg}$, and let G/S be a semi-abelian scheme over S of relative dimension 2 and constant toric rank 1. We write G_{tor} for the union of all the torsion points of the various fibers of $G \to S$. This set G_{tor} is also the set of values at all points of S of the various torsion-sections of the group scheme G/S. Let further $s : S \to G$ be a section of G/S. The image of s is an irreducible algebraic curve $s(S) = W$ in G, defined over $\mathbb{Q}^{alg}$. Pursuing the theme of "unlikely intersections" and relative versions of the Manin-Mumford conjecture (see [33]), we here study the following question, where "strict" means "distinct from G".

Question 1 : *assume that $W \cap G_{tor}$ is infinite (i.e. Zariski dense in W). Must W then lie in a strict subgroup scheme of G/S ?*

Let us review some of the results known along this line :

i) the analogous question has a positive answer when G/S is replaced by an abelian scheme of relative dimension 2 : see [20], [33], Theorem III.12 and §5 for non-simple ones, and [21] for the general case.

ii) Assume that *the scheme G/S is isoconstant*, i.e. isomorphic, after a finite base extension, to a product $G = G_0 \times S$ with $G_0/\mathbb{Q}^{alg}$. Then, the Zariski closure W_0 of the projection of W to G_0 is an algebraic curve (or a point) meeting $G_{0,tor}$ Zariski-densely. By Hindry's generalization of Raynaud's theorem on the Manin-Mumford conjecture, see [17], §5, Thm. 2, W_0 is a torsion-coset of a strict algebraic subgroup H_0 of G_0, and W lies in a translate of $H = H_0 \times S$ by a torsion section. So, a positive answer is known in this case.

Therefore, the first new case occurs when G is a non isoconstant extension over S of an isoconstant elliptic scheme by $\mathbb{G}_m$, i.e. when *G is a semi-constant extension* in the sense of [11]. But as in [11] (though for different reasons), this case turns out to be more delicate, and the question can then have a *negative* answer. A counterexample is given in [8], and we shall refer to the corresponding sections s_R as Ribet sections of G/S and to their images $W_R = s_R(S)$ as Ribet curves. For a formal definition, see the end of this subsection (and [18] for its initial version). In this paper, we will prove that in all other cases, our question has a positive answer; in other words :

Main Theorem . *Let E/S be an elliptic scheme over the curve $S/\mathbb{Q}^{alg}$, and let G/S be an extension of E/S by $\mathbb{G}_{m/S}$. Let further $s : S \to G$ be a section of G/S, with image $W = s(S)$.*

(A) *Assume that $W \cap G_{tor}$ is infinite. Then,*

i) either s is a Ribet section;

ii) or s factors through a strict subgroup scheme of G/S.

(B) *More precisely, $W \cap G_{tor}$ is infinite if and only if s is a Ribet section, or a torsion section, or a non isoconstant section of a strict subgroup scheme of G/S.*

We point out that this statement is invariant under isogenies $G \to G'$ of the ambient group scheme, and under finite base extensions $S' \to S$. Throughout the paper, we will allow ourselves, sometimes tacitly, to perform such isogenies and base extensions.

We now rephrase Part (A) of the theorem according to the various types of extensions G/S and elliptic schemes E/S which can occur, and explain in each case the meaning of "isoconstant" in its Part (B) ; a more concrete discussion of this array of cases is given in §2. Concerning the type of E/S, we recall that the scheme E/S is isoconstant if and only if the j-invariant of its various fibers is constant; performing a finite base extension, we will then assume that E is equal to $E_0 \times S$ for some elliptic curve $E_0/\mathbb{Q}^{alg}$. As for the type of G/S,

- either G is isogenous as a group scheme over S to a direct product $\mathbb{G}_m \times E$. We then say that *the extension G/S is isotrivial* and perform this isogeny. Since W is flat over S, Conclusion (ii) then reads : W lies in a translate of E/S or of $\mathbb{G}_{m/S} = \mathbb{G}_m \times S$ by a torsion section of G/S (and Conclusion (i) does not occur); in this case, the isoconstant sections of the strict subgroup schemes are the translates by torsion sections of the constant sections of $\mathbb{G}_{m/S}$, or of the constant sections of E/S if $E = E_0 \times S$ is constant.

- or the extension G/S is not isotrivial. Conclusion (ii) then reads : W lies in a translate of $\mathbb{G}_{m/S}$ by a torsion section of G/S; in this case, the isoconstant sections are the translates by torsion sections of the constant sections of $\mathbb{G}_{m/S}$.

Now, whether G/S is or is not an isotrivial extension,

- we automatically get Conclusion (ii) if either the scheme E/S is not isoconstant, or it is isoconstant, but E_0 does not admit complex multiplications, or if E_0 is CM, but G is isoconstant (a case already covered by [17], of course).

- in the remaining case where E/S is isoconstant, with a CM elliptic curve $E_0/\mathbb{Q}^{alg}$, and G is not isoconstant (hence in particular, not isotrivial), Ribet sections of G/S do exist, their images W do not lie in strict subgroup schemes of G/S but meet G_{tor} infinitely often, while not all sections s satisfying the hypotheses of the theorem are Ribet sections.

In other words, both conclusions (i) and (ii) of the Main Theorem occur in this last case and are then mutually exclusive. However, there is a way of reconciling them, through the setting of Pink's extension of the André-Oort and Zilber conjectures to mixed Shimura varieties, which we turn to in the next subsection §1.2 (see, e.g., Corollary 1).

Another type of application of the Main Theorem is given in §7, Appendix II of the paper : this concerns the solvability of Pell's equations over polynomial rings, and extends some of the results of [21] to the case of non-separable discriminants.

To conclude this introduction, here is the promised definition of a *Ribet section* $s_R : S \to G$ (see [9], §1 - several other definitions are discussed in [10], and a more concrete, but analytic, characterization is mentioned in Remark 4 of §7, Appendix I, below). In the notations of the Main Theorem, let $\hat{E}/S$ be the dual of E/S: the isomorphism class of the $\mathbb{G}_m$-torsor G over E is given by a section $q : S \to \hat{E}$. Let further $\mathcal{P} \to E \times_S \hat{E}$ be the Poincaré biextension of E and $\hat{E}$ by $\mathbb{G}_m$: by [13], 10.2.13, a section $s : S \to G$ of G/S lifting a section $p : S \to E$ of E/S is entirely described by a trivialisation of the $\mathbb{G}_m$-torsor $(p,q)^*\mathcal{P}$ over S. Assume now that $E = E_0 \times S$ is isoconstant, and admits complex multiplications, and let $f : \hat{E}_0 \to E_0$ be a non-zero antisymmetric isogeny (i.e., identifying E_0 and $\hat{E}_0$, a purely imaginary complex multiplication) which for simplicity, we here assume to be divisible by 2. Then, $(f(q),q)^*\mathcal{P}$ is a trivial torsor in a canonical way, and the corresponding trivialization yields a well-defined section $s = s(f)$ of G/S above $p = f(q)$. When G/S is semiconstant (which means : when q is not constant), any section s_R of G/S a non-zero multiple of which is of the form $s(f)$ for some antisymmetric f will be called a Ribet section of G/S. So, on such a semi-abelian scheme G/S, there exists essentially only one Ribet section s_R (more precisely, all are linearly dependent over $\mathbb{Z}$), and by [8] (see also [10], and Appendix I), its image $W_R = s_R(S)$ meets G_{tor} infinitely often. It follows from Hindry's theorem (see Corollary 3 of §4) that the latter property characterizes Ribet sections among those sections of G/S which project to a section of E/S of the form $p = f(q)$.

1.2 The viewpoint of mixed Shimura varieties : Pink's conjecture

The consequences of the Main Theorem described in this subsection are discussed in [9], which we here summarize for the convenience of the reader.

Let X be a modular curve, parametrizing isomorphism classes of elliptic curves with some level structure, let $\mathcal{E}$ be the universal elliptic scheme over X, with dual $\hat{\mathcal{E}}$, and let $\mathcal{P}$ be the Poincaré bi-extension of $\mathcal{E} \times_X \hat{\mathcal{E}}$ by $\mathbb{G}_m$. This is a mixed Shimura variety of dimension 4, which parametrizes points P on extensions G of elliptic curves E by $\mathbb{G}_m$. A point of $\mathcal{P}(\mathbb{C})$ can be represented by a triple (E, G, P) , and is called special if the attached Mumford-Tate group is abelian, which is equivalent to requiring that E has complex multiplications, that G is an isotrivial extension, and that P is a torsion point on G. Denote by $\mathcal{P}_{sp}$ the set of special points of $\mathcal{P}$. Following [29], we further say that an irreducible subvariety of $\mathcal{P}$ is special if it is a component of the Hecke orbit of a mixed Shimura subvariety of $\mathcal{P}$. Given any irreducible subvariety Z of $\mathcal{P}$, the intersection of all the special subvarieties of $\mathcal{P}$ containing Z is called the special closure of Z. The special subvarieties of $\mathcal{P}$ of dimension 0 are the special points; the special curves of $\mathcal{P}$ are described below; for the full list, see [9], section 3.

Corollary 1. *Let $W/\mathbb{Q}^{alg}$ be an irreducible closed algebraic curve in $\mathcal{P}$. Assume that $W \cap \mathcal{P}_{sp}$ is infinite. Then, W is a special curve.*

To prove this corollary, we distinguish the various cases provided by the projection $\varpi : \mathcal{P} \to X$ and its canonical section (rigidification) $\sigma : X \to \mathcal{P}$, whose image $\sigma(X)$ is made up of points of the type $(E, \mathbb{G}_m \times E, 0) \in \mathcal{P}$:

- either the restriction of ϖ to W is dominant : the corollary then says that W lies in the Hecke orbit of the curve $\sigma(X)$. Indeed, up to Hecke transforms, $\sigma(X)$ is the only one-dimensional (mixed, but actually pure) Shimura subvariety of $\mathcal{P}$ dominating X.

In this case where $\varpi_{|W}$ is dominant, the corollary follows not from our Main Theorem, but from André's theorem [2], p. 12, on the special points of the mixed Shimura variety $\mathcal{E}$ (see also [23], Thm. 1.2).

- or $\varpi(W)$ is a point x_0 of X, necessarily of CM type. In particular, W lies in the fiber $\mathcal{P}_0$ of ϖ above x_0. This fiber $\mathcal{P}_0$ is a 3-dimensional mixed Shimura subvariety of $\mathcal{P}$, which can be identified with the Poincaré biextension of $E_0 \times \hat{E}_0$ by $\mathbb{G}_m$, where E_0 denotes an elliptic curve in the isomorphism class of x_0. An analysis of the generic Mumford-Tate group of $\mathcal{P}_0$ as in [6], p. 52, shows that up to Hecke transforms, there are exactly four types of special curves in $\mathcal{P}_0$: the fiber $(\mathbb{G}_m)_{x_0}$ above $(0,0)$ of the projection $\mathcal{P}_0 \to (\mathcal{E} \times_X \hat{\mathcal{E}})_{x_0} = E_0 \times \hat{E}_0$ and the images $\psi_B(B)$ of the elliptic curves $B \subset E_0 \times \hat{E}_0$ passing through $(0,0)$ such that the $\mathbb{G}_m$-torsor $\mathcal{P}_{0|B}$ is trivial, under the corresponding (unique) trivialization $\psi_B : B \to \mathcal{P}_{0|B}$. There are three types of such elliptic curves B : the obvious ones $E_0 \times 0$ and $0 \times \hat{E}_0$ (whose images we denote by $\psi(E_0 \times 0), \psi(0 \times \hat{E}_0)$), *and* the graphs of the non-zero antisymmetric isogenies from E_0 to $\hat{E}_0$, in which case ψ_B corresponds precisely to a Ribet section of the semi-abelian scheme $\mathcal{G}_0/\hat{E}_0$ defined below.

Corollary 1 now follows from the Main Theorem, by interpreting $\mathcal{P}_0/\hat{E}_0$ as the "universal" extension $\mathcal{G}_0$ of E_0 by $\mathbb{G}_m$, viewed as a group scheme over the curve $S := \hat{E}_0$, so that $\mathcal{P}_{sp} \cap \mathcal{P}_0 \subset (\mathcal{G}_0)_{tor}$. More precisely, suppose that W dominates $\hat{E}_0$: then, it is the image of a multisection of $\mathcal{G}_0/\hat{E}_0$, and after a base extension, the theorem implies that up to a torsion translate, W is the Ribet curve $\psi_B(B)$, or it lies in $\mathbb{G}_{m/\hat{E}_0} = \mathbb{G}_m \times \hat{E}_0$, where a new application of the theorem (or more simply, of Hindry's) shows that it must coincide with a Hecke transform of $\mathbb{G}_m = (\mathbb{G}_m)_{x_0}$ or of $\psi(0 \times \hat{E}_0)$. By biduality (i.e. reverting the roles of $\hat{E}_0$ and E_0), the same argument applies if W dominates E_0. Finally, if W projects to a point of $E_0 \times \hat{E}_0$, then, this point must be torsion, and W lies in the Hecke orbit of $(\mathbb{G}_m)_{x_0}$.

Although insufficient in the presence of Ribet curves, the argument devised by Pink to relate the Manin-Mumford and the André-Oort settings often applies (see the proof of Theorems 5.7 and 6.3 of [29], and [28], Prop. 4.6, as well as [10], for abelian schemes). In the present situation, one notes that given a point (E, G, P) in $\mathcal{P}(\mathbb{C})$, asking that it be special as in Corollary 1 gives 4 independent conditions, while merely asking that P be torsion on G as in the Main Theorem gives 2 conditions. Now, unlikely intersections for a curve W in $\mathcal{P}$ precisely means studying its intersection with the union of the special subvarieties of $\mathcal{P}$ of codimension ≥ 2 (i.e. of dimension ≤ 2), and according to Pink's Conjecture 1.2 of [29], when this intersection is infinite, W should lie in a special subvariety of dimension < 4, i.e. a proper one. Similarly, if W lies in the fiber $\mathcal{P}_0$ of $\mathcal{P}$ above a CM point x_0 and meets infinitely many special curves of this 3-fold, then, it should lie in a special surface of the mixed Shimura variety $\mathcal{P}_0$. In these directions, our Main Theorem, combined with [2], p. 12, and with the relative version of Raynaud's theorem obtained in [20], implies :

Corollary 2. *Let $W/\mathbb{Q}^{alg}$ be an irreducible curve in the mixed Shimura 4-fold $\mathcal{P}$, and let δ_W be the dimension of the special closure of W.*

i) Suppose that $\delta_W = 4$; then, the intersection of W with the union of all the special surfaces of $\mathcal{P}$ dominating X is finite;

ii) Suppose that $\delta_W = 3$; then, the intersection of W with the union of all the special non-Ribet curves of $\mathcal{P}$ is finite.

The proof goes along the same lines as that of Corollary 1; see [9] for more details, and for a discussion on the gap between these corollaries and the full statement of Pink's conjecture 1.2 of [29], which would give a positive answer to :

Question 2 : *Let $W/\mathbb{Q}^{alg}$ be an irreducible curve in the mixed Shimura 4-fold $\mathcal{P}$, and let δ_W be the dimension of the special closure of W. Is the intersection of W with the union of all the special subvarieties of $\mathcal{P}$ of dimension $\leq \delta_W - 2$ then finite ?*

The case where $\delta_W = 2$ is covered by Corollary 1 . The remaining cases would consist in disposing of the restrictions "dominating X" and "non-Ribet" in Corollary 2. These problems are out of the scope of the present paper.

Although the Shimura view-point will not be pursued further, the Poincaré biextension, which has already appeared in the definition of Ribet sections, plays a role in the proof of the Main Theorem. See §5.3, Remark 3.(iii), §3.3, Footnote (3), and §6, Case (**SC2**), where s is viewed as a section of $\mathcal{P}$, rather than of G. See also the sentence concluding §4. We finally mention that very recently, Ziyang Gao [14] has obtained a proof of the André-Oort conjecture for many mixed Shimura varieties. His work implies Corollary 1 above on special points, but probably not Corollary 2 on unlikely intersections.

1.3 Plan of the paper

• In the first sections (§§ 2, 3.1, 3.2), we give a set of notations, present the overall strategy of the proof, borrowed from [20] and based on the same set of preliminary lemmas : large Galois orbits, bounded heights, Pila-Wilkie upper bounds. The outcome is that for a certain real analytic surface $\mathcal{S}$ in $\mathbb{R}^4$ attached to the section $s \in G(S)$:

$W \cap G_{tor}$ **infinite** $\Rightarrow \mathcal{S}$ **contains a semi-algebraic curve.**

• The program for completing the proof is sketched in §3.3, and can be summarized by the following two steps ($\boldsymbol{\alpha}$), ($\boldsymbol{\beta}$). Here $log_G(s)$ is a local logarithm of s, and F_{pq} is a certain Picard-Vessiot extension of K attached to G and to the projection p of s to E. Then, under a natural assumption on p (see Proposition 4),

($\boldsymbol{\alpha}$) $\mathcal{S}$ **contains a semi-algebraic curve** $\Rightarrow log_G(s)$ **is algebraic over** F_{pq}**.**

The proof is thereby reduced to a statement of algebraic independence, which forms the content of our "Main Lemma", cf. middle of §3.3. Notice the similarity between the statements of this Main Lemma and of the Main Theorem, making it apparent that *up to translation by a constant section*,

$$(\boldsymbol{\beta}) \quad log_G(s) \textbf{ is algebraic over } F_{pq} \Rightarrow s \textbf{ is Ribet or factors}$$

as is to be shown.

• As a warm up, we realize these two steps, in Section 4 when G is an isotrivial extension. In Section 5, we go back to the general case G/S, prove the first step $(\boldsymbol{\alpha})$, and comment on the use of Picard-Vessiot extensions.

• Section 6 is devoted to the proof of the Main Lemma $(\boldsymbol{\beta})$. As in §4, we appeal to results of Ax type [3] to treat isoconstant cases, and to Picard-Vessiot theory, in the style of André's theorem [1], for the general case.

• Finally, Appendix I of §7 gives a concrete description of the local logarithm $log_G(s)$, while Appendix II is devoted to an application to polynomial Pell's equations, following the method of [21].

2 Restatement of the Main Theorem

2.1 Introducing q and p

We first repeat the setting of the introduction, with the help of the fundamental isomorphism $Ext_S(E, \mathbb{G}_m) \simeq \hat{E}$ to describe the various cases to be studied.

So, let $\mathbb{Q}^{alg}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Extensions of scalars from $\mathbb{Q}^{alg}$ to $\mathbb{C}$ will be denoted by a lower index $_{\mathbb{C}}$. Let S be an irreducible algebraic curve over $\mathbb{Q}^{alg}$, whose generic point we denote by λ, and let $K = \mathbb{Q}^{alg}(S) = \mathbb{Q}^{alg}(\lambda)$, so $K_{\mathbb{C}} := \mathbb{C}(S_{\mathbb{C}}) = \mathbb{C}(\lambda)$. We use the notation $\overline{\lambda}$ for the closed points of $S_{\mathbb{C}}$, i.e. $\overline{\lambda} \in S(\mathbb{C})$. Let G/S be an semi-abelian scheme over S of relative dimension 2, all of whose fibers have toric rank 1. Making Question 1 more precise, we write G_{tor} for the union of all the torsion points of the various fibers of $G_{\mathbb{C}} \to S_{\mathbb{C}}$, i.e. $G_{tor} = \cup_{\overline{\lambda} \in S(\mathbb{C})} (G_{\overline{\lambda}})_{tor} \subset G(\mathbb{C})$, where $G_{\overline{\lambda}}$ denotes the fiber of G above $\overline{\lambda}$. This set G_{tor} is also the set of values at all $\overline{\lambda} \in S(\mathbb{C})$ of the various torsion-sections of $G_{\mathbb{C}} \to S_{\mathbb{C}}$. Let finally $s : S \to G$ be a section of G/S defined over $\mathbb{Q}^{alg}$, giving a point $s(\lambda) \in G_\lambda(K)$ at the generic point of S, and closed points $s(\overline{\lambda}) \in G_{\overline{\lambda}}(\mathbb{C})$ at the $\overline{\lambda}$'s in $S(\mathbb{C})$.

In the description which follow, we may have to withdraw some points of S, or replace S by a finite cover, but will still denote by S the resulting curve. After a base extension, the group scheme G/S can be presented in a unique way as an S-extension

$$0 \longrightarrow \mathbb{G}_{m/S} \longrightarrow G \xrightarrow{\pi} E \longrightarrow 0$$

of an elliptic scheme E/S by $\mathbb{G}_{m/S} = \mathbb{G}_m \times S$. We denote by $\pi : G \to E$ the corresponding S-morphism.

The extension G is parametrized by a section

$$q \in \hat{E}(S)$$

of the dual elliptic scheme $\hat{E}/S$. We write

$$p = \pi \circ s \in E(S)$$

for the projection to E of the section s. Finally, we denote by $k \subset \mathbb{Q}^{alg}$ the number field over which S is defined, and assume without loss of generality that the generic fibers of G, hence of q, and of s, hence of p, are defined over $k(S) = k(\lambda)$.

Since the algebraic curve $W = s(S) \subset G$ is the image of a section, the minimal strict subgroup schemes H of G which may contain W are flat over S, with relative dimension 1. These can be described as follows :

– either q has infinite order in $\hat{E}(S)$: then, G/S is a non isotrivial extension, and $H \subsetneq G$ is contained in a finite union of torsion translates of $\mathbb{G}_{m/S}$; in particular, $\pi(H)$ is finite over S, and H can contain W only if $p = \pi(s)$ is a torsion section of E/S.

– or q has finite order : in this case, G is isogeneous to the direct product $\mathbb{G}_{m/S} \times_S E$. Since the answer to the question is invariant under isogenies, we can then assume that G is this direct product, i.e. that $q = 0$. Strangely enough, this (easy) case of Question 1 does not seem to have been written up yet. We present it in §4. The answer (cf. Theorem 1 below) is a corollary of Hindry's theorem when E/S is isoconstant, since G/S too is then isoconstant; in an apparently paradoxical way, we will use it to characterize the Ribet sections of semi-constant extensions, cf. Corollary 3.

So, from now on and apart from §4, we could assume that q has infinite order in $\hat{E}(S)$, i.e. that G is a non-isotrivial extension. However, the only hypothesis we will need in our general study of §3.3 and §5 concerns the section $p = \pi(s)$ of E/S, see next subsection.

2.2 Isoconstant issues

In general, given our curve $S/\mathbb{Q}^{alg}$, we say that a scheme X/S is isoconstant if there exists a finite cover $S' \to S$ and a scheme $X_0/\mathbb{Q}^{alg}$ such that $X_{S'} = X \times_S S'$ is isomorphic over S' to the constant scheme $X_0 \times_{\mathbb{Q}^{alg}} S'$. We then say that a section x of X/S is *constant* if after a base change S'/S which makes X constant, the section of $X_{S'}/S'$ which x defines comes from the constant part X_0 of $X_{S'}$. This notion is indeed independent of the choice of S'. See Footnote (4) of §4 below for further conventions in the isoconstant cases.

In these conditions, the hypothesis just announced about p reads :

$p \in E(S)$ **is not a torsion section, and is not constant if** E/S **is isoconstant,**

and will be abbreviated as "the section p is not torsion, nor constant"; in terms to be described in §5.3 and §6, it is better expressed as :

$p \in E(S)$ **does not lie in the Manin kernel** $E^\sharp$ **of** E**.**

The relation to the Main Theorem is as follows. If p is a torsion section, then, a torsion translate of $s \in G(S)$ lies in $\mathbb{G}_m$, so s satisfies Condition (ii) of the Main Theorem. And if $p = p_0$ is a constant (and not torsion) section, then, $p(S) = \{p_0\} \times S$ does not meet E_{tor} at all, so $s(S) \cap G_{tor}$ is empty. In other words, the Main Theorem is trivial in each of theses cases.

A precision on the expression "*G is semi-constant*" is now in order : it appears only if E/S is isoconstant, and means that there exists a finite cover $S' \to S$ such that on the one hand, there exists an elliptic curve $E_0/\mathbb{Q}^{alg}$ such that the pull-back of E/S to S' is isomophic over S' to $E_0 \times S'$, and that on the other hand, the section $q' \in \hat{E}'(S')$ which q defines is given by a section of $\hat{E}_0 \times_{\mathbb{Q}^{alg}} S$ which does *not* come from $\hat{E}_0(\mathbb{Q}^{alg})$. Since the answer to our question is invariant under finite base extension of S, we will assume in this case that E/S is already constant, i.e. $E = E_0 \times S$, and that $q \in \hat{E}_0(S) \setminus \hat{E}_0(\mathbb{Q}^{alg})$; indeed, as just said above, the second condition is then valid for any S'/S. Notice that the condition that q be non-constant forces it to be of infinite order. Consequently, semi-constant extensions are automatically both non isoconstant and non-isotrivial. On the other hand, if $q \in \hat{E}_0(\mathbb{Q}^{alg})$ is constant, we are in the purely constant case of [17] already discussed in the Introduction.

Remark 1 (*traces and images*). - Let $\mathcal{E}_0/\mathbb{Q}^{alg}$ denote the constant part ($K/\mathbb{Q}^{alg}$-trace) of E/S. An innocuous base extension allows us to assume that if E/S is isoconstant, then $\mathcal{E}_0 \neq \{0\}$ (and we then set $\mathcal{E}_0 := E_0$), and E/S is actually isomorphic to $E_0 \times S$. Denote by $G_0/\mathbb{Q}^{alg}$ the constant part ($K/\mathbb{Q}^{alg}$-trace) of G/S, and by $G^0/\mathbb{Q}^{alg}$ the maximal constant quotient ($K/\mathbb{Q}^{alg}$-image) of G/S. Then :

- G/S is an isotrivial extension if and only if q is a torsion section of $\hat{E}/S$. In this case, an isogeny allows us to assume that $q = 0$, i.e. $G = \mathbb{G}_m \times E$. We then have $G_0 = G^0 = \mathbb{G}_m \times \mathcal{E}_0$, and G/S is isoconstant if and only if E/S is isoconstant.

- assume now that G/S is not an isotrivial extension, and that E/S, hence $\hat{E}/S$, is not isoconstant. Then, $G_0 = \mathbb{G}_m$, while $G^0 = \mathcal{E}_0 = \{0\}$.

- finally , assume that G/S is not an isotrivial extension, but $E = E_0 \times S$, hence $\hat{E} = \hat{E}_0 \times S$, is (iso)constant. Then either q is a non constant section of $\hat{E}/S$, in which case G is semi-constant, and we have $G_0 = \mathbb{G}_m, G^0 = \mathcal{E}_0 = E_0$; or q is constant, in which case $G^0 = G_0 \in Ext_{\mathbb{Q}^{alg}}(E_0, \mathbb{G}_m)$, and $G = G_0 \times S$ is itself constant.

For the sake of brevity, we will henceforth say that a section of a group scheme H/S is (iso)constant if it is a constant section of the constant part $H_0 \times S$ of H/S.

2.3 Antisymmetric relations and restatement

The answer to Question 1 of the Introduction, as well as the proofs, depend on possible relations between p and q. For ease of notations, we fix a principal polarization $\psi : \hat{E} \simeq E$ of the elliptic scheme, and allow ourselves to identify q with its image $\psi(q) \in E(S)$. Also, we denote by $\mathcal{O}$ the ring of endomorphisms of E. If E/S is not isoconstant, $\mathcal{O}$ reduces to $\mathbb{Z}$. Otherwise, $\mathcal{O}$ may contain complex multiplications, and we say that the non-torsion and non-constant sections p and q are *antisymmetrically related* if there exists $\alpha \in \mathcal{O} \otimes \mathbb{Q}$ with $\overline{\alpha} = -\alpha$ such that $q = \alpha p$ in $E(K)$ modulo torsion.

Notice that we reserve this expression for sections p, q which are non torsion (and non constant). Therefore, an antisymmetric relation between p and q necessarily involves a non-zero imaginary α, hence $\mathcal{O} \neq \mathbb{Z}$, forcing E/S to be isoconstant. And since q is not constant, the corresponding semi-abelian scheme $G = G_q$ is then semiconstant, and admits a Ribet section s_R projecting to $p \in E(S)$.

For any positive integer m, we set :

$$S_m^G = \{\overline{\lambda} \in S(\mathbb{C}), s(\overline{\lambda}) \text{ has order } m \text{ in } G_{\overline{\lambda}}(\mathbb{C})\},$$

$$S_m^E = \{\overline{\lambda} \in S(\mathbb{C}), p(\overline{\lambda}) \text{ has order } \; m \text{ in } E_{\overline{\lambda}}(\mathbb{C})\},$$

$$S_\infty^G = \cup_{m \in \mathbb{Z}_{>0}} S_m^G \simeq W \cap G_{tor} \;,\; S_\infty^E = \cup_{m \in \mathbb{Z}_{>0}} S_m^E \simeq \pi(W) \cap E_{tor},$$

where the indicated bijections are induced by $S \simeq s(S) = W$, $S \simeq p(S) = \pi(W)$. Clearly, $\cup_{k|m} S_k^G \subset \cup_{k|m} S_k^E$ for all m's, and the points of S_∞^G can be described as those points of $\pi(W) \cap E_{tor}$ ("likely intersections") that lift to points of $W \cap G_{tor}$ ("unlikely intersections").

Our Main Theorem can then be divided into the following three results. We first consider the case when q is torsion, which reduces after an isogeny to the case $q = 0$.

Theorem 1. *Let E/S be an elliptic scheme over the curve $S/\mathbb{Q}^{alg}$, and let $G = \mathbb{G}_m \times E$ be the* trivial *extension of E/S by $\mathbb{G}_{m/S}$. Let further $s : S \to G$ be a section of G/S, with image $W = s(S)$, such that $p = \pi(s)$ has* infinite order *in $E(S)$. Then, S_∞^G is finite (in other words, $W \cap G_{tor}$ is finite) as soon as*

(o) no multiple of s by a positive integer factors through E/S (i.e. the projection of s to the $\mathbb{G}_m$-factor of G is not a root of unity).

The case when q is not torsion can be restated as follows.

Theorem 2. *Let E/S be an elliptic scheme over the curve $S/\mathbb{Q}^{alg}$, and let G/S be a non-isotrivial extension of E/S by $\mathbb{G}_{m/S}$, i.e. parametrized by a section $q \in \hat{E}(S) \simeq E(S)$ of* infinite order. *Let further $s : S \to G$ be a section of G/S, with image $W = s(S)$, such that* $p = \pi(s)$ has infinite order in $E(S)$. *Then, S_∞^G is finite (in other words, $W \cap G_{tor}$ is finite) in each of the following cases :*

(i) E/S is not isoconstant;

(ii) E/S is isoconstant, and p and q are not antisymmetrically related;

(iii) E/S is isoconstant, p and q are non-constant antisymmetrically related sections, and no multiple of s is a Ribet section.

For the sake of symmetry, we recall that in these two theorems, the hypothesis that p is not torsion is equivalent to requiring that no multiple of s by a positive integer factors through $\mathbb{G}_{m/S}$.

Since Ribet sections exist only in Case (iii) of Theorem 2, the conjunction of Theorems 1 and 2 is equivalent to Part (A) of the Main Theorem, giving necessary conditions for $W \cap G_{tor} \simeq S_\infty^G$ to be infinite. That these conditions are (essentially) sufficient, i.e. that Part (B) holds true, is dealt with by the following statement, which we prove right now.

Theorem 3. *Let E/S be an elliptic scheme over the curve $S/\mathbb{Q}^{alg}$, and let G/S be an extension of E/S by $\mathbb{G}_{m/S}$. Let further $s : S \to G$ be a section of G/S. Then,*

i) if s is a Ribet section, S^G_∞ is infinite, and equal to S^E_∞;

ii) if s is a torsion section, then $S^G_\infty = S(\mathbb{C})$;

iii) if s is a non-torsion section factoring through a strict subgroup scheme H/S of G/S, then S^G_∞ is empty if s is an isoconstant section of H/S, and infinite but strictly contained in $S(\mathbb{Q}^{alg})$ if s is not isoconstant.

Proof : (i) is proved in [8], [10], see also Remark 4 in Appendix I below. The second statement is clear. As for (iii), this is an easy statement if the connected component of H is $\mathbb{G}_m$. If (for an isotrivial G), it is isogenous to E, the isoconstant case is again clear, while the non-isoconstant one follows from "torsion values for a single point" as in [33], cf. Proposition 1.(iv) below.

So, we can now concentrate on Theorems 1 and 2.

3 The overall strategy

Our stategy will be exacty the same as in [20].

3.1 Algebraic lower bounds

In this section, we denote by k a number field over which the algebraic curve S, the group scheme G/S and its section s are defined. We fix a embedding of S in a projective space over k, and denote by H the corresponding height on the set $S(\mathbb{Q}^{alg})$ of algebraic points of S. We then have :

Proposition 1. *Let E/S be an elliptic scheme, and let $p : S \to E$ be a section of E/S* of infinite order. *There exist positive real numbers C, C' depending only on $S/k, E/S$ and p, with the following properties. Let $\overline{\lambda} \in S(\mathbb{C})$ be such that $p(\overline{\lambda})$ is a torsion point of $E_{\overline{\lambda}}(\mathbb{C})$, i.e. $\overline{\lambda} \in S^E_\infty$. Then,*

i) the point $\overline{\lambda}$ lies in $S(\mathbb{Q}^{alg})$, i.e. the field $k(\overline{\lambda})$ is an algebraic extension of k;

ii) the height $H(\overline{\lambda})$ of $\overline{\lambda}$ is bounded from above by C;

iii) if $n = n(\overline{\lambda}) \geq 1$ denotes the order of $p(\overline{\lambda})$, then $[k(\overline{\lambda}) : k] \geq C' n^{1/3}$.

iv) the set S^E_∞ is infinite (assuming that if E/S is isoconstant, p is not constant).

Proof : i, ii, iii) In the non-isoconstant case, one can reduce to the Legendre curve, where all is already written in [20], [21], [33], based on diophantine results of Silverman, David and the second author. Notice that the upper bound (ii) on $H(\overline{\lambda})$ is needed to deduce the lower bound (iii) on degrees. In the isoconstant case, the proof is easier as (ii) is not needed, and one can sharpen the lower bound (iii) to n^2 in the non-CM case (non effective), resp. $n/logn$ in the CM case. But as usual, any positive power of n will do.

iv) This is the issue of "torsion values for a single point", an analytic proof of which is given in [33], p. 92, in the non-isoconstant case. If E/S is isoconstant, the second case of this analytic proof does not occur, since we assume in (iv) that p is not constant.

Corollary. *Hypotheses and the notation C' being as in Proposition 1, let G/S be an extension of E/S by $\mathbb{G}_{m/S}$, and let s be a section of G/S lifting the non-torsion section p of E/S. Assume that $\overline{\lambda} \in S^E_\infty$ actually lies in S^G_∞, and let m be the order of the torsion point $s(\overline{\lambda}) \in G_{\overline{\lambda}}(\mathbb{Q}^{alg})$. Then, $[k(\overline{\lambda}) : k] \geq C' m^{1/6}$.*

Proof : if $s(\overline{\lambda})$ has precise order m in $G_{\overline{\lambda}}$ and projects to a point $p(\overline{\lambda})$ of order $n|m$ on $E_{\overline{\lambda}}$, then, $ns(\overline{\lambda})$ is a primitive m/n-th root of unity, so has degree $>> (m/n)^{1-\varepsilon}$ over k, and we can assume that this is larger than $C'(m/n)^{1/3}$ (assume $C' < [k : \mathbb{Q}]^{-1}$). Since the fields of definition of $ns(\overline{\lambda})$ and of $p(\overline{\lambda})$ are contained in $k(\overline{\lambda})$, we get $[k(\overline{\lambda}) : k] \geq C'(sup(n, m/n))^{1/3} \geq C' m^{1/6}$.

The conclusion of this first step is summarized by the implications :

$$\forall \overline{\lambda} \in S(\mathbb{C}), \overline{\lambda} \in S^G_\infty \Rightarrow \overline{\lambda} \in S(\mathbb{Q}^{alg}) \text{ and } H(\overline{\lambda}) \leq C,$$

and

$$\forall \overline{\lambda} \in S(\mathbb{Q}^{alg}), \forall m \geq 1, \overline{\lambda} \in S^G_m \Rightarrow [k(\overline{\lambda}) : k] \geq C' m^{1/6}.$$

In particular, if $W = s(S)$ contains a point $w = s(\overline{\lambda})$ of order m (w.r.t. the group law of its fiber $G_{\overline{\lambda}}$), then W contains at least $C' m^{1/6}$ points of order m (w.r.t. the group laws of their respective fibers) : indeed, since W is defined over k, it contains the orbit of w under $Gal(\mathbb{Q}^{alg}/k)$. However, we will need a sharper version of this statement, involving the archimedean sizes of the conjugates of $\overline{\lambda}$, and the upper bound on $H(\overline{\lambda})$ will again come to help at this stage.

3.2 Transcendental upper bounds

The next step is based on the following theorem of Pila. For the involved definitions and a short history on this type of results, leading to [22] and the accompanying references, we refer to [33], Remark 3.1.1, and to the Amer. J. Math. version of [20]). Dimensions here refer to real dimensions. For any $m \in \mathbb{Z}_{>0}$, we set $\mathbb{Q}_m = \frac{1}{m}\mathbb{Z} \subset \mathbb{Q}$.

Proposition 2. *Let $\mathcal{S}$ be a naive-compact-2-(dimensional)-analytic subset of an affine space $\mathbb{R}^d$. Assume that no semi-algebraic curve of $\mathbb{R}^d$ is contained in $\mathcal{S}$. For any $\varepsilon > 0$, there exists a real number $c = c(\mathcal{S}, \varepsilon) > 0$ with the following property. For each positive integer m, the set $\mathcal{S} \cap \mathbb{Q}^d_m$ contains at most $c\, m^\varepsilon$ points.*

Proof : see [20], Lemma 1.

In §3.3, we will give a precise description of the real surfaces $\mathcal{S}$ to which Proposition 2 is to be applied. See the following §4.1 for a more easily recognizable form when $G = \mathbb{G}_m \times E$. Roughly speaking,

$$\mathcal{S} \sim log_G^B(W) \subset (\mathbb{Z} \otimes \mathbb{C}) \oplus (\mathbb{Z}^2 \otimes \mathbb{R}) \simeq \mathbb{R}^4$$

is the set of logarithms of the various points $s(\overline{\lambda})$, when $\overline{\lambda}$ runs through $S(\mathbb{C})$, but we express these logarithms in terms of *a basis attached to the* $\mathbb{Z}$*-local system of periods* Π_G, i.e. in Betti terms rather than in the de Rham view-point provided by the Lie algebra, hence the upper index B above. In this basis, the logarithms of the torsion points $s(\overline{\lambda})$ of order m are represented by vectors with coordinates in $\mathbb{Q}_m$, and so $log_G^B(s(S_m^G)) \subset \mathbb{Q}_m^4$. Thanks to the "zero estimate" discussed at the end of the present subsection, which compares the **graph** $\tilde{\mathcal{S}} \subset S \times \mathbb{R}^4$ of $log_G^B \circ s : S \to \mathbb{R}^4$ with its projection $\mathcal{S}$ in $\mathbb{R}^4$, Proposition 2 then implies (with a proviso to be explained below for the first conclusion)

- either that for any positive integer m, $exp_G(\tilde{\mathcal{S}}) \sim s(S) = W$ contains at most $c\ m^\varepsilon$ points of order m w.r.t the group law of their respective fibers;

- or that $\mathcal{S}$ contains a real semi-algebraic curve, where algebraicity refers to the real affine space $\mathbb{R}^4$ associated to the above mentioned basis.

In §§4 and 5-6, we will prove that in all cases considered in Theorems 1 and 2, the surface $\mathcal{S}$ contains no semi-algebraic curve. The first conclusion must then hold true. Combined with the conclusion of §3.1, this implies that the orders m of the torsion points lying on W are uniformly bounded, and so, there exists a positive integer $N = N(k, S, G, s)$ such that $W \cap G_{tor} \subset \cup_{\overline{\lambda} \in S(\mathbb{Q}^{alg})} G_{\overline{\lambda}}[N]$. The latter set is the union of the values at all $\overline{\lambda} \in S(\mathbb{Q}^{alg})$ of the torsion sections of G/S of order dividing N, whose images form a finite union of curves in G. As soon as p is not a torsion section, neither is s and $W = s(S)$ intersects this finite union in a finite number of points. Hence, $W \cap G_{tor}$ is finite, and this concludes the proof of Theorems 1 and 2. (NB : as done in [20], this conclusion can be reached in a faster way via the inequalities $m^{1/6} << [k(\overline{\lambda}) : k] << \sharp(S_m^G) << m^\epsilon$, $H(\overline{\lambda}) << 1$, and the Northcott property.)

However, two points must be modified for the above discussion to hold.

• We need a uniform determination of the logarithms of the points $s(\overline{\lambda})$, and this requires fixing from the start a (any) simply connected pointed subset $(\Lambda, \overline{\lambda}_0)$ of the Riemann surface S^{an} attached to $S(\mathbb{C})$; in particular, our surface

$$\mathcal{S} = \mathcal{S}_\Lambda := log^B_{G,\overline{\lambda}_0}(s(\Lambda)),$$

and the graph $\tilde{\mathcal{S}}_\Lambda \subset \Lambda \times \mathbb{R}^4$ of $log^B_{G,\overline{\lambda}_0} \circ s : \Lambda \to \mathbb{R}^4$ will depend on a choice of Λ. Furthermore, the surfaces $\mathcal{S}$ studied by Proposition 2 must be compact, so, Λ too must be compact. Consequently, $exp_G(\tilde{\mathcal{S}}_\Lambda) = s(\Lambda) \subset W(\mathbb{C})$ is truly smaller than $W(\mathbb{C})$, and the desired "first conclusion" is reached in a slightly different way : as in [20] (see Lemma 6.2 of AJM version, Lemma 8.2 of Math. Ann.), one first attaches to the height bound C a finite union Λ_C of simply-connected compact pointed sets $(\Lambda_i, \overline{\lambda}_i)$ such that for any point

$\overline{\lambda} \in S(\mathbb{Q}^{alg})$ of sufficiently large degree with $H(\overline{\lambda}) \leq C$, a positive proportion (say half, i.e. independent of $\overline{\lambda}$) of the conjugates of $\overline{\lambda}$ over k lies in Λ_C [1] . Letting $\tilde{\mathcal{S}}_{\Lambda_C}$ be the finite union of the graphs of the maps $log^B_{G,\overline{\lambda}_i} \circ s$, we deduce that for *any* $\overline{\lambda} \in S^G_m$, a similar positive proportion of the orbit of $s(\overline{\lambda}) = w$ under $Gal(\mathbb{Q}^{alg}/k)$ lies in $exp_G(\tilde{\mathcal{S}}_{\Lambda_C}) = s(\Lambda_C) \subset W$. Proposition 2, combined with the subsequent zero estimate, then implies that this orbit has at most $c'm^\varepsilon$ points, and one can conclude as above (or via the Northcott property on $\overline{\lambda}$). In what follows, we fix one of the Λ_i's, call it Λ and write $log^B_G \circ s$ for $log^B_{G,\overline{\lambda}_i} \circ s$.

• Proposition 2 provides an upper bound for the *image* of $S^G_m \cap \Lambda$ in $\mathcal{S} \cap \mathbb{Q}^4_m \subset \mathbb{R}^4$ under the map $log^B_G \circ s : \Lambda \to \mathbb{R}^4$, while we need an upper bound for $S^G_m \cap \Lambda$ itself. In other words, we must show that not too many points $\overline{\lambda}$ of S^G_m can be sent by $log^B_G \circ s$ onto the same point of $\mathbb{Q}^4_m$. Clearly, it suffices to show that (the Betti presentation of) the projection

$$u(\lambda) := d\pi(log_G(s(\lambda))) = log_E(\pi(s(\lambda))) = log_E(p(\lambda))$$

of $log_G(s)$ under the differential of $\pi : G \to E$ satisfies this separation property. The gap can now be filled by appealing to the "zero estimate" of [20], Lemma 7.1 of AJM, or Lemma 9.1 of Math. Ann., as follows. The elliptic Betti notations introduced here will be repeated and developed in §3.3 and §4.

Let Λ be a compact and simply connected subset of the Riemann surface $S(\mathbb{C})$, which, without loss of generality, we may assume to be homeomorphic to a closed disk in $\mathbb{C}$. We recall that given an analytic sheaf $\mathcal{F}$ on $S(\mathbb{C})$, a section $\sigma \in \mathcal{F}(\Lambda)$ of $\mathcal{F}$ over Λ is by definition analytic on a neighbourhood of Λ in $S(\mathbb{C})$. Let now E/S be an elliptic scheme over S, and let $\omega_1(\lambda), \omega_2(\lambda)$ be the analytic functions on a neighbourhood of Λ in $S(\mathbb{C})$ expressing its periods relative to a given global differential form of the first kind on E/S. Fix a determination log_E of the corresponding elliptic logarithm on $E(\Lambda)$. For any analytic section $\mathbf{p} \in E(\Lambda)$, there then exists unique real analytic functions $\beta_1, \beta_2 : \Lambda \to \mathbb{R}$ such that $log_E(\mathbf{p}(\lambda)) = \beta_1(\lambda)\omega_1(\lambda) + \beta_2(\lambda)\omega_2(\lambda)$. We call $\{\beta_1, \beta_2\}$ the Betti coordinates of $\mathbf{p}$, set $log^B_E(\mathbf{p}(\lambda)) = (\beta_1(\lambda), \beta_2(\lambda)) \in \mathbb{R}^2$ and (extending a well-known notion for the Legendre family) say that $\mathbf{p}$ is a *Picard-Painlevé section* of E/Λ if its Betti coordinates β_1, β_2 are *constant*. We then have :

Proposition 3. (Zero estimate). *Let E/S and the simply-connected compact subset Λ be as above, and let p be a regular section of E/S. Assume that p is not a torsion section, and if E/S is isoconstant, that it is not a constant section. There exists an integer C''*

[1] In the present case, let us first remove from S the finite set consisting of the points of bad reduction and those where the section is not defined (or any finitely many ones that may cause trouble along the way, possibly none...). Now remove "small" open disks around each of these points; what remains is a compact set in S. We want them small enough so that at most half of the conjugates of the relevant $\overline{\lambda}$ fall in their union: this may be achieved because $\overline{\lambda}$ has bounded height. In fact, if "many" conjugates fall into a same small disk, then the corresponding contribution to the height is too big. In turn this follows for instance on looking at the difference $f(\overline{\lambda}) - f(\overline{\lambda}_0)$ where $\overline{\lambda}_0$ is the center of the disk and f is a suitable nonconstant coordinate on S. Using the coordinate reduces the verification to the case of algebraic numbers (rather than algebraic points). Having chosen these small enough disks, we cover the said compact set with finitely many simply connected domains where the logs are locally defined.

depending only on $E/S, \Lambda$ and p such that for any Picard-Painlevé section $\mathbf{p}$ *of* E/Λ*, the set*

$$\{\overline{\lambda} \in \Lambda, log_E(p(\overline{\lambda})) = log_E(\mathbf{p}(\overline{\lambda})\})$$

has at most C'' elements.

In other words, if we set $u(\lambda) = log_E(p(\lambda)) = b_1(\lambda)\omega_1(\lambda) + b_2(\lambda)\omega_2(\lambda)$, then for any real numbers β_1, β_2, the equation $u(\overline{\lambda}) = \beta_1\omega_1(\overline{\lambda}) + \beta_2\omega_2(\overline{\lambda})$ has at most C'' solutions $\overline{\lambda} \in \Lambda$. So, the statement above is just a fancy translation of [20], Amer. JM, p. 12, and follows from Lemma 7.1 there in exactly the same way if E/S is not isoconstant. The constant case is even easier. Notice that we need Proposition 3 only for β_1, β_2 running in $\mathbb{Q}$, i.e. for torsion Picard-Painlevé sections $\mathbf{p}$, and that the Painlevé equation may bring a new view point on the computation of the bound C''.

In our applications, p is not a torsion section of E/S. And in the isoconstant case, we have assumed without loss of generality that p is not constant. So, the map $log_E^B \circ p$ separates the points of Λ up to the bounded error C''; a fortiori, so does its lift $log_G^B \circ s$, and the gap between its image $\mathcal{S}$ and its graph $\tilde{\mathcal{S}}$ is now filled.

3.3 What remains to be done

In view of the previous discussion, the proof of Theorems 1 and 2 is now reduced to defining the real surface $\mathcal{S}$ properly, and to showing that under each of their hypotheses, $\mathcal{S}$ contains no semi-algebraic curve. This is dealt with as follows.

The real surface $\mathcal{S}$

Fix a simply connected and compact subset $\Lambda \in S(\mathbb{C})$, homeomorphic to a closed disk, as well as a point $\overline{\lambda}_0$ in Λ, and a point U_0 in $Lie(G_{\overline{\lambda}_0}(\mathbb{C}))$ such that $exp_{G_{\overline{\lambda}_0}}(U_0) = s(\overline{\lambda}_0) \in G_{\overline{\lambda}_0}(\mathbb{C})$. We henceforth denote by λ the general element[2] of Λ, and (sometimes) by an upper index ^{an} the analytic objects over the Riemann surface S^{an} attached to our schemes over S.

We first repeat the definition of the real surface $\mathcal{S}$ in more precise terms. The group scheme G/S defines an analytic family G^{an} of Lie groups over the Riemann surface S^{an}. Similarly, its relative Lie algebra $(LieG)/S$ defines an analytic vector bundle $LieG^{an}$ over

[2] A remark may be in order about the meaning of the notation λ. In the first paragraphs, it represented the generic point of $S_{\mathbb{C}}$, i.e. we set $\mathbb{C}(S) = \mathbb{C}(\lambda)$ (notice that, from now on, we are over $\mathbb{C}$, so, dropping the lower index $\mathbb{C}$, we will write $K = \mathbb{C}(S)$). But it now represents the "general element" of the simply connected domain $\Lambda \subset S^{an}$, which can have many analytic automorphisms. It is understood that we here consider only a global λ. Such a λ may require several algebraically dependent parameters to be expressed. Of course, when Λ is small enough, we can work with a chart t on Λ such that $\mathbb{C}(\lambda)$ is an algebraic extension of $\mathbb{C}(t)$. The results of functional algebraic independence we appeal to do not require such reduction.

S^{an}, of rank 2. The $\mathbb{Z}$-local system of periods of G^{an}/Λ is the kernel of the exponential exact sequence

$$0 \longrightarrow \Pi_G \longrightarrow LieG^{an} \overset{exp_G}{\longrightarrow} G^{an} \longrightarrow 0 \ ,$$

over S^{an}. Its sections over Λ form a $\mathbb{Z}$-module $\Pi_G(\Lambda) \subset LieG^{an}(\Lambda)$ of rank 3. Indeed, on using similar notations for the group schemes E/S and $\mathbb{G}_m \times S$, the canonical projection $\pi : G \to E$ over S induces at the Lie algebra level an exact sequence

$$0 \longrightarrow Lie\mathbb{G}_m^{an} \longrightarrow LieG^{an} \overset{d\pi}{\longrightarrow} LieE^{an} \longrightarrow 0 \ .$$

From the compatibility of the exponential morphisms, we deduce an exact sequence of $\mathbb{Z}$-local systems of periods

$$0 \longrightarrow \Pi_{\mathbb{G}_m} \longrightarrow \Pi_G \overset{d\pi}{\longrightarrow} \Pi_E \longrightarrow 0 \ ,$$

with $\Pi_{\mathbb{G}_m}(\Lambda)$ and $\Pi_E(\Lambda)$ of respective ranks 1 and 2 over $\mathbb{Z}$.

There exists a unique analytic section $U := \log_{G,\overline{\lambda}_0}$ of $Lie(G^{an})/\Lambda$ such that

$$U(\overline{\lambda}_0) = U_0 \ \text{ and } \ \forall \lambda \in \Lambda, exp_{G^{an}_{\overline{\lambda}}}(U(\lambda)) = s(\lambda).$$

Since Λ is fixed and $\overline{\lambda}_0$ plays no role in what follows, we will forget about them and will just write $U = log_{G,\Lambda} = log_G$, i.e.

$$\forall \lambda \in \Lambda, U(\lambda) = log_G(s(\lambda)).$$

We call $U = log_G(s)$ "the" logarithm of the section s. Its projection $p = \pi(s) \in E(S)$ admits as logarithm

$$log_E(p) := u = d\pi(U) = d\pi(log_G(s)) = log_E(\pi(s)).$$

We describe these logarithms in terms of classical Weierstrass functions in §4 for the (iso)trivial case $G = \mathbb{G}_m \times E$, and in Appendix I for the general case. These explicit expressions are not needed, but will provide the interested reader with a translation of the algebraic independence results in more classical terms.

Now, we rewrite U in terms of a conveniently chosen basis of the $\mathbb{Z}$-local system of periods Π_G/Λ of G^{an}/Λ. We call $U^B(\lambda) = log_G^B(s(\lambda))$ the resulting expression. For this, we choose a generator $\varpi_0 = 2\pi i$ of $\Pi_{\mathbb{G}_m}$, and a $\mathbb{Z}$-basis $\{\omega_1, \omega_2\}$ of $\Pi_E(\Lambda)$. At each point $\overline{\lambda} \in \Lambda$, the latter generate over $\mathbb{R}$ the $\mathbb{C}$-vector space $Lie(E_{\overline{\lambda}})$. Consequently (and as already said before Proposition 3), there exist uniquely defined real analytic functions $b_1, b_2 : \Lambda \to \mathbb{R}^2$ such that

$$\forall \lambda \in \Lambda, \ u(\lambda) = b_1(\lambda)\omega_1(\lambda) + b_2(\lambda)\omega_2(\lambda). \qquad (\mathfrak{R}_u)$$

We call $u^B = (b_1, b_2) : \Lambda \to \mathbb{R}^2$ the Betti presentation of the logarithm $u = log_E(p)$.

Now, choose at will lifts $\{\varpi_1, \varpi_2\}$ of $\{\omega_1, \omega_2\}$ in $\Pi_G(\Lambda)$. Then, $U - b_1\varpi_1 - b_2\varpi_2$ lies in the kernel $Lie\mathbb{G}_m(\Lambda)$ of $d\pi$, which is generated over $\mathbb{C}$ by ϖ_0. Therefore, there exist a unique real analytic function $a : \Lambda \to \mathbb{C} = \mathbb{R}^2$ such that

$$U = a\varpi_0 + b_1\varpi_1 + b_2\varpi_2.$$

In conclusion, there exist uniquely defined real-analytic functions $a : \Lambda \to \mathbb{C}, b_1 : \Lambda \to \mathbb{R}, b_2 : \Lambda \to \mathbb{R}$ such that $U = log_G(s)$ satisfies the relation :

$$\forall \lambda \in \Lambda, U(\lambda) = a(\lambda)\varpi_0(\lambda) + b_1(\lambda)\varpi_1(\lambda) + b_2(\lambda)\varpi_2(\lambda). \qquad (\mathfrak{R}_U)$$

We call the *real analytic* map

$$U^B = (a, b_1, b_2) : \Lambda \to \mathbb{C} \times \mathbb{R}^2 = \mathbb{R}^4,$$

the Betti presentation of the logarithm U of $log_G(s)$. Its image $\mathcal{S} = \mathcal{S}_\Lambda := U^B(\Lambda) = log_G^B(s(\Lambda)) \subset \mathbb{R}^4$ is the real surface to be studied. Since Π_G is the kernel of the exponential morphism, it is clear that for any $\overline{\lambda} \in S_m^G$, $U^B(\overline{\lambda})$ lies in $\mathbb{Q}_m \times \mathbb{Q}_m^2 \subset \mathbb{Q}_m^4 \subset \mathbb{R}^4$.

Reducing to algebraic independence.

To complete their proofs, we must show that under the hypotheses of Theorems 1 and 2, the surface $\mathcal{S}$ contains no semi-algebraic curve of the ambient affine space $\mathbb{R}^4$. This will be done in two steps, as follows. But before we describe them, we point out that since $log_G(s), log_E(p), log_E(q), \omega_i, \varpi_i, ...$ are local sections of the globally defined vector bundles $(LieG)/S, (LieE)/S$, it makes sense to speak of the minimal extension $K(log_G(s)), ...$ of $K = \mathbb{C}(\lambda)$ they generate in the field of meromorphic functions over a neighbourhood Λ' of Λ. A similar remark applies to the field $K(a), ...$ generated by the real analytic functions $a, ...$ in the fraction field of the ring of real analytic functions over Λ'.

• *Step* ($\boldsymbol{\alpha}$). - Let $F_{pq}^{(1)} = K(\omega_1, \omega_2, \log_E(p), log_E(q))$ be the field generated over K by $\omega_1, \omega_2, log_E(p) = u$ and a logarithm $v = log_E(q)$ of the section $q \in \hat{E}(S) \simeq E(S)$ parametrizing the extension G. We recall that p is not a torsion section, and assume as usual that if E/S is isoconstant, then p is not constant. We then claim that *if $\mathcal{S}$ contains a semi-algebraic curve, then $log_G(s)$ is algebraic over $F_{pq}^{(1)}$*. In fact, we will only need a corollary of this result, involving the universal vectorial extensions of E and of G, where the base field $F_{pq}^{(1)}$ is replaced by a **differential field** $F_{pq} := F_{pq}^{(2)}$ containing $F_{pq}^{(1)}$, which, inspired by the theory of one-motives, we can call the *field of generalized periods of* $\{E, p, q\}$. (In more classical terms, the upper indexes (1) and (2) here stand for elliptic integrals of the first and second kinds.) In these conditions, and under *no* assumption on q, we will prove :

Proposition 4. *Assume that p is not torsion and not constant, and that $\mathcal{S}$ contains a semi-algebraic curve. Then, $log_G(s)$ is algebraic over the field F_{pq} of generalized periods of $\{E, p, q\}$.*

• *Step* ($\boldsymbol{\beta}$). - The desired contradiction is then provided by the following Main Lemma, whose proof is the object of §6 (and §4 for $q = 0$). This is a statement of "Ax-Lindemann" type, but with logarithms replacing exponentials, in the style of André's theorem [1] (see also [7]) for abelian schemes. For results on semi-abelian surfaces close to this Main Lemma, see [6], Propositions 4.a, 4.b and Theorem 2. For a broader perspective on algebraic independence of relative periods, and on the role of the constant part and image of G, see Ayoub [4], proof of Theorem 2.57, and his recent work on the Kontsevich-Zagier conjecture.

Main Lemma. *With $S/\mathbb{C}$, let G/S be an extension by $\mathbb{G}_m$ of an elliptic scheme E/S, parametrized by a section q of $\hat{E}/S$, and let G_0 be the constant part of G. Let further s be a section of G/S, with projection $p = \pi \circ s$ to E/S, and let F_{pq} be the field of generalized periods of $\{E, p, q\}$.*

(A) *Assume that $log_G(s)$ is algebraic over F_{pq}. Then, there exists a constant section $s_0 \in G_0(\mathbb{C})$ such that*

i) either $s - s_0$ is a Ribet section;

ii) or $s - s_0$ factors through a strict subgroup scheme of G/S.

(B) *More precisely, $log_G(s)$ is algebraic over F_{pq} if and only if there exists a constant section $s_0 \in G_0(\mathbb{C})$ such that $s - s_0$ is a Ribet section, or a torsion section, or factors through a strict subgroup scheme of G/S projecting onto E/S.*

The analogy with the Main Theorem is clear, except perhaps for the last conclusion of Part (B) of the Main Lemma (which forces an isotrivial $G \simeq \mathbb{G}_m \times E$). This is due to the fact that even in this isotrivial case, the roles of $\mathbb{G}_m$ and E are here not symmetric, because of the occurence of p in the base fields $F_{pq}^{(1)}, F_{pq}$. On the contrary ("torsion values for a single point" on a group scheme of relative dimension 1 over a curve), they played similar roles for relative Manin-Mumford.

As was pointed out in §1.3, these steps ($\boldsymbol{\alpha}$), ($\boldsymbol{\beta}$) imply the desired conclusion only up to translation by a constant section. We now show how to replace constant by torsion sections, thereby concluding the proof of the Main Theorem. This reduction is achieved through the case-by-case description of the constant part G_0 of G given in Remark 1, as follows.

Proposition 4 + Main Lemma (A) $\Rightarrow$ Theorems 1 and 2.

Let us first deal with Theorem 1, where $G = \mathbb{G}_m \times E$, with constant part $G_0 = \mathbb{G}_m$, resp. $G_0 \times S = G$, if E is not, resp. is, isoconstant. Assume for a contradiction that Hypothesis (o) holds, but that S_∞^G is infinite. Then, the real surface $\mathcal{S}$ must contain an algebraic curve, and since G admits no Ribet section, Proposition 4, combined with Part (A) of the Main Lemma, implies that a multiple by a non-zero integer of the section s factors through a translate of $H = \mathbb{G}_m$ or of $H = E$ by a constant (non-necessarily torsion) section $s_0 \in G_0(\mathbb{C})$. But the projection p of s to E is by assumption not torsion, and we know that it cannot be constant. So, H must be equal to E, and s projects on the $\mathbb{G}_m$-factor of G to a constant point δ_0. Since $s(S) = W$ contains torsion points, δ_0 must be a root of unity, and s factors through a torsion translate of E, contradicting (o).

In the direction of Theorem 2, we now assume that G is a non-isotrivial extension, so $H = \mathbb{G}_{m/S}$ is the only connected strict subgroup scheme of G/S, and that one of its hypotheses (i), (ii), (iii) holds, but S_∞^G is infinite. The proof above easily adapts to the case when $G \simeq G_0 \times S$ is isoconstant, where again, G admits no Ribet section (in the sense of §1). Now, assume that G is not isoconstant, so $G_0 = \mathbb{G}_m$. If G is not semi-constant, there still are no Ribet sections and Proposition 4, combined with Part (A) of the Main Lemma, implies that a multiple of s factors through a translate of $H = \mathbb{G}_m$ by a constant section $s_0 \in \mathbb{G}_m(\mathbb{C})$, so, since S_∞^G is not empty, s factors through a torsion translate of $\mathbb{G}_m$, and $p = \pi(s)$ is torsion, contradicting the general hypothesis of Theorem 2. So, G must be semi-constant, $E = E_0 \times S$ must be isoconstant (concluding Case (i)), and the argument just described shows that s must satisfy Conclusion (i) of the Main Lemma, for some $s_0 \in \mathbb{G}_m(\mathbb{C})$. The mere existence of a Ribet section $s_R := s - s_0$ of G/S implies that $p = \pi(s) = \pi(s_R)$ and q are antisymmetrically related (concluding Case (ii)). Moreover, by Theorem 3.i, $s_R(\overline{\lambda})$ is a torsion point on $G_{\overline{\lambda}}$ whenever $s(\overline{\lambda})$ is so, since $\pi(s_R(\overline{\lambda})) = \pi(s(\overline{\lambda})) = p(\overline{\lambda})$ is then a torsion point of E_0. There are infinitely many such $\overline{\lambda}$'s, so at least one. Consequently the constant section $s_0 \in \mathbb{G}_m(\mathbb{C})$ is torsion, and (a multiple of) s is a Ribet section of G/S. This concludes Case (iii), and Theorem 2 is established.

Part (B) of the Main Lemma.

Just as for the Main Theorem (cf. Theorem 3), let us right now deal with the "if" side of Part B of the Main Lemma.

The periods Π_G of G are defined over the subfield F_q of F_{pq} (see §5.1, or the explicit formula given in Appendix I, §7.1, or the footnote [3] below), so, clearly, $log_G(s)$ lies in F_{pq} if $s - s_0$ is a torsion section. When $s - s_0 := s_R$ is a Ribet section, an explicit formula for $log_G(s_R)$ in terms of u and $\zeta(u)$ is given in Appendix I, Remark 4, from which the rationality of $log_G(s_R)$ over F_{pq} immediately follows. In fact, we will prove this in a style closer to Manin-Mumford issues in Lemma 3 of §6. The last case considered in Part B forces G to be an isotrivial extension. In the notations of §4, we then have $s - s_0 = (\delta, p) \in G(S)$, with δ a root of unity, so $log_G(s)$ is rational over the field F_p.

As for the "only if" side of Part (B) not covered by Part (A), we must show that if (a multiple by a positive integer of) $s - s_0$ is a non constant section δ of $\mathbb{G}_m(S)$, then $\ell := log_{\mathbb{G}_m}(\delta)$ is transcendental over F_{pq}. But then, $p - \pi(s_0)$ is a torsion section of E/S, so $F_{pq} = F_q$, and the statement follows from Lemma 1 of §4, with q playing the role of p.

In conclusion, we have reduced the proof of the Main Theorem (more specifically, of

[3] More intrinsically, concerning the field of definition of Π_G : the Cartier dual of the one-motive $[0 \to G]$ is the one-motive $[\mathbb{Z} \to \hat{E}]$ attached to $q \in \hat{E}(S)$, so their fields of (generalized) periods coincide, and $K(\Pi_G) = F_G^{(1)} \subset F_G^{(2)} = F_q$ (in the notations of the later §5.1). Concerning $log_G(s_R)$: in the notations of §1.2, it suffices to consider the generic Ribet section s_R of the semi-abelian scheme $\mathcal{P}_0$, viewed as an extension $\mathcal{G}_0$ of E_0 by $\mathbb{G}_m$, over the base $\hat{E}_0$. As mentioned there, its image W_R is a special curve of the mixed Shimura variety $\mathcal{P}_0$. Therefore, the inverse image of W_R in the uniformizing space of $\mathcal{P}_0$ is an algebraic curve. In the notations of §6, the statement amounts to the vanishing of τ_{s_R}, and could alternatively be deduced from the self-duality of the one-motive $[M_{s_R} : \mathbb{Z} \to G]$ attached to the Ribet section, cf. [10].

Theorems 1 and 2) to defining the field F_{pq}, proving Proposition 4, and proving Part (A) of the Main Lemma.

4 A warm up : the case of direct products

In this Section, we perform the above-mentioned tasks under the assumption that G is an isotrivial extension, thereby establishing Theorem 1, as stated in §2. Without loss of generality, we assume that $G = \mathbb{G}_m \times E$, i.e. $q = 0$ (so, the field $F_{pq} = F_{p0}$ will coincide with F_p). Of course, if E/S is isoconstant, say $E = E_0 \times S$, then $G = G_0 \times S$ with $G_0 = \mathbb{G}_m \times E_0/\mathbb{Q}^{alg}$, and Theorem 1 follows from Hindry's theorem [17]; in this isoconstant case, the strategy we are here following reduces to that of [26].

As announced in §3.2, we first rewrite in concrete terms the logarithms U, u and their Betti presentations, under no assumption on the elliptic scheme E/S nor on its section p. We fix global differential forms[4] of the first and second kind ω, η for E/S, and for any $\lambda \in \Lambda$, we let

$$\wp_\lambda \ , \ \zeta_\lambda \ , \ \sigma_\lambda$$

be the usual Weierstrass functions attached to the elliptic curve $E_\lambda/\mathbb{C}$ and its differential forms $\omega_\lambda, \eta_\lambda$. We also fix an elliptic logarithm of the point $p(\overline{\lambda}_0)$, and extend it to an analytic function $u(\lambda) = log_E(p(\lambda)) = Arg\wp_\lambda(p(\lambda))$ on Λ. Similarly, we fix a basis of periods and quasi-periods for $E_{\overline{\lambda}_0}$, and extend them to analytic functions $\omega_1(\lambda), \omega_2(\lambda), \eta_1(\lambda), \eta_2(\lambda)$ (of hypergeometric type if E/S is the Legendre curve). There then exist uniquely defined real-analytic functions b_1, b_2 with values in $\mathbb{R}$ such that

$$\forall \lambda \in \Lambda, u(\lambda) = b_1(\lambda)\omega_1(\lambda) + b_2(\lambda)\omega_2(\lambda), \qquad (\mathfrak{R}_u)$$

and the Betti presentation of $log_E(p(\lambda)$ is given by

$$u^B(\lambda) := log_E^B(p(\lambda)) = (b_1(\lambda), b_2(\lambda)) \in \mathbb{R}^2.$$

We now go to $G = \mathbb{G}_m \times E$ over Λ. The section $s : \Lambda \to G$ has two components (δ, p), where $\delta : S \to \mathbb{G}_{m/S}$ is expressed by a rational function on S. We fix a classical logarithm of $\delta(\overline{\lambda}_0)$ and extend it to an analytic function $\ell(\lambda) := log_{\mathbb{G}_m}(\delta(\lambda))$ on Λ. With these notations, the section $log_G \circ s$ of $(LieG^{an})/\Lambda$ is represented by the analytic map

$$\Lambda \ni \lambda \mapsto log_G(s(\lambda)) := U(\lambda) = \begin{pmatrix} \ell(\lambda) \\ u(\lambda) \end{pmatrix} \in \mathbb{C}^2 = (LieG)_\lambda.$$

The $\mathbb{Z}$-local system of periods Π_G admits the basis

$$\varpi_0(\lambda) = \begin{pmatrix} 2\pi i \\ 0 \end{pmatrix}, \varpi_1(\lambda) = \begin{pmatrix} 0 \\ \omega_1(\lambda) \end{pmatrix}, \varpi_2(\lambda) = \begin{pmatrix} 0 \\ \omega_2(\lambda) \end{pmatrix},$$

[4] When the modular invariant $j(\lambda)$ is constant, i.e. when E/S is isoconstant, we tacitly assume that $E = E_0 \times S$, with $E_0/\mathbb{C}$, and that the chosen differential of first and second kind ω, η are *constant* (i.e. come from $E_0/\mathbb{C}$). In particular, the periods ω_1, ω_2 and quasi-periods η_1, η_2 are constant. The Weierstrass functions are those of E_0, and we drop the index λ from their notation.

and the Betti presentation of $log_G(s(\lambda))$ is given by

$$\Lambda \ni \lambda \mapsto U^B(\lambda) := log_G^B(s(\lambda)) = (a(\lambda), b_1(\lambda), b_2(\lambda)) \in \mathbb{C} \times \mathbb{R}^2 = \mathbb{R}^4,$$

where a, b_1, b_2 are the unique real analytic functions on Λ satisfying

$$\forall \lambda \in \Lambda, \begin{pmatrix} \ell(\lambda) \\ u(\lambda) \end{pmatrix} = a(\lambda) \begin{pmatrix} 2\pi i \\ 0 \end{pmatrix} + b_1(\lambda) \begin{pmatrix} 0 \\ \omega_1(\lambda) \end{pmatrix} + b_2(\lambda) \begin{pmatrix} 0 \\ \omega_2(\lambda) \end{pmatrix}. \quad (\mathfrak{R}_{\ell,u})$$

We then set $\mathcal{S} = U^B(\Lambda) \subset \mathbb{R}^4$ as usual.

Since $q = 0$, the tower of function field extensions of $K = \mathbb{C}(\lambda)$ to be considered take here the following simple forms :

$$F^{(1)} = K(\omega_1, \omega_2) \quad , \quad F^{(1)}_{p0} := F^{(1)}_p = F^{(1)}(u) = K(\omega_1, \omega_2, u),$$

while their differential extensions

$$F^{(2)} = F^{(1)}(\eta_1, \eta_2) \quad , \quad F^{(2)}_{p0} := F^{(2)}_p = F^{(1)}_p(\zeta_\lambda(u)) = K(\omega_1, \omega_2, \eta_1, \eta_2, u, \zeta_\lambda(u))$$

involve the Weierstrass ζ function, and can be rewritten as

$$F := F^{(2)} = K(\omega_1, \omega_2, \eta_1, \eta_2) \quad , \quad F_p := F^{(2)}_p = F(u, \zeta_\lambda(u)).$$

We point out that since it contains the field of definition F of the periods of ω and η, the field F_p depends only on the section p, not on the choice of its logarithm u, so, the notation is justified. Furthermore, let $\alpha \in \mathcal{O} = End(E)$ be a non-zero endomorphism of E, and let $p_0 \in \mathcal{E}_0(\mathbb{C})$ be a constant section of E. Then, the section $p' = \alpha p + p_0$ yields the same field $F_{p'} = F_p$ as p. In particular, $F_p = F$ if p is a torsion or a constant section of E/S.

Step ($\boldsymbol{\alpha}$). - Proof of Proposition 4 when $q = 0$.

Suppose for a contradiction that $\mathcal{S}$ contains a real semi-algebraic curve $\mathcal{C}$, and denote by $\Gamma \subset \Lambda$ the inverse image of $\mathcal{C}$ in Λ under the map U^B (all we will need is that Γ has an accumulation point inside Λ , but it is in fact a real curve). We are going to study the restrictions to Γ of the functions

$$a, b_1, b_2, u, \ell, \omega_1, \omega_2.$$

Recall that all these are functions of $\lambda \in \Lambda$. In view of the defining relation $(\mathfrak{R}_{\ell,u})$, the transcendence degree of the functions u, ℓ over the field $\mathbb{C}(\omega_1, \omega_2, a, b_1, b_2)$ is at most 0. When restricted to Γ, the latter field has transcendence degree ≤ 1 over $\mathbb{C}(\omega_1, \omega_2)$, since $U^B(\Gamma) = (a, b_1, b_2)(\Gamma)$ is the algebraic curve $\mathcal{C}$. So, the restrictions to Γ of the two functions u, ℓ generate over $\mathbb{C}(\omega_{1|\Gamma}, \omega_{2|\Gamma})$ a field of transcendence degree $\leq 1+0 = 1$, and are therefore algebraically dependent over $\mathbb{C}(\omega_{1|\Gamma}, \omega_{2|\Gamma})$. Since Γ is a real curve of the complex domain

Λ, the complex analytic functions u, ℓ are still algebraically dependent over the field of Λ-meromorphic functions $\mathbb{C}(\omega_1, \omega_2)$, i.e.

$$tr.deg._{\mathbb{C}(\omega_1(\lambda),\omega_2(\lambda))}\, \mathbb{C}\big(\omega_1(\lambda), \omega_2(\lambda), u(\lambda), \ell(\lambda)\big) \leq 1.$$

Now, assume as in Proposition 4 that p has infinite order, and if E/S is isotrivial, that p is not constant. Then, André's Theorem 3 in [1] (see also [5], Thm. 5) implies that $u(\lambda)$ is transcendental over the field $F^{(1)} = K\big(\omega_1(\lambda), \omega_2(\lambda)\big)$. The previous inequality therefore says that the function $\ell(\lambda)$ is algebraic over the field $F_p^{(1)} = K\big(\omega_1(\lambda), \omega_2(\lambda), u(\lambda)\big)$, or equivalently, that the field of definition of $log_G(s(\lambda)) = (\ell(\lambda), u(\lambda))$ is algebraic over $F_{p0}^{(1)}$, hence also over $F_{p0} = F_{p0}^{(2)}$, and Proposition 4 is proved when $q = 0$.

Step (β). - Proof of Main Lemma (A) when $q = 0$

Before giving this proof, let us point out that the advantage of the fields $F = F^{(2)}, F_{pq} = F_{pq}^{(2)}$ over their first kind analogues is that they are closed under the derivation $' = \partial/\partial\lambda$. Moreover, by Picard-Fuchs theory, they are Picard-Vessiot (i.e. differential Galois) extensions of K. Since $\ell(\lambda)$ satisfies a K-rational DE of order 1, $K(\ell)$ and $F_p(\ell) = F_p(log_G(s))$ too are Picard-Vessiot extension of K.

Lemma 1. *Let Λ be a ball in $\mathbb{C}$, let $\{\wp_\lambda, \lambda \in \Lambda\}$ be a family of Weierstrass functions, with invariants g_2, g_3 algebraic over $\mathbb{C}(\lambda)$ and periods ω_1, ω_2, and let u be an analytic function on Λ such that u, ω_1, ω_2 are linearly independent over $\mathbb{Q}$. If $j(\lambda)$ is constant, we assume that g_2, g_3 too are constant, and that u is not constant. Let further ℓ be a* non-constant *analytic function on Λ. Assume that $\wp_\lambda(u(\lambda))$ and $e^{\ell(\lambda)} := \delta(\lambda)$ are algebraic functions of λ, and consider the tower of differential fields $K \subset F \subset F_p$, where $K = \mathbb{C}(\lambda), F = K(\omega_1, \omega_2, \eta_1, \eta_2), F_p = F(u, \zeta_\lambda(u))$. Then,*

$$tr.deg._F F_p\big(\ell(\lambda)\big) = 3.$$

In particular, $\ell(\lambda)$ is transcendental over F_p, i.e. Part (A) of the Main Lemma holds true when $q = 0$.

This last statement is indeed equivalent to Part (A) of the Main Lemma when $G \simeq \mathbb{G}_m \times E$ is a trivial extension (or more generally, an isotrivial one), with constant part $G_0 = \mathbb{G}_m$, resp. $G_0 \times S = G$, if E is not, resp. is, isoconstant. Indeed, with $s = exp_G(\ell, u) = (\delta, p)$ as above, we then have $F_p = F_{pq}$, and $F_p(\ell) = F_p(log_G(s))$. Lemma 1 then says that if $log_G(s)$ is algebraic over F_{pq}, then either p is a torsion point (so, a multiple of s factors through $\mathbb{G}_m$), or E is isoconstant and $p = p_0$ is constant (so, the constant section $s_0 = (1, p_0) \in G_0(\mathbb{C})$ satisfies $s - s_0 \in \mathbb{G}_m(S)$), or $\delta = \delta_0$ is constant (so, the constant section $s_0 = (\delta_0, 0) \in G_0(\mathbb{C})$ satisfies $s - s_0 \in E(S)$). In all cases, we therefore derive Conclusion (ii) of the Main Lemma.

Proof (of Lemma 1): this is essentially due to André, cf. [1], but not fully stated there (nor in [5]). It is proven in full generality in [7], but one must look at the formula on top of p.

2786 to see that K can be replaced by F in Theorem L . So, it is worth giving a direct proof.

We first treat the case when E/S is not isoconsant. By Picard-Lefchetz, the Picard-Vessiot extension $F = K(\omega_1, \omega_2, \eta_1, \eta_2)$ of $K = \mathbb{C}(\lambda)$ has Galois group SL_2. By [5], the Galois group of $F_p = F(u(\lambda), \zeta_\lambda(u(\lambda))$ over F is a vector group $\mathcal{V} \simeq \mathbb{C}^2$ of dimension 2 (i.e. these two functions are algebraically independent over F), while the Galois group of $K(\ell(\lambda))$ over $K = \mathbb{C}(\lambda)$ is $\mathbb{C}$. Since $\mathbb{C}$ is not a quotient of SL_2, the Galois group of $F(\ell(\lambda))$ over F is again $\mathbb{C}$. Now, SL_2 acts on the former $\mathcal{V} = \mathbb{C}^2$ by its standard representation, and on the latter $\mathbb{C}$ via the trivial representation, so the Galois group of $F(u(\lambda), \zeta_\lambda(u(\lambda), \ell(\lambda))$ over F is a subrepresentation $\mathcal{W}$ of SL_2 in $\mathbb{C}^2 \oplus \mathbb{C}$ projecting onto both factors. Since the standard and the trivial representations are irreducible and non isomorphic, we must have $\mathcal{W} = \mathbb{C}^2 \oplus \mathbb{C}$. Therefore,

$$tr.deg._F \; F(u(\lambda), \zeta_\lambda(u(\lambda)), \ell(\lambda)) = dim\mathcal{W} = 3.$$

We now turn to the case of an isoconstant $E = E_0 \times S$. The field of periods F then reduces to K, and SL_2 disappears. But since the ambient group $G = \mathbb{G}_m \times E$ is now isoconstant, we can appeal to Ax's theorem on the functional version of the Schanuel conjecture. More precisely, since the result we stated involves the ζ-function, we appeal to its complement on vectorial extensions, see [12], Thm. 2.(iii), or more generally, [6], Proposition 1.b, which implies that $tr.deg_K K(u, \zeta(u), \ell) = 3$ as soon as u and ℓ are not constant. (Actually, André's method still applies to the isoconstant case, but requires a deeper argument, involving Mumford-Tate groups : cf. [1], Theorem 1, and [7], §8.2).

Remark 2. - (i) Concerning the proof of Theorem 1 : as pointed out by the fourth author in [33], p. 79, Comment (v), since we are here dealing with a direct product, the torsion points yield torsion points on $\mathbb{G}_m$, which lie on the unit circle, a real-curve. So, in the argument of §3.2, the dimension decreases by 1 a priori, and Bombieri-Pila (real-curves) rather than Pila (real-surfaces) as in Proposition 2 would suffice.

(ii) As shown by the proof of Proposition 4 ($q = 0$) above, it would have sufficed to prove that the (non constant) logarithm ℓ is transcendental over the field $\mathbb{C}(\omega_1, \omega_2, u)$. Adjoining λ leads to $F_p^{(1)}$ as base field, and as already said, differential algebra then forces to consider $F_p^{(2)}$. For a broader perspective on these extensions of the base field, see §5.3 below.

(iii) For the last statement of Lemma 1 to hold, the only necessary hypothesis is that ℓ be non-constant. Indeed, if p is torsion or constant, then $F_p = F$, and u plays no role. But we have preferred to present Lemma 1 and its proof in this way, as an introduction to the general proofs of §5 and §6.

(iv) Conversely, let q be any (non-necessarily torsion or constant) section of E/S, and set $v = log_E(q)$, $F_q = F^{(2)}(v, \zeta_\lambda(v))$ and $F_{pq} = F_p.F_q$, as will be done in §5. The same proof as above shows that

$$\forall p, q \in E(S), \forall \delta \in \mathbb{G}_m(S), \delta \notin \mathbb{G}_m(\mathbb{C})), \;\; \ell := log_{\mathbb{G}_m}(\delta) \; \textit{is transcendental over} \; F_{pq}.$$

Indeed, the only new case is when p and q are linealy independent over $End(E)$ modulo the constant part of E/S. From the same references and argument as above, replacing $\mathcal{V} = \mathbb{C}^2$ by $\mathcal{V} \oplus \mathcal{V}$, we deduce that the transcendence degree of $F_{pq}(\ell)$ over F is equal to 5 , yielding the desired conclusion on the transcendency of ℓ.

A characterization of Ribet sections

We close this section on isotrivial extensions by a corollary to Theorem 1, which plays a useful role in checking the compatibility of the various definitions of Ribet sections : see for instance the equalities $\beta_R = \beta_J$ in [8].

Corollary 3. *Let G/S be an extension of E/S by $\mathbb{G}_m$, and let p be a section of E/S of infinite order and not constant if E/S is isoconstant (equivalently, by Proposition 1.iv, such that the set S_∞^E attached to p is infinite). Let further $s^\dagger$ and s be two sections of G/S such that $\pi \circ s^\dagger = \pi \circ s = p$. Assume that for all but finitely many (resp. for infinitely many) values of $\overline{\lambda}$ in S_∞^E, the point $s^\dagger(\overline{\lambda})$ (resp. $s(\overline{\lambda})$) lies in G_{tor}, i.e. that* $s^\dagger$ (resp. s) "lifts almost all (resp. infinitely many) torsion values of p to torsion points of G". *Then, there exists a torsion section $\delta_0 \in \mathbb{G}_m(\mathbb{C})$ (i.e. a root of unity) such that $s = s^\dagger + \delta_0$.*

Proof : Let $\delta_0 := s - s^\dagger \in \mathbb{G}_m(S)$. We know that $s^\dagger$ lifts almost *all* torsion points $p(\overline{\lambda}) \in E_{tor}$ to points in G_{tor}. If s does so for infinitely many of them, then, so does the section $s_1 := (\delta_0, p)$ of the direct product $G_1 = \mathbb{G}_m \times E/S$. This contradicts Theorem 1, unless the projection $\delta_0(S)$ of $s_1(S)$ to $\mathbb{G}_m$ is reduced to a root of unity.

It is interesting to note that in this way, Theorem 1 on the trivial extension G_1 has an impact on extensions G which need not be isotrivial. For instance, if G is semi-constant and E_0 has CM, Corollary 3, applied to the Ribet section $s^\dagger = s_R$, shows that up to isogenies, s_R is the only section which lifts infinitely many torsion values of $\pi(s_R)$ to torsion points of G. We also point out that since the elliptic scheme $E \simeq E_0 \times S$ is here constant, Hindry's theorem on the constant semi-abelian variety $\mathbb{G}_m \times E_0$ suffices to derive this conclusion.

On the other hand, assume that G is an isotrivial extension. Then, there exists a subgroup scheme $E^\dagger/S$ of G/S such that the restriction $\pi^\dagger$ of $\pi : G \to E$ to $E^\dagger$ is an S-isogeny. Any section $s^\dagger$ of G/S a non zero multiple of which factors through $E^\dagger$ then satisfies the lifting property of the corollary, since $p(\overline{\lambda}) := \pi^\dagger \circ s^\dagger(\overline{\lambda})$ is a torsion point of $E_{\overline{\lambda}}$ if and only if $s^\dagger(\overline{\lambda})$ is a torsion point of $G_{\overline{\lambda}}$. By Corollary 3, such sections $s^\dagger$ are, up to a root of unity, the only section s above $p = \pi \circ s^\dagger$ for which $s(S) \cap G_{tor}$ is infinite. Of course, this is (after an isogeny) just a rephrasing of Theorem 1, but it shows the analogy between these "obvious" sections and the Ribet sections. This is a reflection of the list of special curves of the mixed Shimura variety described in §1.2.

5 The general case

5.1 Fields of periods and the Main Lemma

Apart from the statement of Lemma 2 below, we henceforth make no assumption on the extension G of E/S by $\mathbb{G}_m$. So the section $q \in \hat{E}(S)$ which parametrizes G is arbitrary. Concerning the elliptic scheme E/S, we recall the notations of §4, and in particular, the fields of periods $F^{(1)} = K(\omega_1, \omega_2)$ of E and its differential extension $F := F^{(2)} = F^{(1)}(\eta_1, \eta_2)$. We identify $\hat{E}$ and E in the usual fashion, and denote by $v = log_E(q)$ a logarithm of the section q over Λ. We recall that the field $F_q := F_q^{(2)} = F^{(2)}(v, \zeta_\lambda(v))$ depends only on q, and coincides with $F^{(2)} = F$ when q is a torsion section, i.e. when G is isotrivial. We will also use the notations of §3.3 on the local system of periods $\Pi_G = \mathbb{Z}\varpi_0 \oplus \mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2 \subset LieG^{an}(\Lambda)$ of G^{an}/Λ.

Consider the extension $F_G^{(1)} = K(\varpi_0, \varpi_1, \varpi_2) = K(\varpi_1, \varpi_2)$ of $K = \mathbb{C}(\lambda)$ generated by the elements of Π_G. Since Π_G projects onto Π_E under $d\pi$, whose kernel has relative dimension 1, this field is an extension of $F^{(1)} = K(\omega_1, \omega_2)$ of transcendence degree ≤ 2. So, the field $F_G^{(2)}$ generated by Π_G over $F := F^{(2)}$ has transcendence degree ≤ 2. In fact, the duality argument mentioned in Footnote (3), or more explicitly, the computation given in Appendix I, shows that

$$F_G^{(2)} = F^{(2)}(v, \zeta_\lambda(v)) := F_q,$$

i.e $F_G^{(2)}$ coincides with the differential extension F_q of $F^{(2)} = F$ attached to q. So, $F_G^{(2)}$ is in fact a Picard-Vessiot extension of K.

A more intrinsic way to describe these "fields of the second kind" is to introduce the **universal vectorial extension** $\tilde{E}/S$ of E/S, cf. [11]. This is an S-extension of E/S by the additive group $\mathbb{G}_a$, whose local system of periods $\Pi_{\tilde{E}}$ generates the field $F^{(2)}$. The universal vectorial extension $\tilde{G}/S$ of G/S is the fiber product $G \times_E \tilde{E}$, and its local system of periods $\Pi_{\tilde{G}}$ generates the field $F_G^{(2)}$. Now, for both $\tilde{E}$ and $\tilde{G}$ (and contrary to E and G), these local systems generate the spaces of horizontal vectors of connections $\partial_{Lie\tilde{E}}, \partial_{Lie\tilde{G}}$ on $Lie\tilde{E}/S, Lie\tilde{G}/S$. This explains why the fields $K(\Pi_{\tilde{E}}) = F^{(2)} = F$ and $K(\Pi_{\tilde{G}}) = F_G^{(2)} = F_q$ are Picard-Vessiot extensions of K.

Let now s be a section of G/S, and let $U = log_G(s) \in LieG^{an}(\Lambda)$ be a logarithm of s over Λ. As usual, set $p = \pi(s) \in E(S), u = log_E(p) = d\pi(U) \in LieE^{an}(\Lambda)$. Since $Ker(d\pi)$ has relative dimension 1, the field generated over K by $log_G(s)$ is an extension of $K(log_E(p))$ of transcendence degree ≤ 1, so $log_G(s)$ has transcendence degree ≤ 1 over $F_p = F(u, \zeta_\lambda(u))$. Finally, set

$$F_{pq} := F_p.F_q \ , \ L = L_s := F_{pq}(log_G(s)).$$

The field $F_{pq} = F_{pq}^{(2)}$ is the field of generalized periods of $\{E, p, q\}$ promised in §3.3. Since it contains $F_G^{(2)}$, the field $L = L_s$ depends only on s, not on the choice of its logarithm

$U = log_G(s)$, and is an extension of F_{pq}, of transcendence degree ≤ 1. In fact, the explicit formulae of Appendix I show that for $q \neq 0$ and $p \neq 0, -q$:

$$L_s = F_{pq}\big(\ell_s - g_\lambda(u,v)\big), \text{ where } g_\lambda(u,v) = log_{\mathbb{G}_m} \frac{\sigma_\lambda(v+u)}{\sigma_\lambda(v)\sigma_\lambda(u)}$$

is a Green function attached to the sections $\{p, q\}$ of E/S, and $\ell_s = log_{\mathbb{G}_m}(\delta_s)$ for some rational function $\delta_s \in K^*$ attached to the section s of G/S. This formula implies that L is a differential field, but L is even a Picard-Vessiot extension of K. One way to check this is to relate $log_G(s)$ to an integral of a differential of the 3rd kind on E, with integer, hence *constant* residues, and to differentiate under the integral sign. Another way consists in lifting s to a section $\tilde{s}$ of $\tilde{G}/S$, projecting to $\tilde{p} \in \tilde{E}(S)$. Then, for any choices $\tilde{u} = log_{\tilde{E}}(\tilde{p})$ and $\tilde{U} = log_{\tilde{G}}(\tilde{s})$ of logarithms of $\tilde{p}$ and $\tilde{s}$, the field $F^{(2)}(\tilde{u}) = F(u, \zeta_\lambda(u)) = F_p$ is contained in $F_q(\tilde{U})$, which can therefore be written as $F_{pq}(\tilde{U})$, and the latter field $F_{pq}(\tilde{U})$ coincides with $F_{pq}(U) = L$, since $\tilde{U}$ lifts U and $\tilde{u}$ in the fiber product $Lie\tilde{G} = LieG \times_{LieE} Lie\tilde{E}$. Now, in the notations of [11], [7], $\tilde{U}$ is a solution of the inhomogenous linear system $\partial_{Lie\tilde{G}}(\tilde{U}) = \partial\ell n_{\tilde{G}}(\tilde{s})$, which, on the one hand, is defined over K, and on the other hand, admits $F_q(\tilde{U})$ as field of solutions. So, $L = F_q(\tilde{U})$ is indeed a Picard-Vessiot extension of K. By the same argument, applied to the differential equation $\partial_{Lie\tilde{E}}(\tilde{u}) = \partial\ell n_{\tilde{E}}(\tilde{p})$, we see anew that F_p is a Picard-Vessiot extension of K. Notice, on the other hand, that $F_q(U)$ is in general not a differential extension of F_q (it contains u, but not $\zeta_\lambda(u)$).

The following diagram summarizes these notations (and proposes other natural ones...) :

$$\begin{tikzcd}[column sep=small]
 & L & \\
 & F_{pq} \arrow[u] & \\
F_q \arrow[ur] & & F_p \arrow[ul] \\
 & F \arrow[ul] \arrow[ur] & \\
 & K \arrow[u] &
\end{tikzcd}$$

$$L = F_{pq}(log_G(s)) = F_{\tilde{G}}(log_{\tilde{G}}(\tilde{s})) = F_{pq}\big(\ell_s - g(u,v)\big)$$

$$F_{pq} = F_p.F_q = F(u, \zeta(u), v, \zeta(v))$$

$$F_q = F_G^{(2)} := F_{\tilde{G}} = F_{\tilde{E}}(log_{\tilde{E}}(\tilde{q})) \qquad F_p = F_{\tilde{E}}(log_{\tilde{E}}(\tilde{p}))$$

$$F = F^{(2)} := F_{\tilde{E}} = K(\omega_1, \omega_2, \eta_1, \eta_2)$$

$$K = \mathbb{C}(\lambda)$$

All the notations of the Main Lemma have now been specified, and we can restate its Part (A) in the non-isotrivial case, as follows :

Lemma 2. (= Main Lemma for q non-torsion) *With $S/\mathbb{C}$, let G/S be a* non isotrivial *extension by $\mathbb{G}_m$ of an elliptic scheme E/S, parametrized by a section q of $\hat{E}/S$, and let G_0 be the constant part of G. Let further s be a section of G/S, with projection $p = \pi \circ s$ to E/S, and let $F_{pq} = F_p.F_q \supset F$ be the field of generalized periods of $\{E, p, q\}$. Assume that $log_G(s)$ is algebraic over F_{pq}. Then, there exists a constant section $s_0 \in G_0(\mathbb{C})$ such that*

i) either $s - s_0$ is a Ribet section;
ii) or $s - s_0$ is a torsion section.

In other words, if s is not a constant translate of a Ribet or of a torsion section of G/S, then $g_\lambda(u,v) - \ell_s$ is transcendental over F_{pq}.

Conclusion (ii) of Lemma 2 appears to be stronger than Conclusion (ii) of Part (A) of the Main Lemma, but is in fact equivalent to it when G is not isotrivial. Indeed, in this case, $\mathbb{G}_{m/S}$ is the only connected strict subgroup scheme of G. Now, if a multiple by a non-zero integer N of the section $s - s_0$ factors through $\mathbb{G}_m$, i.e. is of the form δ for some section $\delta \in \mathbb{G}_m(S)$, then, $p = \pi(s)$ is a torsion or a constant section of E/S, so $F_{pq} = F_q$, and $F_{pq}(\log_G(s)) = F_q(\ell)$, where $\ell = log_{\mathbb{G}_m}(\delta)$. By assumption, ℓ is then algebraic over F_q, and Lemma 1 of §4 implies that $\delta = \delta_0 \in \mathbb{G}_m(\mathbb{C})$ is a constant section. Considering the constant section $s_0' = s_0 - \frac{1}{N}\delta_0$ of G/S, we derive that $s - s_0'$ is a torsion section, i.e. that Conclusion (ii) of Lemma 2 is fulfilled.

5.2 Reducing the Main Theorem to the Main Lemma

In view of §4, we could now restrict to the case of a non isotrivial extension G, prove Proposition 4 in this case, and finally prove Lemma 2, thereby concluding the proof of Theorem 2. However, as announced above, we will remain in the general case, and make no assumption on q.

We now perform Step ($\boldsymbol{\alpha}$), i.e. prove Proposition 4 in the general case, extending the pattern of proof of §4. We recall the notations of Proposition 4, including the fundamental assumption that $p = \pi(s)$ is neither a torsion nor a constant section of E/S. By Lemma 1 , this condition implies that F_p has transcendence degree 2 over F, hence that $u(\lambda) = log_E(p(\lambda))$ *is transcendental over the field F.*

Consider the following tower of fields of functions on Λ, where the lower left (resp. upper right) ones are generated by complex (resp. real) analytic functions. The inclusions which the NE-arrows represent come from the definition of a, b_1, b_2 in terms of $log_E(p(\lambda)) = u(\lambda), log_G(s(\lambda)) = U(\lambda)$, cf. Relations $(\mathfrak{R}_u), (\mathfrak{R}_U)$ of §3.3; the inclusions of the NW-arrows on the left come from the definition of the fields of periods F, F_{pq}; those of the NW-arrows on the right are obvious.

$$\begin{tikzcd}
& F_{pq}(a,b_1,b_2) & & & (\mathfrak{R}_U): \quad log_G s = a\varpi_0 + b_1\varpi_1 + b_2\varpi_2 \\
F_{pq}(log_G(s)) \arrow[ur] & & F_{pq}(b_1,b_2) \arrow[ul] & & \\
& F_{pq} \arrow[ul] \arrow[ur] & & F(b_1,b_2) \arrow[ul] & . \\
& & F(log_E(p)) \arrow[ul] \arrow[ur] & (\mathfrak{R}_u): \quad u := log_E p = b_1\omega_1 + b_2\omega_2 & \\
& & F \arrow[u] & &
\end{tikzcd}$$

Now, assume for a contradiction that the real surface $\mathcal{S}$ contains a semi-algebraic curve $\mathcal{C}$. As in §4, consider the real curve $\Gamma = (U^B)^{-1}(\mathcal{C}) \subset \Lambda \subset S(\mathbb{C})$, and denote by a lower index Γ the restrictions to Γ of the various functions of λ appearing above; similar notation $F_{|\Gamma}, F_{pq|\Gamma}$, etc, for the fields they generate. For instance, since $(a_{|\Gamma}, b_{1|\Gamma}, b_{2|\Gamma})$ parametrize the algebraic curve $\mathcal{C}$, these three functions generate a field of transcendence degree 1 over $\mathbb{R}$, and $tr.deg._{F_{|\Gamma}} F_{|\Gamma}(b_{1|\Gamma}, b_{2|\Gamma}) \leq 1$. But by the result recalled above and the principle of isolated zeroes, $u_{|\Gamma}$ is transcendental over $F_{|\Gamma}$. Therefore, the restriction to Γ of the field $F(b_1, b_2)$ is an algebraic extension of the restriction to Γ of the field $F(u)$. We may abbreviate this property by saying that $F(u)$ and $F(b_1, b_2)$ are essentially equal over Γ. Going up north-west in the tower, we deduce that the fields F_{pq} and $F_{pq}(b_1, b_2)$ are essentially equal over Γ.

Notice that $b_{1|\Gamma}$ and $b_{2|\Gamma}$ are not both constant since $F_{|\Gamma}(b_{1|\Gamma}, b_{2|\Gamma}) := \big(F(b_1, b_2)\big)_{|\Gamma} \supset \big(F(u)\big)_{|\Gamma}$ is a transcendental extension of $F_{|\Gamma}$. So, $a_{|\Gamma}$ must be algebraic over $\mathbb{R}(b_{1|\Gamma}, b_{2|\Gamma})$. Therefore, $\big(F_{pq}(a, b_1, b_2)\big)_{|\Gamma}$ is an algebraic extension of $\big(F_{pq}(b_1, b_2)\big)_{|\Gamma}$, hence of the essentially equal field $\big(F_{pq}\big)_{|\Gamma}$, and we deduce that the intermediate field $\big(F_{pq}(log_G(s))\big)_{|\Gamma}$ is algebraic over $\big(F_{pq}\big)_{|\Gamma}$. But $log_G(s(\lambda)) = U(\lambda)$ is a complex analytic map, so, by isolated zeroes, $F_{pq}(log_G(s(\lambda))$ must also be algebraic over F_{pq}. This concludes the proof of Proposition 4.

5.3 The role of K-largeness

The change of base field from K to F_{pq} can be viewed as the "logarithmic" equivalent of the passage from K to the field $K^\sharp_{\mathbf{G}}$ generated by the Manin kernel $\mathbf{G}^\sharp := Ker(\partial\ell n_{\mathbf{G}})$ of an algebraic D-group $\mathbf{G}/K$, which one encounters in the study of the exponentials of algebraic sections of $Lie(\mathbf{G})/S$, as in [11]. A Manin kernel has in fact already appeared in the present paper at the level of the D-group $\tilde{E}$: in this case, the group of K^{alg}-points of $\tilde{E}^\sharp$ projects onto $E_{tor} \oplus \mathcal{E}_0(\mathbb{C})$, and the recurrent hypothesis made on the section p (that it be neither torsion nor constant) exactly means that none of its lifts $\tilde{p}$ to $\tilde{E}(S)$ lies in $\tilde{E}^\sharp$.

For elliptic curves, the field of definition $K^\sharp_{\tilde{E}}$ of $\tilde{E}^\sharp$ is always algebraic over K, and one says that the D-group $\tilde{E}$ is K-large. This is the hypothesis required on $\mathbf{G}$ for the Galois theoretic approach to the proof of the relative Lindemann-Weierstrass theorem of [11], §6. But the third author has checked that it can be extended to non K-large groups (those with $K^\sharp_{\mathbf{G}}$ transcendental over K), yielding some new cases of the following conjecture. See [24] - and [7], §8.1 and §4.3 of S. & C. in the abelian case.

Conjecture : *Let $\mathbf{G}$ be an almost semi-abelian algebraic D-group over K, let $a \in Lie\mathbf{G}(K)$, and let $y = exp_{\mathbf{G}}(\int a) \in \mathbf{G}(K^{diff})$ be a solution of the equation $\partial\ell n_{\mathbf{G}}(y) = a$. Then, $tr.deg.(K^\sharp_{\mathbf{G}}(y)/K^\sharp_{\mathbf{G}})$ is the smallest among dimensions of connected algebraic D-groups H of $\mathbf{G}$ defined over K such that $a \in LieH + \partial\ell n_{\mathbf{G}}(\mathbf{G}(K))$, equivalently such that $y \in H + \mathbf{G}(K) + \mathbf{G}^\sharp$. Moreover, $H^\sharp(K^{diff})$ is the Galois group of $K^\sharp_{\mathbf{G}}(y)$ over $K^\sharp_{\mathbf{G}}$.*

By $\int a$, we here mean any $x \in Lie\mathbf{G}(K^{diff})$ such that $\partial_{Lie\mathbf{G}}x = a$. When x lies in $Lie\mathbf{G}(K)$, this leads to results of "Ax-Lindemann" type (as used in [26], [24]), whereas at least in the non-isoconstant case, the transcendence results required by the present strategy (i.e. that of [20]) concern the equation $\partial_{Lie\mathbf{G}}x = b$, with $b = \partial\ell n_{\mathbf{G}}(y)$ for some $y \in \mathbf{G}(K)$. Notice that when $\mathbf{G}$ is the universal vectorial extension $\tilde{G}$ of our semi-abelian scheme $G = G_q$, the analogous field $K^\sharp_{Lie\mathbf{G}}$ is precisely the field of periods $F_q = F_{\tilde{G}}$ of $\tilde{G}$; see [7], §2 of S & C, for a justification of this analogy, and Remark 3.ii below for the adjunction of F_p in the base field.

More generally, given $(x, y) \in (Lie\mathbf{G} \times \mathbf{G})$, analytic over a ball $\Lambda \subset S(\mathbb{C})$ and linked by the relation $\partial\ell n_{\mathbf{G}}(y) = \partial_{Lie\mathbf{G}}x$ (i.e. essentially $y = exp_{\mathbf{G}}(x)$), one may wonder which extension of the Ax-Schanuel theorem holds for the transcendence degree *over* $K^\sharp_{Lie\mathbf{G}}.K^\sharp_{\mathbf{G}}$ of the point (x, y). The case when x is algebraic over $K^\sharp_{Lie\mathbf{G}}$ includes the study of Picard-Painlevé sections; in this direction, see [15], Lemma 3.4. The case when x has transcendence degree ≤ 1 over $K^\sharp_{Lie\mathbf{G}}$ would be of particular interest, as it occurs when the Betti coordinates of x parametrize an algebraic curve, and this may pull the present strategy back into the "exponential" frame-work of Ax-Lindemann-Weierstrass.

Remark 3.- i) Just as in §4, all we need to know for relative Manin-Mumford statements is the transcendence degree of $log_G \circ s$ over the field of generalized periods F_{pq} of $\{E, p, q\}$. More clearly put, the transcendence degree (i.e. the differential Galois group) of F_{pq} over K plays no role. Of course, $Gal(F_{pq}/K)$ will come as a help during the proof of the Main Lemma, in parallel with the role of SL_2 in §4. But in the notations of the diagram of §6, we must merely compute $Gal(L/F_{p,q}) = Im(\tau_s) \subset \mathbb{C}$, and show that under the hypotheses of the Main Lemma, τ_s vanishes only if one of its conclusions (i), (ii) is satisfied.

ii) Adjoining the field F_p to the base field F_q comes in naturally, since the Picard-Vessiot extension $F_q(\tilde{U})$ of K automatically contains it. But another advantage of the compositum F_{pq} is that the roles of p and q become symmetric in the statement of the Main Lemma. The explicit formula for L given by the Green function makes this apparent. More intrisically, the field $L_s = F_{pq}(log_G(s))$ is the field of periods of the smooth one-motive $M = [M_s : \mathbb{Z} \to G]$ over S, in the sense of [13], attached to the section $s \in G(S)$, with $p = \pi(s)$ and $q \in \hat{E}(S)$ parametrizing the extension G. By biduality, p parametrizes an extension G' of $\hat{E}$ by $\mathbb{G}_m$, and s may be viewed as a section s' of G' above the section q of $\hat{E}(S)$. The Cartier dual of M is the one-motive $M' = [M'_{s'} : \mathbb{Z} \to G']$ attached to this section s', and its field of periods $F_{M'} = L'_{s'} = F_{qp}(log_{G'}(s'))$ coincides with F_M, since they are the Picard-Vessiot extensions of two adjoint differential systems. So, although log_G and $log_{G'}$ have no direct relations, the fields which $log_G(s)$ and $log_{G'}(s')$ generate **over** F_{pq} are the same. Similarly, the structures of G and G' usually differ a lot, but the conclusions (i) and (ii) of the Main Lemma turn out to be invariant under this duality. See Case (**SC2**) of §6 for a concrete implementation of this remark.

iii) The above symmetry is best expressed in terms of the Poincaré bi-extension $\mathcal{P}$, resp. $\mathcal{P}'$, of $E \times_S \hat{E}$, resp. $\hat{E} \times_S E$, by $\mathbb{G}_m$. As recalled in the Introduction (cf. [13]), a section s of $G = G_q$ above p corresponds to a trivialization of the $\mathbb{G}_m$-torsor $(p, q)^*\mathcal{P} \simeq (q, p)^*\mathcal{P}'$.

Then, the inverse image ς of this trivialization under the uniformizing map

$$\mathbb{C}^3 \times \tilde{S} \to \mathcal{P}^{an}$$

of $\mathcal{P}$ generates L over F_{pq} (here, $\tilde{S}$ denotes the universal cover of S^{an}, for instance the Poincaré half-plane when $S = X$ is a modular curve). This view-point turns the Main Lemma into a statement about the transcendency of ς over F_{pq}, and explain why the various types of special curves of the mixed Shimura variety $\mathcal{P}/X$, as encountered in §1.2, occur in its conclusion.

iv) (*Autocritique* on differential extensions).- It would be interesting to pursue the study of this uniformizing map further, as it may lead to a simplification of the present proof of the Main Theorem, where the appeal to differential extensions would be replaced by an Ax-Lindemann statement, extending the recent results of Ullmo-Yafaev [32] and Pila-Tsimerman [25] to mixed Shimura varieties. This has actually just been achieved by Gao, see [14], Theorem 1.2, whose proof is based on o-minimality, but also on a monodromy argument (Theorem 8.1) close to our Main Lemma. See [30] for a perspective on both approaches.

6 Proof of the Main Lemma

We finally perform Step ($\boldsymbol{\beta}$) in the general case. The arguments will be of the same nature as in §4, appealing to Ax type results for constant groups, and to representation theory otherwise. As mentioned before the enunciation of the Main Lemma in §3.3, similar results appear in [6], Propositions 4.a, 4.b and Theorem 2, but it seems better to gather them here into a full proof.

Consider the tower of Picard-Vessiot extensions drawn on the left part of the following picture :

$$\begin{tikzcd}[column sep=small]
 & L & \\
 & F_{pq} \arrow[u] & \\
F_q \arrow[ur] & & F_p \arrow[ul] \\
 & F \arrow[ul] \arrow[ur] & \\
 & K \arrow[u] &
\end{tikzcd}
\quad , \quad \rho_{G,s}(\gamma) = \begin{pmatrix} 1 & {}^t\xi_q(\gamma) & \tau_s(\gamma) \\ 0 & \rho_E(\gamma) & \xi_p(\gamma) \\ 0 & 0 & 1 \end{pmatrix}, \quad
\begin{array}{r}
\tau_s : Gal_\partial(L/F_{pq}) \hookrightarrow \mathbb{C} \\
{}^t\xi_q : Gal_\partial(F_q/F) \hookrightarrow \mathbb{C}^2 \simeq \hat{\mathcal{V}} \\
\xi_p : Gal_\partial(F_p/F) \hookrightarrow \mathbb{C}^2 \simeq \mathcal{V} \\
\rho_E : Gal_\partial(F/K) \hookrightarrow SL_2(\mathbb{C})
\end{array}$$

For convenience, we recall from §5.1 that the field $F = K(\omega_1, \omega_2, \eta_1, \eta_2)$ is the Picard-Vessiot extension of K given by the Picard-Fuchs equation $\partial_{Lie\tilde{E}}(*) = 0$ for E/S, whose set of solutions we denote by $\mathcal{V} \simeq \mathbb{C}^2$. If E/S is isoconstant, $F = K$, while $Gal_\partial(F/K) = SL_2(\mathbb{C})$ otherwise. The field $F_p = F(u, \zeta(u))$ corresponds to the inhomogeneous equation attached to p (given by $\partial_{Lie\tilde{E}}(\tilde{u}) = \partial\ell n_{\tilde{E}}(\tilde{p})$, for any choice of a lift $\tilde{p} \in \tilde{E}(S)$ of p), while

$F_q = F(v, \zeta(v))$ is the field of periods of the semi-abelian scheme G/S, generated by the solutions of $\partial_{Lie\tilde{G}}(*) = 0$. As already said, its resemblance with F_p reflects a duality, witnessed by the dual $\hat{\mathcal{V}}$ of $\mathcal{V}$. We fix a polarization of E/S, allowing us to identify E and $\hat{E}$ (and in particular, q to a section of E/S), but will keep track of this duality. The field of generalized periods of $\{E, p, q\}$ is the compositum F_{pq} of F_p and F_q. Finally,

$$L = F_{pq}(log_G s(\lambda)) = F_{pq}(log_{\tilde{G}}(\tilde{s}(\lambda))$$

is the Picard-Vessiot extension generated by the solutions of the 3rd order inhomogeneous equation $\partial_{Lie\tilde{G}}(\tilde{U}) = \partial\ell n_{\tilde{G}}(\tilde{s})$, where $\tilde{s}$ is the pullback of $\tilde{p}$ to $\tilde{G}$ over s. The corresponding 4th order homogeneous system can be described as the Gauss-Manin connection attached to the smooth one-motive M over S given by the section $s \in G(S)$. The drawing in the middle is a representation $\rho_{G,s}$ of the differential Galois group of L/K. The right part expresses that the coefficients of this representation become injective group homomorphisms on the indicated subquotients of $Gal_\partial(L/K)$.

We will again distinguish between several cases, depending on the position of p and q with respect to the projection $E^\sharp$ to E of the Manin kernel $\tilde{E}^\sharp$ of $\tilde{E}$. So,

$$E^\sharp = E_{tor} + \mathcal{E}_0(\mathbb{C}) = \begin{cases} E_{tor} & \text{if } E \text{ is not isoconstant;} \\ E_0(\mathbb{C}) & \text{if } E \simeq E_0 \times S \end{cases}$$

depending on whether the $K/\mathbb{C}$-trace $\mathcal{E}_0$ of E vanishes or not. (In fact, $E^\sharp$ is the Kolchin closure of E_{tor}, and is also called the *Manin kernel of* E.) We denote by $\hat{p}, \hat{q}$ the images of p, q in the quotient $E/E^\sharp$. Notice that the ring $\mathcal{O} = End(E/S)$ still acts on this quotient.

We recall that we must here merely prove Part (A) of the Main Lemma. By contraposition, we assume that no constant translate of s is a Ribet section, or factors through a strict subgroup scheme of G/S, and must deduce that $log_G(s)$, or equivalently, $log_{\tilde{G}}(\tilde{s})$, is transcendental over F_{pq}.

Case (SC1) : $\hat{q} = 0$

Assume first that E/S is not isoconstant. Then, this vanishing means that q is a torsion section, and after an isogeny, $G = G_q$ is isomorphic to $\mathbb{G}_m \times E$. We have already proven Part (A) of the Main Lemma in this case, see Lemma 1 and the lines which follow. So, we can assume that $E = E_0 \times S$ is constant, and the relation $\hat{q} = 0$ now means that q is constant. So, $G = G_0 \times S$ is a constant semi-abelian variety, and we can apply to its (constant) universal vectorial extension $\tilde{G}_0$ the slight generalization of Ax's theorem given in [6], Proposition 1.b. Since we are assuming that no constant translate $s - s_0, s_0 \in G_0(\mathbb{C})$, of s factors through a strict subgroup scheme H of G, the relative hull G_s of s in the sense of [6], §1, is equal to G, and Proposition 1.b of *loc. cit.* implies that

$$tr.deg.(K(log_{\tilde{G}}(\tilde{s}))/K) = dim(\tilde{G}) = 3.$$

Now, $F_q = F = K$ since E and q are constant, while $K(\tilde{U}) = K(\tilde{u}, U) = F_p(log_G(s))$ has transcendence degre ≤ 1 over F_p, which has transcendence degree ≤ 2 over K. So, both transcendence degrees must be maximal, and $log_G(s)$ is indeed transcendental over $F_p = F_{pq}$.

Case (SC2) : $\hat{p} = 0$

This case is dual to the previous one, and the following preliminary remarks will simplify its study. The hypothesis made on s implies that p is not a torsion section (otherwise, a multiple of s factors through $\mathbb{G}_m$). So, we can assume that $E = E_0 \times S$ is constant, and that $p = p_0$ is a constant non-torsion section of E. In view of Case (**CS1**), we can also assume that $\hat{q} \neq 0$, i.e. that q is not constant[5] (and in particular, not torsion). We now consider the smooth one-motive $M = [M_s : \mathbb{Z} \to G]$ attached to the section s above $p = p_0$, and its Cartier dual $M' = [M'_{s'} : \mathbb{Z} \to G']$, where G' is the extension of $\hat{E}$ by $\mathbb{G}_m$ parametrized by p_0. In particular, G' is a constant and non isotrivial semi-abelian variety. By Remark 3.ii of §5, the field $F_{p_0 q}(log_G(s)) = F_q(log_G(s))$ coincides with the field $F_{qp_0}(log_{G'}(s')) = F_q(log_{G'}(s'))$. But the section s' of G' project to q in $\hat{E}$, which is not constant, so no constant translate of s' factors through $\mathbb{G}_m$. Finally, $\mathbb{G}_m$ is the unique connected subgroup scheme of G', since G' is non isotrivial. So, the constant semi-abelian variety G' and its section s' satisfy all the hypotheses of Case (**SC1**). Therefore, $log_{G'}(s')$ is transcendental over $F_q = F_{qp}$, or equivalenty, $log_G(s)$ is transcendental over F_{pq}.

In the next two cases, the proof of our transcendence claim can be derived from the following simple observation : the Lie algebra $\mathfrak{u}_s$ of the unipotent radical of the image of $\rho_{G,s}$ consists of matrices of the form X indicated below, where $({}^t y, x) \in Im\big(({}^t\xi_q, \xi_p)\big) \subset \hat{\mathcal{V}} \times \mathcal{V}$, and $t \in \mathbb{C}$, and for two such matrices

$$X = \begin{pmatrix} 0 & {}^t y & t \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, X', \text{ we have } [X, X'] = \begin{pmatrix} 0 & 0 & t(X, X') \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $t(X, X') = < y|x' > - < y'|x >$ depends only on the vectors x, y, x', y'. Here, the transposition and the scalar product represent the canonical antisymmetric pairing $\mathcal{V} \times \mathcal{V} \to \mathbb{C}$ provided by the chosen principal polarisation on E/S. Now,

Case (SC3) : $\hat{p}$ **and** $\hat{q}$ **are linearly independent over** $\mathcal{O}$

As mentioned in Remark 2.(iv) of §4, the argument leading to Lemma 1, i.e. the sharpened form of André's theorem [1] given in [7] (or alternatively, if E/S is isoconstant, the sharpened forms of Ax's theorem given in [12], Thm. 2, and in [6], Prop. 1.b) implies in this case that $u, \zeta(u), v, \zeta(v)$ are algebraically independent over F. In other words, the homomorphism $({}^t\xi_q, \xi_p) : Gal_\partial(F_{pq}/F) \to \hat{\mathcal{V}} \times \mathcal{V} \simeq \mathbb{C}^4$ is bijective, and any couple $({}^t y, x)$ occurs

[5] It is worth noticing that this case (**SC2**) is the logarithmic analogue of the counterexample studied in [11], §5.3. It does not provide a counterexample to the Main lemma, whose "exponential" analogue would amount, in the notations of [11], to the equality $tr.deg.(K(y)/K) = 1$. In fact, the work [24], combined with Lemma 1 and with the conclusion of Case (**SC2**), implies that y is transcendental over $K_G^\sharp$.

in the Lie algebra $\mathfrak{u}_s$. Consequently, there exists $X, X' \in \mathfrak{u}_s$ such that $t(X, X') \neq 0$, and $\mathfrak{u}_s$ contains matrices all of whose coefficients, *except* the upper right one, vanish. Therefore, the homomorphism τ_s is bijective, $Gal_\partial(L/F_{pq}) \simeq \mathbb{C}$, and $tr.deg(L/F_{pq}) = 1$. (Notice that this yields $tr.deg(L/F) = 1 + 4 = 5$.)

So, from now on, we can assume that $\hat{q}$ and $\hat{p}$ are linked by a unique relation over $\mathcal{O}$, which, considering multiples if necessary, we write in the shape

$$\hat{q} = \alpha\hat{p}, \alpha \in \mathcal{O}, \hat{p} \neq 0, \alpha \neq 0.$$

We denote by $\overline{\alpha}$ the complex conjugate of α, which represents the image of $\alpha \in End(E/S)$ under the Rosati involution attached to the chosen polarization. We first deal with non antisymmetric relations, in the sense of §2.3.

Case (SC4) : $\hat{q} = \alpha\hat{p}$, where $\overline{\alpha} \neq -\alpha$ and $\hat{p} \neq 0$

Lifted to E and up to an isogeny, this relation reads $q = \alpha p + p_0$, where $p_0 \in E_0(\mathbb{C})$ is a constant section, equal to 0 is E/S is not isoconstant. Then, $F_q = F_p$, and more precisely, $\xi_q = \alpha \circ \xi_p$. In other words, the coefficients of the matrices X in $\mathfrak{u}_s$ satisfy the relation $y = \alpha x$, for the natural action of $\mathcal{O}$ on $\mathcal{V}$. Since $p \notin E_0(\mathbb{C})$, Lemma 1 implies that ξ_p is bijective, and any $x \in \mathcal{V}$ occurs in the Lie algebra $\mathfrak{u}_s$. Recall that $< | >$ is antisymmetric, and that the adjoint of the endomorphism of $\mathcal{V}$ induced by an isogeny $\alpha \in \mathcal{O}$ is its Rosati image $\overline{\alpha}$. For any x, x' in $\mathcal{V}$ occurring in matrices X, X' and such that $< x|x' > \neq 0$, we then have :

$$\begin{aligned} t(X, X') &= < \alpha x|x' > - < \alpha x'|x > = < \alpha x|x' > - < x'|\overline{\alpha}x > \\ &= < \alpha x|x' > + < \overline{\alpha}x, x' > = < (\alpha + \overline{\alpha})x|x' > \neq 0 \end{aligned}$$

since $\alpha + \overline{\alpha}$ is a non-zero integer. We conclude as in Case (**SC3**) that $Gal_\partial(L/F_{pq}) \simeq \mathbb{C}$, and $tr.deg.(L/F_{pq}) = 1$. (Here, this yields $tr.deg(L/F) = 1 + 2 = 3$.)

The remaining cases concerns antisymmetric relations of the type $\hat{q} = \alpha\hat{p}$, with $\hat{p} \neq 0$ and a non zero purely imaginary $\alpha = -\overline{\alpha}$. (In particular, the CM elliptic scheme E/S must be isoconstant and so, Theorem 2 is now already proven under its Conditions (i) or (ii)). We first treat the case when q and p themselves are antisymmetrically related.

Case (SC5) : $q = \alpha p$, where $\overline{\alpha} = -\alpha \neq 0$ and $\hat{p} \neq 0$

This is the only case where a Ribet section of G/S exists above the section $p \in E(S), p \notin E_0(\mathbb{C})$. Denote by s_R this (essentially unique) Ribet section.

Lemma 3. *Let s_R be the Ribet section of G/S. Then, $log_G(s_R)$ is defined over F_{pq}.*

Proof : let $L_R = F_{pq}(log_G(s_R))$ be the field generated over F_{pq} by $log_G(s_R)$. Since the differential Galois group $Gal_\partial(L_R/F_{pq})$ injects via τ_{s_R} into a vectorial group $\mathbb{C}$, $log_G(s_R)$ is either transcendental or rational over F_{pq}. Assume that it is transcendental. Then, by

Proposition 4, the surface $\mathcal{S}$ attached to s_R contains no algebraic curve, and the whole reduction of the Main Theorem to the study of $\mathcal{S}$ given in §3.1 and §3.2 implies that s_R admits only finitely many torsion values. But this contradicts the main result of [8]. Notice that the explicit formula given in §7.1, Remark 4, for $log_G(s_R)$ directly shows that $L_R = F_{pq}$.

We now come back to our section s, which we assumed *not* to be a Ribet section of G, and more accurately, such that no constant translate $s - s_0$ of s is a Ribet section. Since s and s_R project to the same section p of E/S, there exists a section $\delta \in \mathbb{G}_m(S)$, i.e. a rational function in K^*, such that $s = s_R + \delta$. The assumption on s implies that $\delta \notin \mathbb{C}^*$ is not constant. Set $\ell = log_{\mathbb{G}_m}(\delta)$. Then, $log_G(s) = log_G(s_R) + \ell$, and Lemma 3 implies that that $F_{pq}(log_G(s)) = F_{pq}(\ell)$, which is equal to $F_p(\ell)$, since $\hat{q} = \alpha\hat{p}$. By Lemma 1, $F_p(\ell)$ has transcendence degree 1 over F_p, so $log_G(s)$ too is transcendental over F_{pq}.

Case (SC6) : $q = \alpha p + p_0$**, where** $\overline{\alpha} = -\alpha \neq 0$ **and** $p_0 \in E_0(\mathbb{C}), p_0 \notin E_{0,tor}$

Fixing an α-division point p_0' of p_0 in $E_0(\mathbb{C})$, we have $q = \alpha(p - p_0')$, and there exists an (essentially unique) Ribet section s_R' of G/S above $p' := p - p_0'$. Then, the section $s' := s - s_R'$ of G/S projects to $\pi(s') = p - p' = p_0'$ in $E_0(\mathbb{C}) \subset E(S)$. Furthermore, $F_p = F_{p'} = F_q$ since p_0' is constant, and since $log_G(s_R')$ is defined over $F_{p'q} = F_{pq} = F_q$ by Lemma 3, we deduce that $log_G(s') = log_G(s) - log_G(s_R')$ generates over the field $F_{pq} = F_{p'q} = F_q = F_{p_0'q}$ the same field as $log_G(s)$. We are therefore reduced to showing that given $\hat{q} \neq 0$ and a section s' of G/S projecting to a constant non-torsion section $p_0' \in E_0(\mathbb{C})$, then $log_G(s')$ is transcendental over $F_{p_0'q}$. But this is exactly what we proved in Case (**SC2**) !

This concludes the proof of the Main Lemma, hence of the Main Theorem.

7 Appendices

7.1 Analytic description of the semi-abelian logarithm

Let G/S be a *non isotrivial* extension of an elliptic scheme E/S by $\mathbb{G}_m$ and let s be a section of G/S. The aim of this Appendix is to give an explicit formula for its local logarithm $log_G(s)$ in terms of the Weierstrass functions $\wp_\lambda, \zeta_\lambda, \sigma_\lambda$, in parallel with that of §4 for products. We recall the notations of §5.1. In particular, we set $p = \pi(s) \in E(S)$, $u = log_E(p)$, For simplicity, we will work over the generic point of S, consider G as a semi-abelian variety over the field $K = \mathbb{C}(S)$, and often drop the variable λ indexing the Weierstrass functions and their (quasi-)periods $\omega_1, \omega_2, \eta_1, \eta_2$.

By Weil-Rosenlicht-Barsotti, the algebraic group G, viewed as a $\mathbb{G}_m$-torsor, defines a line bundle over E of degree 0, admitting a rational section β with divisor $(-q) - (0) \in \hat{E}$, which we identify with the point $q \in E$ (the sign is admittedly not standard, but it will make the formulae symmetric in p and q). By assumption, q is not a torsion point, and we set $v = log_E(q)$. We further assume that $p \neq 0$ and $p + q \neq 0$.

The rational section β provides a birational isomorphism $G \dashrightarrow \mathbb{G}_m \times E$ and (after a shift away from 0) an isomorphism $LieG \simeq Lie\mathbb{G}_m \oplus LieE$. The 2-cocycle which describes the group law on the product ([31], VII.5) is a rational function on $E \times E$, expressed in terms of σ-functions by $\frac{\sigma(z+z'+v)\sigma(z)\sigma(z')\sigma(v)}{\sigma(z+z')\sigma(z+v)\sigma(z'+v)}$. Therefore, the exponential morphism exp_G is represented by the map

$$(LieG)^{an}(\Lambda) \ni \begin{pmatrix} t(\lambda) \\ z(\lambda) \end{pmatrix} \mapsto \begin{pmatrix} f_{v(\lambda)}(z(\lambda))\ e^{t(\lambda)} \\ \wp_\lambda(z(\lambda)) \end{pmatrix} \in G^{an}(\Lambda)$$

where

$$f_v(z) = \frac{\sigma(v+z)}{\sigma(v)\sigma(z)} e^{-\zeta(v)z}$$

is a meromorphic theta function for the line bundle $\mathcal{O}_E\big((-q) - (0)\big)$, whose factors of automorphy are given by $e^{-\kappa_v(\omega_i)}$ (opposite of the multiplicative quasi-periods), with

$$\kappa_v(\omega_i) = \zeta(v)\omega_i - \eta_i v, \text{ for } i = 1, 2.$$

The occurence of the trivial theta function $e^{-\zeta(v)z}$ in f_v is due to the condition $d_0(exp_G) = id_{LieG}$. The logarithmic form $\frac{df_v}{f_v} = \big(\zeta(v+z) - \zeta(v) - \zeta(z)\big)dz = \frac{1}{2}\frac{\wp'(z)-\wp'(v)}{\wp(z)-\wp(v)}dz$ is the pullback under exp_E of the standard differential form of the 3rd kind on E with residue divisor $-1.(0) + 1.(-q)$.

Under this description, the section s of G/S under study and its logarithm $log_G(s)$ are given by

$$s = \begin{pmatrix} \delta_s \\ p \end{pmatrix},\ U := log_G(s) = \begin{pmatrix} -g(u,v) + \zeta(v)u + \ell_s \\ u \end{pmatrix}$$

where $\delta_s := s - \beta(p) \in K^*$ is a rational function on S, depending only on s (and on the choice of the section β), for which we set $\ell_s = log_{\mathbb{G}_m}(\delta_s)$, and (cf. the formulae in [6], up to signs) :

$$g_\lambda(u,v) = log\big(\frac{\sigma_\lambda(u+v)}{\sigma_\lambda(v)\sigma_\lambda(u)}\big).$$

This is the Green function mentioned in §5.1.

The $\mathbb{Z}$-local system of periods Π_G of G^{an}/Λ which was introduced in §3.3 admits the basis

$$\varpi_0(\lambda) = \begin{pmatrix} 2\pi i \\ 0 \end{pmatrix}, \varpi_1(\lambda) = \begin{pmatrix} \kappa_{v(\lambda)}(\omega_1(\lambda)) \\ \omega_1(\lambda) \end{pmatrix}, \varpi_2(\lambda) = \begin{pmatrix} \kappa_{v(\lambda)}(\omega_2(\lambda)) \\ \omega_2(\lambda) \end{pmatrix}.$$

We can now describe the various extensions of $F = K(\omega_1, \omega_2, \eta_1, \eta_2)$ appearing in §5 for a non isotrivial extension G. In view of the Legendre relation

$$2\pi i = \eta_1\omega_2 - \eta_2\omega_1 \in K^*,$$

the periods of G generate over F the field

$$F_G = F_G^{(2)} = F(\kappa_v(\omega_1), \kappa_v(\omega_2)) = F(v, \zeta(v)) := F_q,$$

while the field generated over F_{pq} by $log_G(s)$ satisfies

$$L = F_{pq}(log_G(s)) = F(u, \zeta(u), v, \zeta(v), -g(u,v) + \zeta(v)u + \ell_s) = F_{pq}\big(\ell_s - g(u,v)\big)$$

Remark 4 (Analytic description of the Ribet sections). - Assume that $E = E_0 \times S$ is a constant elliptic scheme with complex multiplications by $\mathcal{O}$, that q is not constant, and that p and q are antisymmetrically related in the sense of §2. So, their logarithms $u(\lambda), v(\lambda)$ are non constant holomorphic functions on Λ, and up to an isogeny, we can assume that $v = \alpha u$ for a totally imaginary non zero complex multiplication $\alpha \in 2\mathcal{O}$. In these conditions, the semi-abelian scheme G/S parametrized by q is semi-constant, and admits a Ribet section s_R lifting p, given as above by a couple $s_R = \begin{pmatrix} \delta_{s_R} \\ p \end{pmatrix}$, with logarithm $U_R := log_G(s_R) = \begin{pmatrix} -g(u,v) + \zeta(v)u + \ell_{s_R} \\ u \end{pmatrix}$, where $\delta_{s_R} \in K^*$ and $\ell_{s_R} = log_{\mathbb{G}_m}(\delta_{s_R})$. By the theory of complex multiplication (see for instance [19], Appendix I), there exists an algebraic number s_2 such that the quasi-periods of ζ satisfy : $\eta_2 - s_2\omega_2 = \overline{\tau}(\eta_1 - s_2\omega_1)$, where $\omega_2 = \tau\omega_1$. One can then show that up to a root of unity, $\delta_{s_R} = \frac{\sigma(u+v)}{\sigma(v)\sigma(u)} e^{-s_2 uv}$. Consequently, the first coordinate of U_R is given by $\zeta(v)u - s_2 uv$. This makes it apparent that $log_G(s_R)$ lies in the field F_{pq}, as already proven in Lemma 3. Furthermore, this expression, combined with the CM and Legendre relations, implies that for any $\overline{\lambda} \in S$ such that $p(\overline{\lambda})$ is a torsion point of E_0, say of order n, the point $s_R(\overline{\lambda})$ is a torsion point of $G_{\overline{\lambda}}$, of order dividing n^2. For algebraic proofs of this property, see [8], §3, and [10].

7.2 Application to Pell equations

In [21], the relative Manin-Mumford conjecture (RMM) is proven for simple abelian surface schemes, and this is shown to imply the following corollary : consider the family of sextic polynomials $D_\lambda(x) = x^6 + x + \lambda$, where λ is a complex parameter. Then, there are only finitely many $\overline{\lambda} \in \mathbb{C}$ such that the functional Pell equation $X^2 - D_{\overline{\lambda}}(x)Y^2 = 1$ admits a solution in polynomials $X, Y \in \mathbb{C}[x]$, $Y \neq 0$; see also [33], III. 4.5 for connections with other problems and a proof of the deduction from RMM . The involved abelian surface $A/\mathbb{C}(\lambda)$ is the jacobian of the (normalized) relative hyperelliptic curve $C : y^2 = x^6 + x + \lambda$, and RMM is applied to the section s of A defined by the linear equivalence class of the relative divisor $(\infty_+) - (\infty_-)$ on C.

Following a suggestion of the second and fourth authors, we may treat in the same way the case of a sextic $D_\lambda(x) = (x - \rho(\lambda))^2 Q_\lambda(x)$ having a *squared linear factor*, i.e. a generic double root $\rho(\lambda)$ for some algebraic function $\rho(\lambda)$, now applying the Main Theorem of the present paper to a quotient $G = G_\rho$ of the generalized jacobian of the corresponding semi-stable relative sextic curve C. This G_ρ is an extension by $\mathbb{G}_m$ of an elliptic curve E (over $\mathbb{C}(\lambda)$), where E is the jacobian of the (normalized) relative quartic $\tilde{C}$ with equation $v^2 = Q_\lambda(u)$, and RMM may be applied to the section s of G_ρ defined by the class of the relative divisor $(\infty_+)-(\infty_-)$ on $\tilde{C}$, for the strict linear equivalence attached to the node of C at $x=\rho(\lambda)$ (see [31], V.2, [8], Appendix). As a concrete example, we will here consider the family of quartics

$$Q_\lambda(x) = x^4 + x + \lambda.$$

From the analysis in [21], [33] recalled below, we derive that the set Λ_Q of complex numbers $\overline{\lambda}$ such that the Pell equation $X^2 - Q_{\overline{\lambda}}(x)Y^2 = 1$ has a solution in polynomials $X, Y \in \mathbb{C}[x]$, $Y \neq 0$, is infinite. The solutions for each such $\overline{\lambda}$ form a sequence $(X_{\overline{\lambda},n}, Y_{\overline{\lambda},n})_{n\in\mathbb{Z}}$ of polynomials in $\mathbb{C}[x]$. Our result is that for the $\rho(\lambda)$ considered in the following statements, only finitely many of the polynomials $Y_{\overline{\lambda},n}(x)_{\overline{\lambda}\in\Lambda_Q, n\in\mathbb{Z}}$ admit $x=\rho(\overline{\lambda})$ among their roots. In other words :

Theorem 4. *i) (Case $\rho(\lambda)=0$.) There are only finitely many complex numbers $\overline{\lambda}$ such that the equation $X^2 - x^2 Q_{\overline{\lambda}}(x)Y^2 = 1$ admits a solution in polynomials $X, Y \in \mathbb{C}[x], Y \neq 0$.*

ii) (Case $\rho(\lambda) = \frac{4y(\frac{1}{2}p_W+e_3)-1}{8x(\frac{1}{2}p_W+e_3)} \in \mathbb{C}(\lambda)^{alg}$, with notations explained below.) There are only finitely many complex numbers $\overline{\lambda}$ such that the equation $X^2 - (x-\rho(\overline{\lambda}))^2 Q_{\overline{\lambda}}(x)Y^2 = 1$ admits a solution in polynomials $X, Y \in \mathbb{C}[x], Y \neq 0$.

In spite of their similarity, these two statements cover different situations : in (i), the extension G_ρ is not isotrivial, and the theorem is a corollary of Theorem 2. On the contrary, (ii) illustrates the case of an isotrivial extension G_ρ, and follows from Theorem 1. In fact, we believe that on combining these two cases of our Main Theorem, Theorem 4 will hold for *any* choice of ρ. We mention in this direction a recent work of the fourth author on the isotrivial case, which implies that Theorem 4 holds for all but finitely many algebraic functions $\rho(\lambda)$.

The specific function $\rho(\lambda)$ of Case (ii) can be described as follows. Consider the Weierstrass model $(W_E) : y^2 = 4x^3 - \lambda x + (1/16)$ of the elliptic curve $E/\mathbb{C}(\lambda)$, which is therefore not isoconstant, and the relative point $p_W = (0, -\frac{1}{4})$ on (W_E), which is generically of infinite order (it can be rewritten as the point $p_{\tilde{W}} = (0,-1)$ on the curve $(\tilde{W}_E) : Y^2 = X^3 - 4\lambda X + 1$, and is not torsion at $\overline{\lambda} = \frac{1}{4}$). Then, the 2-division points of p_W are the four points of (W_E) :

$$\text{“}\frac{1}{2}p_W\text{”} = (\frac{1}{8}m_\lambda^2, -\frac{1}{8}m_\lambda^3 + \frac{1}{4}), \quad \text{where } m_\lambda \text{ is a root of } m^4 - 8m + 16\lambda = 0.$$

Choose one of the two roots m_λ which is real when $\overline{\lambda} = \frac{1}{4}$, and call the corresponding point $\frac{1}{2}p_W(\lambda)$. Further, choose one of the two points of order 3 on W_E which is real when $\overline{\lambda}$ is real, and call it $e_3(\lambda)$. Computing the x and y coordinates of the relative point $\frac{1}{2}p_W + e_3$ on W_E then provides the function $\rho(\lambda)$ appearing in Case (ii).

In a more enlightening way, let in general $p(\lambda)$ be the section of E defined by the class of the divisor $(\infty_+) - (\infty_-)$ on $\tilde{C}$, for the standard linear equivalence of divisors. Then,

p is the projection to E of the section s of G_ρ defined (via strict equivalence) above, and one checks that p is not a torsion section. By [21], [33] (see also [16], Prop. 3.1), the Pell equation for $Q_{\overline{\lambda}}(x)$ has a non trivial solution if and only if $p(\overline{\lambda})$ is a torsion point on $E_{\overline{\lambda}}$, i.e. $\overline{\lambda} \in S_\infty^E$ in the notations of §2. Similarly, the Pell equation for $(x - \rho(\overline{\lambda}))^2 Q_{\overline{\lambda}}(x)$ has a non trivial solution if and only if $s(\overline{\lambda})$ is a torsion point of $G_{\rho(\overline{\lambda})}$, i.e. $\overline{\lambda} \in S_\infty^{G_\rho}$. Furthermore, the section $q(\lambda)$ of $\hat{E}$ parametrizing the extension G_ρ is represented the (standard) equivalence class of the divisor $(q_+) - (q_-)$ on $\tilde{C}$, where $q_\pm(\lambda)$ is the section $(\rho(\lambda), \pm Q_\lambda^{1/2}(\rho(\lambda))$ of $\tilde{C}$. Now, in the first case $\rho(\lambda) = 0$, q is a non torsion section, i.e. G_ρ is a non isotrivial extension of the non isoconstant elliptic scheme E, and since p is not torsion, Theorem 2.(i) implies that $S_\infty^{G_\rho}$ is finite. On the other hand, Case (ii) is built up in such a way that q has finite order (equal to 3), so that G_ρ is now isogenous to $\mathbb{G}_m \times E$. But one can check (by specializing at the real number $\overline{\lambda} = \frac{1}{4}$) that the projection of the section s to the $\mathbb{G}_m$ factor is not a root of unity, so s does not factor through a translate of E. Since p is not torsion either, Theorem 1 now provides the finiteness of $S_\infty^{G_\rho}$.

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