MOTIVES FOR PERFECT PAC FIELDS WITH PRO-CYCLIC GALOIS GROUP

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Abstract. Denef and Loeser defined a map from the Grothendieck ring of sets definable in pseudo-finite fields to the Grothendieck ring of Chow motives, thus enabling to apply any cohomological invariant to these sets. We generalize this to perfect, pseudo algebraically closed fields with pro-cyclic Galois group.

In addition, we define some maps between different Grothendieck rings of definable sets which provide additional information, not contained in the associated motive. In particular we infer that the map of Denef-Loeser is not injective.

Contents

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1. Preliminaries

1.1. **Introduction.** To understand definable sets of a theory, it is helpful to have invariants with nice properties. For a fixed pseudo-finite field K, there are two well-known invariants of definable sets: the dimension (see [3]), and the measure (see [2]).

In a slightly different setting, Denef and Loeser constructed a much stronger invariant: they do not fix a pseudo-finite field; instead they consider definable sets in the theory of all pseudo-finite fields of characteristic zero. To each such set X they associate an element $\chi_c(X)$ of the Grothendieck group of Chow motives (see [4], [5]). In particular, this implies that all the usual cohomological invariants (like Euler characteristic, Hodge polynomial) are now applicable to arbitrary definable sets.

Note that the work of Denef and Loeser does not directly imply the existence of the measure of [2]: the motive associated to a set defined by a formula ϕ is invariant under equivalence in the theory of pseudo-finite fields, but not under equivalence in the theory of a fixed pseudo-finite field.

The dimension defined in [3] exists for a much larger class of fields and in [10], Hrushovski asked whether one can also generalize the measure. This question has been answered in [9]: it is indeed possible to define a measure for any perfect, pseudo algebraically closed (PAC) field with pro-cyclic Galois group. A natural question is now: Can the work of Denef-Loeser also be generalized to this setting? More precisely, fix an infinite pro-cyclic group Gal, and consider the theory of perfect PAC fields with absolute Galois group Gal. Then to any definable set X in that theory we would like to associate a virtual motive $\chi_c(X)$. The first goal of this article is to do this.

One reason this result seems interesting to me is the following: the map χ_c exists for pseudo-finite fields (by Denef-Loeser) and for algebraically closed fields (by quantifier elimination). The case of general pro-cyclic Galois groups is a common generalization of both and thus a kind of interpolation.

Comparing those maps χ_c for different Galois groups, one gets the feeling that they are closely related. Indeed, given an inclusion of Galois groups $Gal_2 \subset Gal_1$, we will prove the existence of a map θ from the definable sets for Gal_2 to the definable sets for Gal_1 which is compatible with the different maps χ_c.

These maps θ turn out to be interesting in themselves. An open question was whether, using χ_c, one can get *all* the (additive) information about definable sets; in other words: is χ_c injective? We will show that it is not, by giving an example of two definable sets with the same image under χ_c but with different images under one of those maps θ.

We have one more result. In [5], the map χ_c is defined by enumerating certain properties and then existence and uniqueness of such a map is proven. We are able to weaken the conditions needed for uniqueness in the case of pseudo-finite fields. Unfortunately however, we do not get any sensible uniqueness conditions for other pro-cyclic Galois groups.

1.2. **The results in detail.** Let me fix some notation once and for all.

By a "group homomorphism" we will always mean a continuous group homomorphism if there are profinite groups involved.

Let *Gal* be a pro-cyclic group such that there do exist perfect PAC fields having *Gal* as absolute Galois group. This is the case if and only if *Gal* is torsion-free, or equivalently, if it is of the form $\prod_{p \in P} \mathbb{Z}_p$, where P is any set of primes.

"definable" will always mean 0-definable. When we want to consider sets definable with parameters, we will add these parameters to the language. Indeed most of the time we will fix a field k (mostly of characteristic zero) and work in the theory $T_{Gal,k}$ of perfect PAC (pseudo algebraically closed) fields with absolute Galois group *Gal* which contain k.

Models of $T_{Gal,k}$ will be denoted by K; the algebraic closure of a field K will be denoted by $\tilde{K}$.

By "variety", we mean a separated, reduced scheme of finite type. If not stated otherwise, all our varieties will be over k. The reader not so familiar with algebraic geometry can restrict herself to locally closed subsets of $\mathbb{A}^n$, and everything will work out fine.

I will use the notion "definable set" even when there is no model around: By a "definable set (in $T_{Gal,k}$)", I mean a formula up to equivalence modulo $T_{Gal,k}$. In addition, I will permit myself to speak about "definable subsets of (arbitrary) varieties". For affine embedded varieties, it is clear what this should mean. In general, any definable decomposition of a variety V into affine embedded ones yields the same notion of definable subsets of V. For a precise version of this, see e.g. [4]. (Denef and Loeser call this "definable subassignments".)

For the definition of a Chow motive, see e.g. [11]. We do not repeat the definition here, as anyway the only things the reader really needs to believe about them are Theorem 3.5 and Lemma 3.6 and the existence of the map $\chi_c\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Mot}_k)$ mentioned below.

We will work with the following Grothendieck groups: $\mathrm{K}_0(T_{Gal,k})$ is the group generated by the definable sets of $T_{Gal,k}$, modulo the relations $[X_1] = [X_2]$ if there is a definable bijection between X_1 and X_2, and $[X_1] + [X_2] = [X_1 \cup X_2]$ if X_1 and X_2 are disjoint. $\mathrm{K}_0(\mathrm{Var}_k)$ is the group generated by the varieties over k, modulo $[V_1] = [V_2]$ if V_1 and V_2 are isomorphic and $[V] = [U] + [V \setminus U]$ if U is an open subvariety of V. $\mathrm{K}_0(\mathrm{Mot}_k)$ is the group generated by the Chow motives over k, modulo $[m_1] = [m_2]$ if m_1 and m_2 are isomorphic and $[m_1] + [m_2] = [m_1 \oplus m_2]$. All three Grothendieck groups are in fact rings, where the multiplication is induced by $\times$ resp. $\times$ resp. $\otimes$. We will also need to tensor the Grothendieck ring of motives with $\mathbb{Q}$; we denote this by $\mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}} := \mathrm{K}_0(\mathrm{Mot}_k) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Gillet and Soulé [7] and Guillén and Navarro Aznar [8] showed that there exists a unique morphism of rings $\chi_c\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Mot}_k)$ which extends the canonical map on the smooth projective varieties. Our first theorem states that this map can be extended to $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ in such a way that it satisfies one additional property which needs some more definitions to be stated properly. Let me write down the theorem right now anyway; the reader may skip that property for the moment and come back after she has read the necessary definitions in Section 2.

Theorem 1.1. *Suppose $Gal = \prod_{p\in P}\mathbb{Z}_p$ (where P is any set of primes) is a pro-cyclic group and k is a field of characteristic zero. Then there exists a ring homomorphism $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ extending the homomorphism $\chi_c\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Mot}_k)$ with the following property: if $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover such that all prime factors of $|G|$ lie in P, then*

$$(*) \qquad \chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \frac{1}{|G|}\chi_c(V)\,.$$

If $Gal = \hat{\mathbb{Z}}$, then a ring homomorphism with these properties is unique.

In the case $Gal = \hat{\mathbb{Z}}$, this result is almost the same as the one of Denef-Loeser (Theorem 6.4.1 of [5]). However, our condition $(*)$ needed to get uniqueness is a weakening of the one in [5].

As the reader will guess, even if $Gal \neq \hat{\mathbb{Z}}$ we have a precise map χ_c in mind (which we will construct during the proof). Unfortunately we can not prove that condition $(*)$ is strong enough to define χ_c uniquely in the general case, and we do not have any good replacement for $(*)$. The best uniqueness statement we can offer is this: there is a unique map χ_c as in Theorem 1.1 compatible with Definition 3.7 (which in reality is spread over Section 3.4). Below we will give another (equivalent) tentative definition of χ_c.

Our map χ_c will not really depend on the base field k: if we have a second field k' containing k, then there are canonical ring homomorphisms $\mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(T_{Gal,k'})$ and $\mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}} \to \mathrm{K}_0(\mathrm{Mot}_{k'})_{\mathbb{Q}}$, which we will both denote by $\otimes_k k'$. The map χ_c is compatible with these homomorphisms:

Proposition 1.2. *In the setting just described (with χ_c satisfying Definition 3.7) we have, for any definable set X of $T_{Gal,k}$, $\chi_c(X \otimes_k k') = \chi_c(X) \otimes_k k'$.*

In [9], the idea behind the proof of the existence of a measure for definable sets of perfect PAC fields with pro-cyclic Galois group was to reduce the general case to the pseudo-finite case. However, this was just an idea, not a real reduction. For the present problem, it turns out that we really can define maps θ_ι between the different Grothendieck groups which enable us to reduce the existence of a map $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ to the case $Gal = \hat{\mathbb{Z}}$. More generally, we will prove the following theorem.

Theorem 1.3. *Suppose Gal_1 and Gal_2 are two torsion-free pro-cyclic groups, $\iota\colon Gal_2 \hookrightarrow Gal_1$ is an injective map, and k is any field (not necessarily of characteristic zero). Denote the theories $T_{Gal_i,k}$ by T_i for $i = 1, 2$. Then the following defines a ring homomorphism $\theta_\iota\colon \mathrm{K}_0(T_2) \to \mathrm{K}_0(T_1)$: Suppose K_1 is a model of T_1. Then the fixed field $K_2 := \tilde{K}_1^{\iota(Gal_2)}$ is a model of T_2 containing K_1. For any $X_2 \subset \mathbb{A}^n$ definable in T_2, we define $\theta_\iota(X_2)(K_1) := X_2(K_2) \cap K_1^n$.*

To reduce the existence of χ_c to the case $\hat{\mathbb{Z}}$, apply this theorem to $\iota\colon Gal \hookrightarrow \hat{\mathbb{Z}}$, where ι maps Gal to the appropriate factor $\prod_{p\in P}\mathbb{Z}_p$ of $\hat{\mathbb{Z}}$ (such that $\hat{\mathbb{Z}}/Gal$ is torsion-free). Then define χ_c as the composition $\hat{\chi}_c \circ \theta_\iota$, where $\hat{\chi}_c\colon \mathrm{K}_0(T_{\hat{\mathbb{Z}},k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ is the known map in the pseudo-finite case. Verification of the properties of χ_c is not very difficult, as soon as one knows the meaning of $(*)$, and using explicit computations done in the proof of Theorem 1.3.

So in principle, the existence of χ_c is reduced to Theorem 1.3. On the other hand, one has the feeling that it should also be possible to construct this map directly when $Gal \neq \hat{\mathbb{Z}}$, so we will do this. The proof in [4] in the pseudo-finite case uses finite fields a lot, so to generalize it, we first had to "clean it up". in our proof, the only place where finite fields appear is in a proof of Lemma 3.3, which is a kind of qualitative Chebotarev density theorem for $T_{Gal,k}$ (and which might be interesting in itself).

By the way, we now have a second not-so-beautiful definition of χ_c in the general case at hand: in the above reduction, we chose a map $\iota\colon Gal \to \hat{\mathbb{Z}}$, but it can be seen from Theorem 1.3 that θ_ι is independent of this choice (as we fixed the image). So one could also define $\chi_c := \hat{\chi}_c \circ \theta$. We will check that the two different definitions of χ_c are equivalent.

Another interesting application of Theorem 1.3 is the case $Gal_1 = Gal_2 = Gal$, but with a non-trivial injection $\iota\colon Gal \hookrightarrow Gal$. One thus gets endomorphisms of the ring $\mathrm{K}_0(T_{Gal,k})$, which might reveal a lot of information about its structure. Indeed this really gives new information: we will construct a whole family of pairs of definable sets X_1 and X_2 such that $\chi_c(X_1) = \chi_c(X_2)$ but $\chi_c(\theta(X_1)) \neq \chi_c(\theta(X_2))$, thereby proving:

Proposition 1.4. *Let k be a field of characteristic zero and let Gal be a non-trivial torsion-free pro-cyclic group. Then the map $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ is not injective.*

The remainder of the article is organized as follows:

- In Section 2, we define Galois covers, the Artin symbol (in a version suited for this article) and Galois stratifications. After that, the reader will understand condition $(*)$ of Theorem 1.1. At the end of the section, we will state a version of "almost quantifier elimination" from [6] which we will be need several times.
- The construction of the map χ_c of Theorem 1.1 will be done in Section 3, after first proving some lemmas, in particular the version of Chebotarev's density theorem for $T_{Gal,k}$. We also check that the constructed map has the required properties.
- In Section 4, we prove the remaining statements on χ_c, i.e. the uniqueness in the $\hat{\mathbb{Z}}$ case and compatibility with change of the base field k (Proposition 1.2).
- Section 5 is devoted to the maps θ_ι. We prove Theorem 1.3 and, as an example application, Proposition 1.4. In addition, we will check that the maps χ_c for different Galois groups (as constructed in Section 3.4) are compatible with the maps θ_ι for suitable ι.
- Finally Section 6 lists some open problems.

2. Galois stratifications

A standard technique to get hold of definable sets of perfect PAC fields with not-too-large Galois group is the quantifier elimination to Galois formulas. In this section, we define the necessary objects and then, in Section 2.4, state this quantifier elimination result in the version of Fried-Jarden [6].

In most of the article, we will work with parameters in a field k. However, at some point we will need Galois covers defined over rings, so for the definitions in this section assume that all varieties are defined over some fixed ring R.

We also fix a pro-cyclic group $Gal = \prod_{p \in P} \mathbb{Z}_p$ for some set of primes P, and we let $T := T_{Gal,R}$ be the theory of perfect PAC fields with absolute Galois group Gal containing R.

2.1. **Galois covers.**

Definition 2.1. (1) A *Galois cover* consists of two integral and normal varieties V and W (over our ring R) and a finite étale map $f\colon V \to W$ such that for $G := \mathrm{Aut}_W(V)^{\mathrm{opp}}$, we have canonically $W \cong V/G$ (where G acts from the right on V). We denote a Galois cover by $f\colon V \overset{G}{\twoheadrightarrow} W$ and call G the *group* of that cover. The action of G on V will be denoted by $v.g$ (for $v \in V$, $g \in G$).

(2) We say that a Galois cover $f'\colon V' \overset{G'}{\twoheadrightarrow} W$ is a *refinement* of $f\colon V \overset{G}{\twoheadrightarrow} W$, if there is a finite étale map $g\colon V' \twoheadrightarrow V$ such that $f' = f \circ g$.

(3) If W'' is a locally closed subset of W and V'' is a connected component of $f^{-1}(W'')$, then we call $V'' \overset{G''}{\twoheadrightarrow} W''$ the *restriction* of $V \overset{G}{\twoheadrightarrow} W$ to W'', where $G'' := \mathrm{Aut}_{W''}(V'')^{\mathrm{opp}}$.

Remark 2.2. (1) If $f'\colon V' \overset{G'}{\twoheadrightarrow} W$ is a refinement of $f\colon V \overset{G}{\twoheadrightarrow} W$, then we have a canonical surjection $\pi\colon G' \twoheadrightarrow G$.

(2) If $V'' \overset{G''}{\twoheadrightarrow} W''$ is a restriction of $V \overset{G}{\twoheadrightarrow} W$, then we have a canonical injection $G'' \hookrightarrow G$. Different choices of the connected component of $f^{-1}(W'')$ yield isomorphic restricted Galois covers.

2.2. **Artin symbols and colorings.**

Using a Galois cover $V \overset{G}{\twoheadrightarrow} W$, we would like to decompose W into subsets according to the Artin symbol of the elements. However, the usual definition of Artin symbol needs a canonical generator of the Galois group Gal (usually the Frobenius of a finite field); the Artin symbol is then the image of the generator under a certain map $\rho\colon\ Gal \to G$ (which is unique only up to conjugation by G). If one does not have such a canonical generator, then one still can consider the image of ρ. This is what one uses as Artin symbol in our case.

Definition 2.3 (and Lemma)**.** Suppose $f\colon V \overset{G}{\twoheadrightarrow} W$ is a Galois cover and $K \models T$ is a model.

(1) Suppose $v \in V(\tilde{K})$ such that $f(v) \in W(K)$. Then there is a unique group homomorphism $\rho\colon\ \mathrm{Gal}(\tilde{K}/K) \to G$ satisfying $\sigma(v) = v.\rho(\sigma)$ for any $\sigma \in \mathrm{Gal}(\tilde{K}/K)$. The *decomposition group* $\mathrm{Dec}(v) := \mathrm{im}\,\rho \subset G$ of v is the image of that homomorphism.

(2) For $w \in W(K)$, let the *Artin Symbol* $\mathrm{Ar}(w)$ of w be the set $\{\mathrm{Dec}(v) \mid v \in V(\tilde{K}), f(v) = w\}$ of decomposition groups of all preimages of w.

$\mathrm{Ar}(w)$ consists exactly of one conjugacy class of subgroups of G, and these subgroups are isomorphic to a quotient of Gal. The quotients of Gal are just the cyclic groups Q such that all prime factors of $|Q|$ lie in P (where P was the set of primes such that $Gal = \prod_{p \in P} \mathbb{Z}_p$). We introduce some notation for this:

Definition 2.4. Given a finite group G, we will call those subgroups of G which are isomorphic to a subgroup of Gal the *permitted subgroups*. We denote the set of all permitted subgroups of G by $\mathrm{Psub}(G)$. If Q is a finite cyclic group, then we denote by $\mathrm{Ppart}(Q)$ the “permitted part of Q”, i.e. the biggest permitted subgroup of Q.

(The "P" in "Psub" and "Ppart" stands for permitted and/or the set P of primes.) The interest of $\mathrm{Ppart}(Q)$ is the following. We will sometimes identify $Gal = \prod_{p\in P} \mathbb{Z}_p$ with the corresponding factor of $\hat{\mathbb{Z}}$ and consider homomorphisms $\rho\colon \hat{\mathbb{Z}} \to G$. Then the image of Gal in G is just $\rho(Gal) = \mathrm{Ppart}(\operatorname{im}\rho)$.

Given a Galois cover $V \overset{G}{\twoheadrightarrow} W$, we now define subsets of W using the Artin symbol:

Definition 2.5. (1) A *coloring* of a Galois cover $V \overset{G}{\twoheadrightarrow} W$ is a subset C of the permitted subgroups of G which is closed under conjugation. A Galois cover together with a coloring is called a *colored Galois cover*.
(2) Given a colored Galois cover $(V \overset{G}{\twoheadrightarrow} W, C)$ and a model $K \models T$, we define the set $X(V \overset{G}{\twoheadrightarrow} W, C)(K) := \{w \in W(K) \mid \mathrm{Ar}(w) \subset C\}$.

Note that $X(V \overset{G}{\twoheadrightarrow} W, C)$ is definable, i.e. there is a formula ϕ such that for any model $K \models T$ we have $\phi(K) = X(V \overset{G}{\twoheadrightarrow} W, C)(K)$.

Remark 2.6. (1) If $(V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover and $V' \overset{G'}{\twoheadrightarrow} W$ is a refinement with canonical map $\pi\colon G' \twoheadrightarrow G$, then we can also refine the coloring: by setting $C' := \{Q \in \mathrm{Psub}(G') \mid \pi(Q) \in C\}$, we get $X(V' \overset{G'}{\twoheadrightarrow} W, C') = X(V \overset{G}{\twoheadrightarrow} W, C)$.
(2) Similarly if $V'' \overset{G''}{\twoheadrightarrow} W''$ is a restriction of $f\colon V \overset{G}{\twoheadrightarrow} W$: in that case, set $C'' := \{Q \in C \mid Q \subset G''\}$. Then we get $X(V'' \overset{G''}{\twoheadrightarrow} W'', C'') = X(V \overset{G}{\twoheadrightarrow} W, C) \cap W''$.

2.3. **Galois stratifications.**

Definition 2.7. A *Galois stratification* $\mathcal{A}$ of a variety W is a finite family $(f_i\colon V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)_{i\in I}$ of colored Galois covers where the W_i form a partition of W into locally closed sub-varieties. We shall say that $\mathcal{A}$ *defines* the following subset $\mathcal{A}(K) \subset W(K)$, where $K \models T$ is a model:

$$\mathcal{A}(K) := \bigcup_{i\in I} X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)(K)$$

The data of a Galois stratification denoted by $\mathcal{A}$ will always be denoted by V_i, W_i, G_i, C_i, and analogously with primes for $\mathcal{A}'$, $\mathcal{A}''$, etc. This will not always be explicitely mentioned.

Definition 2.8. Suppose $\mathcal{A}$ and $\mathcal{A}'$ are two Galois stratifications. We say that $\mathcal{A}'$ is a *refinement* of $\mathcal{A}$, if:

- Each W_i is a union $\bigcup_{j\in J_i} W'_j$ for some $J_i \subset I'$.
- For each $i \in I$ and each $j \in J_i$, the Galois cover $V'_j \overset{G'_j}{\twoheadrightarrow} W'_j$ is a refinement of the restriction of the Galois cover $V_i \overset{G_i}{\twoheadrightarrow} W_i$ to the set W'_j.
- C'_j is constructed out of C_i as described in Remark 2.6, such that $X(V'_j \overset{G'_j}{\twoheadrightarrow} W'_j, C'_j) = X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i) \cap W'_j$.

By the third condition, $\mathcal{A}$ and $\mathcal{A}'$ define the same set.

One reason for Galois stratifications being handy to use is the following well-known lemma:

Lemma 2.9. *If $\mathcal{A}$ and $\mathcal{A}'$ are two Galois stratifications, then there exist refinements $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ of $\mathcal{A}$ resp. $\mathcal{A}'$ which differ only in the colorings.*

2.4. **Quantifier elimination to Galois stratifications.** We now can state the version of quantifier elimination which we will use. It is given in [6], Proposition 25.9. Applied to our situation, that proposition reads:

Lemma 2.10. *Suppose Gal is a torsion-free pro-cyclic group and k is any field. Then each definable set X of $T_{Gal,k}$ is already definable by a Galois stratification $\mathcal{A}$ (over k), i.e. for any $K \models T_{Gal,k}$, we have $X(K) = \mathcal{A}(K)$.*

Note that Proposition 25.9 of [6] requires that K is what Fried-Jarden call a "prefect Frobenius field"; this is indeed the case for any model of $T_{Gal,k}$.

3. Existence of χ_c

In this section we prove the existence of the map χ_c of Theorem 1.1. For the whole section, we fix a torsion-free pro-cyclic group *Gal* and a field k of characteristic zero. We also fix the theory $T := T_{Gal,k}$ we will work in.

3.1. **Overview of the proof.** We follow the ideas of [4]. The proof consists of three parts:

(1) Associate virtual motives to colored Galois covers, and, more generally, to Galois stratifications.
(2) Verify that this only depends on the set defined by the stratification. Using the quantifier elimination result Lemma 2.10, we thus get a map χ_c from the definable sets to the virtual motives.
(3) Check that this map χ_c has all the required properties: that it is invariant under definable bijections and compatible with disjoint union and products (so it defines a ring homomorphism $\mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$) and that it satisfies condition $(*)$ of Theorem 1.1.

To associate a virtual motive to a colored Galois cover, we use results of [1] where an "equivariant version" of the map $\chi_c \colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Mot}_k)$ is given. In this definition we have to be careful if $Gal \neq \hat{\mathbb{Z}}$; otherwise, the motive will really depend on the Galois cover and not only on the set defined by the cover. We will do this in Section 3.4.

In Section 3.5, we will extend the definition of χ_c to arbitrary Galois stratifications and check (2). The most important ingredient for this is that different colorings of a fixed Galois cover always define different sets. This is essentially a qualitative version of Chebotarev's density theorem for models of $T = T_{Gal,k}$, which we will prove in Section 3.3.

Finally, the properties (3) of χ_c will be verified in Section 3.6. The most laborious one is that χ_c is invariant under definable bijections. For this, we will first prove an additional lemma in Section 3.2, which replaces the quantitative part of Chebotarev's density theorem.

3.2. **Some fiber sizes.**

Lemma 3.1. *Suppose $f\colon V \overset{G}{\twoheadrightarrow} W$ is a Galois cover, $K \models T$ a model, $w \in W(K)$, and $C = \mathrm{Ar}(w)$ is the Artin symbol of w. Fix a group $Q \in C$. Then the number*

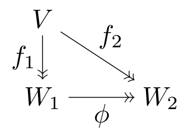
$$\#\left\{v \in V(\tilde{K}) \mid f(v) = w \wedge \mathrm{Dec}(v) = Q\right\}$$

of elements in the fiber of w with decomposition group Q is $\frac{|G|}{|C|}$.

Proof. For $Q \in C$, write $F(Q) := \{v \in V(\tilde{K}) \mid f(v) = w \wedge \mathrm{Dec}(v) = Q\}$ for the part of the fiber with decomposition group Q. We will show that for any two $Q, Q' \in C$, $F(Q)$ and $F(Q')$ have the same cardinality. As the whole preimage of w consists of $|G|$ elements, the lemma follows.

Given $Q, Q' \in C$, there exists a $g \in G$ such that $Q' = Q^g$. Suppose $v \in F(Q)$, i.e. the homomorphism $\rho\colon \mathrm{Gal}(\tilde{K}/K) \to G$ which satisfies $\sigma(v) = v.\rho(\sigma)$ (for all $\sigma \in \mathrm{Gal}(\tilde{K}/K)$) has image $\mathrm{im}\,\rho = Q$. Then $\sigma(v).g^{-1} = v.\rho(\sigma)g^{-1} = v.g^{-1}\rho(\sigma)^g$. Now $\mathrm{im}(\rho^g) = Q'$, so $v.g^{-1} \in F(Q')$, which means that g^{-1} maps $F(Q)$ to $F(Q')$. By the same argument, g maps $F(Q')$ to $F(Q)$, so we have a bijection. $\square$

Lemma 3.2. *Suppose we have the following diagram, where the maps $f_1\colon V \to W_1$ and $f_2\colon V \to W_2$ are Galois covers with groups G_1 resp. G_2. Note that we have naturally $G_1 \subset G_2$.*



Suppose additionally that C_1 is a coloring of $f_1\colon V \overset{G_1}{\twoheadrightarrow} W_1$ consisting of a single conjugacy class of groups and $C_2 := C_1^{G_2}$ is the induced coloring of $f_2\colon V \overset{G_2}{\twoheadrightarrow} W_2$. Then the image under ϕ of $X_1 := X(V \overset{G_1}{\twoheadrightarrow} W_1, C_1)$ is $X_2 := X(V \overset{G_2}{\twoheadrightarrow} W_2, C_2)$. In addition, ϕ restricts to a bijection $X_1 \to X_2$ if and only if $\frac{|G_1|}{|C_1|} = \frac{|G_2|}{|C_2|}$.

Proof. "$\phi(X_1) \subset X_2$": Suppose $K \models T$ is a model, $w_1 \in X_1(K)$ and $v \in f_1^{-1}(w_1)$ is a preimage in $V(\tilde{K})$. Then $\mathrm{Dec}(v) \in C_1$. But v is also a preimage of $\phi(w_1)$, so $\mathrm{Ar}(\phi(w_1))$ contains $\mathrm{Dec}(v)$ and is therefore equal to $C_1^{G_2}$.

"$\phi(X_1) \supset X_2$": Suppose $w_2 \in X_2(K)$ for some model $K \models T$. As $\mathrm{Ar}(w_2)$ contains C_1, we may choose a preimage $v \in V(\tilde{K})$ of w_2 with $\mathrm{Dec}(v) \in C_1$. In particular, $\mathrm{Dec}(v) \subset G_1$, which means that $\mathrm{Gal}(\tilde{K}/K)$ fixes $v.G_1 \in (V/G_1)(K) \cong W_1(K)$. So the image $f_1(v)$ lies in $W_1(K)$. As $\mathrm{Ar}(f_1(v))$ contains $\mathrm{Dec}(v)$, we have $\mathrm{Ar}(f_1(v)) = C_1$, so $f_1(v)$ is a preimage of w_2 lying in $X_1(K)$.

It remains to check that the fibers of $\phi\restriction_{X_1}$ have size one if and only if $\frac{|G_1|}{|C_1|} = \frac{|G_2|}{|C_2|}$. Indeed, we prove that the size of the fibers is $\frac{|G_2|\cdot|C_1|}{|C_2|\cdot|G_1|}$. Fix a group $Q \in C_1$. For any model $K \models T$ and any $w_2 \in X_2(K)$, consider the set $F_2 := \{v \in f_2^{-1}(w_2) \mid \mathrm{Dec}(v) = Q\}$. By Lemma 3.1, this set has $\frac{|G_2|}{|C_2|}$ elements.

For any $v \in F_2$, $f_1(v)$ lies in $X_1(K) \cap \phi^{-1}(w_2)$ (lying in $X_1(K)$ holds true by the same argument as in "$\phi(X_1) \supset X_2$"). So the sets $F_1(w_1) := \{v \in f_1^{-1}(w_1) \mid \mathrm{Dec}(v) = Q\}$, $w_1 \in X_1(K) \cap \phi^{-1}(w_2)$, form a partition of F_2. By Lemma 3.1, each set $F_1(w_1)$ has $\frac{|G_1|}{|C_1|}$ elements, so w_2 has $\frac{|G_2|\cdot|C_1|}{|C_2|\cdot|G_1|}$ preimages in $X_1(K)$. $\square$

3.3. **A version of Chebotarev's density theorem.** We will need the following (qualitative) version of Chebotarev's density theorem. Remember that we fixed a torsion-free pro-cyclic group *Gal*, a field k of characteristic zero, and the theory $T = T_{Gal,k}$. Concerning the permitted subgroups Psub, remember Definition 2.4.

Lemma 3.3. *For any Galois cover $V \overset{G}{\twoheadrightarrow} W$ and any conjugacy class $C \subset \mathrm{Psub}(G)$ of permitted subgroups of G, there exists a model $K \models T$ such that $X(V \overset{G}{\twoheadrightarrow} W, C)(K)$ is Zariski dense in $W(K)$.*

Remark 3.4. As K is PAC, $X(V \overset{G}{\twoheadrightarrow} W, C)(K)$ is even dense in $W(\tilde{K})$.

Proof of Lemma 3.3. First note that the condition "$X(V \overset{G}{\twoheadrightarrow} W, C)(K)$ is dense in $W(K)$" can be expressed by an infinite number of first order sentences: for every open subset $W_0 \subset W$, take the sentence which says that $X(V \overset{G}{\twoheadrightarrow} W, C) \cap W_0$ is not empty. Let T' be the theory T with these sentences added. Using the usual density theorem of Chebotarev, we will show that T' is finitely satisfyable; then, the claim follows by compactness. (In algebraic terms: we will construct a lot of fields satisfying some of the properties we would like to have; then an appropriate ultraproduct of these fields will do the job.)

Fix once and for all a ring $R \subset k$ which is finitely generated over $\mathbb{Z}$ and such that $V \overset{G}{\twoheadrightarrow} W$ can be defined over R. Choose a version $V_R \overset{G}{\twoheadrightarrow} W_R$ of $V \overset{G}{\twoheadrightarrow} W$ over R, i.e. such that $V = V_R \otimes_R k$, etc.

Let A be any finite subset of k, and let $W_0 \subset W_R \otimes_R R[A]$ be any open subset defined over $R[A]$. Out of this, we will construct a field $K' = K'_{A,W_0}$ together with a ring homomorphism $\phi\colon R[A] \to K'$ with the following properties: K' is an algebraic extension of a finite field, its Galois group is Gal, and $X(V_R \overset{G}{\twoheadrightarrow} W_R, C)(K') \cap W_0(K')$ is not empty, where we use ϕ to interpret the parameters of this formula (which lie in $R[A]$) in K'.

Before we give the construction of K', let us check that those fields indeed verify any finite part of T'. Of course, the constants for the elements of $R[A]$ in the language of T' are interpreted using ϕ and the remaining constants $k \setminus R[A]$ are interpreted anyhow.

All fields K'_{A,W_0} are perfect and have Galois group Gal. By choosing W_0 small enough, we can make K'_{A,W_0} satisfy any finite part of "$X(V \overset{G}{\twoheadrightarrow} W, C)$ is dense in W". To get a field K'_{A,W_0} which satisfies a finite part of the atomic diagram of k, we just have to choose A big enough; in particular to get $\phi(a) \neq \phi(a')$ for $a, a' \in k$, we put $\frac{1}{a-a'}$ into A. To get a finite part of PAC, by the Lang-Weil estimates we just have to ensure that K'_{A,W_0} is big enough. This can be done by adding elements of the form $\frac{1}{p}$ to A.

Now the only thing left to do is to construct the field K_{A,W_0} for any given A and W_0 as above. Let us start by specifying the objects we want to apply the usual density theorem of Chebotarev to. Tensor the Galois cover $V_R \overset{G}{\twoheadrightarrow} W_R$ with $\otimes_R R[A]$, restrict it to W_0, and denote the result by $V_0 \overset{G}{\twoheadrightarrow} W_0$. This is again a Galois cover, and both V_0 and W_0 are finitely generated over $\mathbb{Z}$. Define $\hat{C} := \{g \in G \mid \mathrm{Ppart}(\langle g \rangle) \in C\}$.

We apply Chebotarev's density theorem (in the version of [12], Theorem 7) to $V_0 \overset{G}{\twoheadrightarrow} W_0$ and $\hat{C}$. This yields the existence of a finite field F and an element $w \in W_0(F)$ such that the usual Artin symbol of w lies in $\hat{C}$. Note that here W_0 is interpreted as a scheme over $\mathbb{Z}$, so "$w \in W_0(F)$" means that we get a map $R[A] \to F$, and $W_0(F)$ has to be interpreted using this map.

To get our desired field K', we identify Gal with a subgroup of $\hat{\mathbb{Z}} \cong \mathrm{Gal}(\tilde{F}/F)$ in such a way that $\hat{\mathbb{Z}}/Gal$ is torsion-free. Then we define K' to be the subfield of

$\tilde{F}$ fixed by $Gal \subset \mathrm{Gal}(\tilde{F}/F)$. As promised it is an algebraic extension of a finite field, it has Galois group Gal, and it comes with a map $\phi\colon R[A] \to F \hookrightarrow K'$. It remains to check that the element w, interpreted as an element of $W_0(K')$, has Artin symbol C in our sense.

Denote the usual Frobenius automorphism $x \mapsto x^{|F|}$ in $\mathrm{Gal}(\tilde{F}/F)$ by σ_0. Choose a preimage $v \in V_0(\tilde{F})$ of w and let $\rho\colon \mathrm{Gal}(\tilde{F}/F) \to G$ be the group homomorphism satisfying $\sigma(v) = v.\rho(\sigma)$ for all $\sigma \in \mathrm{Gal}(\tilde{F}/F)$. Then the image of ρ is $\langle\rho(\sigma_0)\rangle$, and the image of $\rho{\restriction}_{Gal}$ is $\rho(Gal) = \mathrm{Ppart}(\langle\rho(\sigma_0)\rangle)$ (see the remark after Definition 2.4). As $\rho(\sigma_0)$ lies in $\hat{C}$, $\mathrm{im}(\rho{\restriction}_{Gal})$ lies in C, which is what we had to show. □

3.4. **Assigning a virtual motive to a colored Galois cover.** Gillet and Soulé [7] and Guillén and Navarro Aznar [8] have defined a map $\chi_c\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Mot}_k)$ assigning a virtual motive to any variety. As in [4], we use the following "equivariant version" of this result from [1].

For a finite group G, denote by $R(G)$ the group of virtual characters of G (i.e. $\mathbb{Z}$-linear combinations of irreducible characters of G) and by $\mathrm{K}_0(G, \mathrm{Var}_k)$ the Grothendieck ring of G-varieties. (As for the Galois covers, we let G act from the right.) The following is Theorem 6.1 of [1]:

Theorem 3.5. *There exists a unique map χ_c which associates to each finite group G, each G-variety V and each character $\alpha \in R(G)$ a virtual motive $\chi_c(G \looparrowright V, \alpha) \in \mathrm{K}_0(\mathrm{Mot}_k)$ such that:*

1. *For any G and α, the induced map $\chi_c(G \looparrowright \_, \alpha)\colon \mathrm{K}_0(G, \mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Mot}_k)$ is a group homomorphism.*
2. *For any G and V, $\chi_c(G \looparrowright V, \_)\colon R(G) \to \mathrm{K}_0(\mathrm{Mot}_k)$ is a group homomorphism.*
3. *If α_{reg} is the character of the regular representation of G, then $\chi_c(G \looparrowright V, \alpha_{\mathrm{reg}}) = \chi_c(V)$.*
4. *Suppose V is projective and smooth and $\alpha \in R(G)$ is irreducible and of dimension n_α. Then $\chi_c(G \looparrowright V, \alpha)$ is the class of the motive (V, p_α), where $p_\alpha := \frac{n_\alpha}{|G|}\sum_{g\in G} \alpha(g^{-1})[g] \in \mathbb{Q}[G]$ is the idempotent corresponding to α.*

We shall also need the following properties of χ_c.

Lemma 3.6. *Let G be a finite group, $H \subset G$ be a subgroup and V be a G-variety.*

1. *Suppose H is normal in G and denote the projection $G \twoheadrightarrow G/H$ by π. Then, for any character $\alpha \in R(G/H)$,*
$$\chi_c(G/H \looparrowright V/H, \alpha) = \chi_c(G \looparrowright V, \alpha \circ \pi)\,.$$
2. *For any character $\alpha \in R(H)$,*
$$\chi_c(G \looparrowright V, \mathrm{Ind}_H^G \alpha) = \chi_c(H \looparrowright V, \alpha)\,.$$
3. *Suppose V is isomorphic as a G-variety to $\dot{\bigcup}_{s\in G/H} Us$ for a H-variety U. Then, for any character $\alpha \in R(G)$,*
$$\chi_c(G \looparrowright V, \alpha) = \chi_c(H \looparrowright U, \mathrm{Res}_H^G \alpha)\,.$$
4. *Suppose that for $i = 1, 2$ we have a finite group G_i, a G_i-variety V_i, and a character α_i of G_i. Then*
$$\chi_c(G_1 \times G_2 \looparrowright V_1 \times V_2, \alpha_1 \otimes \alpha_2) = \chi_c(G_1 \looparrowright V_1, \alpha_1) \otimes \chi_c(G_2 \looparrowright V_2, \alpha_2)\,.$$

Proof. (1) to (3) are Proposition 3.1.2 of [4]. (4) is a straightforward computation using the definition of χ_c in [1]. □

$\chi_c(G \looparrowright V, \_)$ naturally extends to a map $R(G) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}} = \mathrm{K}_0(\mathrm{Mot}_k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the same properties.

Using this map χ_c we will now associate to any colored Galois cover $(V \overset{G}{\twoheadrightarrow} W, C)$ a motive $\chi_c(V \overset{G}{\twoheadrightarrow} W, C)$. This will be done by associating a virtual character $\alpha_C \in R(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ to any coloring C.

In the case of pseudo-finite fields, in [4] Denef-Loeser defined α_C to be 1 on the set $\{g \in G \mid \langle g \rangle \in C\}$ and 0 elsewhere. Just copying this definition does not work when the Galois group is not $\hat{\mathbb{Z}}$. The reason is that in a certain sense, the meaning of "$Q \in C$" depends on the Galois group. For example, "$\{1\} \in C$" means "just a little part of W" when $Gal = \hat{\mathbb{Z}}$, whereas when Gal is trivial, it means "the whole of W".

To get a working definition for α_C in the non-$\hat{\mathbb{Z}}$-case, one has to remember that the Artin symbol is the image of a certain map $\rho\colon\ Gal \to G$. Then one views Gal as a subgroup of $\hat{\mathbb{Z}}$ and considers extensions of ρ to $\hat{\mathbb{Z}}$, as described in the remark after Definition 2.4 and as it was done in the proof of Lemma 3.3. In this way one naturally gets the following definition:

$$\alpha_C(g) := \begin{cases} 1 & \text{if } \mathrm{Ppart}(\langle g \rangle) \in C \\ 0 & \text{otherwise}. \end{cases}$$

The first thing we have to check is that indeed α_C lies in $R(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ (and not just in $R(G) \otimes_{\mathbb{Z}} \tilde{\mathbb{Q}}$). This follows from the fact that $\alpha_C(g) = \alpha_C(g')$ if $\langle g \rangle = \langle g' \rangle$: we get that α_C can be written as $\mathbb{Q}$-linear combination of characters induced by trivial representations of cyclic subgroups of G.

Now we can associate a virtual motive to any colored Galois cover:

Definition 3.7. Given a colored Galois cover $(V \overset{G}{\twoheadrightarrow} W, C)$, define

$$\chi_c(V \overset{G}{\twoheadrightarrow} W, C) := \chi_c(G \looparrowright V, \alpha_C)\,.$$

3.5. **Assigning a virtual motive to a definable set.** We define the motive associated to a Galois stratification $\mathcal{A}$ in the following way:

$$\chi_c(\mathcal{A}) := \sum_{i \in I} \chi_c(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)$$

Lemma 3.8. *If two Galois stratifications $\mathcal{A}$ and $\mathcal{A}'$ of the same variety W define the same subset of W, then the corresponding motives $\chi_c(\mathcal{A})$ and $\chi_c(\mathcal{A}')$ are equal.*

Proof. By Lemma 2.9, there exist refinements $\tilde{\mathcal{A}}$ resp. $\tilde{\mathcal{A}}'$ of $\mathcal{A}$ resp. $\mathcal{A}'$ which differ only in the colorings, and the refinements define the same sets as the originals. So it is enough to check that the associated motives are invariant under refinement of the stratification and that if $\mathcal{A}$ and $\mathcal{A}'$ differ only in the colorings and define the same set, then even the colorings are the same.

Suppose first that $\mathcal{A} = (f_i\colon V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)_{i \in I}$ and $\mathcal{A}' = (f_i\colon V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i')_{i \in I}$ differ only in the colorings and that both define the same set. Consider a conjugacy class D of permitted subgroups of one of the groups G_i. By Lemma 3.3 there exists a model $K \models T$ and an element $w \in W_i(K)$ such that $\mathrm{Ar}(w) = D$. So as $\mathcal{A}(K)$ and $\mathcal{A}'(K)$ do not differ in w, C_i and C_i' do not differ in D.

To check that the virtual motive associated to a Galois stratification does not change under refinement, it is enough to verify the two following special cases:

(1)
$$\chi_c(f\colon V \overset{G}{\twoheadrightarrow} W, C) = \sum_{i\in I} \chi_c(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)$$

where $(f\colon V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover, (W_i) a decomposition of W into finitely many locally closed subsets, and $(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)$ is the restriction of $(V \overset{G}{\twoheadrightarrow} W, C)$ to W_i.

(2)
$$\chi_c(V \overset{G}{\twoheadrightarrow} W, C) = \chi_c(V' \overset{G'}{\twoheadrightarrow} W, C')$$

where $(V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover and $(V' \overset{G'}{\twoheadrightarrow} W, C')$ is a refinement.

Proof of (1):

$$\begin{aligned}\chi_c(V \overset{G}{\twoheadrightarrow} W, C) &= \chi_c(G \looparrowright V, \alpha_C)\\ &= \sum_i \chi_c(G \looparrowright f^{-1}(W_i), \alpha_C) \qquad \text{by Theorem 3.5 (1)}\\ &= \sum_i \chi_c(G_i \looparrowright V_i, \operatorname{Res}^G_{G_i} \alpha_C) \qquad \text{by Lemma 3.6 (3)}\\ &= \sum_i \chi_c(G_i \looparrowright V_i, \alpha_{C_i}) = \sum_i \chi_c(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)\end{aligned}$$

Proof of (2): By Lemma 3.6 (1), it is enough to check that $\alpha_{C'} = \alpha_C \circ \pi$. But indeed we have, for any $g' \in G'$:

$$\begin{aligned}\alpha_{C'}(g') = 1 &\iff \operatorname{Ppart}(\langle g'\rangle) \in C' \iff \pi(\operatorname{Ppart}(\langle g'\rangle)) \in C\\ &\iff \operatorname{Ppart}(\langle \pi(g')\rangle) \in C \iff \alpha_C(\pi(g')) = 1\,.\end{aligned}$$

□

Using this lemma (and Lemma 2.10 about the quantifier elimination), we are now able to associate virtual motives to definable sets:

Definition 3.9. Let X be a definable set of T. Then define the associated virtual motive $\chi_c(X)$ by choosing any Galois stratification $\mathcal{A}$ defining X and by letting $\chi_c(X) := \chi_c(\mathcal{A})$.

3.6. **The properties of χ_c.** We now check all the properties of χ_c we need for Theorem 1.1: χ_c should induce a ring homomorphism $\mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$, i.e. it has to be invariant under definable bijections and compatible with disjoint unions and products; and we want χ_c to satisfy condition $(*)$ of the theorem.

Compatibility with disjoint unions is clear. (Use a Galois stratification which is sufficiently fine to define all involved sets.) Let us now quickly verify compatibility with products and $(*)$, so that afterwards the only thing remaining is compatibility with definable bijections.

Lemma 3.10. *Suppose X_1 and X_2 are two definable sets. Then $\chi_c(X_1 \times X_2) = \chi_c(X_1) \otimes \chi_c(X_2)$.*

Sketch of proof. It is enough to verify the statement if $X_i = X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)$ for $i = 1, 2$.

Using the definition of the Artin symbol and by choosing a generator of Gal, one easily verifies that

$$X_1 \times X_2 = X(V_1 \times V_2 \overset{G_1 \times G_2}{\twoheadrightarrow} W_1 \times W_2, C)\,,$$

where $C = \{\langle (g_1, g_2)\rangle \subset G_1 \times G_2 \mid \langle g_1\rangle \in C_1, \langle g_2\rangle \in C_2\}$.

One then computes $\alpha_C((g_1, g_2)) = \alpha_{C_1}(g_1) \cdot \alpha_{C_1}(g_1)$ for $g_1 \in G_1$ and $g_2 \in G_2$. For this, the following formula is helpful. Let m be the product of all prime factors of $|G_1 \times G_2|$ (with multiplicity) not lying in P. Then for any $g \in G_1 \times G_2$ we have $\mathrm{Ppart}(\langle g\rangle) = \langle g^m\rangle$.

Finally, apply Lemma 3.6 (4). □

Lemma 3.11. *Suppose $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover such that all prime factors of $|G|$ lie in P (where P is the set of primes such that $Gal = \prod_{p\in P} \mathbb{Z}_p$). Then the map χ_c of Definition 3.9 satisfies*

$$(*) \qquad \chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \frac{1}{|G|}\chi_c(V)\,.$$

Proof. Remember the definition: $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \chi_c(G \looparrowright X, \alpha_{\{1\}})$ where $\alpha_{\{1\}}(g) = 1$ if $\mathrm{Ppart}(\langle g\rangle) = \{1\}$ and $\alpha_{\{1\}}(g) = 0$ otherwise. As all prime factors of $|G|$ lie in P, we have $\mathrm{Ppart}(\langle g\rangle) = \langle g\rangle$, so $\alpha_{\{1\}} = \frac{1}{|G|}\alpha_{\mathrm{reg}}$, where α_{reg} is the character of the regular representation of G. By Theorem 3.5 (2) and (3), we get $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, C)) = \frac{1}{|G|}\chi_c(V)$. □

The last piece of work is compatibility with definable bijections:

Lemma 3.12. *Suppose $X \subset \mathbb{A}^n$ and $X' \subset \mathbb{A}^{n'}$ are two definable sets and suppose there is a definable bijection ϕ from X to X'. Then the associated virtual motives $\chi_c(X)$ and $\chi_c(X')$ are the same.*

Proof. We have definable bijections between the graph of ϕ and X and X', induced by the projections $\mathbb{A}^{n+n'} \twoheadrightarrow \mathbb{A}^n$ and $\mathbb{A}^{n+n'} \twoheadrightarrow \mathbb{A}^{n'}$. So it is enough to check the statement when the bijection itself is induced by a projection.

Choose a Galois stratification $\mathcal{A} = (V_i \overset{G_i}{\twoheadrightarrow} W_i)_{i\in I}$ defining X. By compatibility of χ_c with disjoint union, it is enough to show the statement for all the restricted bijections $W_i \to \phi(W_i)$, so we can suppose that X is a subset of a variety W and defined by a single colored Galois cover: $X = X(f\colon V \overset{G}{\twoheadrightarrow} W, C)$.

Now choose a Galois stratification $\mathcal{A}'$ of X'. Again it is enough to show the statement for all restricted bijections $\phi^{-1}(W'_i) \cap W \to W'_i$, so we can suppose that X' is defined by a single colored Galois cover: $X' = X(f'\colon V' \overset{G'}{\twoheadrightarrow} W', C')$. Note that X is still defined by a single Galois cover as we can simply restrict the previous one to $\phi^{-1}(W'_i) \cap W$. ϕ will now denote the map from W to W'.

By induction, we can remove parts of W' of smaller dimension (and restrict the two Galois covers appropriately). We will do this in the next paragraph.

The next thing we claim is that we can suppose $V = V'$. By Lemma 3.3 (our version of Chebotarev) and Remark 3.4, there exists a model $K' \models T$ such that $X'(K')$ is dense in $W'(\tilde{K}')$. As the image of ϕ contains $X'(K')$, ϕ is dominant. Similarly there is a model $K \models T$ such that $X(K)$ is dense in $W(\tilde{K})$; as ϕ is bijective on $X(K)$, we get $\dim W = \dim W'$, and ϕ is generically finite. Therefore, after cutting away parts of smaller dimension, we can refine both Galois covers such that we get $V = V'$ and $G \subset G'$.

Decompose the set C into conjugacy classes C_i. To each $X(V \overset{G}{\twoheadrightarrow} W, C_i) \subset X$ we apply Lemma 3.2. The result is that ϕ restricted to $X(V \overset{G}{\twoheadrightarrow} W, C_i)$ has image $X(V \overset{G'}{\twoheadrightarrow} W', C_i^{G'})$, so we may again cut ϕ into parts and suppose that C consists of a single conjugacy class and $C' = C^{G'}$. Lemma 3.2 also yields $\frac{|C|}{|G|} = \frac{|C'|}{|G'|}$. This equation in particular implies that for any $Q \in C$ we have $\mathrm{N}_{G'}(Q) \subset G$.

In view of Lemma 3.6 (2), we now only have to check that $\alpha_{C'} = \mathrm{Ind}_G^{G'} \alpha_C$.

Set

$$\hat{C} := \{\langle g \rangle \subset G \mid \alpha_C(g) = 1\} = \{\langle g \rangle \subset G \mid \mathrm{Ppart}(\langle g \rangle) \in C\} \quad \text{and}$$
$$\hat{C}' := \{\langle g' \rangle \subset G' \mid \alpha_{C'}(g') = 1\} = \{\langle g' \rangle \subset G \mid \mathrm{Ppart}(\langle g' \rangle) \in C'\}\,.$$

We want to understand the relation between $\hat{C}$ and $\hat{C}'$. For this, consider the map $\eta\colon \hat{C}' \to C', Q \mapsto \mathrm{Ppart}(Q)$. It maps $\hat{C}$ to C. We claim that $\hat{C}$ is exactly the preimage of C under η. For this, we have to verify that for any group $Q \in \hat{C}'$ with $\mathrm{Ppart}(Q) \in C$, we already have $Q \subset G$. Indeed: Q is abelian, so it is contained in $\mathrm{N}_{G'}(\mathrm{Ppart}(Q))$, and $\mathrm{N}_{G'}(\mathrm{Ppart}(Q))$ is contained in G.

Now using that C consists of a single conjugacy class and that η commutes with conjugation, we arrive at two conclusions: $\hat{C}' = \hat{C}^{G'}$ and $\frac{|\hat{C}'|}{|C'|} = \text{fibersize of } \eta = \frac{|\hat{C}|}{|C|}$.

Using this, we can finally compute $\mathrm{Ind}_G^{G'} \alpha_C$. For any $g' \in G'$, we have

$$\mathrm{Ind}_G^{G'} \alpha_C(g') = \frac{1}{|G|} \#\{h \in G' \mid \langle h g' h^{-1} \rangle \in \hat{C}\}\,.$$

This is zero if $\langle g' \rangle \notin \hat{C}^{G'} = \hat{C}'$. Otherwise:

$$\cdots = \frac{1}{|G|} \cdot |\hat{C}| \cdot |\,\mathrm{N}_{G'}(\langle g' \rangle)| = \frac{|\hat{C}|}{|G|} \cdot \frac{|G'|}{|\hat{C}'|} = 1\,.$$

(In the last equality, we combine $\frac{|C|}{|G|} = \frac{|C'|}{|G'|}$ and $\frac{|\hat{C}'|}{|C'|} = \frac{|\hat{C}|}{|C|}$.) □

4. Other proofs on χ_c

In this section we prove the remaining statements on the map χ_c, namely the uniqueness statement of Theorem 1.1 in the pseudo-finite case and compatibility of the map χ_c with change of the base field k.

4.1. **The uniqueness statement.** We will use following properties of χ_c: it extends the known map $\chi_c\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Mot}_k)$, it is invariant under definable bijections, it is compatible with disjoint unions, and for any Galois cover $V \overset{G}{\twoheadrightarrow} W$, the equality

$$(*) \qquad \chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \frac{1}{|G|} \chi_c(V)$$

holds.

Note that we will not need that χ_c is compatible with products.

Proof of uniqueness in Theorem 1.1. By the almost quantifier elimination (Lemma 2.10) and compatibility with disjoint unions, it is enough to prove uniqueness for definable sets of the form $X(V \overset{G}{\twoheadrightarrow} W, C)$, where $(V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover and $C = Q^G$ consists of a single conjugacy class of cyclic subgroups of G.

We proceed by induction on $|G|$ and $|Q|$. (We will suppose that the statement is true for G of the same size and Q smaller and vice versa.)

Suppose first that Q is not normal in G. Let $G' := \mathrm{N}_G(Q)$ be its normalizer and $W' := V/G'$. Note that $C' := Q^{G'} = \{Q\}$. By induction, we know $\chi_c(X(V \overset{G'}{\twoheadrightarrow} W', C'))$. We have $\frac{|G'|}{|C'|} = \frac{|G|}{|C|}$, so Lemma 3.2 implies that the map $W' \to W$ induces a bijection $X(V \overset{G'}{\twoheadrightarrow} W', C') \to X(V \overset{G}{\twoheadrightarrow} W, C)$. So by assumption $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, C)) = \chi_c(X(V \overset{G'}{\twoheadrightarrow} W', C'))$.

Now suppose Q is normal in G (and in particular $C = \{Q\}$). Let $G' := G/Q$ and $V' := V/Q$. We know $\chi_c(X(V' \overset{G'}{\twoheadrightarrow} W, \{1\}))$ by $(*)$, and we have $X(V' \overset{G'}{\twoheadrightarrow} W, \{1\}) = X(V \overset{G}{\twoheadrightarrow} W, C_1)$, where $C_1 = \{Q_1 \in \mathrm{Psub}(G) \mid Q_1 \subset Q\}$ consists of all (cyclic) subgroups of G contained in Q. But for any strict subgroup $Q_1 \subsetneq Q$, we know $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, Q_1^G))$ by induction. So $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, Q))$ is the only (up to now) unknown term in the equation

$$\chi_c(X(V \overset{G}{\twoheadrightarrow} W, C_1)) = \sum_{\substack{C_2 \subset C_1 \\ C_2 \text{ one conjugacy class}}} \chi_c(X(V \overset{G}{\twoheadrightarrow} W, C_2))\,.$$

□

4.2. **Change of the base field k.** In this section, we prove Proposition 1.2, i.e. that our map χ_c is essentially independent of the base field k. So suppose we have a field k' containing k and remember that we have canonical homomorphisms $\otimes_k k'\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(T_{Gal,k'})$ and $\otimes_k k'\colon \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}} \to \mathrm{K}_0(\mathrm{Mot}_{k'})_{\mathbb{Q}}$.

Proof of Proposition 1.2. The claim of the proposition is $\chi_c(X \otimes_k k') = \chi_c(X) \otimes_k k'$ for any definable set X of $T_{Gal,k}$. It is enough to check the statement for generators of $\mathrm{K}_0(T_{Gal,k})$, i.e. we may suppose that X is of the form $X = X(V \overset{G}{\twoheadrightarrow} W, C)$ where $(V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover defined over k.

By tensoring with k' and choosing one irreducible component V' of $V \otimes_k k'$, we get a Galois cover $V' \overset{G'}{\twoheadrightarrow} W'$ over k', where G' is a subgroup of G. One easily verifies $X \otimes_k k' = X(V' \overset{G'}{\twoheadrightarrow} W', C')$ where $C' := \{Q \in C \mid Q \subset G'\}$.

Using Lemma 3.6 we get

$$\begin{aligned}\chi_c(X(V' \overset{G'}{\twoheadrightarrow} W', C')) &= \chi_c(G' \looparrowright V', \alpha_{C'}) = \chi_c(G' \looparrowright V', \mathrm{Res}^G_{G'} \alpha_C) \\ &= \chi_c(G \looparrowright V \otimes_k k', \alpha_C) = \chi_c(G \looparrowright V, \alpha_C) \otimes_k k' \\ &= \chi_c(X(V \overset{G}{\twoheadrightarrow} W, C)) \otimes_k k'\,.\end{aligned}$$

□

5. Maps between Grothendieck rings

In this section we first prove the existence of the map θ_ι between the different Grothendieck rings $\mathrm{K}_0(T_{Gal,k})$ (Theorem 1.3) and then apply this to get Proposition 1.4. Finally we check a compatibility between the maps θ_ι and the maps χ_c.

5.1. **Existence of the maps θ_ι.** Remember the statement of the theorem. We have a field k and an inclusion of torsion-free pro-cyclic groups $\iota\colon Gal_2 \hookrightarrow Gal_1$. For simplicity, we will now identify Gal_2 with $\iota(Gal_2) \subset Gal_1$. Denote by $T_i := T_{Gal_i,k}$ the theory of perfect PAC fields with Galois group Gal_i and which contain k.

The map $\theta := \theta_\iota\colon \mathrm{K}_0(T_2) \to \mathrm{K}_0(T_1)$ was defined as follows. Any model K_1 of T_1 yields a model $K_2 := \tilde{K}_1^{Gal_2}$ of T_2. For any $X_2 \subset \mathbb{A}^n$ definable in T_2, we defined $\theta(X_2)(K_1) = X_2(K_2) \cap K_1^n$.

What we have to check is:

1. $X_2(K_2) \cap K_1^n$ is definable (uniformly for all K_1).
2. If there is a definable bijection $X_2 \to X_2'$ in T_2, then there is also a definable bijection $\theta(X_2) \to \theta(X_2')$ in T_1.
3. θ is a ring homomorphism, i.e. compatible with disjoint unions and products.

The third statement is clear by definition.

(1) Any definable set X_2 of T_2 can be written as disjoint union of sets of the form $X(f\colon V \overset{G}{\twoheadrightarrow} W, C_2)$, where C_2 is a conjugacy class of permitted subgroups of G, so it is enough to prove that θ maps such sets to definable ones. We claim: $\theta(X(V \overset{G}{\twoheadrightarrow} W, C_2)) = X(V \overset{G}{\twoheadrightarrow} W, C_1)$, where C_1 is defined as follows: Let M be the set of homomorphisms $\rho_1\colon\ Gal_1 \to G$ such that $\rho_1(Gal_2) \in C_2$. Then C_1 is the set of images of these homomorphisms M. In a formula:

$$C_1 = \{\operatorname{im}\rho_1 \mid \rho_1\colon\ \mathrm{Gal}_1 \to G, \rho_1(\mathrm{Gal}_2) \in C_2\}.$$

We have to check: For any model K_1 of T_1 and any element $w \in W(K_1)$, we have $w \in X(V \overset{G}{\twoheadrightarrow} W, C_1)(K_1)$ if and only if $w \in X(V \overset{G}{\twoheadrightarrow} W, C_2)(K_2)$, where $K_2 = \tilde{K}_1^{Gal_2}$ as above.

Choose an element $v \in V(\tilde{K_1})$ with $f(v) = w$. We get a homomorphism $\rho_1\colon\ Gal_1 \to G$ defined by $\sigma(v) = v.\rho_1(\sigma)$ for any $\sigma \in Gal_1$. Of course the restriction $\rho_2 := \rho_1\restriction_{Gal_2}$ satisfies the same property. By definition, we have $w \in X(V \overset{G}{\twoheadrightarrow} W, C_1)(K_1)$ if and only if $\operatorname{im}\rho_1 \in C_1$ and $w \in X(V \overset{G}{\twoheadrightarrow} W, C_2)(K_2)$ if and only if $\operatorname{im}\rho_2 = \rho_1(\mathrm{Gal}_2) \in C_2$. So we have to check that for any $\rho_1\colon\ Gal_1 \to G$ we have $\operatorname{im}\rho_1 \in C_1$ if and only if $\rho_1(\mathrm{Gal}_2) \in C_2$.

"$\Leftarrow$" is clear by the definition of C_1.

"$\Rightarrow$": Suppose $Q_1 := \operatorname{im}\rho_1 \in C_1$. By the definition of C_1, there is a homomorphism $\rho_1' \in M$ with $\operatorname{im}\rho_1' = Q_1$. As Gal_1 is pro-cyclic, homomorphisms $Gal_1 \to Q_1$ are determined by the image of a generator, so we can write $\rho_1 = \alpha \circ \rho_1'$ for some automorphism $\alpha \in \operatorname{Aut}(Q_1)$. As Q_1 is cyclic, all its subgroups are characteristic subgroups, so $\rho_1(Gal_2) = \alpha(\rho_1'(Gal_2)) = \rho_1'(Gal_2) \in C_2$. This implies $\rho_1 \in C_1$.

(2) Suppose $X_2 \subset \mathbb{A}^n$ and $X_2' \subset \mathbb{A}^{n'}$ are two definable sets in T_2 and $f\colon X_2 \to X_2'$ is a definable bijection. We have to show that there is a T_1-definable bijection $\theta(X_2) \to \theta(X_2')$. Indeed, we will check that $\theta(f)$ is such a bijection. In other words we have to verify the following statement:

Let K_1 be any model of T_1 and $K_2 = \tilde{K}_1^{Gal_2}$. Then for any $x \in X_2(K_2)$ and $x' := f(x) \in X_2'(K_2)$, we have $x \in K_1^n$ if and only if $x' \in K_1^{n'}$.

Suppose $x \notin K_1^n$. Then there exists a $\sigma \in \mathrm{Gal}(K_2/K_1)$ moving x. But $\sigma(X_2(K_2)) = X_2(K_2)$, so $\sigma(x) \in X_2$. As f is injective on $X_2(K_2)$, this implies $\sigma(f(x)) = f(\sigma(x)) \neq f(x)$, so $f(x) \notin K_1^{n'}$.

The other direction works analogously. □

5.2. χ_c **is not injective.** As an example application of the maps θ_ι, we will now prove Proposition 1.4. To this end, we will construct a pair of definable sets X_1 and X_2 such that $\chi_c(X_1) = \chi_c(X_2)$ but $\chi_c(\theta_\iota(X_1)) \neq \chi_c(\theta_\iota(X_2))$ for a suitable map $\iota\colon Gal \hookrightarrow Gal$. (In fact, we will construct a whole bunch of such pairs.)

Proof of Proposition 1.4. Remember that Gal is a non-trivial subgroup of $\hat{\mathbb{Z}}$, i.e. $Gal = \prod_{p \in P} \mathbb{Z}_p$, where P is a non-empty set of primes.

For $n \in \mathbb{N}_{\geq 1}$, consider the group homomorphism $\iota\colon Gal \hookrightarrow Gal, \sigma \mapsto \sigma^n$. Applying Theorem 1.3 to this map gives an endomorphism θ_n of $\mathrm{K}_0(T_{\hat{\mathbb{Z}},k})$, which can be explicitly computed on sets defined by Galois covers as follows. Let $(V \overset{G}{\twoheadrightarrow} W, C_2)$ be a colored Galois cover. The computation in the proof of Theorem 1.3 shows that $\theta_n(X(V \overset{G}{\twoheadrightarrow} W, C_2)) = X(V \overset{G}{\twoheadrightarrow} W, C_1)$, where $C_1 = \{Q \in \mathrm{Psub}(G) \mid Q^n \in C_2\}$ consists of those permitted subgroups of G whose subgroups of n-th powers lie in C_2.

Note that θ_n is interesting only if n has prime factors which lie in P; otherwise, n and $|Q|$ are coprime for any permitted subgroup $Q \subset G$, which implies $Q = Q^n$, $C_1 = C_2$, and $\theta_n = \mathrm{id}$.

Now let $V \overset{G}{\twoheadrightarrow} W$ be any non-trivial Galois cover such that all prime factors of $|G|$ lie in P, and define $X := X(V \overset{G}{\twoheadrightarrow} W, \{\mathrm{id}\})$. By condition $(*)$ of Theorem 1.1, we have $\chi_c(X) = \frac{1}{G}\chi_c(V)$, so $\chi_c(X \times G) = \chi_c(V)$. (Here G is interpreted as a discrete set.) However, we will see that for $n = |G|$, we have $\chi_c(\theta_n(X \times G)) \neq \chi_c(\theta_n(V))$.

As θ_n is the identity on $\mathrm{K}_0(\mathrm{Var}_k)$, we have $\theta_n(V) = [V]$. On the other hand, the subgroup of n-th powers of any cyclic subgroup of G is trivial, so $\theta_n(X) = [X(V \overset{G}{\twoheadrightarrow} W, \mathrm{Psub}(G))] = [W]$ and $\theta_n(X \times G) = [W \times G]$. But V and $W \times G$ are two varieties with a different number of irreducible components, so $\chi_c(\theta_n(X \times G)) \neq \chi_c(\theta_n(V))$. □

5.3. **Compatibility of χ_c and θ_ι.** We prove the following compatibility statement:

Proposition 5.1. *Suppose k is a field of characteristic zero and $Gal_2 \subset Gal_1$ are two torsion-free pro-cyclic groups such that $Gal_1 \,/\, Gal_2$ is torsion-free, too. We use the following notation: $T_i := T_{Gal_i,k}$ (for $i = 1, 2$) are the corresponding theories, $\chi_c^i\colon \mathrm{K}_0(T_i) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ are the maps of Theorem 1.1, and $\theta\colon \mathrm{K}_0(T_2) \to \mathrm{K}_0(T_1)$ is the map provided by Theorem 1.3 applied to the inclusion $Gal_2 \subset Gal_1$. Then we have:*

$$\chi_c^2 = \chi_c^1 \circ \theta\,.$$

Proof. For $i = 1, 2$ let P_i be the set of primes such that $Gal_i = \prod_{p \in P_i} \mathbb{Z}_p$. We have $P_2 \subset P_1$, and Gal_2 is just the factor of Gal_1 corresponding to P_2. We will write Psub_i resp. Ppart_i for the permitted subgroups and the permitted part to distinguish between the two Galois groups.

We only have to verify the statement for sets of the form $X(V \stackrel{G}{\twoheadrightarrow} W, C_2)$, where $(V \stackrel{G}{\twoheadrightarrow} W, C_2)$ is a colored Galois cover for T_2. By the proof of Theorem 1.3, we have $\theta(X(V \stackrel{G}{\twoheadrightarrow} W, C_2)) = X(V \stackrel{G}{\twoheadrightarrow} W, C_1)$, where C_1 consists of the images of those maps $\rho\colon\ Gal_1 \to G$ which satisfy $\rho(Gal_2) \in C_2$. As Gal_2 is a direct factor of Gal_1, we get $C_1 = \{Q \in \mathrm{Psub}_1(G) \mid \mathrm{Ppart}_2(Q) \in C_2\}$.

Now remember Definition 3.7: $\chi_c^i(X(V \stackrel{G}{\twoheadrightarrow} W, C_i)) = \chi_c(G \looparrowright V, \alpha_{C_i})$, where

$$\alpha_{C_i}(g) := \begin{cases} 1 & \text{if } \mathrm{Ppart}_i(\langle g\rangle) \in C_i \\ 0 & \text{otherwise.} \end{cases}$$

But $\mathrm{Ppart}_1(\langle g\rangle) \in C_1$ if and only if $\mathrm{Ppart}_2(\mathrm{Ppart}_1(\langle g\rangle)) = \mathrm{Ppart}_2(\langle g\rangle) \in C_2$, so $\alpha_{C_1} = \alpha_{C_2}$, and the claim is proven. □

6. Open problems

6.1. **Uniqueness of χ_c.** In the case of pseudo-finite fields, the conditions given in Theorem 1.1 are enough to render χ_c unique. One would like to have a similar uniqueness statement in the other cases. Unfortunately, the condition

$$(*) \qquad \chi_c(X(V \stackrel{G}{\twoheadrightarrow} W, \{1\})) = \frac{1}{|G|}\chi_c(V)$$

is false in general if $|G|$ has prime factors not in P (where $Gal = \prod_{p\in P}\mathbb{Z}_p$). For algebraically closed fields for example, we have $\chi_c(X(V \stackrel{G}{\twoheadrightarrow} W, \{1\})) = \chi_c(W)$, which is not equal to $\frac{1}{|G|}\chi_c(V)$ unless G is trivial.

The first question is: is the weak version of $(*)$ (when one requires all prime factors of $|G|$ to lie in P) enough to get uniqueness? And if not: is there some other nice condition rendering χ_c unique? One fact suggesting that the weak condition could already be strong enough is that this is true indeed for algebraically closed fields.

6.2. **From motives to measure.** The parallels between the definitions of the virtual motive associated to a definable set and the measure of such a set ([2], [9]) suggest that one should be able to extract the measure from the motive. More precisely, fix a perfect PAC field K of characteristic zero with pro-cyclic Galois group Gal. Note that there are two theories around now: $T_{Gal,K}$, the theory of pseudo-finite fields containing K (which is not complete) and $\mathrm{Th}(K)$, the (complete) theory of K itself.

Denote by $\dim\colon\ \mathrm{K}_0(\mathrm{Th}(K)) \to \mathbb{N}$ the dimension of [3] (which needs not coincide with the usual dimension for varieties: only components "visible over K" are considered) and by $\mu\colon\ \mathrm{K}_0(\mathrm{Th}(K)) \to \mathbb{Q}$ the measure of [9]. The question is whether a dotted map in the following diagram exists making the diagram commutative.

$$\begin{array}{ccc} \mathrm{K}_0(T_K) & \twoheadrightarrow & \mathrm{K}_0(\mathrm{Th}(K)) \\ \downarrow \chi_c & & \downarrow (\dim, \mu) \\ \mathrm{K}_0(\mathrm{Mot}_K)_{\mathbb{Q}} & \dashrightarrow & \mathbb{N} \times \mathbb{Q} \end{array}$$

If K is algebraically closed, then this is obviously true: In this case $\mu(V)$ is just the number of irreducible components of V, and both this and the dimension of V (which is the usual one in this case) can be seen in the corresponding motive.

If K is pseudo-finite, this is true, too: Let X be a definable set of $T_{Gal,K}$. Then it makes sense to speak about $X(F)$ for finite fields F of almost all characteristics.

Lemma 3.3.2 of [4] states that for almost all characteristics, the number of points $|X(F)|$ is encoded in the motive. (Not very surprisingly, it is the trace of the Frobenius automorphism on the motive.) The dimension and the measure of X in K can be computed from these cardinalities.

The way one extracts the dimension and the measure from the motive seems quite different in the two above cases. This suggests that one might get interesting new insights by generalizing this to arbitrary procyclic Galois groups.

6.3. **Larger Galois groups for the maps θ_ι.** The quantifier elimination result of [6] does not only work for fields with pro-cyclic Galois groups, but for some larger Galois groups as well. (The Galois group has to satisfy what Fried-Jarden call the "embedding property".) It seems plausible that Theorem 1.3 should be generalizable to this context as well. However the proof will need some modifications. Indeed for $Gal_1 = \hat{\mathbb{Z}} * \hat{\mathbb{Z}} = \langle a, b\rangle$ and $Gal_2 = \langle a\rangle \subset Gal_1$, one can construct a T_2-definable set $X = X(V \overset{G}{\twoheadrightarrow} W, C)$ such that $\theta(X)$ is not definable using the same Galois cover $V \overset{G}{\twoheadrightarrow} W$.

6.4. **Larger Galois groups for the maps χ_c.** Another natural question is whether the map χ_c can also be defined for fields with larger Galois group. However, in [9] we already showed that the measure of [2] does not extend to this generality. Indeed, no measure exists for example if the Galois group is $\hat{\mathbb{Z}} * \hat{\mathbb{Z}}$. This suggests that it is neither possible to associate motives to definable sets of such theories. Probably, $T_{\hat{\mathbb{Z}}*\hat{\mathbb{Z}},k}$ contains too many definable bijections so that the corresponding Grothendieck ring gets too small. One might even hope to show that $\mathrm{K}_0(T_{\hat{\mathbb{Z}}*\hat{\mathbb{Z}},k})$ is trivial.

6.5. **What exactly do we know about $\mathrm{K}_0(T_{Gal,k})$?** We showed that the maps χ_c do not yield the full information about the definable sets and we showed how additional information can be obtained using the maps θ_ι. The question is now: do we get all additive information using both maps together? More precisely, a strong version of this would be:

Question 6.1. *Suppose X_1 and X_2 are two definable sets in $T_{Gal,k}$, and suppose that for any injective endomorphism $\iota\colon Gal \hookrightarrow Gal$ we have $\chi_c(\theta_\iota(X_1)) = \chi_c(\theta_\iota(X_2))$. Does this imply $[X_1] = [X_2]$ in $\mathrm{K}_0(T_{Gal,k})$?*

This is of course a very difficult question (and I am not so sure if I really believe that the answer is positive), as in particular it would imply that χ_c is injective for algebraically closed fields. In fact the question I would prefer to ask is: is Question 6.1 true "up to varieties", or, "up to whatever is missing in the case of algebraically closed fields"? The first open problem here is to give a precise meaning to this last question.

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