DECIDABILITY OF THE THEORY OF MODULES OVER COMMUTATIVE VALUATION DOMAINS

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ABSTRACT. We prove that, if V is an effectively given commutative valuation domain such that its value group is dense and archimedean, then the theory of all V-modules is decidable.

1. INTRODUCTION

A classical strategy to prove the decidability of the theory T_R of all modules over a given ring R is to 'eliminate quantifiers', that is, to translate uniformly any sentence σ in the language of R-modules into a simpler equivalent sentence σ' without quantifiers or where quantifiers are as less as possible, such that checking the truth of σ' in R-modules becomes almost trivial.

This is exactly the way Wanda Szmielew used to prove her capital result opening this line of research [8]: the theory of abelian groups (that is, $\mathbb{Z}$-modules) is decidable. A famous Baur–Monk theorem (see [4, Cor. 2.13]) gives a good push in a general case, over an arbitrary ring R: every sentence is equivalent in T_R to a boolean combination of 'invariant' sentences (which are $\forall\exists$ sentences, so that we have an elimination of quantifiers down to $\forall\exists$ level). Unfortunately the structure of invariant sentences can be extremely complicated, which often makes a further syntactical analysis incredibly hard.

A more modern and powerful way to prove decidability of the theory of all modules over a ring is to use the Ziegler spectrum of R, Zg_R, a topological space whose points are indecomposable pure-injective R-modules. A good account of this approach and an overview of existing results can be found in [4, Ch. 17] or unpublished M. Prest's notes [5]. Although these ideas have been circulated for quite a while, there are a few examples where this approach was put to the full force. The problem is that to make it work

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we should collect a substantial amount of information about the Ziegler spectrum of R, both about points and topology. Even for relatively moderate rings this is a problem of scaring complexity.

If V is a commutative valuation domain, a complete classification of points of Zg_V is known from [9], and a satisfactory description of the topology is also available (see for example [6]). Thus it seems reasonable to expect that a characterization of (countable) commutative valuation domains with a decidable theory of modules should not be a very hard problem. For instance, if V is finite, then it has a finite representation type, hence T_V is a decidable theory. It follows from Szmielew's result [8] that, for every prime p, the theory of all modules over the localization $\mathbb{Z}_{(p)}$ (which is a commutative valuation domain) is decidable.

An easy generalization of this result (that can be also derived from [1]) is that the theory of all modules over an effectively given noetherian commutative valuation domain is decidable. Thus the answer is known for (effectively given) commutative valuation domains whose value group is isomorphic to the ordered group $(\mathbb{Z}, +, \leq)$ (that is, for discrete rank one commutative valuation domains).

In this paper we consider an opposite case: when the value group of a rank one commutative valuation domain is densely ordered (say, orderly isomorphic to the rationals). Additional difficulties we encounter in this case are that the Ziegler spectrum of V is uncountable, and even (see [7, Thm. 12.12]) there exists a super-decomposable pure-injective V-module.

We show that none of these appearing obstacles affects decidability. Namely, we prove that, if V is a commutative valuation domain with a densely ordered archimedean value group, and V is effectively given in a sense we are going to explain later, then the theory of all modules over V is decidable.

As it should be, the proof of this result relies on the Ziegler spectrum approach as it was outlined in [9] or [4]. We also have in mind (though suppress in proofs) a geometrical interpretation of positive-primitive types over commutative valuation domains as in [7, Ch. 12]. Thus to decide whether a given sentence holds true in the theory of all V-modules, we should answer a question about a configuration of rectangles and lines on the plane. If the value group of V is densely ordered and archimedean this approach provides us with a clear picture convertible (though with some technicalities) into a formal proof. Drawing diagrams backs most proofs of this paper, and we doubt that they could have been worked out or understood otherwise.

We separate our proof of decidability in two cases: when the residue field of V is infinite or finite. The proof in the infinite case is more conceptual and relies mostly on the usage of Ziegler topology. As it is quite common, the finite case is essentially more difficult, because a combinatorics of finite invariants comes in play. Luckily we show that, if the value group of V is dense, then finite invariants are rather rare, hence the proofs are still bearable.

An ideal answer we would expect in general case (that is, for arbitrary countable commutative valuation domains) is the following: the theory T_V of all V-modules is decidable if and only if some questions in the first order theory of V (as a ring) can be answered effectively. Indeed this is what happens in the dense archimedean case, as the condition 'V effectively given' just has this content. Anyhow we show that, if a value group of V is non-archimedean, then some non first-order parts of the theory of V can be encoded in the (first order) theory of V-modules.

Thus the case, when the value group of V is not archimedean (or not dense), appears to be essentially more difficult and may require tremendous combinatorial efforts.

2. Valuation domains

All rings in this paper will be commutative rings with unity and all modules will be unitary (usually right) modules.

A ring V is said to be a *valuation ring*, if the lattice of ideals of V is a chain. This is the same as for every $a, b \in V$ there exists $c \in V$ such that either $ac = b$ or $bc = a$. A valuation ring without zero divisors is called a *valuation domain*.

For instance, $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at a prime ideal $p\,\mathbb{Z}$, is a valuation domain. Note that $\mathbb{Z}_{(p)} \supset p\,\mathbb{Z}_{(p)} \supset p^2\,\mathbb{Z}_{(p)} \supset \ldots \supset 0$ is a complete list of ideals of $\mathbb{Z}_{(p)}$, in particular this ring is noetherian.

Every valuation domain V is a local ring: the set of non-invertible elements of V forms a unique maximal ideal $\operatorname{Jac}(V)$, the Jacobson radical of V. We will consider only infinite valuation domains which are not fields, hence $\operatorname{Jac}(V)$ is always nonzero. The factor $F = V/\operatorname{Jac}(V)$ is a field called a *residue field* of V. For instance, if $V = \mathbb{Z}_{(p)}$, then $\operatorname{Jac}(V) = p\,\mathbb{Z}_{(p)}$, hence $V/\operatorname{Jac}(V) \cong \mathbb{Z}/p\,\mathbb{Z}$ is a finite field of p elements.

If V is a valuation domain, then $Q = Q(V)$ will denote the *field of quotients* of V, and $U = U(V) = V \setminus \operatorname{Jac}(V)$ is the *group of units* of V. Clearly every element of $Q \setminus V$ is of the form j^{-1} for some $0 \neq j \in \mathrm{Jac}(V)$. Let $\Gamma = \Gamma(V)$ be a collection of all cosets qU, $0 \neq q \in Q$. Then (see [3, Ch. 1]) Γ is a linearly ordered abelian group called the *value group* of V. Namely, given $0 \neq q, q' \in Q$, we define $qU + q'U = qq'U$ and set $qU \leq q'U$ if and only if $q^{-1}q' \in V$.

The map $v : Q \setminus \{0\} \to \Gamma(V)$ given by $v(q) = qU$ is called a *valuation* of Q (corresponding to V). This map is usually extended to a map $Q \to \Gamma(V) \cup \infty$ by sending 0 to ∞. In particular, $v(qq') = v(q) + v(q')$ and $v(q + q') \geq \min\{v(q), v(q')\}$ for all $q, q' \in Q$. Also, if $a, b \in V$, then $v(a) \leq v(b)$ if and only if $b \in aV$, that is, $bV \subseteq aV$. For more on valuations of fields and valuation domains see [3].

Recall that, by Krull's theorem (see [3, Thm. 3.4]), for every linearly ordered abelian group Γ and every field F there exists a valuation domain V whose value group is isomorphic to Γ, and whose residue field is isomorphic to F.

The following is a particular case of Krull's construction.

Example 2.1. *(see* [3, p. 12]*) Let $R = F[x_q, q \in \mathbb{Q}]$ be the ring of polynomials over a field F. If $0 \neq \alpha \in F$ and $q_i \in \mathbb{Q}$, we define $v(\alpha x_{q_1}^{k_1} \cdot \ldots \cdot x_{q_n}^{k_n}) = k_1 q_1 + \cdots + k_n q_n$ and $v(\sum \alpha x_{q_1}^{k_1} \cdot \ldots \cdot x_{q_n}^{k_n}) = \min v(\alpha x_{q_1}^{k_1} \cdot \ldots \cdot x_{q_n}^{k_n})$. Then the set of fractions $\{f/g \mid f, g \in R$ and $v(f) \geq v(g)\}$ is a valuation domain whose value group is $\mathbb{Q}$ and whose residue field is F.*

We say that a linearly ordered abelian group $(\Gamma, +, \leq)$ is *archimedean* if, for all positive $a, b \in \Gamma$, there exists a positive integer n such that $na \geq b$. By Hölder's theorem (see [3, Prop. 2.2]) Γ is archimedean if and only if it is isomorphic to an additive subgroup of the reals $(\mathbb{R}, +, \leq)$. In this case either Γ is isomorphic to $(\mathbb{Z}, +, \leq)$, or Γ is *dense*, that is, for every $a < b \in \Gamma$ there is $c \in \Gamma$ such that $a < c < b$. For example, the rationals $(\mathbb{Q}, +, \leq)$ are a dense archimedean linearly ordered abelian group.

An ideal P of a valuation domain V is said to be *prime*, if $ab \in P$ for $a, b \in V$ implies $a \in P$ or $b \in P$. For instance, $\{0\}$ is a prime ideal of V so as $\mathrm{Jac}(V)$. For every prime ideal $P \neq \mathrm{Jac}(V)$, the quotient V/P is an infinite valuation domain.

The following fact is a part of Hölder's theorem.

Fact 2.2. $\mathrm{Jac}(V)$ *is the only nonzero prime ideal of V if and only if $\Gamma(V)$ is archimedean.*

Thus we can 'surround' any ideal of V by two elements of V whose distance can be made arbitrarily small.

Corollary 2.3. *Let V be a valuation domain with a dense archimedean value group. If I is a nonzero ideal of V and $c \in \mathrm{Jac}(V)$, then there are $a, b \in V$ such that $a \notin I$, $b \in I$ and $v(a^{-1}b) < v(c)$, that is, $c \in a^{-1}b\,\mathrm{Jac}(V)$.*

Proof. Let P consist of elements $r \in V$ such that $dr \in I$ for some $d \in V \setminus I$. It is easily checked that P is an ideal of V and $I \subseteq P$. Moreover, P is a prime ideal. Indeed, if $ab \in P$, then $d \cdot ab \in I$ for some $d \notin I$. Then either $da \in I$, hence $a \in P$, or $da \notin I$, therefore $b \in P$.

Since $\Gamma(V)$ is archimedean, Fact 2.2 implies that $P = \mathrm{Jac}(V)$. Since $\Gamma(V)$ is dense, there is $r \in \mathrm{Jac}(V)$ such that $v(r) < v(c)$. Now choose $a \in V \setminus I$, such that $ar \in I$ and put $b = ar$. Clearly a and b work. □

3. Decidability. Preliminaries

Recall that a (countable) theory T is said to *decidable*, if there is an algorithm that decides, for any sentence φ, whether $\varphi \in T$ or not. We will stick with this informal definition throughout the paper. A more rigorous definition is that the set of all theorems of T is recursive.

The following is a standard setup for decidability of the theory of all modules over a ring (see [4, Ch. 17]). We introduce it in the particular framework of a countable valuation domain V. In the sequel T_V will denote the first order theory of all V-modules (that is, the set of all first order sentences that are true in every V-module).

We say that a countable valuation domain is *effectively given*, if the elements of V can be listed (with repetitions)) as $r_0 = 0$, $r_1 = 1$, $r_2, \ldots$ such that the following holds.

1) There is an algorithm which, given $a, b \in V$, produces $a + b$, $-a$, and ab.

2) There is an algorithm that, given $a, b \in V$, decides whether $a = b$ or not.

3) There is an algorithm that, given $a \in V$, decides whether a is a unit or not.

Note that (see [4, Sect. 17.1]) 1) shows that a standard system of axioms of T_V can be arranged into an effective list, that is, T_V is recursively enumerable. Then 2) and 3) are necessary to ensure the decidability of T_V. Indeed, if $a, b \in V$, then it is easily checked that $T_V \models \forall\, x\, (xa = xb)$ if and only if $a = b$. Thus, a decision algorithm of T_V, when restricted to the sentences

$\forall\, x\, (xa = xb)$ with $a, b \in V$, provides 2). Similarly 3) can be encoded in T_V by the sentence $\forall\, x\, (xa = 0 \rightarrow x = 0)$.

For instance, V can be given as a factor ring of the ring of polynomials $R = F[x_1, x_2, \dots]$ over a finite field F, that is, $V \cong R/I$ for some ideal I of R. In this case a standard enumeration of polynomials $f_0 = 0, f_1 = 1, \dots$ gives an effective presentation of V, if the question $f_i = f_j$ in V (that is, $f_i - f_j \in I$) can be decided effectively.

From now on V will be an effectively given valuation domain.

Remark 3.1. *There is algorithm that, given $a \in V$, decides whether a is a unit, and if it is, produces the inverse a^{-1}.*

Proof. The first part is just 3). If a is a unit, then due to 1) we make a list $ar_0, ar_1, \dots$ and at each step we compare ar_i with the unity of V using 2). Since a is a unit, this process terminates on r_i such that $ar_i = 1$, and then $r_i = a^{-1}$. $\square$

Remark 3.2. *There is an algorithm which, given $a, b \in V$ finds $c \in V$ such that $ac = b$, or decides that such an element c does not exist.*

Proof. We make two parallel lists: $ar_0, ar_1, \dots$; $br_0, br_1, \dots$, and at each step compare elements of the first list with b, and of the second list with a. Since V is a valuation domain, we will find $r_i \in V$ such that either $ar_i = b$ or $br_i = a$.

If $ar_i = b$ we are done. Otherwise $br_i = a$. Using 3) we decide whether r_i is a unit. If it is, we will find r_i^{-1} using Remark 3.1, and then $ar_i^{-1} = b$. Otherwise $r_i \in \mathrm{Jac}(V)$, hence $b \notin aV$. $\square$

As a consequence, there is an algorithm which, given a, b in V, decides whether $v(a) = v(b)$, or $v(a) > v(b)$, or $v(a) < v(b)$.

A *positive-primitive formula* (pp-formula) $\varphi(x)$ is a formula $\exists\, \bar{y}\, (\bar{y}A = x\bar{b})$, where $\bar{y} = (y_1, \dots, y_k)$ is a set of (quantified) variables, A is a $k \times l$ matrix over V, and $\bar{b} = (b_1, \dots, b_l)$ is a row of elements of V. We will abbreviate this formula as $A \mid x\bar{b}$ (A divides $x\bar{b}$).

Let M be a V-module and let $m \in M$. We say that m *satisfies* φ *in* M, and write $M \models \varphi(m)$, if there exists a tuple $\overline{m} = (m_1, \dots, m_k) \in M$ such that $\overline{m}A = m\bar{b}$. Then $\varphi(M) = \{m \in M \mid M \models \varphi(m)\}$ is a *positive-primitive subgroup* (pp-subgroup) of M. Moreover, since V is commutative, $\varphi(M)$ is a submodule of M.

For instance, if $a \in V$, then $a \mid x \doteq \exists\, y\,(ya = x)$ is a *divisibility formula*, and $(a \mid x)(M) = Ma$. Also, if $b \in V$, then $xb = 0$ is an *annihilator formula*, and $(xb = 0)(M) = \mathrm{ann}(M)(b) = \{m \in M \mid mb = 0\}$.

Given pp-formulae $\varphi(x)$ and $\psi(x)$, we say that φ implies ψ, $\varphi \to \psi$ if, for every module M, $\varphi(M) \subseteq \psi(M)$. For instance, given $a, a' \in V$, it is easily checked that $a \mid x \to a' \mid x$ if and only if $a \in a'V$, that is, $v(a') \leq v(a)$. Similarly $xb = 0 \to xb' = 0$ for $b, b' \in V$ if and only if $b' \in bV$, that is, $v(b) \leq v(b')$.

We say that pp-formulae $\varphi(x)$ and $\psi(x)$ are *equivalent*, written $\varphi \leftrightarrow \psi$, if $\varphi(M) = \psi(M)$ for every module M. For instance, $a \mid x \leftrightarrow a' \mid x$ if and only if $aV = a'V$, that is, $v(a) = v(a')$. Similarly, $xb = 0 \leftrightarrow xb' = 0$ if and only if $bV = b'V$, and hence, again, if and only if $v(b) = v(b')$. Also recall that the sum of two pp-formulae $\varphi(x)$ and $\psi(x)$ is defined as the formula $(\varphi + \psi)(x) \doteq \exists\, y\,(\varphi(y) \wedge \psi(x - y))$. It is easily seen that $\varphi + \psi$ can be expressed as a pp-formula.

Lemma 3.3. *There exists an algorithm that, given a pp-formula $\varphi(x)$ over V, produces a formula $\wedge_{i=1}^{n} c_i \mid x + xd_i = 0$, $c_i, d_i \in V$, equivalent to φ.*

Proof. Every matrix over a valuation domain can be diagonalized using elementary row and column operations. By Remark 3.2 we can execute these operations effectively.

Thus, if φ is $A \mid x\bar{b}$, we will find invertible matrices U and W such that UAW is a diagonal matrix. Clearly φ is equivalent to $UAW \mid x\bar{b}W$, that is, we may assume that $A = \mathrm{diag}(a_1, \ldots, a_n)$ is a diagonal matrix. Then φ is equivalent to a conjunction of pp-formulae $a_i \mid xd_i$, $a_i, d_i \in V$. If $d_i \in a_iV$ (and we can decide this effectively), then $a_i \mid xd_i$ is a trivial formula, hence equivalent to the formula $1 \mid x$. Otherwise (using Remark 3.2) we will find $c_i \in V$ such that $a_i = d_ic_i$. Then $a_i \mid xd_i$ is $c_id_i \mid xd_i$ which is equivalent to $c_i \mid x + xd_i = 0$. ☐

Corollary 3.4. *Every pp-formula over V is effectively equivalent to a formula $\sum_{i=1}^{n} a_i \mid x \wedge xb_i = 0$, $a_i, b_i \in V$.*

$\sum$ refers here to sum of pp-formulae.

Proof. By Lemma 3.3 and elementary duality (see [4, Ch. 8]) which is clearly effective. ☐

4. The Ziegler spectrum

A module M is said to be *pure-injective*, if it is injective with respect to pure embeddings. Our main interest will be indecomposable pure-injective modules over a valuation domains V. It is known (see [9, p. 161]) that every indecomposable pure-injective V-module is isomorphic to the pure-injective envelope of a module A/B, where $B \subseteq A$ are fractional ideals of V (that is, V-submodules of Q). We will use a somewhat different description of indecomposable pure-injective V-modules using pp-types.

A *positive-primitive pp-type* (pp-type) p over V is a maximal consistent set of pp-formulae and their negations. For instance, if m is an element of a module M, then the set $\{\varphi(x) \mid M \models \varphi(m)\}$ is a pp-type (of m in M), $pp_M(m)$. For every pp-type p there exists a pure-injective module M and an element $m \in M$ such that $pp_M(m) = p$ and M is 'minimal' among all pure-injective modules realizing m (see [4, Ch. 4] for precise definitions). M is unique up to an isomorphism fixing m and called a *pure-injective envelope* of p, $\mathrm{PE}(p)$. A pp-type p is said to be *indecomposable*, if $\mathrm{PE}(p)$ is an indecomposable module.

Although the pure-injective envelope of a pp-type is uniquely determined, different pp-types may have isomorphic pure-injective envelopes (that is, different elements of an indecomposable pure-injective module may have different pp-types). Thus to classify indecomposable pure-injective V-modules we describe indecomposable pp-types, and an equivalence relation corresponding to the isomorphism of their pure-injective envelopes.

By Lemma 3.3, every pp-type is uniquely determined by the set of pp-formulae $a \mid x + xb = 0 \in p$, $a, b \in V$. Moreover (see [7, Cor. 11.7]), p is indecomposable if and only if $a \mid x + xb = 0 \in p$ implies $a \mid x \in p$ or $xb = 0 \in p$. Thus indecomposable pp-types are uniquely determined by their divisibility and annihilator formulae. Namely, set $I(p) = \{b \in V \mid xb = 0 \in p\}$ and $J(p) = \{a \in V$ such that $a \mid x \notin p\}$. If p is nonzero, then $I = I(p)$ and $J = J(p)$ are (proper) ideals of V, and p is uniquely determined by the pair (I, J). Moreover, for every pair (I, J) of ideals of V there exists a unique indecomposable pp-type $p = p(I, J)$ such that $I = I(p)$ and $J = J(p)$. Let $\mathrm{PE}(I, J)$ denote the pure-injective envelope of p (hence $\mathrm{PE}(I, J)$ is an indecomposable pure-injective module).

The following fact states that two pairs of ideals of V lead to isomorphic indecomposable pure-injective modules if and only if they can be identified by a shift by an element of V.

Fact 4.1. *(see* [7, L. 11.1]*)* $\mathrm{PE}(I,J) \cong \mathrm{PE}(I',J')$ *if and only if there exist* $r \notin I'$*,* $s \notin J'$ *such that either a)* $Ir = I'$ *and* $J = J'r$*, or b)* $I = I's$ *and* $Js = J'$*.*

The following corollary is almost immediate.

Corollary 4.2. *1) If* $I = bV$ *and* $J = cV$*, then* $\mathrm{PE}(I,J) \cong \mathrm{PE}(I',J')$ *if and only if* $I' = b'V$*,* $J' = c'V$ *for some* $b', c' \in V$ *such that* $v(bc) = v(b'c')$*.*

2) If $I = d\,\mathrm{Jac}(V)$ *and* $J = a\,\mathrm{Jac}(V)$*, then* $\mathrm{PE}(I,J) \cong \mathrm{PE}(I',J')$ *if and only if* $I' = d'\,\mathrm{Jac}(V)$*,* $J' = a'\,\mathrm{Jac}(V)$ *for some* $a', d' \in V$ *such that* $v(a'd') = v(ad)$*.*

Proof. 1) Suppose that $I = bV$ and $J = cV$. If $Ir = I'$ and $J = J'r$ for some $r \notin I'$, then $I' = Ir = brV$, hence take $b' = br$. From $J = cV$ we obtain $J'r = cV$. If $r \in cV$, then $r = cr'$ for some $r' \in V$, hence $J'cr' = cV$ yields $J'r' = V$, a contradiction. Thus $c = rc'$ for some $c' \in \mathrm{Jac}(V)$, hence $J'r = cV$ yields $J' = c'V$. Also $b'c' = brc' = bc$, hence $v(b'c') = v(bc)$.

The proof in case b) and the converse is similar.

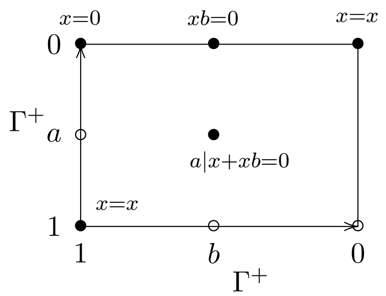
2) Suppose that $I = d\,\mathrm{Jac}(V)$, $J = a\,\mathrm{Jac}(V)$ and $Ir = I'$, $J = J'r$ for some $r \notin I'$. Then $I' = Ir = dr\,\mathrm{Jac}(V)$, hence take $d' = dr$. If $r \in a\,\mathrm{Jac}(V)$, then $r = ar'$ for some $r' \in \mathrm{Jac}(V)$, hence $J'ar' = a\,\mathrm{Jac}(V)$ implies $J'r' = \mathrm{Jac}(V)$. From $r' \in \mathrm{Jac}(V)$ it follows that $j'r' = r'$ for some $j' \in J'$, therefore $j' = 1 \in J'$, a contradiction.

Thus we may assume that $a = ra'$ for some $a' \in V$. Then $J'r = a\,\mathrm{Jac}(V)$ is $J'r = ra'\,\mathrm{Jac}(V)$, hence $a'\,\mathrm{Jac}(V) = J'$. Now $a'd' = a' \cdot dr = a'r \cdot d = ad$ yields $v(a'd') = v(ad)$.

The proof in case b) and the converse is similar. □

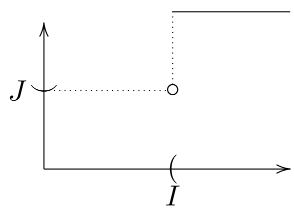
If K is an ideal of V and $c \in V \setminus K$, one defines $(K : c) = \{d \in V \mid cd \in K\}$. Using this notation, a) in Fact 4.1 can be written as $I = (I' : r)$ and $J = J'r$; and b) as $I = I's$ and $J = (J' : s)$.

We will give the following 'geometrical' interpretation of indecomposable pp-types. Let $\Gamma^+ = \Gamma^+(V)$ be the nonnegative part of the value group $\Gamma(V)$ of a valuation domain V. The elements of $\Gamma^+(V)$ are cosets uQ, $u \in V$ thus correspond to principal ideals uV of V. The smallest element of Γ^+ corresponds to $1 \cdot V = V$ and the largest element of Γ^+ (∞) corresponds to $0 \cdot V = 0$. Thus we use elements of V to denote elements of Γ^+. We represent pp-formulae by points on the plane $\Gamma^+ \times \Gamma^+$. Namely, if $a, b \in V$, then we assign to the pp-formula $a \mid x + xb = 0$ the point (b, a) of this plane:

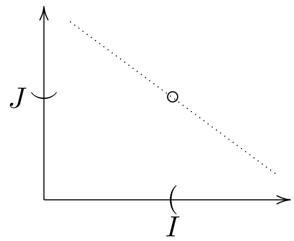


For instance, the point $(b, 0)$ corresponds to the formula $0 \mid x + xb = 0$ which is equivalent to $xb = 0$. Thus, all annihilator formulae live at the upper side of the above square, and the divisibility formulas occupy its left side. There is a singularity in this representation: all points at the lower side and at the right side of the square represent the (trivial) formula $x = x$.

Every ideal I of V defines a cut in Γ^+ in the obvious way: we take all elements from I in the upper part of the cut, and all elements not in I in the lower part of the cut. An indecomposable pp-type $p = p(I, J)$ consists of pp-formulae $a \mid x + xb = 0$ such that $b \in I$ or $a \notin J$, that is, p consists of points (b, a) that are below or on the following one step ladder:



Each dashed line on this diagram is an 'imaginary line', that is, a cut on Γ^+. One can convert this lines into real ones, completing Γ by cuts, but we avoid these unnecessary complications. Still, if $I = bR$ is a principal ideal of V, then the vertical line on the diagram is real (has an equation $x = b$). Thus p can be thought of as an imaginary point (at the junction of two dashed lines in the above diagram) of the plane. If both ideals $I = bV$ and $J = aV$ are principal, then (I, J) represents a real point (b, a). Therefore, due to Fact 4.1, the set of all pp-types realized in a given indecomposable pure-injective module can be considered as an (imaginary) line $v(x)+v(y) = \text{const}$ on the plane (see [7, Ch. 12] for more on that):



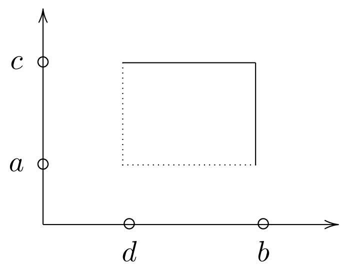
The *Ziegler spectrum* of V, Zg_V is a topological space whose points are the (isomorphism classes of) indecomposable pure-injective V-modules, and a basis of open sets is given by $(\varphi/\psi) = \{M \in \mathrm{Zg}_V \mid \varphi(M)/(\varphi\wedge\psi)(M) \neq 0\}$, where $\varphi(x)$ and $\psi(x)$ are pp-formulae over V. If φ are ψ are (typographically) complicated, we will write $(\varphi \ / \ \psi)$ instead of (φ/ψ). It is known (see [4, Thm. 4.66]) that with respect to this topology Zg_V is a compact space.

Corollary 4.3. *Every (nonempty) basic open set (φ/ψ) in the Ziegler spectrum of a valuation domain V is a finite union of basic open sets $(a \mid x \wedge xb = 0 \ / \ c \mid x + xd = 0)$, where $a, b, c, d \in V$, $v(a) < v(c)$ and $v(d) < v(b)$.*

Proof. By Corollary 3.4, φ is equivalent to $\sum_i \varphi_i$, where $\varphi_i \doteq a_i \mid x \wedge xb_i = 0$, $a_i, b_i \in V$. Also, by Lemma 3.3, ψ is equivalent to $\wedge_j \psi_j$, where $\psi_j \doteq c_j \mid x + xd_j = 0$, $c_j, d_j \in V$. It is easily checked that $(\varphi/\psi) = \cup_{i,j}(\varphi_i/\psi_j)$.

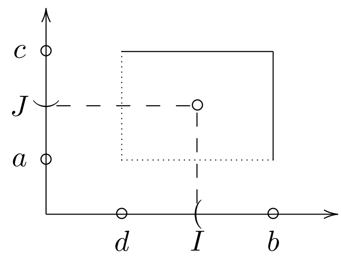
It remains to notice that, if $v(a) \geq v(c)$, then $a \mid x \to c \mid x$, hence $(a \mid x \wedge xb = 0 \ / \ c \mid x + xd = 0)$ is a trivial pair. The same is true, if $v(d) \geq v(b)$, because $xb = 0 \to xd = 0$ in this case. □

We will assign to the pair $(a \mid x \wedge xb = 0 \ / \ c \mid x + xd = 0)$ the following rectangle on the plane $\Gamma^+ \times \Gamma^+$:



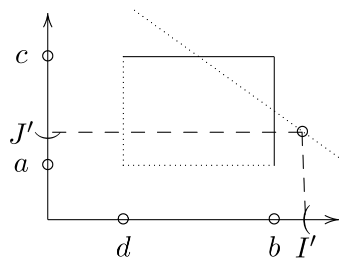
The explanation for this picture is the following. Suppose that I, J are ideals of V such that $d \notin I$, $b \in I$ and $a \notin J$, $c \in J$. Then clearly $\mathrm{PE}(I, J) \in (a \mid x \wedge xb = 0 \ / \ c \mid x + xd = 0)$. Indeed, by the definition of $p = p(I, J)$, $b \in I$ implies $xb = 0 \in p$ and $a \notin J$ yields $a \mid x \in p$. Thus, $a \mid x \wedge xb = 0 \in p$.

On the other hand, $d \notin I$ implies $xd = 0 \notin p$ and $c \in J$ yields $c \mid x \notin p$, hence (since p is indecomposable — see a remark above) $c \mid x + xd = 0 \notin p$. Thus, if m is an element of $\mathrm{PE}(I, J)$ that satisfies p, then m witnesses $\mathrm{PE}(I, J) \in (a \mid x \wedge xb = 0 \;\; / \;\; c \mid x + xd = 0)$. Geometrically, this is true, because the (imaginary) point (I, J) is within the rectangle:



Now we explain why two sides of the rectangle are dashed. Indeed, suppose that a point representing an indecomposable pp-type $p = p(I, J)$ belongs to the left side of the rectangle, hence $I = dV$ and $a \notin J$, $c \in J$. Since $xd = 0 \in p$, the natural realization of p fails to open $(a \mid x \wedge xb = 0 \;\; / \;\; c \mid x + xd = 0)$.

More generally, for any ideals I', J' of V, we have that $\mathrm{PE}(I', J') \in (a \mid x \wedge xb = 0 \;\; / \;\; c \mid x + xd = 0)$ if and only if the (imaginary) line corresponding to $p(I', J')$ intersects the rectangle:



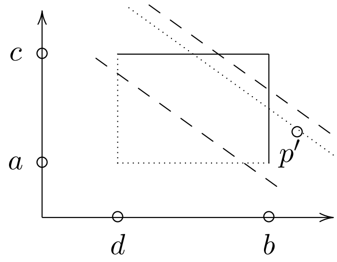
Now we convert this geometrical observation into a formal result.

Lemma 4.4. *Let* $\varphi \doteq a \mid x \wedge xb = 0$, $\psi \doteq c \mid x + xd = 0$, *and let* $p' = p(I', J')$. *Then the following is equivalent:*

1) $\mathrm{PE}(p') \in (\varphi / \psi)$*;*

2) there are $d' \notin I'$, $b' \in I'$ *and* $a' \notin J'$, $c' \in J'$ *such that* $v(a) + v(d) \leq v(a') + v(d')$ *and* $v(b') + v(c') \leq v(b) + v(c)$.

Geometrically this means that we should approximate an imaginary line corresponding to p' by two real lines from the above and from the below such that the first line separates p' from the right upper corner of the rectangle, and the second line separates p' from the left lower corner of the rectangle:



Proof. 1) $\Rightarrow$ 2). There is an element $m \in M = \mathrm{PE}(I', J')$ such that $M \models \varphi(m) \wedge \neg\psi(m)$. Hence, if $p = pp_M(m)$, then $p = p(I, J)$, where $d \notin I$, $b \in I$ and $a \notin J$, $c \in J$. By Fact 4.1, (I, J) is obtained from (I', J') by a shift by a certain element of V. Precisely, there are $r \notin I'$, $s \notin J'$ such that either a) $I = (I' : r)$ and $J = J'r$, or b) $I = I's$ and $J = (J' : s)$.

In case a) we have $d \notin (I' : r)$, $b \in (I' : r)$ and $a \notin J'r$, $c \in J'r$. Take $d' = dr \notin I'$, $b' = br \in I'$, and $c' = cr^{-1} \in J'$ (recall that r^{-1} is the inverse of r in Q). Put $a' = ar^{-1} \notin J'$ if $ar^{-1} \in V$, and $a' = 1 \notin J'$ otherwise. In any case $v(a') \geq v(ar^{-1})$. Then $v(a') + v(d') \geq v(ar^{-1}) + v(dr) = v(ar^{-1}dr) = v(ad) = v(a) + v(d)$ and $v(b') + v(c') = v(brcr^{-1}) = v(b) + v(c)$.

In case b) we have $d \notin I's$, $b \in I's$ and $a \notin (J' : s)$, $c \in (J' : s)$. Take $b' = bs^{-1} \in I'$ and $a' = as \notin J'$, $c' = cs \in J'$. Put $d' = ds^{-1} \notin I'$ if $ds^{-1} \in V$ and $d' = 1 \notin I'$ otherwise. In any case $v(d') \geq v(ds^{-1})$. Then $v(a') + v(d') \geq v(as) + v(ds^{-1}) = v(asds^{-1}) = v(a) + v(d)$ and $v(b') + v(c') = v(bs^{-1}cs) = v(b) + v(c)$.

2) $\Rightarrow$ 1). By Corollary 2.3, there are $d' \notin I'$, $b' \in I'$ and $a' \notin J'$, $c' \in J'$ such that $v(b') - v(d') \leq v(b) - v(d)$ and $v(c') - v(a') \leq v(c) - v(a)$. Moreover, by the assumption (changing a', b', c' and d' if necessarily) we may assume that $v(b') + v(c') \leq v(b) + v(c)$ and $v(a) + v(d) \leq v(a') + v(d')$. This implies that

$$v(a) - v(a'), v(b') - v(b) \leq v(c) - v(c'), v(d') - v(d)$$

and so the following system of inequalities

$$\begin{cases} v(b') - v(b) \leq z \leq v(d') - v(d) \\ v(a) - v(a') \leq z \leq v(c) - v(c') \end{cases}$$

has a solution z in Γ. Choose $r \in Q$ such that $v(r) = z$. Suppose first that z is nonnegative, hence $r \in V$. If $r \in I'$, then $z > v(d')$ (since $d' \notin I'$), and also $z \leq v(d') - v(d)$, hence $z - v(d') \leq -v(d)$. The left part of this inequality is positive, but the right part is not, a contradiction.

Thus $r \notin I'$, and we prove that $\mathrm{PE}(I, J) \in (\varphi/\psi)$, where $I = (I' : r)$ and $J = J'r$. Since (by Fact 4.1), $\mathrm{PE}(I', J') \cong \mathrm{PE}(I, J)$, it would follow that $\mathrm{PE}(p') \in (\varphi/\psi)$.

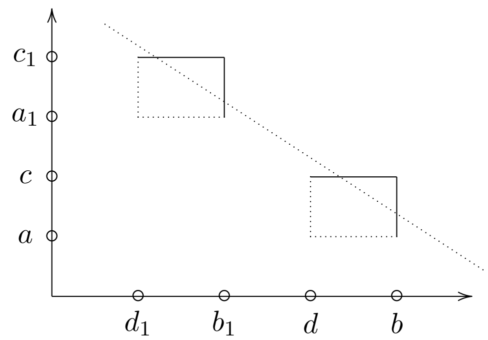
It suffices to check that $d \notin I$, $b \in I$ and $a \notin J$, $c \in J$. Indeed, $v(r) = z \leq v(d') - v(d)$ implies $v(dr) \leq v(d')$. From $d' \notin I'$ it follows that $dr \notin I'$, hence $d \notin I = (I' : r)$. Similarly $v(b') - v(b) \leq z = v(r)$ yields that $v(b') \leq v(br)$. Since $b' \in I'$, we conclude that $br \in I'$, hence $b \in I = (I' : r)$.

By similar arguments, $v(a) - v(a') \leq z$ implies $a \notin J$, and $z \leq v(c) - v(c')$ yields $c \in J$.

If z is negative, then $-z = -v(r) = v(r^{-1})$ is positive, hence $s = r^{-1} \in V$. If $s \in J'$, then $-v(r) = v(s) > v(a')$, and also (from the above inequalities) $v(a) - v(a') \leq v(r^{-1})$, that is, $v(a) \leq v(a') - v(s)$. Looking at the signs of both parts of this inequality we obtain a contradiction. Thus $s \notin J'$.

Now as above it is not difficult to check that, if $I = I's$ and $J = (J' : s)$, then $\mathrm{PE}(I, J) \in (\varphi/\psi)$, hence $\mathrm{PE}(I', J') \cong \mathrm{PE}(I, J)$ is also in (φ/ψ). $\square$

Now we are in a position to analyze an inclusion of two basic open sets $(\varphi/\psi) \subseteq (\varphi_1/\psi_1)$, where $\varphi \doteq a \mid x \wedge xb = 0$, $\psi \doteq c \mid x + xd = 0$ and $\varphi_1 \doteq a_1 \mid x \wedge xb_1 = 0$, $\psi_1 \doteq c_1 \mid x + xd_1 = 0$. Geometrically that means that every line $v(x) + v(y) = \mathrm{const}$ crossing the rectangle (φ/ψ) also intersects the rectangle (φ_1/ψ_1):



An instant look at this diagram suggests an answer: the main diagonal (from the left lower corner to the right upper one) of the rectangle (φ/ψ) should be covered by the main diagonal of the rectangle (φ_1/ψ_1), that is, $v(a_1 d_1) \leq v(ad)$ and $v(bc) \leq v(b_1 c_1)$.

Now we give a formal proof.

Proposition 4.5. *Let* $\varphi \doteq a \mid x \wedge xb = 0$, $\psi \doteq c \mid x + xd = 0$, *and* $\varphi_1 \doteq a_1 \mid x \wedge xb_1 = 0$, $\psi_1 \doteq c_1 \mid x + xd_1 = 0$. *Then the following are equivalent:*

1) $(\varphi/\psi) \subseteq (\varphi_1/\psi_1)$*;*

2) $v(a_1) + v(d_1) \leq v(a) + v(d)$ *and* $v(b) + v(c) \leq v(b_1) + v(c_1)$*.*

Proof. 1) $\Rightarrow$ 2). First we prove that $v(a_1) + v(d_1) \leq v(a) + v(d)$.

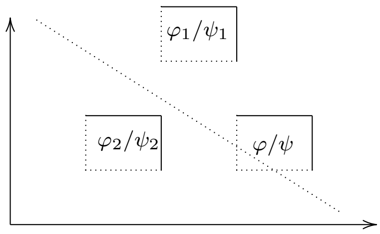
Let $I' = d\,\mathrm{Jac}(V)$ and $J' = a\,\mathrm{Jac}(V)$ (thus $p(I', J')$ represents a line that goes above the left lower corner of the rectangle (φ/ψ) as close as possible). Clearly $\mathrm{PE}(I', J') \in (\varphi/\psi)$, hence, by the assumption, $\mathrm{PE}(I', J') \in (\varphi_1/\psi_1)$. By Lemma 4.4, there are $d' \notin I'$ and $a' \notin J'$ such that $v(a_1)+v(d_1) \leq v(a')+v(d')$. But $a' \notin J' = a\,\mathrm{Jac}(V)$ implies $v(a') \leq v(a)$, and $d' \notin I' = d\,\mathrm{Jac}(V)$ yields $v(d') \leq v(d)$, hence $v(a') + v(d') \leq v(a) + v(d)$. Combining this with $v(a_1)+v(d_1) \leq v(a')+v(d')$ we obtain $v(a_1)+v(d_1) \leq v(a)+v(d)$, as desired.

Now we check that $v(b) + v(c) \leq v(b_1) + v(c_1)$. Indeed, take $I' = bV$ and $J' = cV$ (thus $p(I', J')$ represents a line that goes through the upper right corner of the rectangle (φ/ψ)). Clearly $\mathrm{PE}(I', J') \in (\varphi/\psi)$, hence, by the assumption, $\mathrm{PE}(I', J') \in (\varphi_1/\psi_1)$. By Lemma 4.4, there are $b' \in I'$ and $c' \in J'$ such that $v(b') + v(c') \leq v(b_1) + v(c_1)$. From $b' \in I' = bV$ it follows that $v(b') \geq v(b)$, and $c' \in J' = cV$ implies $v(c') \geq v(c)$, hence $v(b') + v(c') \geq v(b) + v(c)$. Combining this with $v(b') + v(c') \leq v(b_1) + v(c_1)$ we obtain $v(b) + v(c) \leq v(b_1) + v(c_1)$.

2) $\Rightarrow$ 1). Let $\mathrm{PE}(I', J') \in (\varphi/\psi)$. By Lemma 4.4, there are $d' \notin I'$, $b' \in I'$ and $a' \notin J'$, $c' \in J'$ such that $v(a) + v(d) \leq v(a') + v(d')$ and $v(b') + v(c') \leq v(b) + v(c)$. By the assumption, $v(a_1) + v(d_1) \leq v(a) + v(d)$ and $v(b)+v(c) \leq v(b_1)+v(c_1)$. It follows that $v(a_1)+v(d_1) \leq v(a')+v(d')$ and $v(b')+v(c') \leq v(b_1)+v(c_1)$. By Lemma 4.4 again, $\mathrm{PE}(I', J') \in (\varphi_1/\psi_1)$. □

5. Inclusion

In this section we consider a question which is crucial for a proof of decidability: when $(\varphi/\psi) \subseteq \cup_{i=1}^{n}(\varphi_i/\psi_i)$ for basic open sets in the Ziegler spectrum of V? We assume here that $\varphi \doteq a \mid x \wedge xb = 0$, $\psi \doteq c \mid x + xd = 0$ and $\varphi_i \doteq a_i \mid x \wedge xb_i = 0$, $\psi_i \doteq c_i \mid x + xd_i = 0$ for $i = 1, \ldots, n$. Looking at the diagram below we can guess the answer: the main diagonal of the rectangle (φ/ψ) should be covered by the union of main diagonals of rectangles (φ_i/ψ_i), that is, $[v(ad), v(bc)] \subseteq \cup_{i=1}^{n}[v(a_id_i), v(b_ic_i)]$, where $[v(ad), v(bc)]$ refers to the interval $\{z \in \Gamma(V) \mid v(ad) \leq z \leq v(bc)\}$, and similarly for $[v(a_id_i, v(b_ic_i))]$.



In fact the answer is more subtle (mainly because some sides of the rectangles are dashed). But first we consider some examples. If V is an (uncountable!) valuation domain whose value group is $(\mathbb{R}, +, \leq)$, then every ideal I of V is either principal or of the form $r\,\mathrm{Jac}(V)$, $r \in V$. Indeed, every ideal I of V corresponds to a filter (that is, upward closed subset) F of the linearly ordered set of nonnegative reals $\mathbb{R}^+$. If I is not principal, then F is not principal, that is, has no smallest element. Then the set $\mathbb{R}^+ \setminus F$ is bounded from above, hence has a largest element z (since $\mathbb{R}^+$ is Dedekind complete). If $r \in V$ is such that $v(r) = z$, then $r \notin I$ and $I = r\,\mathrm{Jac}(V)$.

On the other hand, if V is a valuation domain whose value group is $(\mathbb{Q}, +, \leq)$, then V has a non-principal ideal that is not of the form $r\,\mathrm{Jac}(V)$ for any $r \in V$. Indeed, take a filter $F = \{q \in \mathbb{Q} \mid q \geq \sqrt{2}\}$. Then F is nonprincipal, and the set $\mathbb{Q}^+ \setminus F$ has no largest element.

Moreover, let I be an ideal of V corresponding to the filter $\{q \in \mathbb{Q} \mid q \geq \sqrt{2}\}$ and let J correspond to the filter $\{q \in \mathbb{Q} \mid q \geq 2 - \sqrt{2}\}$. Then for every $d' \notin I$, $b' \in I$ and $a' \notin J$, $c' \in J$ we have $v(a'd') < 2 < v(b'c')$.

We need the following generalization of this example. The authors are indebted to a (anonymous) referee for pointing out the following lemma that simplifies the proof of the next corollary.

Lemma 5.1. *Let $\Gamma \subseteq \mathbb{R}$ be a countable ordered abelian group. Suppose that $\alpha, \beta, \gamma, \delta, z \in \Gamma$ are such that $\alpha < \gamma$, $\delta < \beta$ and $\alpha + \delta < z < \beta + \gamma$. Then there exists an element $y \in \mathbb{R} \setminus \Gamma$ such that $\alpha < y < \gamma$ and $\delta < z - y < \beta$.*

Proof. Since the sum of open intervals (α, γ) and (δ, β) in $\mathbb{R}$ is an open interval $(\alpha + \delta, \beta + \gamma)$, there must be $\alpha_1 \in (\alpha, \gamma)$ and $\beta_1 \in (\delta, \beta)$ such that $\alpha_1 + \beta_1 = z$. If $\alpha_1 \notin \Gamma$, we set $y = \alpha_1$ and we are done. Otherwise, we choose some $\varepsilon \notin \Gamma$ such that $0 < \varepsilon < \mu = \min\{\gamma - \alpha_1, \beta_1 - \delta\}$; this exists, since Γ is countable but the interval $(0, \mu)$ in $\mathbb{R}$ is not. Then $\alpha_1 + \varepsilon \in (\alpha, \gamma)$, $z - (\alpha_1 + \varepsilon) = \beta_1 - \varepsilon \in (\delta, \beta)$ and $\alpha = \alpha_1 + \varepsilon \notin \Gamma$. $\square$

Corollary 5.2. *Assume that $\Gamma(V)$ is dense and archimedean. Let $a, b, c, d \in V$ be such that $v(d) < v(b)$ and $v(a) < v(c)$, and let $z \in \Gamma(V)$ be such that*

$z \in (v(ad), v(bc))$, that is, $v(ad) < z < v(bc)$. Then there are ideals I, J of V such that $d \notin I$, $b \in I$, $a \notin J$, $c \in J$; and $v(a'd') < z < v(b'c')$ for all $d' \notin I$, $b' \in I$, $a' \notin J$, $c' \in J$.

Proof. Take $\alpha = v(a)$, $\beta = v(b)$, $\gamma = v(c)$, $\delta = v(d)$ and z, and apply Lemma 5.1 to obtain $y \notin \Gamma$. We observe that $z - y \notin \Gamma$, and we set $I = \{r \in V \mid v(r) \geq z - y\} = \{r \in V \mid v(r) > z - y\}$ and $J = \{s \in V \mid v(s) \geq y\} = \{s \in V \mid v(s) > y\}$. It follows that $d \notin I$, $b \in I$ and $a \notin J$, $c \in J$. Further, if $d' \notin I$, $b' \in I$ and $a' \notin J$, $c' \in J$, then $v(d') < z - y < v(b')$ and $v(a') < y < v(c')$ which yields $v(a'd') < z < v(b'c')$. □

Now we are in a position to prove main result of this section.

Proposition 5.3. *Let $\varphi \doteq a \mid x \wedge xb = 0$, $\psi \doteq c \mid x + xd = 0$ and let $\varphi_i \doteq a_i \mid x \wedge xb_i = 0$, $\psi_i \doteq c_i \mid x + xd_i = 0$, $i = 1, \ldots, n$. Then the following are equivalent:*

1) $(\varphi/\psi) \subseteq \cup_{i=1}^n (\varphi_1/\psi_i)$;

2) $(v(ad), v(bc)) \subseteq \cup_{i=1}^n (v(a_i d_i), v(b_i c_i))$;

3) $[v(ad), v(bc)] \subseteq \cup_{i=1}^n [v(a_i d_i), v(b_i c_i)]$ and, if $v(ad) < z < v(bc)$ for some $z = v(a_i d_i) = v(b_j c_j)$, then there is $k = 1, \ldots, n$ such that $v(a_k d_k) < z < v(b_k c_k)$.

Proof. 1) $\Rightarrow$ 2). Suppose that $v(ad) < z < v(bc)$ for some $z \in \Gamma(V)$. By Corollary 5.2, there are ideals I, J such that $d \notin I$, $b \in I$, $a \notin J$, $c \in J$, and $v(a'd') < z < v(b'c')$ for all $d' \notin I$, $b' \in I$, $a' \notin J$, $c' \in J$.

Clearly $\mathrm{PE}(I, J) \in (\varphi/\psi)$. By the assumption, $\mathrm{PE}(I, J) \in (\varphi_i/\psi_i)$ for some i. Then, by Lemma 4.4, there are $d' \notin I$, $b' \in I$, $a' \notin J$, $c' \in J$ such that $v(a_i d_i) \leq v(a'd')$ and $v(b'c') \leq v(b_i c_i)$. From $v(a'd') < z < v(b'c')$ we conclude that $v(a_i d_i) < z < v(b_i c_i)$, hence $z \in (v(a_i d_i), v(b_i c_i))$.

2) $\Rightarrow$ 3) is obvious.

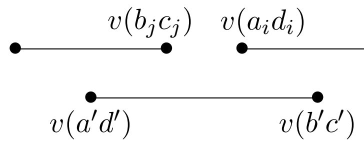
3) $\Rightarrow$ 1). Let p be a pp-type such that $\mathrm{PE}(p) \in (\varphi/\psi)$ and we have to prove that $\mathrm{PE}(p) \in (\varphi_k/\psi_k)$ for some k. Clearly we may assume that $p = p(I', J')$, where $d \notin I'$, $b \in I'$ and $a \notin J'$, $c \in J'$.

First suppose that there are $d' \notin I'$, $b' \in I'$, $a' \notin J'$, $c' \in J'$ such that $v(a_i d_i) \leq v(a'd')$ and $v(b'c') \leq v(b_i c_i)$ for some i. Then, by Lemma 4.4, $\mathrm{PE}(p) \in (\varphi_i/\psi_i)$. Otherwise we may assume that for every i one of the following holds.

a) $v(a_i d_i) > v(a'd')$ for all $d' \notin I'$, $a' \notin J'$, or

b) $v(b'c') > v(b_i c_i)$ for all $b' \in I'$, $c' \in J'$.

Let $S \subseteq \{1, \dots, n\}$ consist of indices i such that a) holds, and let T consist of indices j satisfying b). By the assumption, $S \cup T = \{1, \dots, n\}$. If $S = \emptyset$, then $T = \{1, \dots, n\}$. Since $b \in I$ and $c \in J$, it follows that $v(bc) > v(b_i c_i) > v(a_i d_i)$ for every i, hence $v(bc)$ is not covered by the intervals $[v(a_i d_i), v(b_i c_i)]$, a contradiction. Thus, S is not empty, and by similar arguments T is not empty.



If $i \in S$ and $j \in T$, then $v(a_i d_i) \geq v(b_j c_j)$. Indeed, otherwise $v(a_i d_i) < v(b_j c_j)$, and choose $r \in \mathrm{Jac}(V)$ such that $v(r) = v(b_j c_j) - v(a_i d_i)$. Since $\Gamma(V)$ is dense, there are $s, t \in \mathrm{Jac}(V)$ such that $v(s) + v(t) = v(r)$. Now (using Corollary 2.3) choose $d' \notin I'$, $b' \in I'$, $a' \notin J'$, $c' \in J'$ such that $v(b') - v(d') < v(s)$ and $v(c') - v(a') < v(t)$. Then $v(b_j c_j) < v(b'c')$ (since $j \in T$) and $v(a'd') < v(a_i d_i)$ (since $i \in S$), which implies $v(b_j c_j) - v(a_i d_i) < v(b'c') - v(a'd') < v(s) + v(t) = v(r)$, a contradiction.

Let $z_1 = \max_j v(b_j c_j)$, $j \in T$ and let $z_2 = \min_i v(a_i d_i)$, $i \in S$. By what we have just proved, $z_1 \leq z_2$. If $z_1 < z_2$, then there is a $z \in [v(ad, v(bc))]$ such that $z_1 < z < z_2$, hence $z \notin \cup_k [v(a_k d_k, v(b_k c_k))]$, a contradiction.

Indeed, otherwise $v(a_k d_k) < z < v(b_k c_k)$. Then $v(a_k d_k) < z_2$, hence $k \notin S$. Similarly $v(b_k c_k) > z_1$ shows that $k \notin T$, a contradiction.

Thus $z_1 = z_2$, hence $z_1 = v(b_j c_j) = v(a_i d_i)$ for some $j \in T$ and $i \in S$. By the assumption, there is k such that $v(a_k d_k) < z_1 < v(b_k c_k)$. By the definition of S and T, $v(a'd') < z_1 < v(b'c')$ for all $d' \notin I'$, $b' \in I'$ and $a' \notin J'$, $c' \in J'$. Since the value of $v(b'c') - v(a'd')$ can be made arbitrary small, we may assume that $v(a_k d_k) \leq v(a'd')$ and $v(b'c') \leq v(b_k c_k)$. Then $\mathrm{PE}(p) \in (\varphi_k / \psi_k)$ by Lemma 4.4. $\square$

6. Decidability. Infinite residue field.

In this section we show that Proposition 5.3 implies decidability of the theory T_V of all modules over a valuation domain V with a dense archimedean value group and an infinite residue field. The arguments we use are quite standard and can be found in [4, Sect. 17.3].

Let $\varphi(x)$ and $\psi(x)$ be pp-formulae and let n be a positive integer. Then there exists a first order sentence $\mathrm{Inv}(\varphi, \psi) \geq n$ (called an *invariant sentence*) such that, for every module M, one has $M \models \mathrm{Inv}(\varphi, \psi) \geq n$ if and only if the factor $\varphi(M)/(\varphi \wedge \psi)(M)$ has at least n elements. Then the sentence $\mathrm{Inv}(\varphi, \psi) = n$, that is, $\mathrm{Inv}(\varphi, \psi) \geq n \wedge \neg\,(\mathrm{Inv}(\varphi, \psi) \geq n+1)$, says that there are exactly n elements in the corresponding factor. Similarly $\mathrm{Inv}(\varphi, \psi) \leq n \doteq \neg\,(\mathrm{Inv}(\varphi, \psi) \geq n+1)$ claims that the corresponding factor has at most n elements. To simplify notations, we will write $\mathrm{Inv}(M, \varphi, \psi) \geq n$ instead of $M \models \mathrm{Inv}(\varphi, \psi)$, and similarly for other boolean combinations of invariant sentences. For instance $\mathrm{Inv}(M, \varphi, \psi) = 1$ if and only if $\varphi(M)/(\varphi \wedge \psi)(M)$ is a zero module, that is, $\varphi(M) \subseteq \psi(M)$. We also define $\mathrm{Inv}(M, \varphi, \psi) = \infty$, if $\mathrm{Inv}(M, \varphi, \psi) \geq n$ for every n, that is, if $\varphi(M)/(\varphi \wedge \psi)(M)$ is an infinite group. Thus $\mathrm{Inv}(\varphi, \psi) = \infty$ is an (infinite) conjunction of the sentences $\mathrm{Inv}(\varphi, \psi) \geq n$, $n \geq 1$.

The importance of invariant sentences is backed by the following result.

Fact 6.1. *(Baur-Monk theorem — see* [4, Cor. 2.13]*) Every first order sentence in the language of modules over a given ring is equivalent to a boolean combination of invariant sentences.*

Now we prove the main result of this section.

Theorem 6.2. *Let V be an effectively given valuation domain with an infinite residue field and such that the value group of V is dense and archimedean. Then the theory T_V of all modules over V is decidable.*

Proof. As we have already mentioned, T is recursively axiomatized, that is, there is an effective list of first order sentences *true in every* V-module. Thus what we need is an effective list of first order sentences false in every V-module, equivalently (passing to negations) of the first order sentences *true in some* V-module. This is the same as the existence of an algorithm that, given a sentence σ in the language of V-modules, decides whether there is a module M satisfying σ. We describe this algorithm.

It is not difficult to see that Baur-Monk theorem is effective. Apply this theorem and replace σ by a Boolean combination of invariant sentences. Pulling disjunctions ahead, we represent σ as a disjunction of σ_h, where each σ_h is a conjunction of invariant sentences or their negations. Then 'there exists M such that $M \models \sigma \doteq \vee_h \sigma_h$' is the same as 'for some h there exists M_h such that $M_h \models \sigma_h$'. Thus we may assume that σ is a conjunction of invariant sentences and their negations.

Here is the place to put in use that F is infinite (to simplify the proof). Indeed, since F is infinite, it is easily checked (see Lemma 7.5) that for all pp-formulas $\varphi(x)$, $\psi(x)$ and every V-module M, the factor $\varphi(M)/(\varphi \wedge \psi)(M)$

is either zero or infinite. Thus all sentences $\mathrm{Inv}(\varphi, \psi) \geq n$ for $n \geq 2$ are equivalent (to $\mathrm{Inv}(\varphi, \psi) > 1$). Therefore we may further assume that every conjunct of σ is either $\mathrm{Inv}(\varphi, \psi) > 1$ or $\mathrm{Inv}(\varphi, \psi) = 1$ for some pp-formulas φ and ψ.

By Corollary 3.4, we can (effectively) replace φ by an equivalent formula $\sum_i \varphi_i$, where $\varphi_i \doteq a_i \mid x \wedge xb_i = 0$, and (by Lemma 3.3) we can replace ψ by $\wedge_j \psi_j$, where $\psi_j \doteq \wedge_j c_j \mid x + xd_j = 0$. Clearly $\mathrm{Inv}(\varphi, \psi) > 1$ is equivalent to $\vee_{i,j} \mathrm{Inv}(\varphi_i/\psi_j) > 1$, therefore $\mathrm{Inv}(\varphi/\psi) = 1$ is equivalent to $\wedge_{i,j} \mathrm{Inv}(\varphi_i/\psi_j) = 1$.

Thus, getting rid of disjunctions and separating conjunctions, we may assume that every conjunct of σ is of the form $\mathrm{Inv}(\varphi, \psi) > 1$ or $\mathrm{Inv}(\varphi, \psi) = 1$, where (φ/ψ) are basic open sets in the Ziegler spectrum of V (see Corollary 4.3).

Moreover we can suppose that there exists at most one pp-formula $\mathrm{Inv}(\varphi, \psi) > 1$ among conjuncts of σ; otherwise, for every invariant sentence $\mathrm{Inv}(\varphi, \psi) > 1$, look for a V-module satisfying it and all the sentences involving equality, then form the direct sum of the modules obtained in this way and get a module M as required. Thus we end up with the following question: given basic open sets (φ_i/ψ_i), $i = 1, \ldots, n$ and (φ/ψ) in the Ziegler spectrum of V, does there exists a V-module M such that $\mathrm{Inv}(M, \varphi_i, \psi_i) = 1$ and $\mathrm{Inv}(M, \varphi, \psi) > 1$?

But (see [4, Cor. 4.36]) every module is elementarily equivalent to a direct sum of indecomposable pure-injective modules. Thus (by obvious arguments) we may look for an indecomposable pure-injective M. In this framework our question boils down to the following: is it true that $(\varphi/\psi) \subseteq \cup_{i=1}^n (\varphi_i/\psi_i)$ for basic open sets in the Ziegler spectrum of V? But Proposition 5.3 suggests the answer, and clearly the item 3) of this proposition can be verified effectively. $\square$

A countable field F is said to be *effectively given* if F is listed as $f_0 = 0$, $f_1 = 1$, $f_2, \ldots$ such that all operations of F can be executed effectively, and we can effectively solve the word problem $f_i = f_j$ for F. For instance, every finite field and the field of rationals can be effectively given, as well as their algebraic closures.

Corollary 6.3. *Let V be a valuation domain as in Example 2.1 such that F is effectively given and infinite. Then the theory of V-modules is decidable.*

Proof. Using an effective presentation of F, it is not difficult to obtain an effective presentation for V (since every element of V is a rational function, and we can deal with polynomials effectively). It remains to apply Theorem 6.2. □

7. Finite invariants

In this section we analyze finite invariants of (indecomposable pure-injective) modules. But first we need some preliminaries.

For every module M there exists a 'minimal' pure-injective module $\mathrm{PE}(M)$ containing M as a pure submodule. $\mathrm{PE}(M)$ is called a *pure-injective envelope* of M. By [4, Thm. 4.21] M is an elementary substructure of $\mathrm{PE}(M)$, in particular M is elementary equivalent to $\mathrm{PE}(M)$.

A module M is said to be *uniserial*, if the lattice of submodules of M is a chain. For instance, the field of quotients of a commutative valuation domain V, and V itself, are uniserial V-modules. Also, a vector space over a field is uniserial if and only if it is one-dimensional. Every submodule and every factor module of a uniserial module is uniserial.

Fact 7.1. *Every indecomposable pure-injective module over a commutative valuation domain is the pure-injective envelope of a uniserial module.*

Proof. By Ziegler's analysis of modules over a commutative valuation domain [9, p. 161], every indecomposable pure-injective V-module is isomorphic to the pure-injective envelope of a module A/B, where $B \subseteq A$ are V-submodules of Q, the field of quotients of V. But Q is a uniserial module, hence the same is true for A and A/B. □

However, it is not true that every indecomposable pure-injective module over a valuation domain is uniserial. What we can say is the following.

Fact 7.2. [7, Cor. 11.5] *If M is an indecomposable pure-injective module over a commutative valuation domain, then pp-subgroups of M form a chain, that is, if $\varphi(x)$ and $\psi(x)$ are pp-formulae, then either $\varphi(M) \subseteq \psi(M)$ or $\psi(M) \subseteq \varphi(M)$.*

For instance, if $\varphi(M) \not\subseteq \psi(M)$, then $\psi(M) \subset \varphi(M)$, hence $(\varphi \wedge \psi)(M) = \psi(M)$. We will freely use this fact to simplify notations in what follows. Another obvious consequence of this fact is the following corollary.

Corollary 7.3. *Let M be an indecomposable pure-injective module over a commutative valuation domain V. If $\varphi \doteq \sum_i \varphi$ and $\psi \doteq \wedge_j \psi_j$ are pp-formulae, then* $\mathrm{Inv}(M, \varphi, \psi) = \max_{i,j} \mathrm{Inv}(M, \varphi_i, \psi_j)$.

We need one more fact about uniserial modules.

Fact 7.4. [2, Prop. 5.1] *If M is a uniserial module over a commutative ring, then* $\mathrm{PE}(M)$ *is an indecomposable module.*

Let M be a V-module and let $\varphi(x), \psi(x)$ be pp-formulae. In the following definition for the sake of simplicity we assume that $\psi \to \varphi$. If ψ does not imply φ, one should replace ψ by $\varphi \wedge \psi$ first.

We say that (φ/ψ) is a *minimal pair* (in the theory of M), if $\psi(M) \subset \varphi(M)$ and for every pp-formula $\theta(x)$, $\psi(M) \subseteq \theta(M) \subseteq \varphi(M)$ implies $\psi(M) = \theta(M)$ or $\theta(M) = \varphi(M)$.

For instance, if $M = \mathbb{Z}_{p^2} = \mathbb{Z}/p^2\mathbb{Z}$ considered as a module over $\mathbb{Z}_{(p)}$, then $(xp = 0/x = 0)$ defines the socle of M, hence this pair is minimal.

The next lemma shows that finite minimal pairs cannot occur, if the residue field of V is infinite.

Lemma 7.5. *Let V be a valuation domain with the residue field F. Suppose that M is an indecomposable pure-injective V-module and (φ/ψ) is a minimal pair in the theory of M such that $\varphi(M)/\psi(M)$ is finite. Then $\varphi(M)/\psi(M)$ is a one-dimensional vector space over F, in particular F is finite.*

Proof. Since V is commutative, a multiplication by any $r \in V$ is an endomorphism of M. Let P consists of $r \in V$ which act as non-automorphisms of M. By [9, Thm. 5.4], P is a prime ideal of V, and multiplying by $r \in P$ one properly increases the pp-type of any nonzero element of M. Thus, if $m \in \varphi(M) \setminus \psi(M)$ and $r \in P$, then $mr \in \psi(M)$. Therefore P annihilates $\varphi(M)/\psi(M)$, hence $\varphi(M)/\psi(M)$ is a V/P-module (and V/P is a valuation domain).

Every element of $V \setminus P$ acts on M as an automorphism, hence $\varphi(M)/\psi(M)$ is a vector space over $Q(V/P)$, the quotient field of V/P. If $P \neq \mathrm{Jac}(V)$, then V/P is infinite, hence $\varphi(M)/\psi(M)$ is infinite, a contradiction. Thus $P = \mathrm{Jac}(V)$, $F = V/\mathrm{Jac}(V)$ is a finite field, and $\varphi(M)/\psi(M)$ is a vector space over F.

By Fact 7.1, M is isomorphic to the pure injective envelope $\mathrm{PE}(N)$ of a uniserial V-module N. Since N is an elementary submodule of M, $\varphi(N)/\psi(N)$ is a vector space over F. But $\varphi(N)/\psi(N)$ is a uniserial V-module, hence the dimension of this vector space is one. Since M is elementarily equivalent to N, and elementary equivalent modules have the same finite invariants, $\varphi(M)/\psi(M)$ is also one-dimensional over F. $\square$

The following corollary shows that over valuation domains with a dense valuation group finite invariants are rather rarity.

Corollary 7.6. *Let V is a valuation domain with a dense value group and let $\varphi(x)$, $\psi(x)$ be pp-formulae over V. Let M be an indecomposable pure-injective V-module such that $\varphi(M)/\psi(M)$ is finite. Then $\varphi(M)/\psi(M)$ is a one-dimensional vector space over F, the residue field of V, hence (φ/ψ) is a minimal pair in the theory of M.*

Proof. Since the value group of V is dense, $\mathrm{Jac}(V) = \mathrm{Jac}(V)^2$.

By Lemma 7.5, the result is true if φ/ψ is a minimal pair in the theory of M. Otherwise we may assume that $\psi(M) \subset \theta(M) \subset \varphi(M)$ for some pp-formula θ. Arguing by induction on the length of $\varphi(M)/\psi(M)$ we prove that $\varphi(M) \cdot \mathrm{Jac}(V) \subseteq \psi(M)$. By the induction hypothesis, we may assume that $\varphi(M)\cdot\mathrm{Jac}(V) \subseteq \theta(M)$ and $\theta(M)\cdot\mathrm{Jac}(V) \subseteq \psi(M)$. But then $\varphi(M)\cdot\mathrm{Jac}(V) = \varphi(M) \cdot \mathrm{Jac}^2(V) \subseteq \psi(M)$, as desired.

Thus $\varphi(M)/\psi(M)$ is a vector space over F, and the proof can be completed as in Lemma 7.5. □

Now we decide what pairs of pp-formulae cut out finite invariants on indecomposable pure-injective modules.

Lemma 7.7. *Let V be a valuation domain with a dense value group such that the residue field F of V is finite. Let (φ/ψ), where $\varphi \doteq a \mid x \wedge xb = 0$ and $\psi \doteq c \mid x + xd = 0$, be a basic open set in the Ziegler spectrum of V. Suppose that $M = \mathrm{PE}(I, J)$ is an indecomposable pure-injective V-module and $m \in M$ realizes $p = p(I, J)$. Then the following are equivalent:*

1) $m \in \varphi(M) \setminus \psi(M)$ and $\varphi(M)/\psi(M)$ is finite;

2) $m \in \varphi(M) \setminus \psi(M)$ and $\varphi(M)/\psi(M)$ is a one-dimensional vector space over F;

3) either $I = bV$ and $J = cV$, or $I = d\,\mathrm{Jac}(V)$ and $J = a\,\mathrm{Jac}(V)$.

Proof. 1) is equivalent to 2) by Corollary 7.6.

2) $\Rightarrow$ 3). Suppose first that $J \neq \mathrm{Jac}(V)$.

Choose $j \in \mathrm{Jac}(V) \setminus J$. From $j \notin J$ we obtain $j \mid x \in p$, hence there exists $m' \in M$ such that $m'j = m$. If $q = pp_M(m')$ then (as it easily checked) $q = p(I', J')$, where $I' = Ij$ and $J' = (J : j)$. Since $\mathrm{Jac}(V)$ kills all minimal pairs, we must have $m' \notin \varphi(M)$ (otherwise $m = m'j \in \psi(M)$, a contradiction). It follows that $a \mid x \wedge xb = 0 \notin q$, hence either $b \notin I' = Ij$ or $a \in J' = (J : j)$, that is, $aj \in J$. Since this is true for every $j \in \mathrm{Jac}(V) \setminus J$ (and $b \in I$, $a \notin J$), we obtain that $I = bV$ or $J = a\,\mathrm{Jac}(V)$.

Now suppose that $I \neq \operatorname{Jac}(V)$.

Take any $i \in \operatorname{Jac}(V) \setminus I$. Let $m' = mi$ and $q = pp_M(m')$. Then $q = p(I', J')$, where $I' = (I : i)$ and $J' = Ji$. Again, because $\operatorname{Jac}(V)$ kills (φ/ψ), one gets $m' = mi \in \psi(M)$, that is, $c \mid x + xd = 0 \in q$. Since q is indecomposable, it follows that $c \mid x \in q$ or $xd = 0 \in q$, that is, $c \notin J' = Ji$ or $d \in I' = (I : i)$, that is, $di \in I$. Because this is true for all $i \in \operatorname{Jac}(V) \setminus I$ (and $d \notin I$, $c \in J$), we conclude that $I = d\operatorname{Jac}(V)$ or $J = cV$.

Thus, if $I \neq \operatorname{Jac}(V)$ and $J \neq \operatorname{Jac}(V)$ we got that $I = bV$ or $J = a\operatorname{Jac}(V)$, and $I = d\operatorname{Jac}(V)$ or $J = cV$. Since $\operatorname{Jac}(V) = \operatorname{Jac}(V)^2$, all the ideals $r\operatorname{Jac}(V)$, $0 \neq r \in V$ are not principal. Thus, if $I = bV$, then $I \neq d\operatorname{Jac}(V)$, hence $J = cV$. Otherwise $I \neq bV$, hence $J = a\operatorname{Jac}(V)$. Then $J \neq cV$, therefore $I = d\operatorname{Jac}(V)$.

Suppose that $J = \operatorname{Jac}(V)$, hence $a \notin J$ is invertible. If $I = \operatorname{Jac}(V)$ we can take $a = d = 1$. Thus we may assume that $I \neq \operatorname{Jac}(V)$, hence (see above) either $I = d\operatorname{Jac}(V)$ or $J = cV$. But $J = \operatorname{Jac}(V) = cV$ is not possible, hence $I = d\operatorname{Jac}(V)$ (and we take $a = 1$). Similarly, if $I = \operatorname{Jac}(V)$ (hence d is invertible), then we obtain $J = a\operatorname{Jac}(V)$.

3) $\Rightarrow$ 2). Suppose that $I = bV$ and $J = cV$. Observe that $c \in \operatorname{Jac}(V)$. Let $N = \operatorname{Jac}(V)/cbV$, and let m denote the coset of c in N. Put $p = pp_N(m)$. Since N is uniserial, by Fact 7.4 its pure-injective envelope is indecomposable. In particular, p is an indecomposable pp-type. Looking at the divisibility and the annihilator formulae valid on m in N, we see that $p = p(I, J)$, in particular $\operatorname{PE}(N) \cong M$ and $m \in \varphi(M)$. If $i \in \operatorname{Jac}(V)$, then c divides mi in N, hence $mi \in \psi(M)$. Also, if $j \in \operatorname{Jac}(V) \setminus J$ and $m' \in N$ is such that $m'j = m$, then $m'b \neq 0$ in N, hence $m' \notin \varphi(M)$. It follows that $\varphi(N)/\psi(N)$ is a one-dimensional vector space over F (generated by m), hence the same is true for $M \cong \operatorname{PE}(N)$.

Similarly, if $I = d\operatorname{Jac}(V)$ and $J = a\operatorname{Jac}(V)$, then the projection m of a in the quotient module $N = V/ad\operatorname{Jac}(V)$ realizes $p = p(I, J)$. Then $M \cong \operatorname{PE}(N)$ and $\varphi(M)/\psi(M)$ is a one-dimensional vector space over F generated by m. □

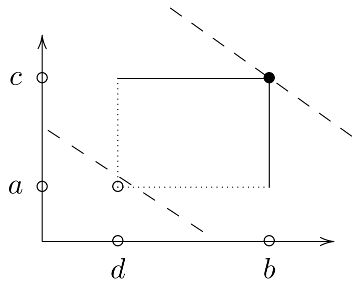
Proposition 7.8. *Let V be a valuation domain with a dense value group such that the residue field F of V is finite. Let (φ/ψ), where $\varphi \doteq a \mid x \wedge xb = 0$ and $\psi \doteq c \mid x + xd = 0$, be a basic open set in the Ziegler spectrum of V. Suppose that I', J' are ideals of V, $p' = p(I', J')$ and $M = \operatorname{PE}(p')$. Then the following are equivalent:*

1) $\varphi(M)/\psi(M)$ is a finite module;

2) $\varphi(M)/\psi(M)$ *is a one-dimensional vector space over* F*;*

3) either a) $I' = b'V$*,* $J' = c'V$ *and* $v(b'c') = v(bc)$*, or b)* $I' = d' \operatorname{Jac}(V)$*,* $J' = a' \operatorname{Jac}(V)$ *and* $v(a'd') = v(ad)$*.*

Geometrical explanation for this proposition is the following. A pair (φ/ψ) cuts out a finite nonzero chunk of M if and only if the line corresponding to $p(I', J')$ either crosses the rectangle (φ/ψ) at precisely one point, its upper right corner (and so it is a real line), or separates the left lower corner from the rest of the rectangle (and so it is an imaginary line):



Proof. 1) is equivalent to 2) by Corollary 7.6.

2) $\Rightarrow$ 3). Choose $m \in \varphi(M) \backslash \psi(M)$ and let $p = pp_M(m)$, hence $p = p(I, J)$ for some ideals I, J of V. By Lemma 7.7, we obtain that either a) $I = bV$ and $J = cV$, or b) $I = d \operatorname{Jac}(V)$ and $J = a \operatorname{Jac}(V)$. Since $M = \operatorname{PE}(p') \cong \operatorname{PE}(p)$, the result follows by Corollary 4.2.

3) $\Rightarrow$ 2). By Corollary 4.2 again, $\operatorname{PE}(I', J') \cong \operatorname{PE}(I, J) = M$. It remains to apply Lemma 7.7. $\square$

In the following lemma we calculate invariants of an indecomposable pure-injective module given by principal ideals. As we will see there are only 3 possibilities for those: 1, $|F|$ and ∞.

Lemma 7.9. *Let* V *be a valuation domain with a dense value group such that the residue field* F *of* V *consists of* p *elements. Let* (φ/ψ)*, where* $\varphi \doteq a \mid x \wedge xb = 0$ *and* $\psi \doteq c \mid x + xd = 0$*, be a basic open set in the Ziegler spectrum of* V*. Let* $I' = b'V$*,* $J' = c'V$ *be principal ideals of* V*,* $p' = p(I', J')$ *and* $M = \operatorname{PE}(p')$*. Then the following holds.*

1) $\operatorname{Inv}(M, \varphi, \psi) > 1$ *if and only if* $v(ad) < v(b'c') \leq v(bc)$*.*

2) $\operatorname{Inv}(M, \varphi, \psi) = n$ *for some* $n \geq 2$ *if and only if* $n = p$ *and* $v(ad) < v(b'c') = v(bc)$*.*

3) $\operatorname{Inv}(M, \varphi, \psi) = \infty$ *if and only if* $v(ad) < v(b'c') < v(bc)$*.*

4) Otherwise $\operatorname{Inv}(M, \varphi, \psi) = 1$*.*

Proof. 1) Suppose $\operatorname{Inv}(M, \varphi, \psi) > 1$, equivalently $M \in (\varphi/\psi)$. Then, by Lemma 4.4, there are $d'' \notin I'$, $b'' \in I'$ and $a'' \notin J'$, $c'' \in J'$ such that $v(ad) \leq v(a''d'')$ and $v(b''c'') \leq v(bc)$. Since $b'' \in I' = b'V$, then $v(b'') \geq v(b')$ and similarly $v(c'') \geq v(c')$. Thus $v(b'c') \leq v(b''c'') \leq v(bc)$.

From $a'' \notin J' = c'V$ it follows that $v(a'') < v(c')$, and similarly $v(d'') < v(b')$. Thus $v(b'c') > v(a''d'') \geq v(ad)$, as desired.

The converse is an immediate consequence of Lemma 4.4.

2) Suppose that $\operatorname{Inv}(M, \varphi, \psi) = n$ for some $n \geq 2$. In particular, $\operatorname{Inv}(M, \varphi, \psi) > 1$, hence, by 1), $v(ad) < v(b'c') \leq v(bc)$. Also, by Corollary 7.6, (φ/ψ) is a minimal pair in the theory of M and $n = p$. Apply Proposition 7.8 and conclude $v(bc) = v(b'c')$.

The converse is similar.

3) and 4) is a consequence of 1), 2) and Corollary 7.6. □

The following lemma is similar (so we omit the proof).

Lemma 7.10. *Let V be a valuation domain with a dense value group such that the residue field F of V consists of p elements. Let (φ/ψ), where $\varphi \doteq a \mid x \wedge xb = 0$ and $\psi \doteq c \mid x + xd = 0$, be a basic open set in the Ziegler spectrum of V. Let $I' = d' \operatorname{Jac}(V)$, $J' = a' \operatorname{Jac}(V)$, $p' = p(I', J')$ and $M = \operatorname{PE}(p')$. Then the following holds.*

1) $\operatorname{Inv}(M, \varphi, \psi) > 1$ *if and only if* $v(ad) \leq v(a'd') < v(bc)$.

2) $\operatorname{Inv}(M, \varphi, \psi) = n$ *for some* $n \geq 2$ *if and only if* $n = p$ *and* $v(ad) = v(a'd') < v(bc)$.

3) $\operatorname{Inv}(M, \varphi, \psi) = \infty$ *if and only if* $v(ad) < v(a'd') < v(bc)$.

4) Otherwise $\operatorname{Inv}(M, \varphi, \psi) = 1$.

8. Decidability. Finite residue field

In this section we prove decidability of the theory of all modules over an effectively given valuation domain V with a dense archimedean value group and a finite residue field F. But first we clarify the notion of effectiveness in this particular case.

Suppose that the theory of all V-modules is decidable, and F is finite. Could we find the number of elements of F effectively? The answer is 'yes'. Indeed, by Corollary 7.6, this number is exactly the integer $n > 1$ such that there exists a V-module M satisfying $\operatorname{Inv}(\varphi, \psi) = n$ for some pp-formulae φ and ψ. In fact (by Proposition 7.7), we can choose (any!) $a, b, c, d \in V$ such that $v(d) < v(b)$, $v(a) < v(c)$ and set $\varphi \doteq a \mid x \wedge xb = 0$, $\psi \doteq c \mid x + xd = 0$. Now we proceed, starting from $n = 2$, answering the question: 'Does there exists a module M such that $\mathrm{Inv}(\varphi, \psi, M) = n$?' (this is equivalent to check that a suitable first order sentence of the language of V-modules is in T_V – see a discussion in Section 6). The first 'yes' we get will determine n.

Thus, aiming to prove decidability of the theory T_V we are required to know the size of F in advance. So let F have exactly p elements. First we prove an auxiliary lemma.

Lemma 8.1. *Let V be an effectively given valuation domain with a dense value group and such that the residue field F of V has p elements. Let $I = bV$, $J = cV$ or $I = d\,\mathrm{Jac}(V)$, $J = a\,\mathrm{Jac}(V)$; $p = p(I, J)$ and $M = \mathrm{PE}(p)$. Suppose that $\varphi(x)$ and $\psi(x)$ are pp-formulae. Then we can effectively calculate $\mathrm{Inv}(M, \varphi, \psi)$, and it is equal to 1, p or ∞.*

Proof. Represent φ as $\sum_i \varphi_i$, where $\varphi_i \doteq a_i \mid x \wedge xb_i = 0$ and ψ as $\wedge_j \psi_j$, where $\psi_j \doteq c_j \mid x + xd_j = 0$. Lemmas 7.9 and 7.10 give an effective way to calculate each $\mathrm{Inv}(M, \varphi_i, \psi_j)$, which is 1, p or ∞. Now, by Corollary 7.3, $\mathrm{Inv}(M, \varphi, \psi) = \max_{i,j} \mathrm{Inv}(M, \varphi_i, \psi_j)$, and the result follows. □

Now we are in a position to prove decidability.

Theorem 8.2. *Let V be an effectively given valuation domain with a dense archimedean value group and such that the residue field F of V consists of p elements. Then the theory T_V of all V-modules is decidable.*

Proof. As in the proof of Theorem 6.2 we may assume that we are given a first order sentence σ which is a conjunction of invariant sentences and their negations, and we should (uniformly) decide if there exists a V-module M satisfying σ.

Without loss of generality we can assume that each conjunct of σ is of one of the following forms:

1) $\mathrm{Inv}(\varphi_1, \psi_1) \geq n_1$ for some $n_1 \geq 2$;

2) $\mathrm{Inv}(\varphi_2, \psi_2) = 1$;

3) $\mathrm{Inv}(\varphi_3, \psi_3) = n_3$, for some $n_3 \geq 2$.

It is also known, that, if such a module M exists, it can be found between finite direct sums of indecomposable pure-injective modules. So we are going to check on potential candidates for modules M of this form.

First assume that 3) is empty. Then proceed as in the proof of Theorem 6.2 with minor changes. Namely, by that proof we can effectively check the existence of a module M satisfying all sentences $\mathrm{Inv}(\varphi_1, \psi_1) > 1$ as in 1), and all sentences in 2). If no such M exists, then σ has no models. Otherwise M^k for some (large enough) k will satisfy 1) and 2).

Thus we may assume that 3) is nonempty. Take a pair (φ_3/ψ_3) from this list. If there exists a module N such that $\operatorname{Inv}(N, \varphi_3, \psi_3) = n_3$, then there exists an indecomposable pure-injective V-module M such that $\operatorname{Inv}(M, \varphi_3, \psi_3) > 1$ is finite, hence equal to p. Moreover we can assume that N itself is a finite direct sum of indecomposable pure-injective modules with this property. In particular, if n_3 is not a power of p, then σ cannot admit any model. Thus we may assume that n_3 is a power of p. (In the same way, we may replace each n_1 in 1) by the smallest power of p which is $\geq n_1$, and assume that n_1 itself is a power of p).

Now we decide whether an indecomposable pure-injective module M_3 such that $\operatorname{Inv}(M_3, \varphi_3, \psi_3) = p$ exists and make a (finite) list of such modules.

Represent $\varphi_3 \doteq \sum_l \varphi_{3l}$, where $\varphi_{3l} \doteq a_l \mid x \wedge xb_l = 0$ and $\psi_3 \doteq \wedge_k \psi_{3k}$, where $\psi_{3k} \doteq c_k \mid x + xd_k = 0$. Recall that $(\varphi_3/\psi_3) = \cup_{kl}(\varphi_{3l}/\psi_{3k})$. By Proposition 7.8, if M_3 exists, then we must have $M_3 \cong \operatorname{PE}(I, J)$, where either $(I, J) = (b_l V, c_k V)$ or $(I, J) = (d_k \operatorname{Jac}(V), a_l \operatorname{Jac}(V)$ for some l and k (only finitely many possibilities). Now, using Lemmas 7.9 and 7.10, we calculate (for each pair (I, J) as above) $\operatorname{Inv}(\operatorname{PE}(I, J), \varphi_3, \psi_3)$, which is 1, p or ∞. If no indecomposable pure-injective modules $\operatorname{PE}(I, J)$ with $\operatorname{Inv}(\operatorname{PE}(I, J), \varphi_3, \psi_3) = p$ exists, then σ has no model. Repeat this procedure for every conjunct in 3).

Otherwise let $M(0), \ldots, M(s)$ be a complete list of indecomposable pure-injective modules whose (φ_3/ψ_3)-invariant is p for every pair $(\varphi_3,\ \psi_3)$ in 3). For each module $M(t)$ with $t \leq s$ we calculate all the invariants from the first and the second list. If $M(t)$ does not satisfy some sentence $\operatorname{Inv}(\varphi_2, \psi_2) = 1$ in 2), then we drop $M(t)$ from consideration (it cannot be a direct summand of any model of σ). Thus we may assume that $\operatorname{Inv}(M(t), \varphi_2, \psi_2) = 1$ for every pair conjunct in 2).

Now we 'subtract' $M(t)$ from a potential candidate on a model of σ and see what happens. Namely, we change our original question to a similar one, but replace $\operatorname{Inv}(\varphi_3, \psi_3) = n_3$ by $\operatorname{Inv}(\varphi_3, \psi_3) = n_3/p$ for each sentence in 3). Further, we drop a sentence $\operatorname{Inv}(\varphi_1, \psi_1) \geq n_1$ from 1) whenever $\operatorname{Inv}(M(t), \varphi_1, \psi_1) = \infty$ (as it has been already satisfied); replace it by $\operatorname{Inv}(\varphi_1, \psi_1) \geq n_1/p$, if $\operatorname{Inv}(\varphi_1, \psi_1) = p$; and leave it unchanged otherwise (i.e., if $\operatorname{Inv}(\varphi_1, \psi_1) = 1$). Further, we leave all the sentences in 2) unchanged.

Applying this procedure, eventually we end up with a list as above, but excluding any sentence as in 3). At this point we go back to the beginning of the proof. $\square$

Corollary 8.3. *Let V be a valuation domain from Example 2.1 , where F is a finite field. Then the theory of all V-modules is decidable.*

9. Conclusions

As we have seen, if the value group of an effectively given valuation domain V is 'nice', then only very simple fragments of the first order theory of V (as a ring) can be interpreted in the theory of V-modules. Now we give an example showing that in general T_V does encode some conditions on V which cannot be expressed by first order sentences.

Recall that, if I is an ideal of V, then the *radical* of I, $\mathrm{rad}(I)$ is the intersection of all prime ideals containing I. Since the ideals of V are linearly ordered, $\mathrm{rad}(I)$ is a prime ideal. It is well known that $b \in \mathrm{rad}(I)$ if and only if $b^n \in I$ for some positive integer n.

Lemma 9.1. *Let V be a valuation domain and let $a, b \in V$. Then the following are equivalent:*

1) $\exists x\, (x \neq 0 \wedge xa = 0) \rightarrow \exists y\, (y \neq 0 \wedge yb = 0)$ holds true in the theory of all V-modules;

2) $b^n \in aV$ for some n;

3) $b \in \mathrm{rad}(aV)$.

Proof. 2) is equivalent to 3) by the above remark.

1) $\Rightarrow$ 2). Otherwise $b \notin P = \mathrm{rad}(aV)$, and P is a prime ideal. Let $M = V/P$ and $m = 1 + P$. Then $m \neq 0$ and $ma = 0$ in M, because $a \in P$ and so a acts as 0 on M. On the other hand, since P is prime and $b \notin P$, $m'b = 0$ implies $m' = 0$ for every $m' \in M$, a contradiction.

2) $\Rightarrow$ 1). Suppose that m is a nonzero element of a module M such that $ma = 0$. If $b^n = as$ for $s \in V$, then $mb^n = mas = 0$. Then there is a non-negative integer $k < n$ such that $mb^k \neq 0$ but $mb^{k+1} = 0$. If $m' = mb^k$, then $m' \neq 0$ but $m'b = 0$. □

If V is a valuation domain with an archimedean value group and $0 \neq a, b \in V$, it is always true that $b^n \in aV$ for some n. But, if the value group of V is not archimedean, checking the decidability of T_V requires to answer a non-trivial non first-order question on V, that is, membership to the radical of aV for $a \in V$.

So we guess that there may exist an effectively given valuation domain V such that the first order theory of V is decidable, but the theory of all V-modules is undecidable.

In particular we suggest the following conjecture.

Conjecture 9.2. *Let V be an effectively given valuation domain such that the value group of V is dense. The the following are equivalent.*

1) The theory of all V-modules is decidable.

2) There is an algorithm that, given $a, b \in V$, answers whether $b^n \in aV$ for some integer $n \geq 1$.

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