EXPANSIONS OF O-MINIMAL STRUCTURES BY DENSE INDEPENDENT SETS

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ABSTRACT. Let $\mathfrak{M}$ be an o-minimal expansion of a densely ordered group and $\mathcal{H}$ be a pairwise disjoint collection of dense subsets of M such that $\bigcup \mathcal{H}$ is definably independent in $\mathfrak{M}$. We study the structure $(\mathfrak{M}, (H)_{H \in \mathcal{H}})$. Positive results include that every open set definable in $(\mathfrak{M}, (H)_{H \in \mathcal{H}})$ is definable in $\mathfrak{M}$, the structure induced in $(\mathfrak{M}, (H)_{H \in \mathcal{H}})$ on any $H_0 \in \mathcal{H}$ is as simple as possible (in a sense that is made precise), and the theory of $(\mathfrak{M}, (H)_{H \in \mathcal{H}})$ eliminates imaginaries and is strongly dependent and axiomatized over the theory of $\mathfrak{M}$ in the most obvious way. Negative results include that $(\mathfrak{M}, (H)_{H \in \mathcal{H}})$ does not have definable Skolem functions and is neither atomic nor satisfies the exchange property.

INTRODUCTION

Broadly speaking, this paper continues to explore the central theme of our earlier works [8, 9], whose origin lies in Miller and Speissegger [26], in which we examine extensions T' of well-behaved first-order theories T extending that of dense linear orders without endpoints (DLO)—typically, o-minimal T—in which good behavior is preserved. The properties that are investigated can be either topological or model theoretic. Every ordered structure comes equipped with the topology on definable sets induced by the order topology. Given this, following [26], it can be asked if the so-called open core of any model of the extended theory—that is, the structure with atomic formulas precisely for those definable open sets in the model of the extended theory—is o-minimal, and if so, whether the open core includes no additional open sets as the open core of the structure in the original language. Model-theoretic properties whose preservation (or lack thereof) are investigated include elimination of imaginaries, strong dependence, existence of atomic models and definable Skolem functions, and the exchange property (with respect to definable closure).

We now begin to make more precise the setting of this paper. Given a theory T extending DLO in a language $L \supseteq \{<\}$, we are interested in understanding (relative to T) extensions T' of T in languages extending L by unary relation symbols that are interpreted in models of T' as sets that are both dense and codense (that is, having empty interior) in the underlying sets of the models. The general goal is to understand the result of allowing various kinds of topological noise to be introduced into models of T. In order to avoid degeneracy, we want the base theory T to be sufficiently well behaved and rich. A natural case for first

investigation is that T be complete, o-minimal and extend the theory of densely ordered groups.[1]

There is a canonically "wild" example, namely, $T = \mathrm{Th}(\mathbb{R}, <, +, \cdot)$ and $T' = \mathrm{Th}(\mathbb{R}, <, +, \cdot, \mathbb{Q})$. The model theory of T is both well understood and quite well behaved; in particular, T is o-minimal, and so every open set (of any arity) definable in any model of T has only finitely many definably connected components. But $\mathbb{Z}$ is interdefinable with $\mathbb{Q}$ over $(\mathbb{R}, <, +, \cdot)$ (by Robinson [32]) and $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$ defines every real Borel set (see Kechris [19, 37.6]), in particular, every open subset of any finite cartesian power of $\mathbb{R}$ and every subset of any finite cartesian power of $\mathbb{Q}$. Thus, while the open sets definable in models of T are as simple as possible relative to the theory of ordered rings, the extension T' has a model where the definable open sets are as complicated as possible, as is the structure induced on the new predicate.

In contrast to the preceding example, if K is any proper real-closed subfield of $\mathbb{R}$ and $T' = \mathrm{Th}(\mathbb{R}, <, +, \cdot, K)$, then no model of T' defines any open set (of any arity) that is not definable in the reduct of the model to L (van den Dries [12, Theorem 5]); as in [8], we abbreviate this property by saying that T **is an open core of** T'. More generally, if $\mathfrak{M}$ is an o-minimal expansion of a densely ordered group, and A is dense in M and the underlying set of a proper elementary substructure of $\mathfrak{M}$, then $(\mathfrak{M}, A)$ is called a **dense pair**[2] (of o-minimal expansions of densely ordered groups). By [12, Lemma 4.1], A is also codense in M. Dense pairs have been been studied extensively (*e.g.*, [8, 12, 26]) and $\mathrm{Th}(\mathfrak{M}, A)$ is well understood relative to T; in particular, if $\mathfrak{M}' \equiv \mathfrak{M}$ and $(\mathfrak{M}', A')$ is a dense pair, then $(\mathfrak{M}, A) \equiv (\mathfrak{M}', A')$ (by [12, 2.5]) and $\mathrm{Th}(\mathfrak{M})$ is an open core of $\mathrm{Th}(\mathfrak{M}, A)$ (see [8, Section 5]).

In this paper, we analyze an orthogonal complement (so to speak) to dense pairs, namely, expansions $(\mathfrak{M}, (H)_{H \in \mathcal{H}})$ of $\mathfrak{M}$ by dense subsets H of M such that $\mathcal{H}$ is pairwise disjoint and $\bigcup \mathcal{H}$ is (definably) independent over $\mathfrak{M}$. The canonical motivating example is $(\mathbb{R}, <, +, H)$, where H is a dense Hamel basis, that is, a dense subset of $\mathbb{R}$ that is a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Remarkably, the analysis of this seemingly quite special case is not significantly any easier than that of the general (thus explaining our use of "H" for dense independent sets).

Throughout: T denotes a complete o-minimal extension of the theory of densely ordered groups with a distinguished positive element[3] in a language $L \supseteq \{<, +, -, 0, 1\}$. We let T^{pair} denote the theory of dense pairs of T.[4] Let $\mathcal{P}$ be a set of unary relation symbols P, none of which belong to L, and put $L_{\mathcal{P}} = L \cup \{P : P \in \mathcal{P}\}$. Let T^{indep} be the $L_{\mathcal{P}}$-extension of T by the axiom schemata expressing that each P is dense and $\mathcal{P}$ is mutually independent over T (that is, $\mathcal{P}$ is pairwise disjoint and $\bigcup \mathcal{P}$ is independent over T). Our main result is that T is an open core of T^{indep}. Once this is established, we analyze T^{indep} and its models.

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[1]Without group structure, differences in kinds of noise tend to disappear, and results tend to degenerate. To illustrate, the theory of the extension of DLO by any given pairwise disjoint family of dense-codense unary predicates is easily seen to be complete.

[2]By convention, "pair" abbreviates "proper elementary pair" in this context.

[3]The assumption of a distinguished positive element is primarily for later technical convenience and is not needed in order to state our main results.

[4]T^{d} is used instead of T^{pair} in [8, 12].

In the large, T^{indep} is rather similar to T^{pair}, especially if T extends the theory of ordered rings, but there is at least one significant difference: T^{indep} eliminates imaginaries (3.13) and T^{pair} does not ([8, 5.5]).

In order to simplify notation, from here on we state and prove results only for the case that $\mathcal{P}$ is a singleton $\{P\}$ (so we write L_P instead of $L_{\mathcal{P}}$, and so on) but some explanation is in order. As each $L_{\mathcal{P}}$-formula involves only finitely members of $\mathcal{P}$, we may reduce to the case that $\mathcal{P}$ is finite. Hence, if T has a pole (*i.e.*, if some model $\mathfrak{M}$ of T $\emptyset$-defines a bijection $f\colon M \to M^{>0}$) then all results (possibly after appropriate rewording) follow easily from the case that $\mathcal{P}$ is a singleton, as then $\mathcal{H}$ can be $\emptyset$-definably encoded as a single dense independent set. (If $H, H' \subseteq M$ are dense and mutually independent, then $f(H) \cup (-f)(H')$ is dense, independent, and $\emptyset$-interdefinable with $\{H, H'\}$ over $\mathfrak{M}$.) But otherwise we see no shortcuts; almost everything has to be done from the start using finitely many new predicates, including redoing in this greater generality many key results from other papers. We shall not provide the details, as this would add considerable length to this paper while simultaneously reducing readability, all in order to deal with a fairly degenerate setting.[5] But neither do we wish to suggest that checking all the details is nothing but a routine exercise for the reader, as it would be rather lengthy. We assure the reader that we have done our best to check all of the needed modifications, and that they are all routine.

Here is an outline of the body of this paper.

Section 1 consists of preliminary material such as global conventions and technical lemmas; several of the latter hold assuming only that T is o-minimal (no assumption of group structure) and is extended by an independent unary predicate (no assumption of density). We then establish in Section 2 as many results about T^{indep} as we reasonably can that require only minor modification of previously-known facts about T^{pair}; in particular, T^{indep} is complete (2.8) and has T as an open core (2.25). Just as with T^{pair}, major steps in the proof of the latter are that T^{indep} admits quantifier elimination down to "special formulas" (see 2.9 for the precise statement) and if $X \subseteq H^n$ is definable in $(\mathfrak{M}, H)$, then there exists $Y \subset M^n$ definable in $\mathfrak{M}$ such that $X = H^n \cap Y$ (2.16). Other similarities to T^{pair} are that T^{indep} is strongly dependent (2.28), does not have definable Skolem functions (2.23), has no atomic model (2.27), and fails to satisfy the exchange property (1.1.1). We also show in Section 2 that the structure induced on H in $(\mathfrak{M}, H)$ is as simple as possible among all structures of the form $(\mathfrak{M}, E)$ where E is dense and codense in M (see 2.29 for the precise statement). We then focus in Section 3 on the main difference between T^{indep} and T^{pair}, namely, T^{indep} eliminates imaginaries (3.13). On the way, we also characterize þ-forking over T^{indep} in terms of "small" closure (3.4). Having thus established all of our main results about T^{indep}, we then proceed in Section 4 to do a detailed comparison of T^{indep} with T^{pair} and certain other examples of complete L_P-extensions of T known to have T as an open core (namely, completions of the extension of T by a generic predicate). Finally, Section 5 consists of discussions of optimality and some open issues.

We conclude this introduction with a survey of how material in this paper relates to and in some cases is inspired by existing literature. The wellspring for much of the basic

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[5]It is known that if T does not have pole, then up to interdefinability, T is the theory of some ordered division ring expanded by a collection of bounded sets; see, *e.g.*, [8, 1.13] for a more detailed discussion.

work here is the previously-mentioned paper [12] by van den Dries on T^{pair}. Several of the results in Section 2 are modifications of assertions from [12], although some are not entirely straightforward, in which instances we provide proofs. For the primary results of Section 2 on completeness and quantifier simplification, we originally had proofs modeled after those in [12], but we later learned that arguments from Berenstein and Vassiliev [3] could be adapted to our setting and yield simpler proofs. Given the specificity of our context, once again not all of the modifications are straightforward and we thus state sharper results and provide detailed proofs where warranted. Our proof of strong dependence of T^{indep} is a minor modification of that for T^{pair}, a result due to Berenstein, Dolich and Onshuus [1, 2.11]. Work of Berenstein, Ealy and Günaydın [2] is important in Section 3. We lastly note that, inspired by our work here in the o-minimal context, Berenstein and Vassiliev study in [4,5] models augmented by a predicate for a dense independent set in the more general setting of geometric theories. While they obtain results similar to ours, given the generality in which they they work, they are unable to reproduce many of our stronger results such as elimination of imaginaries.

1. Preliminaries

In this section, we declare some global notation and conventions, and collect some basic technical results about independent sets to be used later in the paper. We state and prove a number of theses results in greater than we need in the hope that they might find use elsewhere.

The variables j, k, l, m, n range over $\mathbb{N}$ (the non-negative integers) unless indicated otherwise.

Given a set X, its cardinality is denoted by $\operatorname{card} X$ (or $\operatorname{card}(X)$), except when applied to languages or theories, in which case $\operatorname{card} X$ is the cardinality of $X \cup \mathbb{N}$. The n-th cartesian power of X is denoted by X^n, with $X^0 = \{\emptyset\}$. Whenever convenient, we identify $X^m \times X^n$ with X^{m+n}, and $(X^m)^n$ with X^{mn}. Given a set $S \subseteq X^n$, we say that a given property holds for S **in each coordinate** if for every $(x_1, \ldots, x_n) \in S$, the property holds for each of the sets

$$\{\, t \in X : (t, x_2, \ldots, x_n) \in S \,\}$$
$$\{\, t \in X : (x_1, t, x_3 \ldots, x_n) \in S \,\}$$
$$\vdots$$
$$\{\, t \in X : (x_1, \ldots, x_{n-1}, t) \in S \,\},$$

that is, if $\pi(S \cap (\pi^\perp)^{-1}(x))$ has the given property for each coordinate projection $\pi \colon X^n \to X$ and $x \in S$, where $\pi^\perp$ is the orthogonal complement. (To illustrate, $S \subseteq X^2$ is infinite in each coordinate if for every $(x_1, x_2) \in S$, the sets $\{\, t \in X : (x_1, t) \in S \,\}$ and $\{\, t \in X : (t, x_2) \in S \,\}$ are infinite.) Given collections $\mathcal{C}, \mathcal{D}$ of subsets of X, we say that $\mathcal{C}$ is **compatible** with $\mathcal{D}$ if for all $(C, D) \in \mathcal{C} \times \mathcal{D}$, either $C \cap D = \emptyset$ or $C \subseteq D$. If $\mathcal{D}$ is a singleton $\{D\}$, then we tend to say that $\mathcal{C}$ is compatible with D, and similarly if $\mathcal{C}$ is a singleton.

Given a set Y, we identify a function $f \colon X^0 \to Y$ with the constant $f(\emptyset) \in Y$.

Given a function $f\colon X \to Y$ and $S \subseteq X$, we let $f{\restriction}S$ denote the restriction of f to S. We say that a function $f\colon S \to Y$ is **given piecewise** by a collection $\mathcal{F}$ of functions $X \to Y$ if there is a finite $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathrm{graph}(f) \subseteq \bigcup_{g \in \mathcal{G}} \mathrm{graph}(g)$.

Given $S \subseteq X^n$ and $A \subseteq X$, we say that $f\colon S \to X$ is **A-simple** if it is either constant, a coordinate projection or $f(A^n \cap X) \cap A = \emptyset$. A map $S \to X^k$ is A-simple if all of its component functions are A-simple.

We have a similar convention for functions as for sets regarding the general notion of a property holding in each coordinate, *e.g.*, if $S \subseteq X^2$, then a function $f\colon S \to Y$ is injective in each coordinate if for every $(x_1, x_2) \in S$, the functions

$$t \mapsto f(x_1, t)\colon \{\, t \in X : (x_1, t) \in S \,\} \to Y$$
$$t \mapsto f(t, x_2)\colon \{\, t \in X : (t, x_2) \in S \,\} \to Y$$

are injective.

Topological generalities. Given a topological space X, we say that $A \subseteq X$ is: **constructible** if it is a (finite) boolean combination of open sets; **discrete** if all of its points are isolated; **locally closed** if it is open in its closure; **somewhere dense** if its closure has (nonempty) interior; **nowhere dense** if its closure has no interior;[6] **dense in $C \subseteq X$** if the closure of $C \cap A$ is equal to that of C; and **codense in C** if the closure of $C \setminus A$ is equal to that of C. For convenience, we may say that A is **dense-codense** in C if it is both dense in C and codense in C. Whenever $C = X$ and X is understood, we omit "in C". Some basic facts:

(1) A set is constructible if and only if it is a finite union of locally closed sets.
(2) Constructible sets either have interior or are nowhere dense.
(3) Discrete sets are locally closed.
(4) If $A \subseteq X$ is locally closed, Y is topological space and $f_1, \ldots, f_k\colon A \to Y$ are continuous, then the union of their graphs is locally closed in $X \times Y$. If moreover $B \subseteq Y$ is locally closed, then each $f_i^{-1}(B)$ is locally closed.
(5) The previous item holds with "constructible" in place of "locally closed".

Remarks. (i) Neither complements nor unions of locally closed sets need be locally closed: Both $\{\, (x, y) \in \mathbb{R}^2 : y \neq 0 \,\}$ and $\{(0,0)\}$ are locally closed in $\mathbb{R}^2$, their union is not, and the complement of the union is. (ii) Coordinate projections of constructible sets need not be constructible. To illustrate, let $(q_n)_{n \in \mathbb{N}}$ be an enumeration of $\mathbb{Q}$, and regard $A := \{\, (q_n, n) : n \in \mathbb{N} \,\}$ as a subset of $\mathbb{R}^2$. Then A is closed and discrete, but its projection onto the first coordinate is dense and codense. By regarding $A \subseteq \mathbb{Q}^2$, this also shows that a function can have a constructible graph, yet be nowhere continuous.

Model-theoretic generalities. We shall find it practical to have some flexibility in our conventions and notation for points, and how we regard them. We shall often write things like "$\bar{x} \in X$" as an abbreviation for "$x \in X^n$ for some $n \in \mathbb{N}$" if n is not germane. If $\bar{a}, \bar{b} \in X$, then $\bar{a}\bar{b}$ denotes concatenation $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ of the tuples $\bar{a} = (a_1, \ldots, a_m)$ and $\bar{b} = (b_1, \ldots, b_n)$. But we also sometime think of $\bar{x}$ as listing a finite set, in which case $\bar{x} \in X$ means just $x_1, .., x_n \in X$ for some $n \in \mathbb{N}$, and $\bar{a}\bar{b}$ would be the union of the finite sets

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[6]We say "no interior" instead of "nonempty interior".

$\bar{a} = \{a_1, \ldots, a_m\}$ and $\bar{b} = \{b_1, \ldots, b_n\}$. Context should resolve any ambiguities ($e.g.$, $\bar{a} \cap \bar{b}$ indicates that $\bar{a}$ and $\bar{b}$ are being regarded as sets), and in any case, we shall attempt to avoid ambiguity by writing either $x \in X^n$ or $x_1, \ldots, x_n \in X$ as appropriate unless the $\bar{x}$ notation significantly reduces clutter.

A first-order structure on a set A is usually indicated by the corresponding $\mathfrak{A}$, and vice versa. Given $S \subseteq A$, "S-definable (in $\mathfrak{A}$)" means "definable (in $\mathfrak{A}$) with parameters from S". If no ambient space A^n is specified, then "S-definable set" means "S-definable subset of some A^n", while "S-definable function (or map)" means "S-definable partial function (or map) from some $X \subseteq A^m$ into some A^n". Mention of S in the above is often suppressed when S is not germane. We denote the definable closure of S in $\mathfrak{A}$ by $\mathrm{dcl}_{\mathfrak{A}}(S)$, although the subscripted $\mathfrak{A}$ will nearly always be suppressed when $\mathfrak{A}$ is understood. We say that $\mathfrak{A}$ satisfies the **exchange property**—for short, $\mathfrak{A}$ has EP or $\mathfrak{A} \vDash$ EP—if $y \in \mathrm{dcl}(S \cup \{x\})$ for all $S \subseteq A$ and $x, y \in A$ such that $x \in \mathrm{dcl}(S \cup \{y\}) \setminus \mathrm{dcl}(S)$. To be precise, the definition is that of EP *with respect to definable closure*, since EP can be formulated with respect to any pregeometry. The notions of dependence and independence are usually with respect to definable closure, although we also have occasion to refer to dependence or independence of theories in the sense of Shelah. For a theory T_0, we write $T_0 \vDash$ EP if every model of T_0 has EP.

Given structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with common underlying set A, we write $\mathfrak{A}_1 =_{\mathrm{df}} \mathfrak{A}_2$ if every set definable in $\mathfrak{A}_1$ is definable in $\mathfrak{A}_2$, and vice versa.

1.1. *Let $\mathfrak{G}$ be an expansion of an abelian group $(G, +, 0)$ and $A \subseteq G$ be infinite and algebraically independent in $\mathfrak{G}$.*

   (1) $\mathrm{Th}(\mathfrak{G}, A) \nvDash$ EP.
   (2) $\mathrm{Th}(\mathfrak{G}, A)$ *does not have finite dp-rank.*
   (3) $\mathrm{Th}(\mathfrak{G}, A)$ *does not have finite þ-rank.*
   (4) *If the underlying group is also divisible and ordered (and the order is definable), then A is codense in G.*

(See Kaplan, Onshuus and Uzvyatsov [18] for the definition of dp-rank. For details on þ-rank, see Onshuus [27].)

Proof. 1. By passing to a sufficiently saturated elementary extension, we reduce to the case that there exist $a, a' \in A$ that are algebraically independent in $(\mathfrak{G}, A)$. The only solutions in A^2 to $x + y = a + a'$ are (a, a') and (a', a) so $\{a, a'\}$ and $\{a + a'\}$ are interalgebraic in $(\mathfrak{G}, A)$. The result follows.

2. Once again we pass to a sufficiently saturated elementary extension. We use the observation that if $a, b \in A^n$ and $2a_1 + \cdots + 2^n a_n = 2b_1 + \cdots + 2^n b_n$ then $a = b$. We must show that there is a randomness pattern of depth n for all n (see [18]). For $1 \leq i \leq n$, let $\phi_i(x, y)$ be the formula

$$\exists z_1 \ldots z_n \left(\bigwedge_{i=1}^{n} P z_i \wedge \bigwedge_{i \neq j} z_i \neq z_{i+1} \wedge y = z_i \wedge x = 2z_1 + \cdots + 2^n z_n \right).$$

6

Pick distinct $a_i^j \in A$ for $1 \leq i \leq n$ and $j \in \mathbb{N}$. Note that if $j \neq k$ and $1 \leq i \leq n$ then $\phi_i(x, a_i^j) \wedge \phi_i(x, a_i^k)$ is inconsistent but that if $j_1 \ldots j_n \in \mathbb{N}$ then

$$\phi_1(x, a_1^{j_1}) \wedge \cdots \wedge \phi_n(x, a_n^{j_n})$$

is consistent. Thus we have constructed a randomness pattern of depth n as desired.

3. The proof is similar to that of 2.

4. If A has interior, then it contains a nontrivial closed and bounded interval $[a, a']$. But then $(a + a')/2 \in A$, contradicting independence. $\qquad\square$

Remark. Without the passage to a model where A is large enough, it could be that a and a' as in the proof of 1 do not exist, say, if $M = \mathbb{R}$ and A is closed and discrete. We shall see later that this problem does not arise in models of T^{indep}, and so *no* model of T^{indep} has EP.

We use **first-order topological structure** as defined in Pillay [28]. If $\mathfrak{B}$ is a first-order topological structure, then the **open core** of $\mathfrak{B}$, denoted by $\mathfrak{B}^\circ$, is the structure $(B, (U))$ where U ranges over all open sets (of any arity) definable in $\mathfrak{B}$. (Syntactically: For each open set U definable in $\mathfrak{B}$, we introduce a relation symbol P_U of the appropriate arity where it is understood that the map $U \mapsto P_U$ is injective, and then $\mathfrak{B}^\circ$ is obtained by interpreting P_U as U.)

1.2. *Let $\mathfrak{B}$ be a first-order topological structure and $A \subseteq B$ be independent. Suppose that for each locally nonconstant definable $g \colon B \to B$ there exists $N_g \in \mathbb{N}$ such that $\mathrm{card}(A \cap g^{-1}(b)) \leq N_g$ for all $b \in B$. Let $f \colon U \to B$ be definable on some open $U \subseteq B^n$ such that all points in U have pairwise distinct coordinates and f is locally nonconstant in each coordinate. Then there exists $N \in \mathbb{N}$ such that $\mathrm{card}(A^n \cap f^{-1}(b)) \leq N$ for all $b \in B$.*

Proof. Without loss of generality, $(\mathfrak{B}, A)$ is sufficiently saturated. We proceed by induction on $n \geq 1$, with the basis holding by assumption. Let $n > 1$ and assume the result for all $m < n$. Let $b \in B$ and $f \colon U \to B$ be definable on some open $U \subseteq B^n$ such that all points in U have pairwise distinct coordinates and f is locally nonconstant in each coordinate. We must show that $A^n \cap f^{-1}(b)$ is finite. Suppose not, and pick $\bar{w}^i \in A^n \cap U$ for $i \in \omega_1$ so that $f(\bar{w}^i) = b$ for all $i \in \omega_1$. By the Δ-system Lemma (see, *e.g.*, Kunen [21]) and after passing to a subsequence there is a finite set W such that $\{w_1^i, \ldots, w_n^i\} \cap \{w_1^j, \ldots w_n^j\} = W$ for all distinct i and j in ω_1. If $W \neq \emptyset$, then without loss of generality we may assume that for some $k < n$ there are $w_1, \ldots, w_k$ such that $\bar{w}^i = w_1, \ldots, w_k, w_{k+1}^i, \ldots, w_n^i$ for all $i \in \omega_1$. Inductively, we obtain a contradiction, so we reduce to the case that $\bar{w}^i \cap \bar{w}^j = \emptyset$ for all $i \neq j$.

Let $\bar{d} \in B$ be independent such that f is $\bar{d}$-definable. Without loss of generality, the sequence $(\bar{w}^i)$ is indiscernible over $\bar{d}$. We now show that $f^{-1}(b)$ has interior (thus contradicting the assumptions on f) by showing that each $\bar{w}^i$ is independent over $\bar{d}$. If otherwise, then by indiscernibility $\bar{w}^0$ is dependent over $\bar{d}$, so without loss of generality assume that $w_0^0 \in \mathrm{dcl}(w_1^0, \ldots w_n^0 \bar{d})$. Pick $\bar{e} \subseteq \bar{d}$ minimal such that $w_0^0 \in \mathrm{dcl}(w_1^0, \ldots, w_n^0 \bar{e})$. Let $\bar{e} = e_0 \ldots e_l$ and note that $\bar{e} \neq \emptyset$. Hence, $e_0 \in \mathrm{dcl}(\bar{w}^0 e_1, \ldots e_l)$. For ease of notation assume that $e_0 = d_0$ (where $\bar{d} = d_0, \ldots, d_m$). Thus $w_0^1 \in \mathrm{dcl}(w_1^1 \ldots w_n^1 d_1 \ldots d_m \bar{w}^0)$ by indiscernibility. Hence once again without loss of generality we may conclude that $d_1 \in$

$\mathrm{dcl}(\bar{w}^1 d_2 \ldots d_m \bar{w}^0)$. By repeating this argument we find that $d_1 \ldots d_m \in \mathrm{dcl}(\bar{w}^0 \ldots \bar{w}^m)$. Hence, $w_0^{m+1} \in \mathrm{dcl}(w_1^{m+1}, \ldots, w_n^{m+1} \bar{w}^0 \ldots \bar{w}^m)$, which is impossible. $\qquad\square$

Induced structure. Given sets B and $A \subseteq B^m$, a **trace** on A (with respect to B) is a set of the form $A^n \cap S$ for some $n \in \mathbb{N}$ and $S \subseteq B^{mn}$. If moreover S is definable in a structure $\mathfrak{B}$, then we call it a trace on A in $\mathfrak{B}$. We define the **structure induced on A in $\mathfrak{B}$**, denoted by $A(\mathfrak{B})$, to be the structure $(A, (X))$ where X ranges over all traces on A in $\mathfrak{B}$. (Syntactically: For each trace X on A in $\mathfrak{B}$, we introduce a relation symbol P_X of the appropriate arity where it is understood that the map $X \mapsto P_X$ is injective, and then $A(\mathfrak{B})$ is obtained by interpreting P_X as X.) We tend to reduce parentheses in usage, *e.g.*, we write $A(B, <)$ instead of $A((B, <))$, and $A(\mathfrak{B}, A)$ instead of $A((\mathfrak{B}, A))$. If A is definable in $\mathfrak{B}$, then $A(\mathfrak{B}, A) = A(\mathfrak{B})$. Note that: every definable set of $A(\mathfrak{B})$ is $\emptyset$-definable; every quantifier-free definable set of $A(\mathfrak{B})$ is a trace on A in $\mathfrak{B}$, and so is defined by an atomic formula; and if A is definable in $\mathfrak{B}$, then every set definable in $A(\mathfrak{B})$ is a trace on A in $\mathfrak{B}$. If $\mathfrak{B}$ is a first-order topological structure, then we regard $A(\mathfrak{B})$ as first-order topological structure via the induced topology.

1.3. *Let $\mathfrak{B}$ be a first-order topological structure and $A \subseteq B^m$ be definable in $\mathfrak{B}$.*

(1) *Open sets definable in $A(\mathfrak{B})$ are traces on A of open sets definable in $\mathfrak{B}$.*
(2) *Locally closed sets definable in $A(\mathfrak{B})$ are traces on A of locally closed sets definable in $\mathfrak{B}$.*
(3) *Constructible sets definable in $A(\mathfrak{B})$ are traces on A of constructible sets definable in $\mathfrak{B}$.*
(4) *Projections of constructible sets definable in $A(\mathfrak{B})$ are finite unions of projections of traces on A of locally closed sets definable in $\mathfrak{B}$.*

Proof. If $U \subseteq B^{mn}$ is open and $A^n \cap U$ is definable, then $A^n \cap U = A^n \cap V$, where V is the union of all open boxes (of the definable topology) $U' \subseteq B^{mn}$ such that $A^n \cap U' \subseteq A^n \cap U$. Observe that V is definable in $\mathfrak{B}$. The argument for "locally closed" instead of "open" is essentially the same. By Dougherty and Miller [10], every definable constructible set is a finite union of definable locally closed sets. $\qquad\square$

Remark. The above can easily fail if A is not definable in $\mathfrak{B}$. To illustrate, the set $2\mathbb{Z}$ of even integers is open and definable in $\mathbb{Z}(\mathbb{R}, <, +)$—the induced topology is also the discrete topology—but by o-minimality, $2\mathbb{Z}$ is not the trace on $\mathbb{Z}$ of any open set definable in $(\mathbb{R}, <, +)$.

It is worth rephrasing 1.3 in terms of the open core:

1.4. *Let $\mathfrak{B}$ be a first-order topological structure and $A \subseteq B^m$ be definable in $\mathfrak{B}$.*

(1) *The atomically definable sets of $A(\mathfrak{B})^\circ$ are the traces on A in $\mathfrak{B}^\circ$ of the atomically definable sets of $\mathfrak{B}^\circ$.*
(2) *The quantifier-free definable sets of $A(\mathfrak{B})^\circ$ are the traces on A in $\mathfrak{B}^\circ$ of the quantifier-free definable sets of $\mathfrak{B}^\circ$.*
(3) *The quantifier-free definable sets of $A(\mathfrak{B})^\circ$ are its constructible definable sets.*

(4) *The existentially definable sets of $A(\mathfrak{B})^\circ$ are the projections of its constructible definable sets, equivalently, the finite unions of projections of its definable locally closed sets.*

Expansions of linear orders. Let $(B, <)$ be a linear order without endpoints. We adjoin formally the endpoints $-\infty$ and $+\infty$ to B in the usual fashion. For our purposes, **interval** always means *nondegenerate* interval, that is, an *infinite* convex $I \subseteq B$ such that both $\inf I$ and $\sup I$ exist in $B \cup \{\pm\infty\}$. The usual notation is employed for the various kinds of intervals, but given $b \in B$, we often write $B^{>b}$ instead of $(b, +\infty)$. Each cartesian power B^n is equipped with the product topology induced by the interval topology on B. A **box** in B^n is an n-fold product of open intervals, and a **closed box** is a product of closed intervals.

We say that $X \subseteq B^n$ is **regular** if it is convex in each coordinate, and that X **strongly regular** if it is regular and all points in X have pairwise distinct coordinates. A map $(f_1, \ldots, f_k) \colon X \to B^k$ is **regular** if X is regular and each f_i is, in each coordinate, either constant or strictly monotone and continuous; and f is **strongly regular** if X is strongly regular, f is continuous and each f_i is strictly monotone in each coordinate.

Recall that an expansion $\mathfrak{B}$ of $(B, <)$ is **o-minimal** if every subset of B definable in $\mathfrak{B}$ is a finite union of points and intervals. Recall also (Pillay and Steinhorn [30, 31]) that the study of such structures resolves into those of o-minimal expansions of DLOs and o-minimal expansions of discrete linear orders, and the latter subject is demonstrably trivial. Hence, throughout this paper, o-minimality always includes the assumption that the underlying order is dense. We assume the reader to be familiar with the essential model theory of o-minimal structures; standard references are [11, 13, 20, 29]. **Cells** and **decompositions** relative to $\mathfrak{B}$ are defined as in the o-minimal setting ([13, Chapter 3]). We regard B^0 as an open cell. Cells are locally closed. Regular (and strongly regular) cells and decompositions are defined as expected, as are A-simple cells and decompositions (given $A \subseteq B$). As an immediate consequence of [13, p. 58], we have:

1.5 (regular decomposition). *Let $\mathfrak{B}$ be o-minimal and $S \subseteq B$.*

   (1) *If $\mathcal{X}$ is finite collection of S-definable subsets of B^n, then there is a decomposition $\mathcal{C}$ of B^n into S-definable cells such that $\mathcal{C}$ is compatible with $\mathcal{X}$ and every open $C \in \mathcal{C}$ is strongly regular.*

   (2) *If $X \subseteq B^n$ and $f \colon X \to B$ is S-definable, then there is a decomposition $\mathcal{C}$ of B^n as above for $\mathcal{X} = \{X\}$ such that $f{\restriction}C$ is regular for each open $C \in \mathcal{C}$.*

We also have

1.6 (A-simple decomposition). *Let $\mathfrak{B}$ be o-minimal, $S \subseteq B$ and $A \subseteq B$ be independent with respect to $\mathfrak{B}$.*

   $(\mathrm{I})_n$ *If $\mathcal{X}$ is a finite collection of S-definable subsets of B^n, then there is an A-simple decomposition $\mathcal{C}$ of B^n into $(A \cup S)$-definable cells that is compatible with $\mathcal{X}$.*

   $(\mathrm{II})_n$ *If $X \subseteq B^n$ and $f \colon X \to B$ is S-definable, then there is an A-simple decomposition $\mathcal{C}$ of B^n into $(A \cup S)$-definable cells that is compatible with X and such that $f{\restriction}C$ is continuous and A-simple for each $C \in \mathcal{C}$.*

Sketch of proof. $(\mathrm{I})_1$ is trivial and $(\mathrm{II})_1$ is immediate from the monotonicity theorem. By the usual arguments, it suffices now to let $n > 0$ and show that $(\mathrm{II})_n$ follows from $(\mathrm{I})_1, \ldots,$

$(I)_n$ and $(II)_1, \ldots, (II)_{n-1}$. It suffices to consider the case that f is A_0-definable for some finite $A_0 \subseteq A$. (If $A' \supseteq A$ and f is A'-simple, then f is A-simple.) By cell decomposition and the inductive assumptions, we reduce to the case that X is an open cell, all points of X have pairwise distinct coordinates, $f(X) \cap A_0 = \emptyset$, f is nowhere locally constant and $f(x) \neq x_i$ for all $x \in X$ and $i = 1, \ldots, n$. Then $f(A^n \cap X) \cap A = \emptyset$ by independence, and so f is A-simple. $\square$

As an immediate corollary of 1.2 and the Monotonicity Theorem,

1.7. *Let $\mathfrak{B}$ be o-minimal and $A \subseteq B$ be independent. Let $f \colon U \to B$ be definable on some open $U \subseteq B^n$ such that all points in U have pairwise distinct coordinates and f is locally nonconstant in each coordinate. Then $A^n \cap f^{-1}(b)$ is finite for each $b \in B$.*

1.8. *Let $\mathfrak{B}$ be o-minimal; $U, V \subseteq B^n$ be open and strongly regular such that all points of U and V have the same order type; and $f \colon U \to B$ and $g \colon V \to B$ be definable and strongly regular. If $A \subseteq B$ is a dense dcl-basis for $\mathfrak{B}$ and $f(A^n \cap U) = g(A^n \cap V)$, then $f = g$.*

Proof. As $B = \mathrm{dcl}(A)$, there exist $\bar{a}, \bar{a}' \in A$ such that $\bar{a}$ is independent and f is $\bar{a}$-definable, and $\bar{a}'$ is independent and g is $\bar{a}'$-definable. Let $\bar{c} \in A^n \cap U$ be such that $\bar{c} \cap \bar{a}\bar{a}' = \emptyset$. By assumption, there exists $\bar{d} \in A^n \cap V$ such that $g(\bar{d}) = f(\bar{c})$. By 1.7, we have $\bar{c} \subseteq \mathrm{dcl}(\bar{a}\bar{a}'\bar{d})$, and so $\bar{c} \subseteq \bar{a}\bar{a}'\bar{d}$; then $\bar{c} \subseteq \bar{d}$ by independence, and so $\bar{c} = \bar{d}$ since all points of U have pairwise distinct coordinates and the same order type. We have now shown that the set

$$C := \{\, \bar{c} \in A^n \cap U : \bar{c} \cap \bar{a}\bar{a}' = \emptyset \,\}$$

is contained in V and $g \upharpoonright C = f \upharpoonright C$. As C is dense in U and both U and V are convex in each coordinate, we have $U \subseteq V$ and, by continuity, $g \upharpoonright U = f$. Symmetrically, $V \subseteq U$, so $f = g$. $\square$

1.9 (*cf.* [12, 4.2 and 4.3]). *Let $\mathfrak{B}$ be o-minimal, $A \subseteq B$ be independent, $g = (g_1 \ldots g_k) \colon B^m \to B^k$ be definable, and $S \subseteq B^m$. If $\mathfrak{B}$ has definable Skolem functions, then:*

   (1) *There is a definable $S' \subseteq S$ such that $A^m \cap S \cap g^{-1}(A^k) = A^m \cap S'$.*
   (2) *$g(A^n \cap S)$ is a finite union of sets of the form $f(A^j \cap C)$ (for various j), where $C \subseteq B^j$ is a strongly regular A-simple open cell and $f \colon C \to B^k$ is definable, strongly regular and A-simple.*

Proof. 1. Repeat verbatim the proof of [12, 4.2] up through the end of the third sentence of the third paragraph. Let $n = 0$. By A-simple cell decomposition and the inductive assumptions we reduce to the case that g is A-simple; then either $S' = S$ or $S' = \emptyset$ will do. Thus, we may now assume that $n > 0$ and finish verbatim as for [12, 4.2].

2. Put $X = g(A^m \cap S)$ and then argue similarly as in the proof of [12, 4.3], using 1, 1.5 and 1.6. $\square$

Remark. The assumption of definable Skolem functions is perhaps unnecessary—the conclusion certainly holds if $\mathfrak{B}$ is just a DLO—but existence of definable Skolem functions is used in the proof of [12, 4.2].

1.10. *Let $\mathfrak{B}$ be an expansion of $(B, <)$.*

(1) *If $A \subseteq B$ is independent in $\mathfrak{B}$, then there exist $\mathfrak{B}^* \succeq \mathfrak{B}$ and $A^* \supseteq A$ such that A^* is densely contained in B^* and independent in $\mathfrak{B}^*$.*

(2) $\mathrm{Th}(\mathfrak{B})$ *extended by any family of mutually independent dense unary predicates is consistent.*

Proof. 1. Let $\mathfrak{B}_1 \succeq \mathfrak{B}$ be $(\mathrm{card}\, B)^+$-saturated. It is routine to construct a subset A_1 of B_1 that contains A, is independent in $\mathfrak{B}_1$ and intersects every open interval of B_1 having endpoints in B. By iteration, we obtain chains $\mathfrak{B} = \mathfrak{B}_0 \preceq \mathfrak{B}_1 \preceq \cdots \preceq \mathfrak{B}_n \preceq \ldots$ and $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \ldots$ such that $A_n \subseteq B_n$, and A_{n+1} is independent in $\mathfrak{B}_{n+1}$ and intersects every open interval in B_{n+1} having endpoints in B_n. Put $\mathfrak{B}^* = \bigcup \mathfrak{B}_n$ and $A^* = \bigcup A_n$.

2. Apply 1 with $A = \emptyset$ and then iterate. $\qquad\square$

2. The basic results

In this section, we establish as many facts about T^{indep} as we reasonably can that require only minor modification of already-known results about T^{pair}; in particular, we establish that T^{indep} is complete and has T as an open core. (See [8] for general information on theories with open cores and why we are interested in them, and also [24, 26] for some special attention to the case that $M = \mathbb{R}$.) Many of our proofs are minor or routine modifications of arguments to be found in other papers (especially [3] and [12]), and details will often be omitted. We abbreviate *mutatis mutandis* by m.m.

For the rest of this paper: $\mathfrak{M}$ denotes an arbitrary model of T and H a dense independent subset of M (thus, $(\mathfrak{M}, H)$ is an arbitrary model of T^{indep}).

Before launching into abstractions we establish an easy, but very important, technical extension of the codensity of H (which we already have by 1.1.4).

2.1. *Let $g_i \colon M^{m_i} \to M$ be definable in $\mathfrak{M}$ for $1 \leq i \leq n$. Then $\bigcup_{i=1}^n g_i(H^{m_i})$ is codense in $\mathrm{dcl}(H) \setminus H$.*

Proof. We first dispose of a special case: the functions $x \mapsto \sum_{i=1}^m x_i$ for $m > 1$. Let (a, b) be an open interval of M with $b \in M$. By density of H, there exist $h \in H \cap (a, b)$ and $0 < h_1 < \cdots < h_{n-1} \in H \setminus \{h\}$ such that $h_1 + \cdots + h_{n-1} < b - h$. Then $h + \sum_{i=1}^{n-1} h_i \in (h, b) \subseteq (a, b)$. By independence, $h + \sum_{i=1}^{n-1} h_i \notin H$.

Now we deal with the general case. Suppose to the contrary that I is an open interval of M and $I \cap (\mathrm{dcl}(H) \setminus H) \subseteq \bigcup_{i=1}^n g_i(H^{m_i})$. There is a finite $S \subseteq M$ such that $H \cup S$ is independent and each g_i is $(H \cup S)$-definable. Put $N = 2 + \sum_{i=1}^n m_i + \mathrm{card}\, S$. By the preceding paragraph, there exist pairwise distinct $h_1 \ldots h_N \in H$ such that $\sum_{i=1}^N h_i \in I$; without loss of generality, we thus have $\sum_{i=1}^N h_i = g_1(k_1, \ldots, k_{m_1})$ for some $k_1, \ldots, k_{m_1} \in H$. Let $1 \leq i \leq N$. Evidently,

$$h_i \in \mathrm{dcl}(\{\, h_j : j \neq i \,\} \cup \{k_1, \ldots, k_{m_1}\} \cup S),$$

so $h_i \in \{\, h_j : j \neq i\} \cup \{k_1, \ldots, k_{m_1}\} \cup S$ by independence of $H \cup S$. As

$$\mathrm{card}(\{k_1, \ldots, k_{m_1}\} \cup S) \leq m_1 + \mathrm{card}(S) < N,$$

there exist $1 \leq i < j \leq N$ such that $h_i = h_j$, a contradiction. $\qquad\square$

11

It is immediate from 1.10 that

2.2. T^{indep} *is consistent.*

Remark. Consistency of T^{indep} is easy to verify more concretely in many cases of interest. For example, if T has an archimedean model, then it has a model $\mathfrak{R}$ over $\mathbb{R}$ (Laskowski and Steinhorn [22]). If moreover $\mathfrak{R}$ is not finitely generated as a model of T (in particular, if T is countable), then $\mathbb{R}$ contains a dense independent set (and similarly for families of mutually independent dense sets, subject to some obvious restrictions on the cardinality of the family). These considerations lead to the more general question of existence of prescribed models: When does any particular model T expand to a model of T^{indep}? (Some will not, *e.g.*, no finitely generated model of T expands to a model of T^{indep}.) We shall not pursue this matter in this paper, as the question is equivalent to asking whether the model contains a dense independent set, which is best checked for cases of interest as they arise.

Our next major goal is to show that T^{indep} is complete; for this and later developments, we now need a few more

Global conventions. As T has definable Skolem functions ([13, p. 94]), we assume for technical convenience that T admits elimination of quantifiers and is universally axiomatizable. Hence, substructures of models of T are elementary substructures. The default for dcl is with respect to T, while dcl_P refers to definable closure with respect to T^{indep}. Independence is with respect to dcl unless stated otherwise. For $A, B, C \subseteq M$, we write $A \underset{C}{\downarrow} B$ to denote that A is independent from B over C, and $A \underset{C}{\not\downarrow} B$ if otherwise. If $C = \emptyset$, then the subscript is omitted. We say that A is ***P*-independent** if $A \underset{A \cap H}{\downarrow} H$. In order to reduce notation, we often indicate union by juxtaposition: We may write AB for $A \cup B$, $A\bar{x}$ instead of $A \cup \bar{x}$ for $\bar{b} \in M$, and so on. For $\bar{a} \in M$, $\mathrm{tp}(\bar{a})$ refers to the L-type of $\bar{a}$, while $\mathrm{tp}_P(\bar{a})$ refers to the L_P-type. Similarly, $\mathrm{qftp}(\bar{a})$ and $\mathrm{qftp}_P(\bar{a})$ denote the corresponding quantifier-free types.

Definitions. (*cf.* [3, 2.3 and 2.4]) Given $E \subseteq M$, we say that $(\mathfrak{M}, E)$:
  — has the **coheir property** if, whenever $A \subseteq M$ is finite dimensional and algebraically closed and $q \in S_1(A)$ is non-algebraic, there is $e \in E$ such that $e \vDash q$.
  — has the **generalized coheir property** if, whenever $A \subseteq M$ is finite dimensional and $q \in S_n(A)$ is n-dimensional, then there is $e \in E^n$ realizing q.
  — has the **extension property** if, whenever $A \subseteq M$ is finite dimensional and algebraically closed and $q \in S_1(A)$ is non-algebraic, there is $a \in M$ such that $a \vDash q$ and $a \notin \mathrm{dcl}(AE)$.
  — has the **generalized extension property** if, whenever $A \subseteq M$ is finite dimensional and $q \in S_n(A)$, there exists $a \in M^n$ realizing q such that $a \underset{A}{\downarrow} AE$.
  — is **chivalrous** it has the coheir and extension properties.
(None of the above require that $\mathfrak{M}$ be anything but a structure on the set M.)

2.3. *If $(\mathfrak{M}, H)$ is $(\mathrm{card}\, T)$-saturated, then $(\mathfrak{M}, H)$ satisfies the coheir property. If $(\mathfrak{M}, H)$ is $(\mathrm{card}\, T)^+$-saturated, then $(\mathfrak{M}, H)$ satisfies the extension property.*

Proof. The first statement is immediate just from density of H and o-minimality of T, while the second follows from 2.1 and o-minimality of T. □

Hence,

2.4. *If $(\mathfrak{M}, H)$ is $(\operatorname{card} T)^+$-saturated, then $(\mathfrak{M}, H)$ is chivalrous.*

2.5 (*m.m.* [3, 2.4]). *$(\mathfrak{M}, H)$ is chivalrous if and only if it satisfies the generalized coheir and extension properties.*

2.6 (*cf.* [3, 2.8]). *Let $(\mathfrak{M}_0, H_0)$ and $(\mathfrak{M}_1, H_1)$ be chivalrous models of T^{indep}. Let $\bar{a}$ and $\bar{b}$ be finite P-independent tuples of the same length from M_0 and M_1 such that $\operatorname{qftp}_P(\bar{a}) = \operatorname{qftp}_P(\bar{b})$. Then $\operatorname{tp}_P(\bar{a}) = \operatorname{tp}_P(\bar{b})$.*

Proof. Let f be a partial L_P map from M_0 to M_1 taking $\bar{a}$ to $\bar{b}$. We show that if $\bar{c} \in M_0^n$ then we may extend f so that its domain includes $\bar{c}$. Note that without loss of generality we may assume that $\bar{c}$ is disjoint from $\bar{a}$. To begin with, by extending $\bar{c}$ if necessary we may assume that $\bar{a}\bar{c}$ is P-independent. Let $\bar{c}_1 = H_0 \cap \bar{c}$ and let $\bar{c}_2 = \bar{c} \setminus \bar{c}_1$. Let $p = \operatorname{tp}(\bar{c}_1/\bar{a})$. By the P-independence of $\bar{a}$ we have $\bar{a} \underset{H_0 \cap \bar{a}}{\downarrow} \bar{c}_1$. Let $p' \in S(\bar{b})$ be the f-image of p. Note that p' is free over $H_1 \cap \bar{b}$. Also notice that since $\bar{c}$ and $\bar{a}$ are disjoint, it follows that p has dimension $\operatorname{card} \bar{c}_1$, and thus the same holds for p'. Hence, we may apply the generalized coheir property to p' to find $\bar{d}_1 \in H_1$ so that $\bar{d}_1 \models p'$.

Claim. $\operatorname{qftp}_P(\bar{c}_1\bar{a}) = \operatorname{qftp}_P(\bar{d}_1\bar{b})$.
We must show that if t is a term, then $t(\bar{c}_1\bar{a}) \in H_0$ if and only if $t(\bar{d}_1\bar{b}) \in H_1$; we do only the reverse implication (the other is similar). Let $b^* = t(\bar{d}_1\bar{b})$. Since $\bar{b}$ is P-independent we have $\bar{b} \underset{H_1 \cap \bar{b}}{\downarrow} b^*\bar{d}_1$ and thus $b^* \underset{(H_1 \cap \bar{b})\bar{d}_1}{\downarrow} \bar{b}\bar{d}_1$. Hence $b^* \in \operatorname{dcl}(\bar{d}_1(H_1 \cap \bar{b}))$ and so $b^* \in \{\bar{d}_1(H \cap \bar{b})\}$ and the result follows.

Now let $\hat{f}$ extend f and take $\bar{c}_1\bar{a}$ to $\bar{d}_1\bar{b}$. Let $q = \operatorname{tp}(\bar{c}_2/\bar{a}\bar{c}_1)$ and let $q' \in S(\bar{b}\bar{d}_1)$ be its image under $\hat{f}$. By the generalized extension property there is $\bar{d}_2 \models q'$ so that $\bar{d}_2 \underset{\bar{b}\bar{d}_1}{\downarrow} H_1\bar{d}_1\bar{b}$.

Claim. $\operatorname{qftp}_P(\bar{c}_2\bar{c}_1\bar{a}) = \operatorname{qftp}_P(\bar{d}_2\bar{d}_1\bar{b})$.
Again, we need to show that if t is a term, then $t(\bar{c}_2\bar{c}_1\bar{a}) \in H_0$ if and only if $t(\bar{d}_2\bar{d}_1\bar{b}) \in H_1$; this follows much as the previous claim.
Thus, we now extend $\hat{f}$ by sending $\bar{c}_2$ to $\bar{d}_2$, completing the proof. $\qquad\square$

As a corollary of the proof, all chivalrous models of T^{indep} are back-and-forth equivalent. Hence,

2.7 (*cf.* [3, 2.9]). *All chivalrous models of T^{indep} are elementarily equivalent.*

When combined with 2.4, all $(\operatorname{card} T)^+$-saturated models of T^{indep} are elementarily equivalent. Hence,

2.8. *T^{indep} is complete.*

Remark. Due to the many good model-theoretic properties of T, we need only axiomatize the independence and density of P to get a completion of T as an L_P-theory. Compare this with the case of lovely pairs ([3, 2.10]), which requires explicit axiomatization of the coheir and extension properties.
Next is an important quantifier-simplification result.

2.9 (*m.m.* [3, 3.2], *cf.* [12, Theorem 1]). *Each L_P-formula $\psi(\bar{y})$ is T^{indep}-equivalent to a Boolean combination of formulae of the form $\exists \bar{x}(P\bar{x} \wedge \phi(\bar{x}, \bar{y}))$ where $\phi(\bar{x}, \bar{y})$ is an L-formula.*

Remark. Allowing boolean combinations is necessary, indeed, T^{indep} is not model complete:

2.10. *The formula $\varphi(w) := \exists uv(Pu \wedge Pv \wedge u + v = w)$ is not T^{indep}-equivalent to a universal L_P-formula.*

Proof. We show that φ is not preserved downward among models of T^{indep}. By removing one element of H, we reduce to the case that there exists $a \in M \setminus \text{dcl}(H)$. Let $\mathfrak{M}' \succeq \mathfrak{M}$ be such that M' contains some $b > M$. It is easy to check that $H \cup \{b, a-b\}$ is independent in $\mathfrak{M}'$. By 1.10, there exists $(\mathfrak{M}'', H'') \vDash T^{\text{indep}}$ such that $(\mathfrak{M}, H) \subseteq (\mathfrak{M}'', H'')$ and $b, a-b \in H''$; then $(\mathfrak{M}'', H'') \vDash \varphi(a)$ and $(\mathfrak{M}, H) \nvDash \varphi(a)$. $\square$

For the next several results (up through 2.25, essentially) we switch from [3] to [12] for templates of results and proofs. The development is quite similar to that in [12], so we proceed rapidly and without much discussion.

2.11 (*m.m.* [12, 2.3]). *If $(\mathfrak{M}', H') \succeq (\mathfrak{M}, H)$, then $H' \underset{H}{\downarrow} M$.*

Next is a generalization of our 2.6.

2.12. *For $i = 1, 2$, let $(\mathfrak{B}_i, A_i) \subseteq (\mathfrak{M}_i, H_i) \vDash T^{\text{indep}}$ be such that $H_i \underset{A_i}{\downarrow} B_i$. Then $\text{tp}(B_1) = \text{tp}(B_2)$ if and only if $\text{tp}_P(B_1) = \text{tp}_P(B_2)$.*

Proof. By 2.4 and 2.11, we may assume without loss of generality that $(\mathfrak{M}_i, H_i)$ are chivalrous. The result is now immediate from 2.6. $\square$

2.13 (*cf.* [12, 2.7]). *If $(\mathfrak{M}, H) \subseteq (\mathfrak{M}', H') \vDash T^{\text{indep}}$ and $M \underset{H}{\downarrow} H'$, then $(\mathfrak{M}, H) \preceq (\mathfrak{M}', H')$.*

Proof. By assumptions on T, we have $\mathfrak{M} \preceq \mathfrak{M}'$, and trivially $H \underset{H}{\downarrow} M$. Hence, 2.12 applies, and the result follows. $\square$

2.14 (*cf.* [12, 2.8]). *Let $A \subseteq M$ be independent. For $i = 1, 2$, let $(M, A) \subseteq (\mathfrak{M}_i, H_i) \vDash T^{\text{indep}}$ be such that $M \underset{A}{\downarrow} H_i$. If $a_i \in H_i^n$ realize the same type over M in $\mathfrak{M}_i$, then they realize the same type over M in $(\mathfrak{M}_i, H_i)$.*

Proof. We have $\text{tp}(Ma_1) = \text{tp}(Ma_2)$, and $M \underset{Aa_1}{\downarrow} H_1$ and $M \underset{Aa_2}{\downarrow} H_2$. By 2.12, $\text{tp}_P(Ma_1) = \text{tp}_P(Ma_2)$. $\square$

2.15 (*cf.* [12, 2.9]). *For $i = 1, 2$, let $(\mathfrak{M}_i, H_i) \vDash T^{\text{indep}}$. Let $C \subseteq H_1 \cap H_2$ be such that $\text{tp}^{\mathfrak{M}_1}(C) = \text{tp}^{\mathfrak{M}_2}(C)$. Let $b_i \in M_i \setminus \text{dcl}(H_i)$ be such that $\text{tp}(b_1/C) = \text{tp}(b_2/C)$. Then $\text{tp}_P(b_1/C) = \text{tp}_P(b_2/C)$.*

Proof. Observe that $\text{tp}(b_1 C) = \text{tp}(b_2 C)$ and $H_i \underset{C}{\downarrow} b_i$. By 2.12, $\text{tp}_P(b_1 C) = \text{tp}_P(b_2 C)$. $\square$

2.16 (*m.m.* [12, Theorem 2, (1)$\Rightarrow$(2)]). *If $X \subseteq H^n$ is definable in $(\mathfrak{M}, H)$, then X is a trace on H of a set definable in $\mathfrak{M}$.*

(It follows that $H(\mathfrak{M}, H)$ is weakly o-minimal—that is, every unary definable set is a finite union of convex definable sets—but we give a much more detailed statement at the end of this section. See, *e.g.*, Macpherson, Marker and Steinhorn [23] for basic information on weak o-minimality.)

2.17 (*cf.* [12, 3.1]). $\mathrm{dcl}_P(H) = \mathrm{dcl}(H)$.

Proof. Let $b \in M \setminus \mathrm{dcl}(H)$. We show that $b \notin \mathrm{dcl}_P(H)$. Let $(\mathfrak{M}^*, H^*) \succeq (\mathfrak{M}, H)$ be sufficiently saturated; then $\mathrm{dcl}(H^*) \neq M^*$, so $\mathrm{dcl}(H^*)$ is codense in M^* by 2.1. Thus, there there exists $b^* \in M^* \setminus \mathrm{dcl}(H^*)$ such that b and b^* are in the same cut over $\mathrm{dcl}(H)$ and $b \neq b^*$. Then $\mathrm{tp}(b/H) = \mathrm{tp}(b^*/H)$, and so $\mathrm{tp}_P(b^*/H) = \mathrm{tp}_P(b/H)$ by 2.15. Hence, $b \notin \mathrm{dcl}_P(H)$. $\qquad\square$

Remark. It follows that there exist dcl_P-independent $a, b \in H$, so $(\mathfrak{M}, H) \nvDash$ EP by the proof of 1.1.1; in other words, no model of T^{indep} has EP. Compare this with the note following [8, 5.7].

2.18 (*cf.* [12, 3.3]). *If* $(\mathfrak{M}, H) \subseteq (\mathfrak{M}^*, H^*) \vDash T^{\mathrm{indep}}$ *and* $M \underset{H}{\downarrow} H^*$, *then* M *is the definable closure of* M *in* $(\mathfrak{M}^*, H^*)$.

Proof. Note that $H \neq M$. Let $\mathfrak{M}\langle H^* \rangle$ be the submodel of $\mathfrak{M}^*$ generated by M and H^*. It is immediate from 2.13 that $(\mathfrak{M}\langle H^* \rangle, H^*) \preceq (\mathfrak{M}^*, H^*)$ and thus that $\mathrm{dcl}_P(M) \subseteq \mathrm{dcl}(H^* M)$. Follow the proof of [12, 3.3] up to Case 2. Put $H' = H a_1^* \ldots a_{k-1}^*$ and $M' = \mathrm{dcl}(M a_1^*, \ldots, a_{k-1}^*)$. Observe that H' is independent and then finish as in [12, 3.3]. $\qquad\square$

Following (but also extending) [12], we say that $X \subseteq M^k$ is **H-small** (relative to $\mathfrak{M}$) if it is definable in $(\mathfrak{M}, H)$ and there is a map $f: M^n \to M^k$ definable in $\mathfrak{M}$ such that $X \subseteq f(H^n)$.

2.19 (*m.m.* [12, 3.4]). *If* $F: M \to M$ *is definable in* $(\mathfrak{M}, H)$ *then there exist an H-small* $X \subseteq M$ *and* $\hat{F}: M \to M$ *definable in* $\mathfrak{M}$ *such that* $F {\restriction} (M \setminus X) = \hat{F} {\restriction} (M \setminus X)$.

2.20 (*m.m.* 4.3, via 1.9.2). *If* $X \subseteq M^l$ *is H-small, then X is a finite union of sets of the form* $f(H^m \cap U)$ *(for various m) where U is an open cell in M^m and* $f: U \to M^l$ *is continuous and definable in* $\mathfrak{M}$.

2.21 (*m.m.* [12, 3.5]). *If* $S \subseteq M$ *is definable in* $(\mathfrak{M}, H)$, *then* $S \setminus X = S' \setminus X$ *for some H-small* $X \subseteq M$ *and* $S' \subseteq M$ *definable in* $\mathfrak{M}$.

It is immediate from 2.1 that

2.22 (*cf.* [12, 4.1]). *No interval of M is H-small.*

We digress to point out that T^{indep} fails to have definable Skolem functions, and in a transparent and uniform way:

2.23 (*m.m.* [8, 5.4] via 2.19 and 2.22). $(\mathfrak{M}, H)$ *defines no Skolem function for the formula* $x < y \wedge Py$.

2.24 (*m.m.* [12, Theorem 4] via 2.20, 2.21 and 2.22). *If* $X \subseteq M$ *is H-small, then there is a partition* $-\infty = b_0 < b_1 < \cdots < b_k < b_{k+1} = \infty$ *of M such that for each $i \in \{0, \ldots, k\}$ either $X \cap (b_i, b_{i+1}) = \emptyset$ or X is dense-codense in (b_i, b_{i+1}).*

If $S \subseteq M$ *is definable in* $(\mathfrak{M}, H)$, *then there is a partition* $-\infty = b_0 < b_1 < \cdots < b_k < b_{k+1} = \infty$ *of M such that for each $i \in \{0, \ldots, k\}$ either $S \cap (b_i, b_{i+1}) = \emptyset$, $(b_i, b_{i+1}) \subseteq S$ or S is dense-codense in (b_i, b_{i+1}).*

We arrive at the main result of this paper:

2.25. *T is an open core of T^{indep}.*

Proof. It is immediate from 2.24 that every open subset of M definable in $(\mathfrak{M}, H)$ is a finite union of open intervals. By [12, 4.6] *m.m.*, every cofinitely continuous unary function definable in $(\mathfrak{M}, H)$ is definable in $\mathfrak{M}$. Now apply [8, 4.14]. $\qquad\square$

We now proceed to analyze finer properties of T^{indep} and its models.

First, we show that dcl_P is easily calculated relative to dcl (something that is not known for dense pairs).

2.26. *If $X \subseteq M$ is finite and $Y \subseteq H$ is minimal such that $X \underset{Y}{\bigcup} H$, then $\mathrm{dcl}_P(X) = \mathrm{dcl}(X \cup Y)$.*

Proof. Let $\mathfrak{M}_0 \preceq \mathfrak{M}$ be generated by $X \cup Y$ and consider the substructure $(\mathfrak{M}_0, \mathfrak{M}_0 \cap H)$ of $(\mathfrak{M}, H)$. We have $(X \cup Y) \underset{Y}{\bigcup} H$, and so $M_0 \underset{M_0 \cap H}{\bigcup} H$. By 2.18, M_0 is definably closed in $(\mathfrak{M}, H)$. Thus, $\mathrm{dcl}(X \cap Y) = \mathrm{dcl}_P(X \cap Y)$, and so $\mathrm{dcl}_P(X) \subseteq \mathrm{dcl}(X \cup Y)$.

That $\mathrm{dcl}(X \cup Y) \subseteq \mathrm{dcl}_P(X)$ is essentially immediate from 1.7, 2.25 and the minimality of Y. $\qquad\square$

2.27 (*cf.* [8, 5.2]). *$(\mathfrak{M}, H)$ is not atomic.*

Proof. By 2.26 and independence of H, we have $\mathrm{dcl}_P(\emptyset) = \mathrm{dcl}(\emptyset) \neq M$; by 2.25 and [8, 4.7], $(\mathfrak{M}, H)$ is not atomic. $\qquad\square$

We have now established analogues of essentially all results about T^{pair} from [12] and [8, §5] except for failure of elimination of imaginaries [8, 5.5] (we show in the next section that T^{indep} does eliminate imaginaries). We note one more similarity with T^{pair}:

2.28. *T^{indep} is strongly dependent.*

See Berenstein, Dolich and Onshuus [1] for a definition of "strongly dependent". The proof is a routine modification of [1, 2.11]: Replace the assumption there that the interpretation of P be algebraically closed with the observation that if $c, d_1, \dots, d_n \in H$ and $c \in \mathrm{dcl}(d_1, \dots, d_n)$, then $c \in \{d_1, \dots, d_n\}$.

We close this section by making precise the notion that $H(\mathfrak{M}, H)$ (the structure induced on H in $(\mathfrak{M}, H)$, as defined in Section 1) is as simple as possible among all $E(\mathfrak{M}, E)$ where E is dense-codense in M. It is immediate from definitions that if $C \subseteq M^n$ is an E-simple cell that intersects E^n, then E^n dense in C (and also codense in C if C is not a point). Thus, every decomposition $\mathcal{C}$ of M^n into E-simple cells of $\mathfrak{M}$ induces something very much like a cell decomposition of E^n in $E(\mathfrak{M}, E)$, namely, the collection of traces $\{ E^n \cap C : C \in \mathcal{C} \}$. Hence, it is fair to say that $E(\mathfrak{M}, E)$ is as simple as possible if the obvious modification of cell decomposition holds. This is true for $E = H$:

2.29.   (1) *If $\mathcal{X}$ is a finite collection of subsets of H^n definable in $H(\mathfrak{M}, H)$, then there is an H-simple decomposition $\mathcal{C}$ of M^n such that $\{ H^n \cap C : C \in \mathcal{C} \}$ is compatible with $\mathcal{X}$.*

16

(2) *If $X \subseteq H^n$ and $f \colon X \to H$ is definable in $H(\mathfrak{M}, H)$, then there is an H-simple decomposition $\mathcal{C}$ of M^n compatible with X such that $f \restriction (H^n \cap C)$ is either constant or a coordinate projection for each $C \in \mathcal{C}$ such that $C \subseteq X$.*

Proof. Item 1 is immediate from 2.16 and 1.6, and 2 is immediate from 1. $\qquad\square$

3. þ-FORKING AND ELIMINATION OF IMAGINARIES

In the previous section, we focussed primarily on similarities between T^{indep} and T^{pair}. We now deal with the major difference: T^{indep} eliminates imaginaries. In the course of proving this, and of interest in its own right, we give an explicit description of þ-forking in T^{indep} that depends visibly and uniformly on T. (This stands is in marked contrast to þ-forking in T^{pair}, where no such result is yet known.) We refer frequently to [2]; expressions of the form [2, n] always mean "the result numbered n in [2]" (as opposed to "[2] and [n]").

First, it is immediate from 1.1.3 and [2, Theorem 3] that

3.1. T^{indep} *has þ-rank ω.*

In particular, T^{indep} is rosy, so all relevant results about þ-forking from [27] are available; we use them freely throughout.

Some further conventions for this section: $(M, H)^{\mathrm{eq}}$ denotes the underlying set of $(\mathfrak{M}, H)^{\mathrm{eq}}$, Z ranges over subsets of $(M, H)^{\mathrm{eq}}$, and $\mathrm{dcl}^{\mathrm{eq}}$ denotes definable closure in $(\mathfrak{M}, H)^{\mathrm{eq}}$. Recall that T eliminates imaginaries [13, p. 94].

3.2. *If $X \subseteq M^n$, then X is Z-definable in $(\mathfrak{M}, H)^{\mathrm{eq}}$ and definable in $\mathfrak{M}$ if and only if it is $(M \cap \mathrm{dcl}^{\mathrm{eq}}(Z))$-definable in $\mathfrak{M}$.*

Proof. Without loss of generality $(\mathfrak{M}, H)$ is sufficiently saturated. Let $\bar{c}$ be canonical parameters for X in $\mathfrak{M}$ and σ be an automorphism of $(\mathfrak{M}, H)^{\mathrm{eq}}$ fixing Z. Then $\sigma(X) = X$, and so $\sigma(\bar{c}) = \bar{c}$, thus yielding $\bar{c} \in M \cap \mathrm{dcl}^{\mathrm{eq}}(Z)$. (The reverse implication is trivial.) $\qquad\square$

Remark. The proof uses nothing beyond that T eliminates imaginaries.

The point is that Z-definability of X in $(\mathfrak{M}, H)^{\mathrm{eq}}$ gives us some control on how we may choose the parameters that define X in $\mathfrak{M}$ (provided that X is definable in $\mathfrak{M}$). These two equivalent conditions will come up so often in this section that we declare a temporary abbreviation: We say that $X \subseteq M^n$ is **Z-definable in $\mathfrak{M}$** if it is $(M \cap \mathrm{dcl}^{\mathrm{eq}}(Z))$-definable in $\mathfrak{M}$, if and only if it is Z-definable in $(\mathfrak{M}, H)^{\mathrm{eq}}$ and definable in $\mathfrak{M}$.

Given $\emptyset \neq X \subseteq M$, we define the **small closure of Z over X**, denoted by $\mathrm{scl}_X(Z)$, to be the union of all images $f(H^n \times X^m)$ such that $f \colon M^{n+m} \to M$ is Z-definable in $\mathfrak{M}$. For $X = \emptyset$, we suppress the subscript and take $m = 0$, that is, $\mathrm{scl}(Z)$ is the union of all $f(H^n)$ such that $f \colon M^n \to M$ is Z-definable in $\mathfrak{M}$. (*Note:* On the face of it, our definition appears to be different than that used in [2], but we shall soon establish that they are equivalent.)

We have an important equivalent description of scl_X:

3.3. *If $X \subseteq M$, then $a \in \mathrm{scl}_X(Z)$ if and only if $a \in f(\bar{c}, X^m)$ for some $f \colon M^{n+m} \to M$ that is Z-definable in $\mathfrak{M}$ and $\bar{c} \in H \cap \mathrm{dcl}^{\mathrm{eq}}(aXZ)$.*

Proof. Let $a \in \mathrm{scl}_X(Z)$. Let n be minimal such that $a = f(\bar{c}, x)$ for some $m \in \mathbb{N}$, function $f \colon M^{n+m} \to M$ that is Z-definable in $\mathfrak{M}$, $x \in X^m$ and $\bar{c} \in H^n$ having pairwise distinct coordinates. As f is definable in $\mathfrak{M}$, there is a ZX-definable neighborhood U of $\bar{c}$ such that U is disjoint from all diagonals and $f{\upharpoonright}U$ is strictly monotone in each coordinate. By 1.7, the set of all $\bar{h} \in H^n \cap U$ such that $f(\bar{h}, x) = b$ is finite, thus yielding the result. $\qquad\square$

Remark. The proof uses only the o-minimality of $\mathfrak{M}$ and the independence of H.

For *finite* $S \subseteq (M, H)^{\mathrm{eq}}$, we let $\dim^{\mathrm{eq}}(S/Z)$ be the minimal cardinality of all finite $S' \subseteq (M, H)^{\mathrm{eq}}$ such that $S \subseteq \mathrm{dcl}^{\mathrm{eq}}(S'Z)$.

We are now ready to state the first main result of this section.

3.4 ($\flat$-forking in T^{indep})**.** *Let $(\mathfrak{M}, H)$ be a sufficiently saturated model of T^{indep} and $a, \bar{b} \in M$.*

    (1) *If $a \in H$, then $a \mathop{\not\smile}\limits^{\flat}_{Z} \bar{b}$ if and only if $a \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b}Z) \setminus \mathrm{dcl}^{\mathrm{eq}}(Z)$.*

    (2) *If $a \in \mathrm{scl}(Z)$, then $a \mathop{\not\smile}\limits^{\flat}_{Z} \bar{b}$ if and only if $\dim^{\mathrm{eq}}(\bar{c}/Z\bar{b}) < \dim^{\mathrm{eq}}(\bar{c}/Z)$ for all $\bar{c}$ as in 3.3.*

    (3) *If $a \notin \mathrm{scl}(Z)$ then $a \mathop{\not\smile}\limits^{\flat}_{Z} \bar{b}$ if and only if $a \in \mathrm{scl}_{\bar{b}}(Z)$.*

Throughout the proof (up through the end of 3.8), we assume that $(\mathfrak{M}, H)$ is sufficiently saturated. In the course of the proof, we establish two auxiliary results that help highlight the usefulness of 3.4: in words, $\mathrm{dcl}^{\mathrm{eq}}$ has EP for H over subsets of $(M, H)^{\mathrm{eq}}$, and our definitions of scl_X and H-small are equivalent to those given in [2].

Proof of 3.4.1. Let $a \in H$. Suppose that $a \notin \mathrm{dcl}^{\mathrm{eq}}(Z\bar{b})$. Let $a \in W \subseteq M$, where W is $Z\bar{b}$-definable. Without loss of generality, $W = I \cap H$ for some interval I. If follows from density of H and o-minimality of $(\mathfrak{M}, H)^{\circ}$ that $I \cap H$ does not $\flat$-fork. (The reverse implication is trivial.) $\qquad\square$

Proof of 3.4.2. Let $a \in \mathrm{scl}(Z)$, $\bar{b} \in M$ and $\bar{c}$ be as in 3.3 (with $X = \emptyset$). Then a is interdefinable with $\bar{c}$ over Z in $(\mathfrak{M}, H)^{\mathrm{eq}}$ and, by 3.4.1, we have $\bar{c} \mathop{\not\smile}\limits^{\flat}_{Z} \bar{b}$ if and only if $\bar{c} \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b}Z) \setminus \mathrm{dcl}^{\mathrm{eq}}(Z)$. The result follows. $\qquad\square$

We pause to note that rosiness of T^{indep} together with 3.4.1 yield exchange for $\mathrm{dcl}^{\mathrm{eq}}$ when restricted to H:

3.5. *If $h, h' \in H$ and $h \in \mathrm{dcl}^{\mathrm{eq}}(h'Z) \setminus \mathrm{dcl}^{\mathrm{eq}}(Z)$, then $h' \in \mathrm{dcl}^{\mathrm{eq}}(hZ)$.*

Next is half of 3.4.3.

3.6. *If $a \in \mathrm{scl}_{\bar{b}}(Z) \setminus \mathrm{scl}(Z)$ and $\bar{b} \in M$, then $a \mathop{\not\smile}\limits^{\flat}_{Z} \bar{b}$.*

Proof. By induction and rosiness of T^{indep} we may reduce to the case that $\bar{b}$ is a singleton b. We have $a = f(\bar{c}, b)$ for some $\bar{c} \in H$ and f Z-definable in $\mathfrak{M}$. By exchange in $\mathfrak{M}$, $b \in \mathrm{scl}_a(Z)$. If $b \in \mathrm{dcl}^{\mathrm{eq}}(aZ)$, then $b \notin \mathrm{dcl}^{\mathrm{eq}}(Z)$, since otherwise $a \in \mathrm{scl}(Z)$. It follows that $b \mathop{\not\smile}\limits^{\flat}_{Z} a$ and we have our desired result by symmetry. Hence, we now assume that $b \notin \mathrm{dcl}^{\mathrm{eq}}(aZ)$.

We have that $\mathrm{tp}_P(a/bZ) \vdash \exists \bar{y}(P\bar{y} \wedge x = f(\bar{y}, b))$. It suffices now to show that the set

$$\Gamma := \{\, \exists \bar{y}(P\bar{y} \wedge x = f(\bar{y}, b^*)) : b^* \vDash \mathrm{tp}(b/aZ) \,\}.$$

Γ is l-inconsistent for some l. Suppose otherwise; then we have b_i with $i \in \omega$ and $\bar{h}_i \in H$ so that $c = f(\bar{h}_i, b_i)$ for some fixed c. By 3.3, there is $\bar{c}_i \in H \cap \mathrm{dcl}^{\mathrm{eq}}(b_i a Z)$ such that $b_i = h(\bar{c}_i, a)$ for some function h that is Z-definable in $\mathfrak{M}$. Thus, $c = f(\bar{a}_i, h(\bar{c}_i, a))$ for all $\bar{a}_i \bar{c}_i$, and we have a function g that is aZ-definable in $\mathfrak{M}$ and such that $g(\bar{d}) = c$ for infinitely many $\bar{d} \in H$. By compactness, there are a function $\tilde{g} \colon M^m \to M$ definable in $\mathfrak{M}$ and a sequence of distinct tuples $\bar{e}_i \in H$ for $i \in \mathbb{N}$ such that $\tilde{g}(\bar{e}_i) = c$ for all $i \in \mathbb{N}$. If we assume that m is chosen to be minimal, then we violate 1.7, a contradiction that finishes the proof. $\qquad\square$

In order to finish the proof of 3.4, it suffices now to show that if $a \notin \mathrm{scl}_{\bar{b}}(Z)$, then $a \underset{Z}{\overset{\mathrm{b}}{\smile}} \bar{b}$, which would be immediate from [2, 45] except that our definitions of scl_X and H-small appear to be different from those used [2]. Thus, it suffices now to show that

3.7 (cf. [2, 33 and 34]). *The following are equivalent for $a, \bar{b} \in M$:*

   (1) $a \in \mathrm{scl}_{\bar{b}}(Z)$;
   (2) $a \in \mathrm{scl}_{M \cap \mathrm{dcl}^{\mathrm{eq}}(\bar{b}Z)}(\emptyset)$;
   (3) *a lies in a $\bar{b}Z$-definable H-small set.*

The above needs only an appropriate version of 3.2 for unary H-small sets:

3.8. *If $X \subseteq M$ is H-small and Z-definable in $\mathfrak{M}$, then $X \subseteq g(H^n)$ for some $g \colon M^n \to M$ that is Z-definable in $\mathfrak{M}$.*

Proof. We first show that $X \subseteq \mathrm{scl}(Z)$. As X is H-small there exist $f \colon M^{n+m} \to M$ $\emptyset$-definable in $\mathfrak{M}$ and $\bar{c} \in M^m$ such that $X \subseteq f(H^n, \bar{c})$. If $m = 0$, then $X \subseteq \mathrm{dcl}(H)$, and we are done.

Now suppose that $m > 0$ and, toward a contradiction, that $X \not\subseteq \mathrm{scl}(Z)$. Put $\bar{d} = c_1 \ldots c_{m-1}$. Since T^{indep} is rosy, there exists $a \in X \setminus \mathrm{scl}(Z)$ such that $a \underset{Z}{\overset{\mathrm{b}}{\smile}} \bar{c}$. Thus, $a \underset{Z\bar{d}}{\overset{\mathrm{b}}{\smile}} \bar{c}$. But $a \in \mathrm{scl}_{\bar{c}}(\emptyset)$, so in particular $a \in \mathrm{scl}_{\bar{c}}(Z)$ and hence by 3.6 $a \in \mathrm{scl}_{\bar{d}}(Z)$. Continuing, we conclude that $a \in \mathrm{scl}(Z)$, a contradiction. Thus $X \subseteq \mathrm{scl}(Z)$.

By compactness, there is a function $g \colon M^n \to M$ Z-definable in $\mathfrak{M}$ such that $X \subseteq g(H^n)$. By elimination of imaginaries for $\mathfrak{M}$, we may assume that g is $\bar{e}$-definable in $\mathfrak{M}$, where $\bar{e}$ are canonical parameters for g. Since g is Z-definable, any automorphism σ of $(\mathfrak{M}, H)^{\mathrm{eq}}$ fixing Z fixes the graph of g and fixes $\bar{e}$ pointwise. Hence, $\bar{e} \in M \cap \mathrm{dcl}^{\mathrm{eq}}(Z)$ as required. $\qquad\square$

We have now established 3.4. Next, we work toward elimination of imaginaries for T^{indep}. Much of the work parallels the analysis of imaginaries in [2] (at least, in spirit), but we must also revisit several our earlier results and carefully track parameters.

3.9 (cf. 2.16). *If $X \subseteq H^n$ is $\bar{b}$-definable in $(\mathfrak{M}, H)$, then X is a trace on H of a set that is $\mathrm{dcl}_P(\bar{b})$-definable in $\mathfrak{M}$.*

Proof. It suffices to show that if $\bar{a}_0, \bar{a}_1 \in H^n$ and $\mathrm{tp}(\bar{a}_0 / \mathrm{dcl}_P(\bar{b})) = \mathrm{tp}(\bar{a}_1 / \mathrm{dcl}_P(\bar{b}))$ then $\mathrm{tp}_P(\bar{a}_0 / \mathrm{dcl}_P(\bar{b})) = \mathrm{tp}_P(\bar{a}_1 / \mathrm{dcl}_P(\bar{b}))$. Fix $\bar{a}_0$ and $\bar{a}_1$. Without loss of generality, $(\mathfrak{M}, H)$ is sufficiently saturated and by 2.26 we have that $\bar{b}$ is P-independent. Thus $\bar{b}\bar{a}_0$ and $\bar{b}\bar{a}_1$ are P-independent. It suffices now by 2.6 to show that $\mathrm{qftp}_P(\bar{a}_0 \bar{b}) = \mathrm{qftp}_P(\bar{a}_1 \bar{b})$. Thus we need only let t be an L-term such that $t(\bar{b}\bar{a}_0) \in H$, and show that $t(\bar{b}\bar{a}_1) \in H$. We have

19

$\bar{b}\bar{a}_0 \underset{\bar{b}\bar{a}_0 \cap H}{\bigcup} H$, and thus $t(\bar{b}\bar{a}_0) \underset{\bar{b}\bar{a}_0 \cap H}{\bigcup} H$. As $t(\bar{b}\bar{a}_0) \in H$, we have $t(\bar{b}\bar{a}_0) \in \mathrm{dcl}(\bar{b}\bar{a}_0 \cap H)$. By independence, $t(\bar{b}\bar{a}_0) \in \bar{b}\bar{a}_0 \cap H$, and so $t(\bar{b}\bar{a}_1) \in \bar{b}\bar{a}_1 \cap H$ as required. □

3.10 (*cf.* 1.9.1). *Let $S \subseteq M^m$ and $g = (g_1 \ldots g_k) \colon M^m \to M^k$ be Z-definable in $\mathfrak{M}$. Then there exists $S' \subseteq S$ that is Z-definable in $\mathfrak{M}$ and such that*

$$H^m \cap S \cap g^{-1}(H^k) = H^m \cap S'.$$

Proof. By 3.2, there exists $\bar{b} \in M \cap \subseteq \mathrm{dcl}^{\mathrm{eq}}(Z)$ such that S and g are $\bar{b}$-definable in $\mathfrak{M}$. As in the proof of 1.9.1 we reduce to the case that $k = 1$. Let $\tilde{g} \colon H^m \to H$ be given by

$$\tilde{g}(x) = \begin{cases} g(x), & \text{if } g(x) \in H \\ x_1, & \text{if } g(x) \notin H. \end{cases}$$

Note that $\tilde{g}$ is $\bar{b}$-definable in $(\mathfrak{M}, H)$. By 2.29, $\tilde{g}$ is given piecewise by constants and projections; let $c_1 \ldots c_l \in H$ be the possible constant values. Note that each $c_i \in \mathrm{dcl}_P(\bar{b})$. For $1 \le i \le l$, let $U_i = g^{-1}(c_i)$, and for $1 \le j \le m$, let $V_j = \{ x \in M^m : g(x) = x_j \}$. Each of these sets is $\mathrm{dcl}_P(\bar{b})$-definable in $\mathfrak{M}$. We now have:

$$H^m \cap S \cap g^{-1}(H) = \bigcup_{i=1}^{l} (H^m \cap (S \cap U_i) \cap g^{-1}(H)) \cup \bigcup_{i=1}^{m} (H^m \cap (S \cap V_i) \cap g^{-1}(H))$$

$$= \bigcup_{i=1}^{l} (H^m \cap (S \cap U_i)) \cup \bigcup_{i=1}^{m} (H^m \cap (S \cap V_i))$$

$$= H^m \cap (\bigcup_{i=1}^{l} (S \cap U_i) \cup \bigcup_{i=1}^{m} (S \cap V_i)).$$

Setting $S' = \bigcup_{i=1}^{l}(S \cap U_i) \cup \bigcup_{i=1}^{m}(S \cap V_i)$ yields the desired result follows, as the U_i and V_j are all Z-definable in $\mathfrak{M}$. □

3.11 (*cf.* 2.20). *If $X \subseteq M^l$ is H-small and Z-definable in $(\mathfrak{M}, H)^{\mathrm{eq}}$, then X is a finite union of sets of the form $f(H^m \cap U)$, where U is a strongly regular open cell in M^m and $f \colon U \to M^l$ is strongly regular and Z-definable in $\mathfrak{M}$.*

Proof. By 3.8, $X \subseteq f(H^m)$ for some $f \colon M^m \to M^l$ that is Z-definable in $\mathfrak{M}$. Observe that $H^m \cap f^{-1}(X)$ is Z-definable in $(\mathfrak{M}, H)^{\mathrm{eq}}$. Via 3.9 and 3.10, the rest of the proof is essentially the same as that of 2.20. □

3.12 (*cf.* [2, 50]). *Let $\bar{d} \in M^n$, E be an equivalence relation $\emptyset$-definable in $(\mathfrak{M}, H)$, $\pi \colon M^n \to M^n/E$ be the canonical projection map, and $e := \bar{d}/E \in (\mathfrak{M}, H)^{\mathrm{eq}}$. Then $\mathrm{scl}(e) \cap \pi^{-1}(e) \ne \emptyset$.*

Proof. Let D_1 be the set of all x_1 for which there are $x_2, \ldots, x_n$ such that $\pi(x_1, \ldots, x_n) = e$. If D_1 is H-small, then pick $d_1 \in D_1$ and note that $d_1 \in \mathrm{scl}(e)$ by 3.7. If D_1 is not H-small, then by 2.21 there is an interval J such that $J \cap (M \setminus D_1)$ is H-small; then $\mathrm{scl}(e) \cap J \nsubseteq (M \setminus D_1)$ by 2.1. Pick $d_1 \in J \cap \mathrm{scl}(e)$ with $d_1 \in D_1$. Now continue in this manner to recursively construct $\bar{d}$. Specifically, let D_2 be the set of all x_2 for which there are $x_3, \ldots, x_n$ such that $\pi(d_1, x_2, \ldots, x_n) = e$, and continue as above. □

20

3.13. T^{indep} *eliminates imaginaries.*

Proof. Let e be an imaginary of $(\mathfrak{M}, H)$. By 3.7 and 3.12 there is an H-small e-definable set $X \subseteq M^n$ such that $X \cap \pi^{-1}(e) \neq \emptyset$ (where π is as in 3.12). Since X is e-definable and H-small, so is $X \cap \pi^{-1}(e)$. By 3.11, $X \cap \pi^{-1}(e) = \bigcup_{i=1}^{l} X_i$ where each X_i is e-definable and of the form $f(H^{m_i} \cap U)$ where U is a strongly regular open cell of $\mathfrak{M}$ and $f: U \to M^n$ is Z-definable in $\mathfrak{M}$ and strongly regular. If $\bar{a}$ is a canonical parameter for X_1, then it follows easily that e and $\bar{a}$ are $\emptyset$-interdefinable in $(\mathfrak{M}, H)^{\text{eq}}$. Thus, we now need only show that if U is a strongly regular open cell of $\mathfrak{M}$ and $f: U \to M^n$ is Z-definable in $\mathfrak{M}$ and strongly regular, then $f(H^m \cap U)$ has a canonical parameter. By replacing $\mathfrak{M}$ with $\text{dcl}(H)$ (regarded as a model of T), we reduce to the case that $M = \text{dcl}(H)$. By elimination of imaginaries for $\mathfrak{M}$, let $\bar{a}$ be a canonical parameter for the graph of f. By 1.8, every automorphism of $(\mathfrak{M}, H)$ that fixes $f(H^m \cap U)$ setwise fixes the graph of f pointwise, hence also $\bar{a}$. As $f(H^m \cap U)$ is $\bar{a}$-definable, $\bar{a}$ is a canonical parameter for $f(H^m \cap U)$ as required. $\qquad\square$

We have now established all of our main results about T^{indep}; we next examine them in the context of the wider program described in the introduction.

4. Comparison with other examples

Let T^{noise} be the L_P-theory of the extension of T by a dense-codense unary predicate. In this section, we compare T^{indep} with the other currently-known examples of complete L_P-extensions of T^{noise} that have T as an open core, namely, T^{pair} and the completions of the extension T^{gen} of T by a generic unary predicate. (See [8, §6] for the definition of T^{gen}, denoted there by T^{g}, and a proof that T is an open core.)

First, we summarize the preservation status of some good model-theoretic properties satisfied by T.

	T^{indep}	T^{pair}	completions of T^{gen}
strong dependence	yes	yes	no (independent)
elimination of imaginaries	yes	no	yes
exchange	no	rarely	yes
atomic model	no	no	almost no
definable Skolem functions	no	no	no

(We assume the reader knows that all of the above properties hold for T except possibly for *strong* dependence, which follows from the even stronger property of dp-minimality; see Dolich, Goodrick and Lippel [7] for information). For T^{indep}, recall 2.28, 3.13, 1.1.1, 2.27 and 2.23. For T^{pair}, strong dependence is by [1, 2.11], and the other results are from [8, §5]. (It is possible for T^{pair} to satisfy exchange, but only under degenerate circumstances, in particular, T cannot interpret an ordered ring; see [8, 5.9] for more information.) As for T^{gen}: (i) independence and exchange is by Chatzidakis and Pillay [6, §2]; (ii) elimination of imaginaries is by Fratarcangeli [15]; (iii) see [8, 6.3] for failure of definable Skolem functions; (iv) by combining [6, 2.6.1] with [8, 6.6.1], there are at least continuum-many completions of T^{gen}, at most one of which has an atomic model.

Now we summarize þ- and dp-ranks:

rank	T	T^{indep}	T^{pair}	completions of T^{gen}
þ	1	ω	ω (pole)	1
dp	1	ω	ω (pole)	∞

By [27], T has þ-rank 1; by [7], it has dp-rank 1. We have already shown that T^{indep} has þ-rank ω (3.1), and it is immediate from 1.1 and 2.28 that T^{indep} has dp-rank ω. Preservation of þ-rank in passing to T^{gen} follows from Ealy and Onshuus [14, 4.1.2] and its proof, while independence of T^{gen} rules out ordinal dp-rank. Once again, there is a split in the behavior of T^{pair}. By [2, Theorem 3] and [1, 2.11], T^{pair} has both þ-rank and dp-rank bounded by ω. If T has a pole, then T also extends the theory of real-closed ordered fields, so T^{pair} has þ-rank ω by [2, Theorem 3]; this is also true for dp-rank, as we now show. Let $(\mathfrak{M}, A) \vDash T^{\mathrm{pair}}$ be saturated, and let $(e_n)_{n>0}$ be sequence of independent elements of M. For ease of notation (and without loss of generality), we assume that the addition of the field structure is the same as $+$. For each n, the map $x \mapsto \sum_{i=1}^n e_i x_i$ is injective on A^n by independence. If T does not have a pole, then the situation is more complicated. There is at least one concrete case of interest where the dp-rank of T^{pair} is finite: By ongoing work of Dolich and Goodrick, $\mathrm{Th}(\mathbb{Q}, <, +, 1)^{\mathrm{pair}}$ has dp-rank 2 (we do not know the þ-rank).

Dense graphs. In any first-order topological structure, the existence of a definable function whose graph is somewhere dense is somehow related to failure of EP, but we do not yet understand this precisely. By [8, 4.15 and 4.16], if T^* is any complete extension of T^{noise} having T as an open core and dcl is not preserved in passing to T^*—in particular, if $T^* \nvDash \mathrm{EP}$—then every model of T^* defines a unary function whose graph is somewhere dense. We do not know if the converse holds, but we do know that if $(\mathfrak{M}, G) \vDash T^{\mathrm{gen}}$, then the graph of each definable function is nowhere dense [9, 6.2]. As for $(\mathfrak{M}, A) \vDash T^{\mathrm{pair}}$: (i) if $T = \mathrm{Th}(\mathbb{Q}, <, +, 0, 1)$, then the graph of each function definable in $(\mathfrak{M}, A)$ is nowhere dense [8, 5.9]; (ii) if T extends the theory of ordered rings, then $(\mathfrak{M}, A)$ explicitly defines, *allowing parameters from M*, a unary function whose graph is somewhere dense (see [12, p. 62]); (iii) we suspect that the graph of every function $\emptyset$-definable in $(\mathfrak{M}, A)$ is nowhere dense, but this is open as yet. For T^{indep}, there is no such variation of behavior: It is easy to see that the union of the sets $\{\, (x, y) : \{x + y, y\} \subseteq H \,\}$ and $\{\, (x, 0) : \{x + y, y\} \nsubseteq H \,\}$ is dense and the graph of a function $M \to M$. Hence, $(\mathfrak{M}, H)$ $\emptyset$-defines a unary function whose graph is dense.

Induced structure. As mentioned in the introduction, T^{noise} can have "wild" models $(\mathfrak{M}, X)$ where $X(\mathfrak{M}, X)$ is as large as possible, while 2.29 shows that T^{noise} always has models $(\mathfrak{M}, X)$ such that $X(\mathfrak{M}, X)$ is as simple as possible. Some intermediate, but still rather simple, behavior is exhibited by T^{pair}: If $(\mathfrak{M}, X) \vDash T^{\mathrm{pair}}$, then $X(\mathfrak{M}, X)$ is interdefinable with the expansion of $\mathfrak{X}$ by all traces on X of open intervals of M ([12, Theorem 2.3]). In all of these examples, the induced structure is easy to describe relative to $\mathfrak{M}$ as some concretely recognizable object, even in the wild case (for any nonempty set X, its expansion by all subsets of each X^n is certainly easy to describe without any information about $\mathfrak{M}$). But this need not be so, even if $X(\mathfrak{M}, X) =_{\mathrm{df}} X(\mathfrak{M})$, as we show next for models of T^{gen}.

Note. The rest of this section has no particular connection to T^{indep} or T^{pair} except for purposes of comparison, and all results hold assuming only that T is o-minimal and $\mathfrak{M}$ defines a unary function that is not definable in $(M, <)$ (we shall use the function $x + 1$). As such, it is something of a side issue to this paper, so we leave a number of routine details to the reader, who we assume to be fairly familiar with genericity over o-minimal structures (see [8], and [26] over the real field). Fix $(\mathfrak{M}, G) \vDash T^{\mathrm{gen}}$. To simply notation, we write $\mathfrak{G}$ instead of $G(\mathfrak{M}, G)$. Recall the basic facts about topology, open cores and induced structures from Section 1.

4.1. *Every open set definable in $\mathfrak{G}$ is a trace on G of an open set definable in $\mathfrak{M}$. Every set definable in $\mathfrak{G}$ is a boolean combination of projections of traces on G of locally closed sets definable in $\mathfrak{M}$.*

Proof. The first statement is immediate from 1.3 and that T is an open core of T^{gen}; the second is immediate from cell decomposition in $\mathfrak{M}$ and that every set definable in $(\mathfrak{M}, G)$ is a boolean combination of preimages of G under functions definable in $\mathfrak{M}$ (see Friedman [16, Theorem 2] or derive from [6, 2.6.4]). $\qquad\square$

Hence,

4.2. $\mathfrak{G} =_{\mathrm{df}} \mathfrak{G}^\circ = G(\mathfrak{M}^\circ).$

The point is that $\mathfrak{G}$ is rather tame in certain ways: It can be regarded as being generated by its open definable sets, which are well understood and mutually well behaved under boolean operations, and there is a uniform (and quite low) syntactic bound on how the definable sets are generated from the open definable sets. Nevertheless, many sets definable in $\mathfrak{G}$ are poorly behaved topologically, as we show next.

4.3. Put $E = \{ (x, y) \in G^2 : y = x + 1 \}$. Observe that E is $\emptyset$-definable in $(\mathfrak{M}, G)$ and closed in G^2. By genericity of G with respect to $\mathfrak{M}$, its projection $\{ x \in G : x + 1 \in G \}$ on the first coordinate is dense-codense in G. Moreover, this is even true locally: Given $e \in E$ and a box $I \times J$ containing e, the projection of $E \cap (I \times J)$ is dense-codense in I. Hence, while every set definable in $\mathfrak{G}$ is a boolean combination of projections of locally closed definable sets, it is possible for images of closed bounded $\emptyset$-definable sets under open continuous $\emptyset$-definable maps to be nowhere locally closed. Note also that the definable partial function $x \mapsto x + 1 \colon \{ x \in G : x + 1 \in G \} \to G$ has dense-codense domain and is a homeomorphism onto its image, but at no point of the complement of its domain do any of the one-sided lower or upper limits exist in $G \cup \{\pm\infty\}$. There is a syntactic manifestation as well:

4.4 (*cf.* 4.1). *$\mathfrak{G}$ is not model complete in the language of $\mathfrak{G}^\circ$.*

Hint of proof. Show by induction on n that if $\varphi(v_0, \dots, v_n)$ is an $L(M)$-formula and

$$(\mathfrak{M}, G) \vDash \forall v_0 \big(\psi(v_0) \to (P v_0 \wedge \neg P(v_0 + 1)) \big),$$

where $\psi(v_0) := \exists v_1 \dots v_n \big(\varphi(v_0, \dots, v_n) \wedge \bigwedge_{i=0}^n P v_i \big)$, then $\psi(\mathfrak{M}, G)$ is finite. (The case $n = 0$ is trivial; the case $n = 1$ contains the main ideas, which are already present in 4.3.) $\qquad\square$

As $\mathfrak{G}^\circ$ defines dense-codense sets, topological analysis of $\mathfrak{G}$ via $\mathfrak{G}^\circ$ is essentially ineffective. Still, we might hope that $\mathfrak{G}$ has reducts that are somehow topologically well behaved and informative about $\mathfrak{G}$. But this also fails:

4.5. *The structure* $\mathfrak{G}_0 := \big(G, <, (G \cap (a,b))_{a,b \in M \cup \{\pm\infty\}}\big)$ *is, up to interdefinability, the unique maximal reduct of* $\mathfrak{G}$ *such that every unary definable set either has interior or is nowhere dense.*

We assume the reader knows or can check that $\mathfrak{G}_0$ has quantifier elimination, and so $\mathfrak{G}_0$ is weakly o-minimal and not much more complicated than $(G, <)$. As for the rest, again we provide only a hint for a proof. First, we define a "quasifunction" from G^n to G to be a subset F of $G^n \times G$ such that for every $x \in G^n$, the set $\{\, y \in G : (x,y) \in F \,\}$ is convex, and either empty or both bounded below and unbounded above. Observe that if F is definable in $\mathfrak{G}$, then F determines a (possibly partial) function f from G^n to M that is definable in $(\mathfrak{M}, G)$, and thus is given piecewise by functions definable in $\mathfrak{M}$ (since dcl is preserved in passing from T to T^{gen}). We say that F has some property if the associated f has the property. Now show by the usual inductive arguments that:

(I$_n$) If $\mathcal{D}$ is a decomposition of M^n into cells of $\mathfrak{M}$ and every unary set definable in

$$(G, <, \{\, G^n \cap D : D \in \mathcal{D} \,\})$$

either has interior or is nowhere dense, then there is a decomposition $\mathcal{C}$ of M^n into cells of $(M, <)$ such that $\{\, G^n \cap C : C \in \mathcal{C} \,\}$ is compatible with $\{\, G^n \cap D : D \in \mathcal{D} \,\}$.

(II$_n$) If $F \colon G^n \to G$ is a quasifunction definable in $\mathfrak{G}$ and every unary set definable in $(G, <, F)$ either has interior or is nowhere dense, then f is given piecewise by constants and coordinate projections.

5. Concluding remarks

Optimality. Density of H can be relaxed, but not by much. To illustrate,

5.1. *If* $E \subseteq M$ *is independent, then the following are equivalent.*

(1) $(\mathfrak{M}, E)^\circ =_{\text{df}} \mathfrak{M}$.
(2) $(\mathfrak{M}, E)^\circ$ *is o-minimal.*
(3) *The topological closure of* E *is definable in* $\mathfrak{M}$.
(4) *The topological closure of* E *is a finite union of points and closed intervals of* M.

Proof. 1⇒2⇒3⇒4 is trivial, and so is 4⇒1 if E is finite. Assume that E is infinite. For 4⇒1, the situation is somewhat similar to that of expanding $\mathfrak{M}$ by collections of mutually independent dense sets. If $\mathfrak{M}$ has a pole, then 1 follows from 2.25 by encoding E as finitely many points and a single dense independent set (although some extra care must be taken if the boundary of the closure of E intersects $\text{dcl}(E) \setminus \text{dcl}(\emptyset)$). But in general, all arguments of results up through 2.25 must be repeated subject to the obvious modifications. This is considerably easier to do here, though, because we still deal with only one new predicate. $\qquad \square$

(Of course, one can easily think of interesting cases where the closure of E is open and convex and $(\mathfrak{M}, E)^\circ$ is not o-minimal, but this goes too far outside the intended scope of this paper.)

We have not said much about expansions of $\mathfrak{M}$ by families of mutually independent dense sets, but omission of "mutually" is easily seen to be problematic. Indeed, things can go quite wild:

5.2. *If T has an archimedean model and* $\mathrm{card}(L) < \mathrm{card}(\mathbb{R})$, *then T has a model $\mathfrak{R}$ over $\mathbb{R}$ and there exist independent $E, H_1, H_2 \subseteq \mathbb{R}$ such that*

   (1) *E is closed and discrete, and $(\mathfrak{R}, E) =_{\mathrm{df}} (\mathbb{R}, +, \cdot, \mathbb{Z})$.*
   (2) *H_1 and H_2 are dense, and $(\mathfrak{R}, H_1, H_2) =_{\mathrm{df}} (\mathbb{R}, +, \cdot, \mathbb{Z})$.*

(This also shows how complicated the situation can get once density is relaxed beyond 5.1.)

Proof. The model $\mathfrak{R}$ exists by [22].

1. There exist independent $a, b, c \in \mathbb{R}$ such that $0 < a < b < c$. Construct sequences (a_n), (b_n) and (c_n) of real numbers such that $\max(|a_n - an|, |b_n - bn|, |c_n - cn|) < 2^{-n}$ for all $n \in \mathbb{N}$ and $\{\, a_n : n \in \mathbb{N} \,\} \cup \{\, b_n : n \in \mathbb{N} \,\} \cup \{\, c_n : n \in \mathbb{N} \,\}$ is independent. Put $A = \{\, a_n : n \in \mathbb{N} \,\}$, $B = \{\, b_n : n \in \mathbb{N} \,\}$ and $C = \{\, c_n : n \in \mathbb{N} \,\}$. Clearly, $\lim_{n \to +\infty}(a_{n+1} - a_n) = a$, $\lim_{n \to +\infty}(b_{n+1} - b_n) = b$ and $\lim_{n \to +\infty}(b_{n+1} - b_n) = c$. By [25, Asymptotic Extraction of Groups], $(\mathfrak{R}, A, B, C)$ defines the cyclic groups $a\mathbb{Z}$, $b\mathbb{Z}$ and $c\mathbb{Z}$. By Hieronymi and Tychonievich [17], $(\mathfrak{R}, a\mathbb{Z}, b\mathbb{Z}, c\mathbb{Z})$ defines multiplication. Hence, it suffices now to put $E = A \cup B \cup C$ and show that $(\mathfrak{R}, E)$ defines each of A, B and C. The set

$$A' := \{\, x \in E : a/2 < \min(E \cap (x, \infty)) - x < (b+a)/2 \,\}$$

is definable in $(\mathfrak{R}, E)$ and $A \setminus A'$ is finite. Hence, $(\mathfrak{R}, E)$ defines A. By arguing similarly with $E \setminus A$ in place of E, $(\mathfrak{R}, E)$ defines B, hence also C.

2. By separabililty, there exist countable dense independent $H_1, H_2 \subseteq \mathbb{R}$ such that $H_1 \cap H_2 = E$. Recall that every real Borel set is definable in $(\mathbb{R}, +, \cdot, \mathbb{Z})$. $\qquad\square$

5.3. It is natural to consider relaxing the o-minimality assumption on T. Given the underlying program of this paper, what comes first to mind is to assume instead that T has an o-minimal open core T_0, and then hope to show that T_0 is an open core of T^{indep}. There is a natural division into two cases, namely, $T \vDash \mathrm{EP}$ or $T \nvDash \mathrm{EP}$. We have relied so heavily in this paper on $T \vDash \mathrm{EP}$ that we simply have no ideas at present on how to get along without it, even over the theory of dense pairs of real-closed fields. Hence, let us assume that $T \vDash \mathrm{EP}$; then T also satisfies uniform finiteness ([8, 1.17]) and thus T is "geometric". This situation is already being studied in much greater abstraction by others (*e.g.*, [4]), but as yet, we do not see how to handle even the most concrete cases that we know of, namely, that $T_0 = \mathrm{Th}(\mathbb{Q}, <, +, 0, 1)$ and T is either T_0^{dense} or a completion of T_0^{gen}. (*Remark:* By [6], definable closure is preserved in passing to generic extensions, so we would always have $(T^{\mathrm{gen}})^{\mathrm{indep}} = T^{\mathrm{gen}} \cup T^{\mathrm{indep}}$.)

5.4. It is also natural to consider T^{noise} over more than one new kind of predicate. To illustrate, let us extend L_P by a new unary relation symbol and extend T^{noise} to the theory of T with two disjoint dense-codense unary predicates, where our intention is that the type of noise associated to the predicates be different. By [8, 9], there are two concrete classes of examples having T as an open core, namely, the extensions $(T^{\mathrm{indep}})^{\mathrm{gen}}$ and $(T^{\mathrm{pair}})^{\mathrm{gen}}$ of T^{indep} and T^{pair} by a generic unary predicate (as to why one might care about these, see the

introduction to [9]). Similarly as in 5.3, we have $(T^{\mathrm{pair}})^{\mathrm{indep}}$ and $(T^{\mathrm{gen}})^{\mathrm{indep}}\ (= T^{\mathrm{gen}} \cup T^{\mathrm{indep}})$ as possibilities, but we do not yet know if T is an open core of either. Perhaps most natural in our setting should be consideration of the theories of structures of the form $(\mathfrak{M}, H, A)$, where $(\mathfrak{M}, H) \vDash T^{\mathrm{indep}}$, $(\mathfrak{M}, A) \vDash T^{\mathrm{pair}}$ and either $\mathrm{dcl}(H) \subseteq A$ or H is independent over A. It seems reasonable to think these could be attacked by appropriate amalgamation of proofs of results for T^{indep} and T^{pair}, but we have not done any work on this.

Questions. We close with a list of questions (most of which we raised in [8] and are still open).

Obviously, the overarching questions are: Which complete extensions T^* of T^{noise} (allowing for extending L_P as well) have T as an open core, and how are the model-theoretic properties of T^* related to those of T? Can such T^* be classified in some useful fashion by the model-theoretic properties that they do or do not possess?

If $\widetilde{T}$ is any theory (in any language) having T as an open core, and some model of $\widetilde{T}$ defines a somewhere dense graph, must EP fail for $\widetilde{T}$? What if $\widetilde{T}$ also extends the theory of ordered fields and has a model over $\mathbb{R}$?

If $\mathfrak{R}$ is an expansion of a DLO having both definable Skolem functions and o-minimal open core, is $\mathfrak{R}$ o-minimal? What if $\mathfrak{R}$ expands an ordered field?

If $\mathfrak{R}$ is a definably complete (see [8] for the definition) expansion of an ordered field such that $\mathrm{Th}(\mathfrak{R})$ has EP and is strongly dependent, is $\mathfrak{R}$ o-minimal?

Early on this paper, we adopted for technical convenience *in proofs* (as opposed to statements of results) that T has quantifier elimination and is universally axiomatizable. If we drop this assumption, then what can be said about expansions of $\mathfrak{M}$ by dense-codense sub*structures*? As the example $(\mathbb{R}, <, +, \cdot, \mathbb{Q})$ makes clear, the result can be be quite wild, so we should feel free to make further assumptions. We give only one concrete question of this sort: If $(K, +, \cdot)$ is a subfield of $(\mathbb{R}, +, \cdot)$ that does not define $\mathbb{Z}$, is every open set definable in $(\mathbb{R}, +, \cdot, K)$ definable in $(\mathbb{R}, +, \cdot)$?

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