ON COMPLETENESS AND LINEAR DEPENDENCE FOR DIFFERENTIAL ALGEBRAIC VARIETIES

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Abstract. In this paper we deal with two foundational questions on complete differential algebraic varieties. We establish, by means of the differential algebraic version of Bertini's theorem and assuming a weaker form of the catenary conjecture, that in order to verify (differential) completeness one can restrict second factors to *zero-dimensional* differential varieties. Then, we extend Kolchin's results from [11] on linear dependence over projective varieties in the constants, to linear dependence over arbitrary complete differential varieties.

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1. Introduction

The study of complete differential algebraic varieties (see Definition 4.1 below) began in the 1960's during the development of differential algebraic geometry, and received the attention of various authors over the next several decades [16, 3, 4, 11]. On the other hand, even though model theory had significant early interactions with differential algebra, it was not until recently that the topic has been the subject of various works using the model-theoretic perspective [18, 8, 22, 17].

Up until the last couple of years, relatively few examples of complete differential algebraic varieties were known. The development of a valuative criterion for completeness [18, 8, in the ordinary and partial case, respectively] led to a variety of new examples. Subsequently, more examples have been discovered [22] using various algebraic techniques in conjunction with the valuative criterion (in Section 4 we present an example of the third author's, which shows that there are zero-dimensional projective differential algebraic varieties which are not complete).

In [8], it was established that every complete differential algebraic variety is zero-dimensional (earlier, this result was established in the ordinary case [18]). Ostensibly, to verify that a given projective differential algebraic variety is complete one has

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to quantify over *all* differential algebraic subvarieties of all products of the given variety with a quasiprojective differential algebraic variety. The first main question we address in this paper is:

Question. *To verify completeness, can one restrict to taking products of the given (zero-dimensional) differential variety with zero-dimensional differential varieties?*

We answer this question affirmatively assuming a very natural conjecture which is a consequence of weak versions of the Kolchin catenary conjecture (we discuss this in detail in Section 3). Our main tool in this endeavour is the differential algebraic version of Bertini's theorem [9, Theorem 0.4]. Besides this, the arguments use elementary differential algebra and model theory.

Differential completeness is a foundational issue in differential algebraic geometry, but, except for [11], there has been no discussion of applications of the idea (outside of foundational issues). The last section of the paper is in this direction. We consider the notion of linear dependence over an arbitrary projective differential algebraic variety. This is a generalization of a notion studied by Kolchin [11] in the case of projective algebraic varieties, which in turn generalizes linear dependence in the traditional sense.

After proving several results generalizing the work of [11], we turn to specific cases. In the case of the field of meromorphic functions (on some domain of $\mathbb{C}$) and the projective variety $\mathbb{P}^n(\mathbb{C})$, Kolchin's results specialize to the classical result: any finite collection of meromorphic functions is linearly dependent over $\mathbb{C}$ if and only if the Wronskian determinant of the collection vanishes. There are generalizations of this in several directions, but we will be interested in the generalization to partial differential fields satisfying some additional criteria (which hold in the example of meromorphic functions).

As for the generalization to multiple variables (i.e., partial differential fields), fully general results on Wronskians and linear dependence of meromorphic functions are relatively recent (coming after Kolchin's work [11], for instance). Roth [21] first established these type of results in the case of polynomials in several variables for use in diophantine approximation. Later his results were generalized to meromorphic functions in some domain of $\mathbb{C}^m$ via [25] and [2, used in Nevanlinna-type uniqueness questions].

The necessity of the vanishing of the generalized Wronskian follows trivially from the assumption of linear dependence, so the difficulty lies entirely in the sufficiency. We will show how sufficiency of this condition under quite general circumstances follows from basic results regarding differential completeness. In fact, as we will see, generalizations of the notion of linear dependence also have necessary and sufficient conditions given by the vanishing of differential algebraic equations.

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2. Projective differential algebraic varieties

We fix a differentially closed field $(\mathcal{U}, \Delta)$ of characteristic zero, where

$$\Delta = \{\delta_1, \ldots, \delta_m\}$$

is the set of m commuting derivations. We assume $\mathcal{U}$ to be a universal domain for differential algebraic geometry; in model-theoretic terms, we are simply assuming that $\mathcal{U}$ is a sufficiently large saturated model of $DCF_{0,m}$. Throughout K will denote a (small) differential subfield of $\mathcal{U}$.

A subset of $\mathbb{A}^n$ is Δ-*closed* if it is the zero set of a collection of Δ-polynomials over $\mathcal{U}$ in n differential indeterminates (these sets are also called affine differential algebraic varieties). When we want to emphasize that the collection of Δ-polynomials defining a Δ-closed set is over K, we say that the Δ-closed is defined over K. For a thorough development of differential algebraic geometry see [12] or [14].

Following the standard convention, we will use $K\{y_0, y_1, \ldots, y_n\}$ to denote the ring of Δ-*polynomials* over K in the Δ-indeterminates $y_0, y_1, \ldots, y_n$.

Definition 2.1. A (non-constant) Δ-polynomial f in $K\{y_0, \ldots, y_n\}$ is Δ-*homogeneous of degree* d if

$$f(ty_0, \ldots ty_n\} = t^d f(y_0, \ldots, y_n),$$

where t is another Δ-indeterminate.

The reader should note that Δ-homogeneity is a stronger notion that homogeneity of a differential polynomial as a polynomial in the algebraic indeterminates $\delta_m^{r_m} \cdots \delta_1^{r_1} y_i$. For instance, for any $\delta \in \Delta$,

$$\delta y - y$$

is a homogeneous Δ-polynomial, but not a Δ-homogeneous Δ-polynomial. The reader may verify that the following is Δ-homogeneous:

$$y_1 \delta y_0 - y_0 \delta y_1 - y_0 y_1.$$

Generally, we can easily homogenize an arbitrary Δ-polynomial in $y_1, \ldots, y_n$ with respect to a new Δ-variable y_0. Let f be a Δ-polynomial in $K\{y_1, \ldots, y_n\}$, then for d sufficiently large $y_0^d f(\frac{y_1}{y_0}, \ldots, \frac{y_n}{y_0})$ is Δ-homogeneous of degree d. For more details and examples see [18].

As a consequence of the definition, the vanishing of Δ-homogeneous Δ-polynomials in $n+1$ variables is well-defined on $\mathbb{P}^n$. In general, the Δ-closed subsets of $\mathbb{P}^n$

defined over K are the zero sets of collections of Δ-homogeneous Δ-polynomials in $K\{y_0, \dots, y_n\}$ (also called projective differential algebraic varieties). Furthermore, Δ-closed subsets of $\mathbb{P}^n \times \mathbb{A}^m$, defined over K, are given by the zero sets of collections of Δ-polynomials in

$$K\{y_0, \dots, y_n, z_1, \dots, z_m\}$$

which are Δ-homogeneous in $\bar{y} = (y_0, \dots, y_n)$.

2.1. **Dimension polynomials for projective differential algebraic varieties.** Take $\alpha \in \mathbb{P}^n$ and let $\bar{a} = (a_0, \dots, a_n) \in \mathbb{A}^{n+1}$ be a representative for α. Choose some index i for which $a_i \neq 0$. The field extensions $K\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)$ and $K\left\langle\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right\rangle$ do not depend on which representative $\bar{a}$ or index i we choose. Here $K\langle\bar{a}\rangle$ denotes the Δ-field generated by $\bar{a}$ over K.

Definition 2.2. With the notation of the above paragraph, the *Kolchin polynomial of α over K* is defined as

$$\omega_{\alpha/K}(t) = \omega_{\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)/K}(t),$$

where $\omega_{\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)/K}(t)$ is the standard Kolchin polynomial of $\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)$ over K (see Chapter II of [12]). The *Δ-type of α* over K is defined to be the degree of $\omega_{\alpha/K}$. By the above remarks, these two notions are well-defined; i.e., they are independent of the choice of the representative $\bar{a}$ and index i.

Let $\beta \in \mathbb{P}^n$ be such that the closure (in the Δ-topology) of β over K is contained in the closure of α over K. In this case we say that β is a *differential specialization* of α over K and denote it by $\alpha \mapsto_K \beta$. Let $\bar{b}$ be a representative for β with $b_i \neq 0$. Then, by our choice of β and α, if $\bar{a}$ is a representative of α, then $a_i \neq 0$; moreover, the tuple $\left(\frac{b_0}{b_i}, \dots, \frac{b_n}{b_i}\right)$ in $\mathbb{A}^{n+1}$ is a differential specialization of $\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)$ over K. When $V \subseteq \mathbb{P}^n$ is an irreducible Δ-closed set over K, then a *generic point* of V over K (when K is understood we will simply say generic) is simply a point $\alpha \in V$ for which $V = \{\beta \,|\, \alpha \mapsto_K \beta\}$. It follows, from the affine case, that every irreducible Δ-closed set in $\mathbb{P}^n$ has a generic point over K, and that any two such generics have the same isomorphism type over K.

Definition 2.3. Let $V \subseteq \mathbb{P}^n$ be an irreducible Δ-closed set. The *Kolchin polynomial* of V is defined to be

$$\omega_V(t) = \omega_{\alpha/F}(t),$$

where F is any differential field over which V is defined and α is a generic point of V over F. It follows, from the affine case, that ω_V does not depend on the choice of F or α. The *Δ-type of V* is defined to be the degree of ω_V.

Remark 2.4. Let $V \subseteq \mathbb{P}^n$ be an irreducible Δ-closed set.

(i) V has Δ-type m if and only if the differential function field of V has positive differential transcendence degree.
(ii) V has Δ-type zero if and only if the differential function field of V has finite transcendence degree.

Definition 2.5. The *dimension* of V, denoted by $\dim V$, is the differential transcendence degree of the differential function field of V.

Thus, by a zero-dimensional differential algebraic variety we mean one of Δ-type less than m (in model-theoretic terms this is equivalent to the Lascar rank being less than ω^m).

In various circumstances it is advantageous to consider $\mathbb{P}^n$ as a quotient of $\mathbb{A}^{n+1}$. For example, if $\mathfrak{p}$ is the Δ-ideal of Δ-homogeneous Δ-polynomials defining $V \subseteq \mathbb{P}^n$ and we let $W \subseteq \mathbb{A}^{n+1}$ be the zero set of $\mathfrak{p}$, then

$$\omega_W(t) = \omega_V(t) + \binom{t+m}{m}, \tag{1}$$

where the polynomial on the left is the standard Kolchin polynomial of W and m is the number of derivations (see §5 of [11]).

3. The catenary problem and related results

The central problem we wish to discuss in this section is known as the *Kolchin catenary problem*:

Problem 3.1. *Given an irreducible differential algebraic variety V of dimension d and an arbitrary point $p \in V$, does there exist a long gap chain beginning at p and ending at V? By a long gap chain we mean a chain of irreducible differential algebraic subvarieties of length $\omega^m \cdot d$. We will call the positive answer to this question the Kolchin catenary conjecture.*

Under stronger assumptions, various authors have established the existence of long gap chains. See [6, pages 607-608] for additional details on the history of this problem. When $p \in V$ satisfies certain nonsingularity assumptions, Johnson [10] established the existence of the chain of subvarieties of X starting at p. In [20], Rosenfeld proves a slightly weaker statement, also with nonsingularity assumptions, perhaps because he was not considering the order type of the chains of differential ideals in full generality. Rosenfeld also expresses the opinion that the nonsingularity hypotheses are non necessary; however, except for special cases, the hypotheses have not been eliminated.

In a different direction, Pong [19] answered the problem affirmatively, assuming that V is an algebraic variety, but assuming nothing about the point p. Pong's proof invokes resolution of singularities (the "nonsingularity" assumptions of [20, 10] are not equivalent to the classical notion of a nonsingular point on an algebraic variety).

It is worth mentioning that even though Pong works in the ordinary case, $\Delta = \{\delta\}$, his approach and results readily generalize to the partial case. It is in this more general setting, $\Delta = \{\delta_1, \dots, \delta_m\}$, that we would like to describe the difficulties of the catenary problem (for algebraic varieties).

3.1. **From commutative algebra to differential commutative algebra.** A differential ring is called a *Keigher ring* if the radical of every differential ideal is again a differential ideal. The rings we will be considering will be assumed to be Keigher rings. Note that every Ritt algebra is a Keigher ring (see for instance [14, §1]).

Given $f : A \to B$ a differential homomorphism of Keigher rings, we have an induced map $f^* : Spec\, B \to Spec\, A$ given by $f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$. We denote by $f^*_\Delta : Spec^\Delta B \to Spec^\Delta A$ the restriction of the map f^* to the differential spectrum. We have the following differential analogs of the going-up and going-down properties:

Definition 3.2.

(1) Suppose we are given some chain $\mathfrak{p}_1 \subseteq \dots \subseteq \mathfrak{p}_n$ with $\mathfrak{p}_i \in Spec^\Delta f(A)$ and any $\mathfrak{q}_1 \subseteq \dots \subseteq \mathfrak{q}_m \in Spec^\Delta B$ such that for each $i \leq m$, $\mathfrak{q}_i \cap f(A) = \mathfrak{p}_i$. We say that f has the *going-up property for differential ideals* if given any such chains $\mathfrak{p}$ and $\mathfrak{q}$, we may extend the second chain to $\mathfrak{q}_1 \subseteq \dots \subseteq \mathfrak{q}_n$ where $\mathfrak{q}_i \in Spec^\Delta A$ such that for each $1 \leq i \leq n$, $\mathfrak{q}_i \cap f(A) = \mathfrak{p}_i$.

(2) Suppose we are given some chain $\mathfrak{p}_1 \supseteq \dots \supseteq \mathfrak{p}_n$ with $\mathfrak{p}_i \in Spec^\Delta f(A)$ and any $\mathfrak{q}_1 \supseteq \dots \supseteq \mathfrak{q}_m \in Spec^\Delta B$ such that for each $i \leq m$, $\mathfrak{q}_i \cap f(A) = \mathfrak{p}_i$. We say that f has the *going-down property for differential ideals* if given any such chains $\mathfrak{p}$ and $\mathfrak{q}$, we may extend the second chain to $\mathfrak{q}_1 \supseteq \dots \supseteq \mathfrak{q}_n$ where $\mathfrak{q}_i \in Spec^\Delta A$ such that for each $1 \leq i \leq n$, $\mathfrak{q}_i \cap f(A) = \mathfrak{p}_i$.

When $(A, \Delta) \subseteq (B, \Delta)$ are integral domains, B is integral over A, and A is integrally closed, then the differential embedding $A \subseteq B$ has the going-down property for differential ideals. Dropping the integrally closed requirement on A, one can still prove the going-up property for differential ideals [19, Proposition 1.1]. In what follows we will see how these results are consequences of their classical counterparts in commutative algebra.

Let us review some developments of differential algebra which are proved in [23]. We will prove the results which we need here in order to keep the exposition self-contained and tailored to our needs. We will also provide some additional comments on a few of the results. Let $f : A \to B$ be a differential homomorphism of Keigher rings. The fundamental idea, which Trushin calls *inheritance*, is to consider one property of such a map f considered only as a map of rings and another property of f as a map of differential rings and prove that the properties are equivalent. Then one might reduce the task of proving various differential algebraic facts to proving corresponding algebraic facts.

Lemma 3.3. *Let $\mathfrak{p} \subset A$ be a prime differential ideal. The following are equivalent:*

(1) $\mathfrak{p} = f^{-1}(f(\mathfrak{p})B)$,
(2) $(f^*)^{-1}(\mathfrak{p}) \neq \emptyset$,
(3) $(f^*_\Delta)^{-1}(\mathfrak{p}) \neq \emptyset$.

Proof. (1) $\Leftrightarrow$ (2) is precisely [1, Proposition 3.16]. (3) $\Rightarrow$ (2) is trivial. To show that (2) $\Rightarrow$ (3), note that $(f^*)^{-1}(\mathfrak{p})$ is homeomorphic to $Spec\, B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. The fact that the fiber is nonempty means that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is not the zero ring. Since it is a Keigher ring, $Spec^\Delta\, B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is nonempty (see [13]) and naturally homeomorphic to $(f^*_\Delta)^{-1}(\mathfrak{p})$. $\square$

The following results are easy applications of the lemma:

Corollary 3.4. *Let $\mathfrak{p} \in Spec^\Delta\, A$. If $(f^*)^{-1}(\mathfrak{p}) \neq \emptyset$, then $(f^*_\Delta)^{-1}(\mathfrak{p}) \neq \emptyset$. Thus, if f^* is surjective, so is f^*_Δ.*

Corollary 3.5. *If f has the going-up property, then f has the going-up property for differential ideals.*

Corollary 3.6. *If f has the going-down property, then f has the going-down property for differential ideals.*

Of course, by applying the two previous corrollaries to integral extensions with the standard additional hypotheses, we get the desired analogs of the classical going-up and going-down properties (see [19] or [23], where the results were reproved).

Proposition 3.7. *Suppose that A is a Ritt algebra, $Spec^\Delta\, A$ is Noetherian, B is a finitely generated differential ring over A, and the map $f : A \to B$ is the embedding map. Then the following are equivalent.*

(1) *f has the going-down property for differential ideals,*
(2) *f^*_Δ is an open map (with respect to the Δ-topology).*

Proof. Let us prove that the second property implies the first. Let $\mathfrak{q} \in Spec^\Delta\, B$ and let $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Since we are interested in differential ideals containing $\mathfrak{q}$, it will be useful to consider the local ring $B_\mathfrak{q}$, and we note that $B_\mathfrak{q} = \varinjlim_{t \in B\setminus\mathfrak{q}} B_t$.

By [1, Exercise 26 of Chapter 3], $f^*(Spec\, B_\mathfrak{q}) = \bigcap_{t\in B\setminus\mathfrak{q}} f^*(Spec\, B_t)$. Now, by Corollary 3.4, surjectivity of f^* implies surjectivity of f^*_Δ, so

$$f^*_\Delta(Spec^\Delta\, B_\mathfrak{q}) = \bigcap_{t\ inB\setminus\mathfrak{q}} f^*_\Delta(Spec^\Delta(B_t)).$$

Since f^*_Δ is an open map, $f^*_\Delta(Spec^\Delta\, B_t)$ is an open neighborhood of $\mathfrak{p}$ and so it contains $Spec^\Delta\, A_\mathfrak{p}$.

We have proved that, for any $\mathfrak{q} \in Spec^\Delta\, B$, the induced map $f^*_\Delta : Spec^\Delta\, B_\mathfrak{q} \to Spec^\Delta A_\mathfrak{p}$ is a surjective map. Since differential ideals contained in $\mathfrak{p}$ correspond to differential ideals in $A_\mathfrak{p}$, we have established the going-down property for differential ideals.

Now we prove that (1) implies (2). Take $\mathfrak{p} \in f^*_\Delta(Spec^\Delta B_t)$ with $f^*_\Delta(\mathfrak{q}) = \mathfrak{p}$. Take some irreducible closed subset $Z \subseteq Spec^\Delta A$ for which $Z \cap f^*_\Delta(Spec^\Delta B_t)$ is nonempty. Now take some $\mathfrak{p}_1 \in Spec^\Delta A$ with $\mathfrak{p}_1 \subset \mathfrak{p}$. By the going-down property for differential ideals, $\mathfrak{p}_1 = f^*_\Delta(\mathfrak{q}_1)$ for some $\mathfrak{q}_1 \in Spec^\Delta B$. Noting that $\mathfrak{q}_1$ is in $Spec^\Delta B_t$, we see that $f^*_\Delta(Spec^\Delta B_t) \cap Z$ is dense in Z. Since the set $f^*_\Delta(Spec^\Delta B_t)$ is constructible [23, Statement 11] and thus contains an open subset of its closure, $Z \cap f^*_\Delta(Spec^\Delta B_t)$ contains an open subset of Z. This holds for arbitrary Z, so $f^*_\Delta(Spec^\Delta B_t)$ is open.

□

3.2. **A stronger form of the catenary problem for algebraic varieties.** We will study the catenary problem in the case that V is an irreducible algebraic variety over K and p is an arbitrary point of V (as we mentioned before this case follows from results of Pong; however, in what follows we propose a different approach to this problem).

Remark 3.8. We echo some general remarks from [19]:

(1) The catenary problem is essentially local; if you can find an ascending chain of irreducible subvarieties of an open set, then taking their closures results in an ascending chain in the given irreducible differential algebraic variety.
(2) As Pong [19, page 759] points out, truncated versions of the differential coordinate rings of singular algebraic varieties do not satisfy the hypotheses of the classical going-down or going-up theorems with respect the ring embedding given by Noether normalization. In [19], this difficulty is avoided using resolution of singularities.

Lemma 3.9. *Let $p \in \mathbb{A}^d$ be an arbitrary point. Then there is an increasing chain of irreducible differential algebraic subvarieties defined over K starting with p and ending with $\mathbb{A}^d$ of order type $\omega^m \cdot d$.*

The following is a specific example which establishes the lemma.

Example 3.10. We produce a family $\{G_r : r \in n \times \mathbb{N}^m\}$ of differential algebraic subgroups of the additive group $(\mathbb{A}^n, +)$ with the following properties. For every $r = (i, r_1, \dots, r_m) \in n \times \mathbb{N}^m$,

$$\omega^m i + \omega^{m-k} r_k + \omega^{m-(k+1)} r_{k+1} + \cdots + \omega r_{m-1} + r_m \le U(G_r) < \omega^m i + \omega^{m-k}(r_k + 1)$$

where k is the smallest such that $r_k > 0$, and if $r, s \in n \times \mathbb{N}^m$ are such that $r < s$, in the lexicographical order, then the containment $G_r \subset G_s$ is strict. Here $U(G_r)$ denotes the Lascar rank of G_r. We refer the reader to [14] for definitions and basic properties of this model-theoretic rank.

For $r = (i, r_1, \dots, r_m) \in n \times \mathbb{N}^m$, let G_r be defined by the homogeneous system of linear differential equations in the Δ-indeterminates $x_0, \dots, x_{n-1}$,

$$\delta_1^{r_1+1} x_i = 0,\ \delta_2^{r_2+1}\delta_1^{r_1} x_i = 0,\ \dots,\ \delta_{m-1}^{r_{m-1}+1}\delta_{m-2}^{r_{m_2}} \cdots \delta_1^{r_1} x_i = 0,\ \delta_m^{r_m}\delta_{m-1}^{r_{m-1}} \cdots \delta_1^{r_1} x_i = 0$$

together with

$$x_{i+1} = 0, \cdots, x_{n-1} = 0.$$

Note that if $r, s \in n \times \mathbb{N}^m$ are such that $r < s$, then $G_r \subset G_s$ is strict. We first show that

$$U(G_r) \geq \omega^m i + \omega^{m-1} r_1 + \omega^{m-2} r_2 + \cdots + \omega r_{m-1} + r_m.$$

We prove this by transfinite induction on $r = (i, r_1, ..., r_m)$ in the lexicographical order. The base case holds trivially. Suppose first that $r_m \neq 0$ (i.e., the succesor ordinal case). Consider the (definable) group homomorphism $f : (G_r, +) \to (G_r, +)$ given by $f(x_i) = \delta_m^{r_m - 1} x_i$. Then the generic type of the generic fibre of f is a forking extension of the generic type of G_r. Since f is a definable group homomorphism, the Lascar rank of the generic fibre is the same as the Lascar rank of $\mathrm{Ker}(f) = G_{r'}$, where $r' = (i, r_1, \ldots, r_{m-1}, r_m - 1)$. By induction,

$$U(G_{r'}) \geq \omega^m i + \omega^{m-1} r_1 + \omega^{m-2} r_2 + \cdots + \omega r_{m-1} + (r_m - 1).$$

Hence,

$$U(G_r) \geq \omega^m i + \omega^{m-1} r_1 + \omega^{m-2} r_2 + \cdots + \omega r_{m-1} + r_m.$$

Now suppose $r_m = 0$ (i.e., the limit ordinal case). Suppose there is k such that $r_k \neq 0$ and that k is the largest such. Let $\ell \in \omega$ and $r' = (i, r_1, \ldots, r_k - 1, \ell, 0, ..., 0)$. Then $G_{r'} \subset G_r$ and, by induction,

$$U(G_{r'}) \geq \omega^m i + \omega^{m-1} r_1 + \omega^{m-2} r_2 + \cdots + \omega^{m-k}(r_k - 1) + \omega^{m-k-1} \ell.$$

Since ℓ was arbitrary,

$$U(G_r) \geq \omega^m i + \omega^{m-1} r_1 + \omega^{m-2} r_2 + \cdots + \omega^{m-k} r_k.$$

Finally suppose that all the r_k's are zero and that $i > 0$. Let $\ell \in \omega$ and $r' = (i - 1, \ell, 0, \ldots, 0)$. Then again $G_{r'} \subseteq G_r$ and, by induction,

$$U(G_{r'}) \geq \omega^m (i - 1) + \omega^{m-1} \ell.$$

Since ℓ was arbitrary,

$$U(G_r) \geq \omega^m i.$$

This completes the induction.

Let k be the smallest such that $r_k > 0$ and let $tp(a_0, \ldots, a_{\ell-1}/K)$ be the generic type of G_r. We now show that if $i = 0$, then Δ-type$(G_r) = m - k$ and Δ-dim$(G_r) = r_k$ (here Δ-dim denotes the typical Δ-dimension, see Chapter II of [12]). As $i = r_1 = \cdots = r_{k-1} = 0$, we have $a_1 = \cdots = a_{n-1} = 0$ and $\delta_1 a_0 = 0, \ldots, \delta_{k-1} a_0 = 0$, $\delta_k^{r_k+1} a_0 = 0$ and $\delta_k^{r_k} a_0$ is Δ_k-algebraic over K where $\Delta_k = \{\delta_{k+1}, \ldots, \delta_m\}$. It suffices to show that $a_0, \delta_k a_0, \ldots, \delta_k^{r_k - 1} a_0$ are Δ_k-algebraically independent over K. Let f be a nonzero Δ_k-polynomial over K in the variables $x_0, \ldots, x_{r_k - 1}$, and let $g(x) = f(x, \delta_k x, \ldots, \delta_k^{r_k - 1} x) \in K\{x\}$. Then g is a nonzero Δ-polynomial over K

reduced with respect to the defining Δ-ideal of G_r over K. Thus, as a is a generic point of G_r over K,

$$0 \neq g(a) = f(a, \delta_k a, \dots, \delta_k^{r_k - 1} a),$$

as desired. Applying this, together with McGrail' s [15] upper bounds for Lascar rank, we get

$$U(G_r) < \omega^{m-k}(r_k + 1).$$

For arbitrary i, the above results show that $U(a_j/K) = \omega^m$ for $j < i$, and $U(a_i/K) < \omega^{m-k}(r_k + 1)$ and $U(a_j/K) = 0$ for $j > i$. Applying Lascar's inequality we get:

$$U(G_r) \leq U(a_0/K) \oplus \cdots \oplus U(a_{n-1}/K) < \omega^m i + \omega^{m-k}(r_k + 1),$$

where $\oplus$ denotes the Cantor sum of ordinals. This proves the other inequality.

In light of Theorem 3.7 and Lemma 3.9, one can see the following question as a stronger version of the Kolchin catenary problem for algebraic varieties:

Question 3.11. *Let $f : V \to \mathbb{A}^d$, where $d = dim\, V$, be a finite open map of irreducible affine algebraic varieties over K. Then, if f_Δ denotes f when regarded as a map of differential algebraic varieties, is f_Δ an open map?*

The following, seemingly open, question would yield, by quantifier elimination and Theorem 3.7, a positive answer to Question 3.11:

Question 3.12. *Let $f : V \to \mathbb{A}^d$, where $d = dim\, V$, be a finite open map of irreducible affine algebraic varieties over K. Let $f_\infty : V_\infty \to \mathbb{A}^d_\infty$ be the induced map on their prolongation spaces. Is f_∞ an open map?*

We do not know the answers to either of these questions in general; however, there is some evidence for the first one. We show that at least, in the context of Question 3.11, the image of every Δ-open set contains a Δ-open set. Let f and f_Δ be as in Question 3.11, and let U be a Δ-open subset of V. Then, the Lascar rank of U is $\omega^m \cdot d$ (where $d = dim\, V$). It follows, from Lascar inequality and the fact that f_Δ has finite fibres, that $f_\Delta(U)$ has Lascar rank $\omega^m \cdot d$. By quantifier elimination, $f_\Delta(U)$ is constructible and so it must contain a Δ-open set. The second question does not appear to be answered in the literature on arc spaces which is pertinent under the additional assumption that the variety V is defined over the constants.

Pong's solution [19] to the Kolchin catenary problem for algebraic varieties avoids the stronger form we have given here. Instead of asking about the general going-down property (for differential ideals) for the map coming from Noether normalization, Pong uses resolution of singularities to reduce the question to smooth varieties.

3.3. **A weaker form of the catenary conjecture.** Let $V \subseteq \mathbb{A}^n$ be an irreducible differential algebraic variety of positive dimension. Let $W \subseteq V$ be a zero-dimensional differential subvariety

Conjecture 3.13. *There is a proper irreducible differential algebraic subvariety V_1 of V such that $V_1 \cap W \neq \emptyset$ and $V_1 \not\subseteq W$.*

This conjecture is a very easy consequence of the catenary conjecture. Indeed, pick $a \in W$ and pick a long gap chain starting at a. Then since the Kolchin polynomials of the sets in the chain are not equal to each other, at some level, the irreducible closed sets in the chain can not be contained in the set W.

Remark 3.14. The conjecture seems inocuous; however, we do not know a proof. Also, by applying the conjecture repeatedly, one can assume that V_1 is also zero-dimensional.

4. Completeness and bounding the quantification

Let us start this section by recalling the definition of (differential) completeness.

Definition 4.1. A Δ-closed $V \subseteq \mathbb{P}^n$ is Δ-*complete* if the second projection

$$\pi_2 : V \times Y \to Y$$

is a Δ-closed map for every quasiprojective differential variety Y. Recall that a quasiprojective differential variety is simply an open subset of a projective differential variety.

We will simply say *complete* rather than Δ-complete. This will cause no confusion with the analogous term from the algebraic category because we will work exclusively in the category of differential algebraic varieties (except for remarks).

Historically, the first differential algebraic varieties for which completeness was established were the constant points of projective algebraic varieties [11]. One might attempt to establish a variety of examples via considering algebraic D-variety structures on projective algebraic varieties; this works, as the following result shows. We refer the reader to [5] for definitions and basic properties of algebraic D-varieties.

Lemma 4.2. *If (V, s) is an algebraic D-variety whose underlying variety V is projective then the sharp points $(V, s)^\#$ is complete.*

Proof. By a nontrivial result of Buium [7], (V, s) is isotrivial; that is, it is isomorphic (as a D-variety) to some (W, s_0) where W is an algebraic variety defined over the constants and s_0 is the zero section $s_0 : W \to TW$. Hence, $(V, s)^\#$ is isomorphic to a projective algebraic variety in the constants. The latter we know is complete, and hence $(V, s)^\#$ is complete. □

In [8], the following fact was established:

Fact 4.3. *Every complete differential variety is zero-dimensional.*

This implies that the completeness question in differential algebraic geometry is one which only makes sense to ask for zero-dimensional projective differential varieties. Thus, it seems natural to inquire whether the notion can be completely restricted to the realm of zero-dimensional differential varieties. A priori, the definition of completeness requires quantification over all differential subvarieties of the product of V with an arbitrary quasiprojective differential variety Y. In light of Fact 4.3, it seems logical to ask if one can restrict to zero-dimensional Y's for the purposes of verifying completeness (assuming the weaker form of the catenary conjecture, we prove this in Theorem 4.7 below). We believe this might help answering the following question:

Question 4.4. *Which projective zero-dimensional differential varieties are complete?*

In general the above question is rather difficult. As the following example shows, even at the level of Δ-type zero differential algebraic varieties one can find incomplete ones.

Example 4.5. Recently, the third author [22] constructed the first known example of a zero-dimensional projective differential algebraic variety which is not complete (in fact of differential type zero). Restrict to the case of a single derivation $(\mathcal{U}, \delta)$. Consider the subset W of $\mathbb{A}^1 \times \mathbb{A}^1$ defined by $x'' = x^3$ and $2yx^4 - 4y(x')^2 = 1$. One may check (by differentially homogenizing $x'' = x^3$ and observing that the point at infinity of $\mathbb{P}^1$ does not lie on the resulting variety) that $x'' = x^3$ is already a projective differential variety. Thus, W is a δ-closed subset of $\mathbb{P}^1 \times \mathbb{A}^1$. A short argument (see [22] for details) establishes that $\pi_2(W)$ is the set $\{y \mid y' = 0 \text{ and } y \neq 0\}$ which is not δ-closed.

Let us point out that this example works because the derivative of $2x^4 - 4(x')^2$ belongs to the δ-ideal generated by $x'' - x^3$ though $2x^4 - 4(x')^2$ itself does not. Since determining membership in a δ-ideal is often difficult, it is quite possible that identifying complete differential varieties is inherently a hard problem. This highlights the need for reductions (such as restricting to zero-dimensional second factors) in the process of checking completeness.

Our main tool to prove Theorem 4.7 below is the following differential version of Bertini's theorem, which appears in [9].

Fact 4.6. *Let $V \subseteq \mathbb{A}^n$ be an irreducible differential algebraic variety of dimension d with $d > 1$, and let H be a generic hyperplane of $\mathbb{A}^n$. Then $V \cap H$ is irreducible of dimension $d - 1$, and its Kolchin polynomial is given by*

$$\omega_{(V \cap H)}(t) = \omega_V(t) - \binom{t+m}{m}.$$

Theorem 4.7. *Assume Conjecture 3.13. Then, a Δ-closed $V \subseteq \mathbb{P}^n$ is complete if and only if $\pi_2 : V \times Y \to Y$ is a Δ-closed map for every quasiprojective zero-dimensional differential variety Y.*

Proof. Suppose that V is not complete, but the second projection $\pi_2 : V \times Y \to Y$ is a Δ-closed map for every quasiprojective zero-dimensional differential variety Y. Then, there must be some Y_1, a positive dimensional quasiprojective differential variety, such that $\pi_2 : V \times Y_1 \to Y_1$ is not Δ-closed. Because the question is local, we can assume that Y_1 is affine and irreducible. We will obtain a contradiction by finding Y_2 of dimension less than the dimension of Y_1 which witnesses incompleteness.

Fix a Δ-closed $X \subseteq V \times Y_1$ such that $\pi_2(X)$ is not closed. We may assume that X is irreducible and, because of our assumption, that $\pi_2(X)^{cl}$, the Δ-closure of $\pi_2(X)$, is positive dimensional. Let W be the Δ-closure of $\pi_2(X)^{cl} \setminus \pi_2(X)$. Since $DCF_{0,m}$ has quantifier elimination, W has strictly smaller Kolchin polynomial than $\pi(X)^{cl}$.

The first case that we consider is that when W is positive dimensional. Let H be a generic hyperplane (generic over a differentially closed field over which everything else is defined) in $\mathbb{A}^n$, where $Y_1 \subseteq \mathbb{A}^n$. By Fact 4.6, $W \cap H \neq \emptyset$. Now consider

$$X \cap V \times (Y_1 \cap H) \subseteq V \times (Y_1 \cap H).$$

We claim that $\pi_2(X \cap V \times (Y_1 \cap H)) = \pi_2(X) \cap H$ is not closed. Suppose it is, then, as $W \cap H \neq \emptyset$, $\pi_2(X) \cap H$ is a closed proper subset of $\pi_2(X)^{cl} \cap H$. The fact that $\emptyset \neq W \cap H \subset \pi_2(X)^{cl} \cap H$ (which follows from Fact 4.6 and noting that intersections with generic hyperplane sections behave predictably with respect to Kolchin polynomials) contradicts irreducibility of $\pi_2(X)^{cl} \cap H$. Thus, $\pi_2(X) \cap H$ is not closed. Further, by Fact 4.6 again, the dimension of $Y_1 \cap H$ is one less that the dimension of Y_1.

The remaining case occurrs when W is zero-dimensional. In this case, apply Conjecture 3.13 to obtain a zero-dimensional and irreducible X_1 which intersects W and $\pi_2(X)^{cl}$ nontrivially. We then obtain a contradiction as before by considering $Y_1 \cap X_1$ and X_1 instead of $Y_1 \cap H$ and $\pi_2(X)^{cl} \cap H$, respectively.

□

5. Linear dependence over differential algebraic varieties

We begin this section by giving a natural definition of linear dependence over an arbitrary projective differential variety.

Definition 5.1. Let $V \subseteq \mathbb{P}^n$ be a projective differential algebraic variety. We say that $\bar{a} = (a_0, \dots, a_n) \in \mathbb{A}^{n+1}$ is *linearly dependent over* V if there is a point $v = [v_0 : \dots : v_n] \in V$ such that $\sum_{j=0}^n v_i x_i = 0$. Similarly, we say that $\alpha \in \mathbb{P}^n$ is linearly dependent over V if there is a representative $\bar{a}$ of α such that $\bar{a}$ is linearly dependent over V. Note that it does not matter which particular representative of α or v we choose when testing to see if v witnesses the V-linear dependence of α.

In [11], Kolchin states the following problem:

Example 5.2. Consider an irreducible algebraic variety V in $\mathbb{P}^n(\mathbb{C})$ for some n. Let $f_0, f_1, \dots, f_n$ be meromorphic functions in some region of $\mathbb{C}$. Ritt once remarked (but seems to have not written down a proof) that there is an ordinary differential polynomial $R \in \mathbb{C}\{y_0, y_1, \dots, y_n\}$ which depends only on V and has order equal to the dimension of V such that a necessary and sufficient condition that there is $c \in V$ with $\sum c_i f_i = 0$ is that $(f_0, f_1, \dots, f_n)$ be in the general solution of the differential equation $R(y_0, y_1, \dots, y_n) = 0$. For a more thorough discussion, see [11].

Question 5.3. *In the previous example, can one replace the algebraic variety V with an arbitrary complete differential algebraic variety?*

Kolchin offers a general solution to the problem by using the property of completeness for differential algebraic varieties; more specifically, the constant points of projective algebraic varieties. The fact that projective algebraic varieties (viewed as zero-dimensional differential algebraic varieties; i.e., inside the constants) are complete in the Kolchin topology turns out to be the key to proving the existence of the above differential polynomial R.

5.1. **A Generalization of Ritt's assertion and Kolchin's proof.** We will extend Kolchin's line of reasoning for proving the assertion in Example 5.2, in the more general context of our above Question 5.3. Namely, we start with a complete differential algebraic variety, rather than the constant points of an algebraic variety.

Theorem 5.4. *Let $V \subset \mathbb{P}^n$ be a complete differential algebraic variety defined and irreducible over the differential field K. Let*

$$ld(V) := \{x \in \mathbb{P}^n \,|\, x \text{ is linearly dependent over } V\}.$$

Then $ld(V)$ is an irreducible differential algebraic subvariety of $\mathbb{P}^n$ defined over K.

Remark 5.5. With the correct hypotheses, Kolchin's proof of the special case essentially goes through here. Similar remarks apply to the strategy of the next proposition and corollary, where Kolchin's original argument provides inspiration. For the proposition, the result seems to require a few new ingredients, mainly doing calculations in the generic fiber of the differential tangent bundle.

Proof. Let $\mathfrak{p} \subseteq K\{\bar{z}\} = K\{z_0, z_1, \dots, z_n\}$ be the differential ideal corresponding to V. Now, let $\bar{y} = (y_0, y_1, \dots y_n)$ and consider the differential ideal $\mathfrak{p}_1 \in K\{\bar{z}, \bar{y}\}$ given by

$$\left[\mathfrak{p}, \sum_{j=0}^{n} y_j z_j\right] : (\bar{y}\bar{z})^\infty$$

which by definition is

$$\left\{ f \in K\{\bar{z}, \bar{y}\} : \, (y_i z_j)^e f \in \left[\mathfrak{p}, \sum y_j z_j\right], 0 \le i, j \le n, \text{ for some } e \in \mathbb{N} \right\}.$$

As $\mathfrak{p}_1$ is differentially bi-homogeneous, it determines a (multi-)projective differential algebraic variety $W \subseteq \mathbb{P}^n \times \mathbb{P}^n$. It is clear that the coordinate projection maps have the form

$$\begin{tikzcd} & W \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V & & ld(V) \end{tikzcd}$$

Further, $ld(V) = \pi_2(W)$, and, since V is complete, $ld(V)$ is closed in the Kolchin topology of $\mathbb{P}^n$ and defined over K.

Next we prove that $ld(V)$ is irreducible over K. Let $\bar{a} = (a_0, \dots, a_n)$ be a representative of a generic point of V over K and fix j such that $a_j \neq 0$. Pick elements $u_k \in \mathcal{U}$ for $0 \leq k \leq n$ and $k \neq j$ which are differential algebraic independent over $K\langle \bar{a} \rangle$. Let

$$u_j = -\sum_{k \neq j} u_k a_j^{-1} a_k,$$

and $\bar{u} = (u_0, \dots, u_n)$. One can see that $(\bar{a}, \bar{u})$ is a representative of a point in $W \subseteq \mathbb{P}^n \times \mathbb{P}^n$, so that $[u_0 : \cdots : u_n] \in ld(V)$.

We claim that $[u_0 : \cdots : u_n]$ is a generic point of $ld(V)$ over K (this will show that $ld(V)$ is irreducible over K). To show this it suffices to show that $(\bar{a}, \bar{u})$ is a generic point of $\mathfrak{p}_1$; i.e., it suffices to show that for every $p \in K\{\bar{z}, \bar{y}\}$ if $p(\bar{a}, \bar{u}) = 0$ then $p \in \mathfrak{p}_1$.

Let $p \in K\{\bar{z}, \bar{y}\}$ be any differential polynomial. By the differential division algorithm there exists $p_0 \in K\{\bar{z}, \bar{y}\}$ not involving y_j such that

$$z_j^e p \equiv p_0 \quad mod \left[\sum_{0 \leq i \leq n} y_i z_i \right]$$

for some $e \in \mathbb{N}$. Thus, we can write p_0 as a finite sum $\sum p_M M$ where each M is a differential monomial in $(y_k)_{0 \leq k \leq n, k \neq j}$ and $p_M \in K\{\bar{z}\}$. Now, as $(u_k)_{0 \leq k \leq n, k \neq j}$ are differential algebraic independent over $K\langle \bar{a} \rangle$, it follows that if $p(\bar{a}, \bar{u}) = 0$ then $p_M(\bar{a}) = 0$ for all M (and hence $p_M \in \mathfrak{p}$, since $\bar{a}$ is a generic point of $\mathfrak{p}$). This implies that, if $p(\bar{a}, \bar{u}) = 0$, $p_0 \in \mathfrak{p} \cdot K\{\bar{z}, \bar{y}\}$ and so $p \in \mathfrak{p}_1$, as desired.

□

For the following proposition we will make use of the following fact of Kolchin's about the Kolchin polynomial of the differential tangent space (we refer the reader to [Kolchin's 2nd book] for the definition and basic properties of differential tangent spaces).

Fact 5.6. *Let V be an irreducible differential algebraic variety defined over K with generic point $\bar{v}$. Then*

$$\omega_{T_{\bar{v}}^{\Delta} V} = \omega_V,$$

where $T_{\bar{v}}^{\Delta} V$ denotes the differential tangent space of V at $\bar{v}$.

In the case when the complete differential algebraic variety V has Δ-type zero, we have the following result on the Kolchin polynomial of $ld(V)$.

Proposition 5.7. *Let $V \subset \mathbb{P}^n$ be a complete differential algebraic variety defined and irreducible over K. If V has constant Kolchin polynomial equal to d, then the Kolchin polynomial of $ld(V)$ is given by*

$$\omega_{ld(V)}(t) = (n-1)\binom{t+m}{m} + d.$$

Remark 5.8. In the ordinary case, the hypotheses of this proposition do not constitute any assumption on V. In that case, every such complete V is finite dimensional. In the partial case, the situation is much less clear. It is not known whether every complete differential algebraic variety has constant Kolchin polynomial (see [8] for more details).

Proof. Let $W \subseteq \mathbb{P}^n \times \mathbb{P}^n$, $\bar{a}$ and $\bar{u}$ be as in the proof of Theorem 5.4. Fix j such that $a_j \neq 0$ and, moreover, assume that $a_j = 1$. Now, write $\bar{a}^*$ and $\bar{u}^*$ for the tuples obtained from $\bar{a}$ and $\bar{u}$, respectively, where we omit the j-th coordinate. Let $W_1 \subseteq \mathbb{A}^n \times \mathbb{A}^{n+1}$ be the differential algebraic variety with generic point $(\bar{a}^*, \bar{u})$ over K. Consider $T^{\Delta}_{(\bar{a}^*,\bar{u})}W_1$, the differential tangent space of W_1 at $(\bar{a}^*, \bar{u})$. Let $(\bar{\alpha}, \bar{\eta})$ be generic of $T^{\Delta}_{(\bar{a}^*,\bar{u})}W_1$ over $K\langle \bar{a}^*, \bar{u}\rangle$.

From the equation $y_j = -\sum_{k\neq j} y_k z_k$ satisfied by $(\bar{a}^*, \bar{u})$, we see that

$$-\sum_{i\neq j} a_i\eta_i - \eta_j = \sum_{i\neq j} u_i\alpha_i.$$

Choose d_1 so that $|\Theta(d_1)|$ is larger than $n \cdot |\Theta(d)|$, where $\Theta(d_1)$ denotes the set of derivative operators of order at most d_1 (similarly for $\Theta(d)$).

For $\theta \in \Theta(d_1)$, we have that

$$\theta\left(-\sum_{i\neq j} a_i\eta_i - \eta_j\right) = \theta\left(\sum_{i\neq j} u_i\alpha_i\right).$$

In the expression on the right $\theta\left(\sum_{i\neq j} u_i\alpha_i\right)$ consider the coefficients of $\theta\alpha_i$ in terms of $\theta\bar{u}^*$, and denote these coefficients by $f(i,\theta',\theta) \in K\langle \bar{u}^*\rangle$; i.e., $f(i,\theta',\theta)$ is the coefficient of $\theta'\alpha_i$ in the equation

$$\theta\left(-\sum_{i\neq j} a_i\eta_i - \eta_j\right) = \theta\left(\sum_{i\neq j} u_i\alpha_i\right).$$

We can thus express this equation in the form

$$\left(\ \theta\left(-\textstyle\sum_{i\neq j} a_i\eta_i - \eta_j\right)\ \right) = F_\theta A$$

where F_θ is the row vector with entries

$$(f(i,\theta',\theta))_{i\neq j,\theta'\in\Theta(d)}$$

and A is the column vector with entries

$$\left(\ \theta'\alpha_i\ \right)_{i\neq j,\theta'\in\Theta(d)}$$

Note that we only have the vectors F_θ and A run through $\theta'\in\Theta(d)$ (instead of all of $\Theta(d_1)$). This is because any derivative of α_i, $i\neq j$, of order higher than d can be expressed as a linear combination of the elements of the vector A. This latter observation follows from the fact that $\omega_{\alpha/K\langle\bar{a}^*,\bar{u}\rangle}=d$, which in turn follows from the facts that α is a generic point of $T^{\Delta}_{\bar{a}^*}V$, our assumptioin on ω_V, and Fact 5.6.

By the choice of d_1 and $\bar{u}^*$ (recall that $\bar{u}^*$ consists of independent differential transcendentals over $K\langle\bar{a}\rangle$), there are $n\cdot|\Theta(d)|$ linearly independent row vectors F_θ. So, we can see (by inverting the nonsingular matrix which consists of $n\cdot|\Theta(d)|$ such F_θ as the rows) that all the elements of the vector A belong to $K\langle\bar{a}^*,\bar{u}\rangle((\theta\bar{\eta})_{\theta\in\Theta(d_1)})$. Thus

$$K\langle\bar{a}^*,\bar{u}\rangle((\theta\bar{\eta})_{\theta\in\Theta(d_1)})=K\langle\bar{a}^*,\bar{u},\bar{\alpha}\rangle((\theta\eta_i)_{i\neq j,\theta\in\Theta(d_1)}).$$

Noting that $\bar{\eta}^*=(\eta_i)_{i\neq j}$ is a tuple of independent transcendentals over $K\langle\bar{a}^*,\bar{u}\rangle$ and that $\bar{\eta}^*\mathop{\smile\!\!\!\!\!\mid}_{K\langle\bar{a}^*,\bar{u}\rangle}\bar{\alpha}$, the above equality means that for all large enough values of t,

$$\omega_{\bar{\eta}/K\langle\bar{a}^*,\bar{u}\rangle}(t)=n\binom{m+t}{t}+d.$$

Finally, since $\bar{\eta}$ is a generic of the differential tangent space at $\bar{u}$ of the K-locus of $\bar{u}$, Fact 5.6 implies that

$$\omega_{\bar{u}/K}=n\binom{m+t}{t}+d, \tag{2}$$

and thus by equation (1) (in Section 2.1), we get

$$\omega_{ld(V)}=(n-1)\binom{m+t}{t}+d,$$

as desired. □

Corollary 5.9. *Let $V\subset\mathbb{P}^n$ be a complete differential algebraic variety defined and irreducible over K, and suppose we are in the ordinary case (i.e., $|\Delta|=1$). If V has constant Kolchin polynomial equal to d, then there exists a unique (up to a nonzero factor in K) irreducible $R\in K\{\bar{y}\}$ of order d such that an element in $\mathbb{A}^{n+1}$ is linearly independent over V if and only if it is in the general solution of the differential equation $R=0$.*

Proof. Let $\mathfrak{p}$ be the differential (homogeneous) ideal of $ld(V)$ over K. Then, by equation (2) in the proof of Proposition 5.7, we get

$$\omega_{\mathfrak{p}} = n(t+1) + d = (n+1)\binom{t+1}{1} - \binom{t-d+1}{1}.$$

Thus, by [12, Chapter IV, §7, Proposition 4], there exists an irreducible $R \in K\{\bar{y}\}$ of order d such that $\mathfrak{p}$ is precisely the general component of R; in other words, an element in $\mathbb{A}^{n+1}$ is linearly independent over V if and only if it is in the general solution of the differential equation $R = 0$. For uniqueness, let R' be another differential polynomial over K of order d having the same general component as R. Then, by [12, Chapter IV, §6, Theorem 3(b)], R' is in the general component of R and so $ord(R') \geq ord(R)$. By symmetry, we get $ord(R) = ord(R')$, and thus R and R' divide each other, as desired. $\square$

Now we discuss how the assertion of Example 5.2 follows from the results of this section.

Definition 5.10. Let V be a differential algebraic variety defined over k and K a differential field extension of k. We say V is k-large with respect to K if $V(k) = V(\bar{K})$ for some (equivalently for every) $\bar{K}$ differential closure of K.

One can characterize the notion of largeness in terms of differential algebraic subvarieties of V:

Lemma 5.11. *V is k-large with respect to K if and only if for each differential algebraic subvariety W of V, defined over K, $W(k)$ is Kolchin-dense in W.*

Proof. Suppose $W(k)$ is Kolchin-dense in W, for each differential algebraic subvariety W of V defined over K. Let $\bar{a}$ be a $\bar{K}$-point of V. Since $tp(a/K)$ is isolated, there is a differential polynomial $f \in K\{\bar{x}\}$ such that every differential specialization $\bar{b}$ of $\bar{a}$ over K satisfying $f(\bar{b}) \neq 0$ is a generic specialization. Let $W \subseteq V$ be the differential locus of $\bar{a}$ over K. By our assumption, there is a k-point $\bar{b}$ of W such that $f(\bar{b}) \neq 0$. Hence, $\bar{b}$ is a generic differential specialization of $\bar{a}$ over K, and so $\bar{a}$ is a k-point.

The converse is clear since for every differential algebraic variety W, defined over $\bar{K}$, $W(\bar{K})$ is Kolchin-dense in W. $\square$

Remark 5.12. Let V be an (infinite) algebraic variety in the constants defined over k, and K a differential field extension of k such that $K^{\Delta} = k^{\Delta}$. Here K^{Δ} and k^{Δ} denote the constants of K and k, respectively.

(1) V is k-large with respect to K if and only if k^{Δ} is algebraically closed. Indeed, if V is k-large, the image of $V(k)$ under any of the Zariski-dominant coordinate projections of V is dense in $\bar{K}^{\Delta}$. Hence, $k^{\Delta} = \bar{K}^{\Delta}$, implying k^{Δ} is algebraically closed. Conversely, if k^{Δ} is algebraically closed, then $k^{\Delta} = \bar{K}^{\Delta}$ (since $k^{\Delta} = K^{\Delta}$). Hence, $V(k^{\Delta}) = V(\bar{K}^{\Delta})$, but since V is in the constants we get $V(k) = V(\bar{K})$.

(2) In the case $k = \mathbb{C}$ and K is the field of meromorphic in some region of $\mathbb{C}$. We have that $K^\Delta = \mathbb{C}$ is algebraically closed, and so V is k-large with respect to K. Hence, in Example 5.2 the largeness condition of V is given implicitly.

Lemma 5.13. *Let $V \subset \mathbb{P}^n$ be a complete differential algebraic variety defined over k, and suppose V is k-large with respect to K. Then the K-points of $\mathbb{P}^n$ that are linearly dependent over V are precisely those that are linearly dependent over $V(k)$.*

Proof. Suppose $a \in \mathbb{P}^n(K)$ is linearly independent over V. Then, since the models of $DCF_{0,m}$ are existentially closed, we can find $v \in V(\bar{K})$, where $\bar{K}$ is a differential closure of K, such that $\sum v_i a_i = 0$. But, by our largeness assumption, $V(\bar{K}) = V(k)$, and thus a is linearly independent over $V(k)$. □

Putting together Theorem 5.4, Remark 5.12, and Lemma 5.13, we see that if V is an irreducible algebraic variety in $\mathbb{P}^n(\mathbb{C})$ and K is a differential field extension of $\mathbb{C}$ with no new constants then there is a projective differential algebraic variety defined over $\mathbb{C}$, namely $ld(V)$, which only depends on V such that for any tuple $f = (f_0, f_1, \dots, f_n)$ from K, f is linearly dependent over V if and only if $f \in ld(V)$.

5.2. **Generalized Wronskians.** It is well-known that a finite collection of meromorphic functions (on some domain of $\mathbb{C}$) is linearly dependent over $\mathbb{C}$ if and only if its Wronskian vanishes. Roth [21] generalized (and specialized) this fact to rational functions in several variables using a generalized notion of the Wronskian. This result was later generalized to the analytic setting [2],[25]. In this section, we will show how generalizations of these results are easy consequences of developments on differential completeness.

Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m_{\geq 0}$, and $|\alpha| = \sum \alpha_i$. Let $f_0, f_1, \dots, f_n \in K$. Fix the (multi-index) notation $\delta^\alpha = \delta_1^{\alpha_1} \dots \delta_m^{\alpha_m}$, and $A = (\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(n)}) \in (Z^m_{\geq 0})^{n+1}$. We call

$$\mathcal{W}_A := \left| \begin{pmatrix} \delta^{\alpha^{(0)}} f_0 & \delta^{\alpha^{(0)}} f_1 & \dots & \delta^{\alpha^{(0)}} f_n \\ \delta^{\alpha^{(1)}} f_0 & \delta^{\alpha^{(1)}} f_1 & \dots & \delta^{\alpha^{(1)}} f_n \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{\alpha^{(n)}} f_0 & \delta^{\alpha^{(n)}} f_1 & \dots & \delta^{\alpha^{(n)}} f_n \end{pmatrix} \right|$$

the Wronskian associated with A.

Theorem 5.14. *If $V = \mathbb{P}^n(\mathcal{U}^\Delta)$, the constant points of $\mathbb{P}^n$, then the projective differential algebraic variety $ld(V)$ is equal to the zero set in $\mathbb{P}^n$ of the collection of generalized Wronskians (i.e., as the tuple A varies).*

Remark 5.15. In [24], Walker determined the least subcollection of generalized Wronskians whose vanishing is sufficient for the linear dependence result. Those interested in the history of the generalized Wronskian result are advised to consult [24], where earlier partial results by Roth and others are discussed. Also, we refer the reader to [12, Chapter II, §1, Theorem 1] for a standard proof of the Wronskian theorem.

Proof. The vanishing of the generalized Wronskians is obviously a necessary condition for linear dependence. We now show that it is also sufficient.

By Theorem 5.7, we know that the differential (homogeneous) ideal

$$\mathfrak{r} \subseteq K\{y_0, \dots, y_n\}$$

of $ld(V)$ has Kolchin polynomial $\omega_{\mathfrak{r}} = \binom{t+m}{m}(n) + n$. Thus, it suffices to show that the zero set in $\mathbb{A}^{n+1}$ of the collection of generalized Wronskians is irreducible with Kolchin polynomial $\binom{t+m}{m}(n) + n$.

The unique generic type of the zero set of the collection of generalized Wronskians is given by choosing n independent differential transcendentals $u_0, \dots, u_{n-1}$, any tuple of independent algebraic transcendentals $\bar{c} \in (\mathcal{U}^\Delta)^{n+1}$, and chosing u_n so that $\sum_{i=0}^n c_i u_i = 0$. This is the unique generic type: if $v_0, \dots, v_{n-1}$ are linearly dependent over $\mathcal{U}^\Delta$, then the coefficients $\bar{c}^*$ witnessing this linear dependence can be chosen to be (differential) specializations of $c_0, \dots, c_{n-1}$, since we chose the c_i's to be independent transcendentals. Then any such chosen $\bar{v}$ is a differential specialization of $\bar{u}$. If $v_0, \dots v_{n-1}$ are linearly independent over $\mathcal{U}^\Delta$, and v_n is chosen via some choice $\bar{c}^*$ of non-independent algebraic transcendentals, then again we can see that $\bar{v}$ is a differential specialization of $\bar{u}$, since $\bar{c}^*$ is a specialization of $\bar{c}$. Only irreducible differential algebraic varieties have unique generic types, and we have given the unique generic type since it specializes to any point contained in the zero set of the collection of generalized Wronskians. Thus, the zero set of the collection of generalized Wronskians is irreducible in the Kolchin topology.

Further, the Kolchin polynomial of this type is at least $\binom{t+m}{m}(n) + n$, since the vanishing of the generalized Wronskians is obviously a necessary condition for linear dependence. To see that a generic point $\bar{u}$ of the zero set of the collection of generalized Wronskians has Kolchin polynomial at most $\binom{t+m}{m}(n) + n$, consider that for a tuple with Kolchin polynomial at least $\binom{t+m}{m}(n) + n$, there must be n coordinates which are indepedent differential transcendentals. Without loss, let these be $u_0, \dots u_{n-1}$. Then one can see by the generalized Wronskian conditions that for any of the derivations, $\delta \in \Delta$ and any $\theta \in \Theta \backslash \{id, \delta, \dots \delta^{n-1}\}$, $\theta u_n \in K\langle u_0, \dots, u_{n-1}\rangle(u_n, \dots, \delta^{n-1} u_n)$. Thus, the Kolchin polynomial of $\bar{u}$ is bounded by $\binom{t+m}{m}(n) + n$, as desired. $\square$

For $k \in \mathbb{Z}_{>0}$, let $\mathcal{T}(k) = \{\alpha \in \mathbb{Z}^m_{\geq 0} \,|\, |\alpha| < k\}$. Let $\mathcal{G}$ be the collection of subsets S of $\mathbb{Z}^m_{\geq 0}$ with $|S| = n$ for some n and $|S \cap \mathcal{T}(k)| \geq k$ for all $1 \leq k \leq n$.

We will be considering the partial order given by $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for each j. A subset $S \subset (Z^m_{\geq 0})^{n+1}$ is called *Young-like* if $\alpha \in S, \beta \in \mathcal{T}$ and $\beta \leq \alpha$ imply that $\beta \in S$ (see [24] for additional details). We will fix n and let $\mathcal{Y}$ be the set of all Young-like sets of cardinality n. There are computational advantages to working with Young-like sets, since the full set of generalized Wronskians grows much faster in (m, n) (where m is the number of derivations and n is the number of functions). Even for small values of (m, n) the difference is appreciable [24, for specifics on the growth of $|\mathcal{Y}|$].

The following lemma is due to Walker [24, Section 3, Lemma 2]:

Lemma 5.16. *The vanishing of the collection of generalized Wronskians is equivalent to the vanishing of the subcollection of those contained in* $\mathcal{Y}$.

The following corollaries show why our results are essentially generalizations of [2, Theorem 2.1], [24, Theorem 3.1] and [25]:

Corollary 5.17. *Let* k *be a differential subfield of* K *and* $f_0, \ldots, f_n \in K$. *If* $\mathbb{P}^n(\mathcal{U}^\Delta)$ *is* k*-large with respect to* K, *then* $f_0, \ldots, f_n$ *are linearly dependent over* $\mathbb{P}^n(k^\Delta)$ *if and only if the subcollection of generalized Wronskians contained in* $\mathcal{Y}$ *vanish on* $(f_0, \ldots, f_n)$.

Since $\mathbb{P}^n(\mathcal{U}^\Delta)$ is $\mathbb{C}$-large with respecto to any field of meromorphic functions on some domain of $\mathbb{C}^m$, we have

Corollary 5.18. *The vanishing of the collection of generalized Wronskians associated with Young-like sets* $Y \in \mathcal{Y}$ *is a necessary and sufficient condition for the linear dependence (over* $\mathbb{C}$*) of a finite collection of meromorphic functions on some domain of* $\mathbb{C}^m$.

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