Elimination of superfluous constants in the language of quasi-analytic classes and description of prime models in these classes

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Abstract. Let $\mathcal{E}$ be a set of real restricted quasi-analytic functions ; we describe a language $\mathcal{L}$ definable from $\mathcal{E}$, $+$, $\times$ and $<$, and a $\mathcal{L}$-theory T, universal and "explicit", such that T is equivalent to the complete theory of $\mathbb{R}$ in $\mathcal{L}$ and T is model-complete. As a consequence, we give a description of the prime model of this theory.

1 Introduction

This article is motivated by the results obtained in [Ra1], and condensed in [Ra2], that we can summarize in this way : let $\mathcal{F}$ be a set of restricted maps from $\mathbb{R}^n$ to $\mathbb{R}$, ie which vanish outside a certain compact box and C^∞ on this same box (this box is not necessarily the same for each map). Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F} \cup \mathbb{R}$ by sums, products, compositions, implicit functions and factorizations by monomials ; in the continuation we only consider sets $\mathcal{F}$ such that $\overline{\mathcal{F}}$ is a quasi-analytic class. In these conditions, we describe a theory T, universal and "explicit", in the language $\mathcal{L}$ of ordered fields with rational powers, enlarged by a symbol of function for each element of $\overline{\mathcal{F}}$; so $\mathcal{L}$ has a symbol of constant for each element of $\mathbb{R}$. We prove at the same time that T is equivalent to the complete theory of $\mathbb{R}$ and that T is model-complete, therefore admits quantifier elimination. (This result is detailed in theorem **??** of this article.)
Consequences are, for example, the o-minimality of the complete theory of $\mathbb{R}$ (proved initially in [RSW]), the cellular decomposition in cells locally C^∞ and the existence of a preparation theorem, in the way of Lion-Rolin (cf. [LR]), for the considered classes.

One of the remarks that we can make is that the language $\mathcal{L}$ potentially uses "too much" constants. Indeed $\mathcal{L}$ contains a constant for each element of $\mathbb{R}$ and it is clear that in many cases, notably if $\mathcal{F}$ is countable, a lot of elements of $\mathbb{R}$ are not definable from $\mathcal{F}$, $+$, $\times$ and $<$.
So the aim of this article is to explicit, for each $\mathcal{F}$, a "minimum" language $\mathcal{L}'$, ie every element of $\mathcal{L}'$ is definable from $\mathcal{F}$, $+$, $\times$ and $<$, and an universal $\mathcal{L}'$-theory T', such that T' is still equivalent to the complete theory of $\mathbb{R}$ in $\mathcal{L}'$ and is still model-complete. The theorem 4.7 establishes this result.

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We obtain as a consequence the corollary 4.8 which describes the prime model of the complete theory $\mathbb{R}$ in $\mathcal{L}'$ and in weaker languages. A particular case of this result deals with restricted analytic classes and notably with the restricted exponential : so we have an "explicit" description of the prime model of $\mathbb{R}$ in the language of ordered rings enlarged by a restricted exponential. (See [vdD2] and [vdDMM] for more details about these classes; [W1] and [W2] can be also consulted to go thoroughly into the study of the real exponential and the restricted real exponential)

We suppose that the reader is familiar to the o-minimality, the basic concepts of quasi-analycity and the results of [Ra1]. We can consult for example [vdD1] or [C] for an introduction to o-minimality (and to real geometry in [C]) and [VP] for a presentation of quasi-analycity.

2 Quasi-analytic classes

Notation 2.1 . *Let $A \subset \mathbb{R}$, $B \subset \mathbb{R}^n$ and $(a_1, ..., a_n) \in \mathbb{R}^n$; in this article, we will use the following notations.*
$A^* = \{x \in A; x \neq 0\}$; $A_+ = \{x \in A; x \geq 0\}$; $A_- = \{x \in A; x \leq 0\}$.
$\overset{\circ}{B}$ *will refer to the interior of B for the topology of $\mathbb{R}^n$.*
$(a_1, ..., a_n) + B$ *will refer to the set* $\{(a_1 + x_1, ..., a_n + x_n); (x_1, ..., x_n) \in B\}$.

Definition 2.2 . *(basic box) Let $n \in \mathbb{N}^*$ and $A \subset \mathbb{R}$; P is a n-basic box over A if there exist two maps $\alpha : \{1, ..., n\} \to (A \cup \mathbb{Q}) \cap \mathbb{R}^*_-$ et $\beta : \{1, ..., n\} \to (A \cup \mathbb{Q}) \cap \mathbb{R}^*_+$ such that $P = \prod_{i=1}^n [\alpha(i), \beta(i)]$.*
The set $\alpha(\{1, ..., n\}) \cup \beta(\{1, ..., n\})$ will be noted $\mathfrak{B}(P)$.
A n-basic box is a n-basic box over $\mathbb{R}$.
If P is a n-basic box, 1_P will refer to the indicator function of P, ie 1_P is the map from $\mathbb{R}^n$ to $\{0, 1\}$ such that if $x \in P$, $1_P(x) = 1$, otherwise $1_P(x) = 0$.

Definition 2.3 . *Let $n \in \mathbb{N}^*$ and P be a n-basic box; we note $\mathcal{Z}(P)$ the set of maps from $\mathbb{R}^n$ to $\mathbb{R}$ which vanish outside P.*
We note $\mathcal{Z}^\infty(P)$ the subset of $\mathcal{Z}(P)$ such that its elements are the maps C^∞ on P, ie C^∞ on the interior of P and whose all partial derivatives have a limit at the boundary of P.
We finally note $\mathcal{Z}^\infty$ the union of $\mathbb{R}$ and of all the sets $\mathcal{Z}^\infty(P)$.

Definition 2.4 . *Let f be a map from $\mathbb{R}^n$ to $\mathbb{R}$ and P be a n-basic box; in this article, we will note $f\rceil_P$ the map from $\mathbb{R}^n$ to $\mathbb{R}$ which is equal to f on P and which vanishes outside P.*
By language abuse, we will say that f is defined on P if f vanishes outside P and is C^∞ -in the sense specified above- on P.

Definition 2.5 . *We note L_{fct} the language which contains a symbol of constant for each element of $\mathbb{R}$ and, for $n \in \mathbb{N}^*$, a symbol of n-ary function for each total map from $\mathbb{R}^n$ to $\mathbb{R}$.*
If c belongs to $\mathbb{R}$ (resp. if f is a map from $\mathbb{R}^n$ to $\mathbb{R}$), we note $^*c$ (resp. $^*f$) the symbol of L_{fct} which corresponds; if $s \in L_{fct}$, we note $\overline{s}$ the interpretation of s in $\mathbb{R}$. Thus if c belongs to $\mathbb{R}$ (res. if f is a map from $\mathbb{R}^n$ to $\mathbb{R}$), $\overline{({}^*c)} = c$ (resp. $\overline{({}^*f)} = f$).

Notation 2.6 . *Let $q \in \mathbb{Q}^*$, we will note π_q the map from $\mathbb{R}$ to $\mathbb{R}$ which associates x^q to x when x^q is defined, 0 otherwise.*
For example, if $x \geq 0$, $\pi_{\frac{1}{2}}(x) = \sqrt{x}$ and if $x < 0$, $\pi_{\frac{1}{2}}(x) = 0$; whereas $\pi_{\frac{1}{3}}(x) = x^{\frac{1}{3}}$ for every $x \in \mathbb{R}$. In the same way , if $x \in \mathbb{R}^*$, $\pi_{-1}(x) = \frac{1}{x}$ and $\pi_{-1}(0) = 0$.

In the continuation, we fix a set $\mathcal{E}$ included in $\mathcal{Z}^\infty$ and A a subset of $\mathbb{R}$.

We define now a certain type of L_{fct}-terms, the regular terms, which will be those used in this article.

Definition 2.7 . *(regular terms) Let $\mathcal{F} \subset \mathcal{Z}^\infty$ and $B \subset \mathbb{R}$; C will be the union of B, of $\mathbb{Q}$ and of $\mathcal{F} \cap \mathbb{R}$. All variables will be noted X_i ($i \in \mathbb{N}^*$); so the countable set $\mathcal{V}$ of the variables is $\{X_i; i \in \mathbb{N}^*\}$. The order of the variables indexing is important.*

We define by induction the set $RT[\mathcal{F}, B](P)$ and $RT_0[\mathcal{F}, B]$, where P is a n-basic box ($n \in \mathbb{N}^*$), in the following way. At the beginning we suppose that all the sets $RT[\mathcal{F}, B](P)$ and $RT_0[\mathcal{F}, B]$ are empty.
Let $n \in \mathbb{N}^*$, P be a n-basic box, $t \in RT[\mathcal{F}, B](P)$, $(a, b) \in RT_0[\mathcal{F}, B]^2$ and σ a strictly increasing map from $\{1, ..., n\}$ to $\mathbb{N}^*$.

1) *(stability by elements of $\mathcal{F}$ and of C)*

If $c \in C$, ${}^*c \in RT_0[\mathcal{F}, B]$. If $f \in \mathcal{F} \cap \mathcal{Z}^\infty(P)$, the term ${}^*f(X_{\sigma(1)}, ..., X_{\sigma(n)})$ belongs to $RT[\mathcal{F}, B](P)$

2) *(stability by permutations and increasing changes of variables)*

if θ is a permutation of $\{1, ..., n\}$, then

$$t(X_{\theta(1)}, ..., X_{\theta(n)}) \in RT[\mathcal{F}, B](P_\theta)$$

where $P_\theta = \{(u_{\theta(1)}, ..., u_{\theta(n)}) \in \mathbb{R}^n; (u_1, ..., u_n) \in P\}$. So P_θ is the permutation of the coordinates of P by θ.
Moreover, $s(X_{\sigma\circ\theta(1)}, ..., X_{\sigma\circ\theta(n)}) \in RT[\mathcal{F}, B](P_\theta)$ if and only if $s(X_{\theta(1)}, ..., X_\theta(n)) \in RT[\mathcal{F}, B](P_\theta)$.

3) *(stability by indicator functions and variables truncated to a basic box)*

If Q is a n-basic box on C, ${}^*1_Q(X1, ..., X_n) \in RT[\mathcal{F}, B](Q)$. In the same way, for $i \in \{1, ..., n\}$, $X_i \, {}^*\!\times \, {}^*1_Q(X_1, ..., X_n)$ is in $RT[\mathcal{F}, B](Q)$

4) *(stability by enlargement)*

If for $p \in \mathbb{N}^*$ and Q a p-basic box over C, h_Q is the map from $\mathbb{R}^{n+p}$ to $\mathbb{R}$, such that for every $(u_1, ..., u_{n+p}) \in P \times Q$, $h_Q(u_1, ..., u_n, u_{n+1}, ..., u_{n+p}) = \overline{t}(u_1, .., u_n)$, and 0 outside $P \times Q$, then ${}^*h_Q \in RT[\mathcal{F}, B](P \times Q)$

5) *(stability of $RT_0[\mathcal{F}, B]$ by field operations and by rational powers)*

If $q \in \mathbb{Q}^*$, the terms $a\ {}^*{+}b$, $a\ {}^*{\times}b$, ${}^*\pi_q(a)$ are in $RT_0[\mathcal{F}, B]$

6*) (stability of $RT_0[\mathcal{F}, B]$ by composition by regular terms)*

If $(a_1, ..., a_n) \in RT_0[\mathcal{F}, B]^n$ and if $s \in RT[\mathcal{F}, B](Q)$, with Q a m-basic box $(m \geq n)$, then :

- if $m = n$, $s(a_1, ..., a_n) \in RT_0[\mathcal{F}, B]$

- if $m > n$, $s(a_1, ..., a_n, X_1, ..., X_{m-n}) \in RT[\mathcal{F}, B](Q')$, where $Q' = \{(u_1, ..., u_{m-n}) \in \mathbb{R}^{m-n}; (a_1, ..., a_n, u_1, ..., u_{m-n}) \in Q\}$

7*) (stability of the regular terms by truncated sums and products)*

If Q is a n-basic box and if $s \in RT[\mathcal{F}, B](Q)$, the terms $(s\ {}^*{+}t)\ {}^*{\times}(\ {}^*1_{P\cap Q})$ and $s\ {}^*{\times}t$ are in $RT[\mathcal{F}, B](P \cap Q)$

8*) (stability of the regular terms by transpositions and products by $RT_0[\mathcal{F}, B]$)*

The terms $(a\ {}^*{+}t)\ {}^*{\times}(\ {}^*1_P)$, $(t\ {}^*{+}a)\ {}^*{\times}(\ {}^*1_P)$, $a\ {}^*{\times}t$ and $t\ {}^*{\times}a$ are in $RT[\mathcal{F}, B](P)$

9*) (stability by compositions)*

If Q_1,...,Q_n are r-basic boxes, if s_1, ..., s_n respectively belong to $RT[\mathcal{F}, B](Q_1)$,..., $RT[\mathcal{F}, B](Q_n)$ and finally, if for all $i \in \{1, ..., n\}$, the following condition is satisfied :

$$\forall (u_1, ..., u_r) \in \bigcap_{j=1}^{n} Q_j \quad (\overline{s_1}(u_1, ..., u_r), ..., \overline{s_n}(u_1, ..., u_r)) \in P$$

then the term $(t(s_1, ..., s_n)\ {}^*{\times}(\ {}^*1_{\bigcap_{j=1}^{n} Q_j})$ belongs to $RT[\mathcal{F}, B](\bigcap_{j=1}^{n} Q_j)$

10*) (stability by rational powers)*

If $\overline{t}(P) \subset \mathbb{R}_+^*$, then for all $q \in \mathbb{Q}^*$, the term ${}^*\pi_q(t) \in RT[\mathcal{F}, B](P)$

11*) (stability at the boundary of the basic boxes)*

If Q and Q' are two n-basic boxes over C, if $s \in RT[\mathcal{F}, B](Q)$, if $(a_1, ..., a_n) \in RT_0[\mathcal{F}, B]^n$ and if $(\epsilon_1, ...\epsilon_n) \in \{-1, 1\}^n$ such that :

$$\forall (u_1, ..., u_n) \in Q' \quad \overline{s}(\overline{a_1} + \epsilon_1.u_1^2, ..., \overline{a_n} + \epsilon_n.u_n^2) \in Q$$

then the term $(s(a_1\ {}^*{+}\ {}^*\epsilon_1\ {}^*{\times}X_1, ..., a_n\ {}^*{+}\ {}^*\epsilon_n\ {}^*{\times}X_n))\ {}^*{\times}\ {}^*1_{Q'} \in RT[\mathcal{F}, B](Q')$.

We note $RT[\mathcal{F}, B]$ the union of $RT_0[\mathcal{F}, B]$ and of all the sets $RT[\mathcal{F}, B](P)$. The elements of $RT[\mathcal{F}, B](P)$ are the regular terms over P, generated by $\mathcal{F}$ and B ; those of $RT_0[\mathcal{F}, B]$ are the regular constants ; and those of $RT[\mathcal{F}, B]$ are the regular terms, generated by $\mathcal{F}$ and B.

Proposition 2.8 . *By induction, we prove the following assertions.*

-There may exist a basic box P such that $RT[\mathcal{F},B](P)$ is empty.

-If P is a n-basic box such that $RT[\mathcal{F},B](P) \neq \emptyset$, the elements of $RT[\mathcal{F},B](P)$ are n-ary terms.

-If $RT[\mathcal{F},B](P)$ is not empty, the interpretation of an element of $RT[\mathcal{F},B](P)$ belongs to $\mathcal{Z}^\infty(P)$.

These terms are called regular because all their subterms have an interpretation in $\mathbb{R}$ which belongs to $\mathcal{Z}^\infty$; moreover we precisely know by advance the basic box where the function of interpretation is defined. Therefore we avoid terms whose interpretation in $\mathbb{R}$ is at the same time C^∞ and « pathological », like for example the term

$$(1+\pi_{\frac{1}{2}}(-2+\pi_{\frac{1}{2}}(1+X)))\times 1_{[-1,1]}$$

(to be clearer, we omit the necessary symbols *). The interpretation of this term is the function equal to 0 on $\mathbb{R}$ and so belongs to $\mathcal{Z}^\infty([-1,1])$. However, the interpretation of some subterms of this term are not C^∞ on $[-1,1]$.
The construction of a regular term is very close to the construction of a standard term, except that, at each step, we have a control on the nature C^∞ of the term and on its basic box of definition. Thereby, the set of regular terms is almost the set of standard terms but is easier to use. We can consult the remark 2.22 for more details on the conditions of closure of $RT[\mathcal{F},B]$.

Definition 2.9 . *$\overline{\mathcal{E}}(A)$ is the smaller set which contains $\mathcal{E}\cup A$ and which satisfies the following conditions. Let P and P' be two n-basic boxes and Q a $(n+1)$-basic box, all the three over $RT_0[\overline{\mathcal{E}}(A),A]$; we suppose that $f\in RT[\overline{\mathcal{E}}(A),A](P)$, $g\in RT[\overline{\mathcal{E}}(A),A](P')$ and $h\in RT[\overline{\mathcal{E}}(A),A](Q)$.*

1*) The interpretation of a term of $RT_0[\overline{\mathcal{E}}(A),A]$ or of $RT[\overline{\mathcal{E}}(A),A]$ is in $\overline{\mathcal{E}}(A)$.*

2*) (Implicit definition) If $\overline{h}(0)=0\neq\frac{\partial\overline{h}}{\partial X_{n+1}}(0)$ then for all n-basic box D and for all 1-basic box Δ such that $D\times\Delta\subset Q$, the map defined by ϕ on the basic box D is in $\overline{\mathcal{E}}(A)$, each time that $\overline{h}(X_1,...,X_n,\phi(X_1,...,X_n))=0$ on D and that $\frac{\partial\overline{h}}{\partial X_{n+1}}\neq 0$ on $D\times\Delta$.*

3*) (Factorization) If there exists $i\in\{1,...,n\}$ such that $\overline{f}(X_1,...,X_n)=X_i.\overline{g}(X_1,...,X_n)$ on $P\cap P'$, $\overline{g}\rceil_{P\cap P'}\in\overline{\mathcal{E}}(A)$.*

Proposition 2.10 . *By construction, the elements of $\overline{\mathcal{E}}(A)$ are in $\mathcal{Z}^\infty$.*

Remark 2.11 . *(partial derivatives) The closure by factorization of $\overline{\mathcal{E}}(A)$ implies the closure of $\overline{\mathcal{E}}(A)$ by partial derivatives (cf. proposition 2.1.6 of [Ra1]) -these both closures are even equivalent-.*

Definition 2.12 . *(quasi-analycity) $\overline{\mathcal{E}}(A)$ is quasi-analytic if every function of $\overline{\mathcal{E}}(A)$, which doesn't vanish around 0, has a partial derivative different from 0 at the point 0.*

In the continuation, we will consider a class $\mathcal{E}$ such that $\overline{\mathcal{E}}(\mathbb{R})$ is quasi-analytic. We stress the fact that it is $\overline{\mathcal{E}}(\mathbb{R})$ which is quasi-analytic and not only

$\overline{\mathcal{E}}(A)$.
We deduce from the definition 2.12, using the translation, that the maps of $\overline{\mathcal{E}}(A) \cap \mathcal{Z}^\infty(P)$ are quasi-analytic at every point of P.

Remark 2.13 . *Thanks to the quasi-analycity, if P is a n-basic box and if $f \in \overline{\mathcal{E}}(A) \cap \mathcal{Z}^\infty(P)$ vanishes on a whole open set included in P, then f vanishes on all $\mathbb{R}^n$.*

Definition 2.14 . *(domain) Thanks to the quasi-analycity, if $f \in \overline{\mathcal{E}}(A) \cap \mathcal{Z}^\infty(P)$ and if f doesn't vanish on P, the basic box P is unique; we will note it dom(f) -domain of f-. If f vanishes on $\mathbb{R}^n$, we pose $dom(f) = \mathbb{R}^n$.*

In the continuation, we will assimilate the symbol ${}^*+$ with the sum $+$, ${}^*\times$ with the product $\times$ (or .). We will use too the symbol $-$ for the subtraction and $|.|$ for the absolute value.
We will note equally ${}^*\pi_{-1}(X)$ or $\frac{1}{X}$ and we will assimilate $\frac{1}{X}$ with the map π_{-1}.

Definition 2.15 . *(language $\mathcal{L}(A)$ and theory $T(A)$) $\mathcal{L}(A)$ is the union of the language of ordered rings, of $\{ {}^*\pi_q; q \in \mathbb{Q}^*\}$ and of $\{ {}^*f; f \in \overline{\mathcal{E}}(A)\}$. So $\mathbb{R}$ is a $\mathcal{L}(A)$-structure.*

$T(A)$ is the $\mathcal{L}(A)$-theory which contains the following axioms. For all n-basic box P, for all term $t \in RT[\overline{\mathcal{E}}(A), A](P)$ and for all $a \in A \cup \mathbb{Q}$,

1) *if $\mathbb{R} \models \forall (u_1, ..., u_n) \in P \;\; t(u_1, ..., u_n) = 0$, the formula*

$$\forall (u_1, ..., u_n) \in P \;\; t(u_1, ..., u_n) = 0$$

is in $T(A)$.

2) *if $\mathbb{R} \models \forall (u_1, ..., u_n) \in P \;\; |t(u_1, ..., u_n)| < {}^*a$, the formula*

$$\forall (u_1, ..., u_n) \in P \;\; |t(u_1, ..., u_n)| < {}^*a$$

is in $T(A)$

3) *the formula $\forall (u_1, ..., u_n) \notin P \;\; t(u_1, ..., u_n) = 0$ is in $T(A)$*

4) *the formula $\forall (u_1, ..., u_n) \in P \;\; {}^*1_P(u_1, ..., u_n) = 1$ is in $T(A)$*

5) *$T(A)$ contains the universal axioms of ordered rings and the universal axioms which define the ${}^*\pi_q$, for $q \in \mathbb{Q}^*$.*

6) *$T(A)$ contains the simple diagram of $\mathbb{R}$ in the language $\mathcal{L}(A)$, ie the quantifier-free formulas satisfied by $\mathbb{R}$.*

Remark 2.16 . *$\mathcal{L}(A)$ without the symbol of the order is a sub-language of L_{fct}; moreover $\mathbb{R}$ is a model of $T(A)$.*

Proposition 2.17 . *With the notations of the previous definition, if $\mathbb{R}$ satisfies the formula $\forall (u_1, ..., u_n) \in P \;\; t(u_1, ..., u_n) > 0$, $T(A)$ also satisfies this formula.*

Proof : as $\overline{t}$ is continuous on P, which is compact, $\overline{t}(P)$ has a minimum, $m \in \mathbb{R}$ and a maximum $M \in \mathbb{R}$; as $\overline{t}$ is strictly positive on P, $m > 0$. Further there exists $(q,a) \in {\mathbb{Q}_+^*}^2$ such that $m > q - a > 0$ and $M < q + a$.
We pose $s = (t - q).\,{}^*1_P$; so s is in $RT[\overline{\mathcal{E}}(A), A](P)$ and $\mathbb{R}$ satisfies $\forall (u_1, ..., u_n) \in P \ \ |s(u_1, ..., u_n)| < a$. Therefore, $T(A)$ proves that t is positive on P. $\square_{proposition}$

Proposition 2.18 . *Let $T_1(A)$ be the $\mathcal{L}$-theory which contains all the formulas of the points* 1, 3, 4, 5 *et* 6 *of the definition 2.15 and the following axioms : if $\mathbb{R} \models \forall (u_1, ..., u_n) \in P \ \ t(u_1, ..., u_n) > 0$, the formula*

$$\forall (u_1, ..., u_n) \in P \ \ t(u_1, ..., u_n) > 0$$

is in $T_1(A)$. $T(A)$ and $T_1(A)$ are equivalent.

Proof : The proposition 2.17 proves that $T(A) \vdash T_1(A)$. Now, let $t \in RT[\overline{\mathcal{E}}(A), A](P)$ and $a \in A \cup \mathbb{Q}$ such that $\mathbb{R} \models \forall (u_1, ..., u_n) \in P \ \ |t(u_1, ..., u_n)| < {}^*a$. Let us consider the two terms $s_1 = -(t - {}^*a) \, {}^*\!\times {}^*1_P$ and $s_2 = (t + {}^*a) \, {}^*\!\times {}^*1_P$. s_1 and s_2 are in $RT[\overline{\mathcal{E}}(A), A](P)$; furthermore, $\mathbb{R}$, and so $T_1(A)$, satisfies that s_1 and s_2 are positive on P. Thus $T_1(A)$ proves $\forall (u_1, ..., u_n) \in P \ \ |t(u_1, ..., u_n)| < {}^*a$. So $T_1(A) \vdash T(A)$.$\square_{proposition}$

Notation 2.19 . *Let $A = \bigcup_{f \in \mathcal{E}^*} \mathfrak{B}(dom(f))$, where $\mathcal{E}^*$ is the set which contains all the maps of $\mathcal{E}$ which are not equal to* 0 *on all $\mathbb{R}^n$, for a given $n \in \mathbb{N}^*$.*
We note $\mathcal{L}(A) = \mathcal{L}$, $T(A) = T$, $\overline{\mathcal{E}}(A) = \overline{\mathcal{E}}$, $RT[\overline{\mathcal{E}}(A), A](P) = RT(P)$, $RT[\overline{\mathcal{E}}(A), A] = RT$ and $\overline{\mathcal{E}}_0 = \overline{\mathcal{E}}(A) \cap \mathbb{R}$.
Moreover, the subset of the n-ary terms of RT will be noted RT_n.

Remark 2.20 . *If $f \in \overline{\mathcal{E}}$, $\mathfrak{B}(dom(f))$ is included in $\overline{\mathcal{E}}_0$.*

The aim of this article is to prove that T is equivalent to the complete theory of $\mathbb{R}$ in the language $\mathcal{L}$ and that T is model-complete. In this purpose we will draw on the following results, which come from [Ra1] -and [Ra2]-.

Theorem 2.21 . *$T(\mathbb{R})$ is equivalent to the complete theory of $\mathbb{R}$ in the language $\mathcal{L}(\mathbb{R})$ and $T(\mathbb{R})$ is model-complete.*

Remark 2.22 . *The language $\mathcal{L}(\mathbb{R})$ and the theory $T(\mathbb{R})$ are different from the language, noted $\mathcal{L}$, and from the theory, noted T, of [Ra1] -notice : the two last symbols $\mathcal{L}$ and T haven't got the same meaning as those used in this article- ; moreover in [Ra1] we don't use the notion of regular term. We leave it to the reader to check that these modifications haven't any impact on the main results of [Ra1] and notably on the theorem* 3.2.3*, which becomes the theorem 2.21 of the present article.*

Indeed, in a first time in [Ra1] -chapter 2.4*-, the results only deal with terms C^∞ around* 0 *and are local (around* 0*), as it is specified at the beginning of the mentioned chapter. Truncating the terms on certain basic boxes, as it is done in the construction of regular terms hasn't an effect on these local properties.*
In particular, we keep the corollary 2.4.9 *of [Ra1] -which establishes a preparation theorem for the terms C^∞- relatively to regular terms (and so the used matrices are composed of regular terms too).*

Then, in the chapter 2.5 *of [Ra1], we deal with all the terms ; there the boundaries of the basic boxes play a part. These problems of boundary are the raison*

d'être of the condition 11 *of closure of regular terms (cf. définition 2.21 of the present article). This condition comes directly from the study of the case* 6 *of the lemma* 2.5.8 *of [Ra1].*
Thus we can transpose the lemma 2.5.8 *to regular terms (and, there too, the considered matrices must be composed of regular terms).*

Finally, thanks to previous remarks, we can transpose too the theorem 2.5.13 *to regular terms ; then it follows the theorem 2.21 of the present article.*

3 $\overline{\mathcal{E}}$-transcendence

Definition 3.1 *($\overline{\mathcal{E}}$-transcendence). Let $n \in \mathbb{N}^*$; $(a_1, ...a_n) \in \mathbb{R}^n$ is $\overline{\mathcal{E}}$-transcendent if, for every n-ary map f, which belongs to $\overline{\mathcal{E}}$ and for every $(q_1, ..., q_n) \in \mathbb{Q}^n$, such that $(a_1 - q_1, ..., a_n - q_n) \in \overset{\circ}{dom}(f)$,*
$(f(a_1 - q_1, ..., a_n - q_n) = 0)$ implies that f vanishes on a whole neighbourhood of $(a_1 - q_1, ..., a_n - q_n)$.

Proposition 3.2 . *Let $n \in \mathbb{N}^*$; let $(a_1, ..., a_n) \in \mathbb{R}^n$ and $y \in \mathbb{R}$ such that $(a_1, ..., a_n)$ is $\overline{\mathcal{E}}$-transcendent but $(a_1, ..., a_n, y)$ is not ; then for all open set U of $\mathbb{R}^{n+1}$ which contains $(a_1, ..., a_n, y)$, there exist $(q_1, ...q_n, q) \in \mathbb{Q}^{n+1}$, S a n-basic box over $\mathbb{Q}$ and $\phi \in \overline{\mathcal{E}} \cap \mathcal{Z}^\infty(S)$ such that :*

-$(a_1 - q_1, ..., a_n - q_n) \in \overset{\circ}{S}$

-$y = q + \phi(a_1 - q_1, ..., a_n - q_n)$

-$\forall (x_1, ..., x_n) \in S \quad (x_1 + q_1, ..., x_n + q_n, q + \phi(x_1, ..., x_n)) \in U$

Proof : as $(a_1, ..., a_n, y)$ is not $\overline{\mathcal{E}}$-transcendent, there exist a $(n + 1)$-ary map $f \in \overline{\mathcal{E}}$ and $(r_1, ..., r_n, r) \in \mathbb{Q}^{n+1}$ such that $(a_1 - r_1, ..., a_n - r_n, y - r) \in \overset{\circ}{dom}(f)$, f doesn't locally vanish around $(a_1 - r_1, ..., a_n - r_n, y - r)$ and $f(a_1 - r_1, ..., a_n - r_n, y - r) = 0$.

As f doesn't vanish on all $\mathbb{R}^{n+1}$, $dom(f)$ is a $(n + 1)$-basic box ; we pose $dom(f) = \Pi_{i=1}^{n+1}[\alpha_i, \beta_i]$. We consider U an open set of $\mathbb{R}^{n+1}$ which contains $(a_1, ..., a_n, y)$.

Let $u \in [\alpha_{n+1}, \beta_{n+1}] \cap \mathbb{Q}$; we pose $g_u(X_1, ..., X_n) = f(X_1, ..., X_n, u)$. So g_u belongs to $\overline{\mathcal{E}} \cap \mathcal{Z}^\infty(\Pi_{i=1}^{n}[\alpha_i, \beta_i])$.
We pose too $h(Y) = f(a_1 - r_1, ..., a_n - r_n, Y)$; so h belongs to $\overline{\mathcal{E}}(\mathbb{R})$.

By hypothesis, h is quasi-analytic around $y - r$. Thus, either h locally vanishes around $y - r$ (so vanishes on all $\mathbb{R}$), or there exists $k \in \mathbb{N}$ such that

$$\frac{\partial f}{\partial Y^k}(a_1 - r_1, ..., a_n - r_n, y - r) \neq 0$$

Case 1 : h vanishes on all $\mathbb{R}$.

In this case, for all $u \in [a_{n+1}, b_{n+1}] \cap \mathbb{Q}$, $f(a_1 - r_1, ..., a_n - r_n, u) = 0$ and so the function g_u always vanishes on all $\mathbb{R}^n$, because $(a_1, ..., a_n)$ is $\overline{\mathcal{E}}$-transcendent. Therefore, by continuity, f vanishes on all $\mathbb{R}^{n+1}$, what is supposed false.

Case 2 : h is not equal to 0 on all $\mathbb{R}$.

In this case, we pose $d = min\{k \in \mathbb{N}; \frac{\partial f}{\partial Y^k}(a_1 - r_1, ..., a_n - r_n, y - r) \neq 0\}$.

d is strictly positive because $f(a_1 - r_1, ..., a_n - r_n, y - r) = 0$. Therefore :

$$\frac{\partial f}{\partial Y^{d-1}}(a_1 - r_1, ..., a_n - r_n, y - r) = 0$$

and

$$\frac{\partial f}{\partial Y^{d}}(a_1 - r_1, ..., a_n - r_n, y - r) \neq 0$$

So there exists a $(n+1)$-basic box D over $\mathbb{Q}$ and $(t_1, ..., t_n, t) \in \mathbb{Q}^{n+1}$ such that :

-$(a_1 - r_1 - t_1, ..., a_n - r_n - t_n, y - r - t) \in \overset{\circ}{D}$

- for every $(u_1, ...u_n, v) \in D$, $(t_1 + r_1 + u_1, ..., t_n + r_n + u_n, t + r + v) \in U$

- $\frac{\partial f}{\partial Y^d}(X_1 + t_1, ..., X_n + t_n, Y + t)$ is different from 0 on D

We pose $\tau = \frac{\partial f}{\partial Y^{d-1}}(t_1, ..., t_n, t)$ (so $\tau \in \overline{\mathcal{E}}_0$) and

$$\theta(X_1, ..., X_n, Y, Z) = (\frac{\partial f}{\partial Y^{d-1}}(X_1 + t_1, ..., X_n + t_n, Y + t) + Z - \tau)\rceil_{D \times [-1-\tau, 1+\tau]}$$

thus θ belongs to $\overline{\mathcal{E}}$ and $\theta(0) = 0$; moreover, $\frac{\partial \theta}{\partial Y}(0) \neq 0$ and $\theta(a_1 - r_1 - t_1, ..., a_n - r_n - t_n, y - r - t, \tau) = 0$

Therefore, there exists a $(n + 1)$-basic box B over $\mathbb{Q}$ and an -implicit- function $\phi \in \overline{\mathcal{E}} \cap \mathcal{Z}(B)$ such that :

-$\forall (x_1, ..., x_n, z) \in B \quad \theta(x_1, ..., x_n, \phi(x_1, ..., x_n, z), z) = 0$

-$(a_1 - r_1 - t_1, ...a_n - r_n - t_n, \tau) \in B$

So we obtain that $y - r - t = \phi(a_1 - r_1 - t_1, ..., a_n - r_n - t_n, \tau).\Box_{proposition}$

Thus, in these conditions, y belongs to the $\mathcal{L}$-structure generated by $(a_1, ..., a_n)$. Besides, by extension to the case « $n = 0$ », we obtain the following proposition.

Proposition 3.3 . *Let $y \in \mathbb{R}$ such that y is not $\overline{\mathcal{E}}$-transcendent then $y \in \overline{\mathcal{E}}_0$.*

Corollary 3.4 . *Let $(x_1, ..., x_p) \in \mathbb{R}^p$ not all in $\overline{\mathcal{E}}_0$; there exist $n \in \{1, ..., p\}$ and a strictly increasing map l from $\{1, ...n\}$ in $\{1, ..., p\}$ such that $(x_{l(1)}, ..., x_{l(n)})$ is $\overline{\mathcal{E}}$-transcendent and satisfies the following conditions.*

For all open set U of $\mathbb{R}^p$ which contains $(x_1,...,x_p)$, there exist $(q_1,...q_p)\in\mathbb{Q}^p$, S a n-basic box over $\mathbb{Q}$ and ϕ_1,...,ϕ_p in $\overline{\mathcal{E}}\cap\mathcal{Z}^\infty(S)$ such that

-$(x_{l(1)}-q_{l(1)},...,x_{l(n)}-q_{l(n)})\in\overset{\circ}{S}$

-$x_i=q_i+\phi_i(x_{l(1)}-q_{l(1)},...,x_{l(n)}-q_{l(n)})$ *for all* $i\in\{1,...,p\}$

-$\forall(u_1,...,u_n)\in S\ \ \forall i\in\{1,...,p\}\ \ (q_1+\phi_1(u_1,...,u_n),...,q_p+\phi_p(u_1,...,u_n))\in U$

Proof : by recurrence, using propositions 3.2 and 3.3.

4 Extensions of structures

Proposition 4.1 . *By construction, $\overline{\mathcal{E}}_0$ is the $\mathcal{L}$-sub-structure of $\mathbb{R}$ generated by the empty set.*

Notation 4.2 . *Let $a\in\mathbb{R}$; we note p_a, the complete 1-type of a in the language $\mathcal{L}$.*

Definition 4.3 . *Let $\mathcal{M}$ be a $\mathcal{L}$-structure ; $\mathcal{M}$ realizes $\mathbb{R}$ if for every $a\in\mathbb{R}$, p_a is realized in $\mathcal{M}$.*

Proposition 4.4 . *Let $\mathcal{N}$ be a model of T and $\mathcal{M}$ be a $\mathcal{L}$-sub-structure of $\mathcal{N}$, such that $\mathcal{M}$ realizes $\mathbb{R}$. There exists an $\mathcal{L}$-embedding from $\mathbb{R}$ to $\mathcal{M}$.*

Proof : we note M the underlying set of $\mathcal{M}$. Let Γ be the set of couples $(\mathcal{C},i)$ where $\mathcal{C}$ is a $\mathcal{L}$-sub-structure of $\mathbb{R}$ and i is an embedding from $\mathcal{C}$ to $\mathcal{M}$; Γ is non-empty because it contains $(\overline{\mathcal{E}}_0,Id)$, where Id is the canonical embedding which associates to an element of $\overline{\mathcal{E}}_0$ its interpretation in $\mathcal{M}$. $(\overline{\mathcal{E}}_0,Id)$ is the minimal element of Γ.
As Γ is inductive for inclusion, there exists a bigger element $(\mathcal{D},j)\in\Gamma$.

If the underlying set of $\mathcal{D}$ is $\mathbb{R}$, we conclude ; otherwise let $y\in\mathbb{R}$ such that y doesn't belong to the underlying set of $\mathcal{D}$, noted D. There exists $\tilde{y}\in M$ such that $\tilde{y}$ realizes the type p_y.
We will use the following lemma ; we can find a proof in [Ri], for example.

Lemma 4.5 . *(embedding of ordered fields) Let k be a field and K and L be two ordered fields such that k is embedded into K by f and in L by g ; let Ω be a non-empty set, σ_1 be a map from Ω to K and σ_2 be a map from Ω to L such that the following condition, noted $(*)$, is satisfied.*

$(*)$ *For every $n\in\mathbb{N}$, for every $P\in k[X_1,...,X_n]$ and for every $(x_1,...,x_n)\in\Omega^n$, if $P(\sigma_1(x_1),...,\sigma_1(x_n))=0$ (resp. $P(\sigma_1(x_1),...,\sigma_1(x_n))>0$) then $P(\sigma_2(x_1),...,\sigma_2(x_n))=0$ (resp. $P(\sigma_2(x_1),...,\sigma_2(x_n))>0$).*

Then there exists an embedding h from the field generated by all the $\sigma_1(x)$ ($x\in\Omega$), into L such that $h(\sigma_1(x))=\sigma_2(x)$ and $h\circ f=g$.

Let $n \in \mathbb{N}^*$ and $\mathcal{T}_n = \{(t, u_1, ..., u_n); t \in RT_n \wedge (u_1, ..., u_n) \in D^n\}$; we note $\mathcal{T}$ the union of $D \cup \{y\}$ and of all the sets $\mathcal{T}_n$. We consider the two following maps σ_1 and σ_2 defined in this way.

1) $\sigma_1 : \mathcal{T} \to \mathbb{R}$ such that

- for all $n \in \mathbb{N}^*$ and for all $(t, u_1, ..., u_n) \in \mathcal{T}$, $\sigma_1(t, u_1, ..., u_n) = \overline{t}(u_1, ..., u_n)$

- for all $u \in D \cup \{y\}$, $\sigma_1(u) = u$

2) $\sigma_2 : \mathcal{T} \to M$ such that

- for all $n \in \mathbb{N}^*$ and for all $(t, u_1, ..., u_n) \in \mathcal{T})$, $\sigma_2(t, u_1, ..., u_n)$ is the interpretation of t in $\mathcal{M}$ applied in $(j(u_1), ..., j(u_n))$

- for all $u \in D$, $\sigma_2(u) = j(u)$

- $\sigma_2(y) = \tilde{y}$

We will show the relation $(*)$ for polynomials over $\mathbb{Q}$ and for the maps σ_1 and σ_2 defined above. We remark that it is sufficient to show that for every regular $(n+1)$-ary term t and for every $(u_1, ..., u_n) \in D^n$, we have the following property :

if the interpretation of t in $\mathbb{R}$ applied in $(u_1, ..., u_n, y)$ is equal to 0 (resp. strictly positive), the interpretation of t in $\mathcal{M}$ applied in $(j(u_1, ..., j(u_n), \tilde{y})$ is equal to 0 (resp. strictly positive).

So let t be a regular $(n+1)$-ary term and $(u_1, ..., u_n) \in D^n$. We can suppose that $\overline{t}$ doesn't vanish identically.

Case 1 : the interpretation of t in $\mathbb{R}$ applied in $(u_1, ..., u_n, y)$ is equal to 0 (ie $\overline{t}(u_1, ..., u_n, y) = 0$).

We note $\tilde{\tau}$ the interpretation of a term τ in $\mathcal{M}$. We can suppose, without restricting the generality, that none of the u_i or y belong to $\mathfrak{B}(dom(\overline{t}))$; indeed we remind that, as $\overline{t} \in \overline{\mathcal{E}}$, the set $\mathfrak{B}(dom(\overline{t}))$ is included in $\overline{\mathcal{E}}_0$.
Thanks to propositions 3.2, 3.3 and to corollary 3.4, as $y \notin D$, there exist a family $\overline{\mathcal{E}}$-transcendent $(v_1, ..., v_p, y)$ with $p \leq n$, $(q_1, ..., q_{n+1}) \in \mathbb{Q}^{n+1}$ and $n+1$ maps ϕ_1,...,ϕ_{n+1} which belong to $\overline{\mathcal{E}} \cap \mathcal{Z}^\infty(P)$ (P is a $(n+1)$-basic box over $\mathbb{Q}$) such that :

- for every $k \in \{1, ..., p\}$, there exists an unique $l(k) \in \{1, ..., n\}$ such that $v_k = u_{l(k)}$.

- $(v_1 - q_{l(1)}, ..., v_{p+1} - q_{l(p+1)}) \in \overset{\circ}{P}$

- for every $(x_1, ..., x_{p+1}) \in P$, $(q_1+\phi_1(x_1, ..., x_{p+1}), ..., q_{n+1}+\phi_{n+1}(x_1, ..., x_{p+1})) \in dom(\overline{t})$

- for every $k \in \{1, .., n\}$, $u_k - q_k = \phi_k(v_1 - q_{l(1)}, ..., v_{p+1} - q_{l(p+1)})$ and $y - q_{n+1} = \phi_{n+1}(v_1 - q_{l(1)}, ..., v_{p+1} - q_{l(p+1)})$

We pose

$$s = (t(q_1 + \phi_1(X_1, ..., X_{p+1}), ..., q_{n+1} + \phi_{n+1}(X_1, ..., X_{p+1}))) \times 1_P$$

(to be clearer, we omit the necessary *).
Thus $\overline{s}(v_1 - q_{l(1)}, ..., v_{p+1} - q_{l(p+1)}) = 0$ and as $(v_1, ..., v_{p+1})$ is $\overline{\mathcal{E}}$-transcendent, $\overline{s}$ vanishes on a whole neighbourhood of $(v_1 - q_{l(1)}, ..., v_{p+1} - q_{l(p+1)})$.

Therefore $\mathcal{N}$ satisfies that $\tilde{s}$ vanishes on all $j(R)$ where R is a $(p+1)$-basic box over $\mathbb{Q}$. As $(\mathcal{D}, j) \in \Gamma$, $(j(u_1), ..., j(u_n), \tilde{y}) \in \{(q_1 + \phi_1(z_1, ..., z_{p+1}), ..., q_{n+1} + \phi_{n+1}(z_1, ..., z_{p+1})); (z_1, ..., z_{p+1}) \in j(R)\}$.
So $\mathcal{N} \models t(j(u_1), ..., j(u_n), \tilde{y}) = 0$ and thus $\mathcal{M} \models t(j(u_1), ..., j(u_n), \tilde{y}) = 0$.

Case 2 : the interpretation of t in $\mathbb{R}$ applied in $(u_1, ..., u_n, y)$ is strictly positive (ie $\overline{t}(u_1, ..., u_n, y) > 0$).

By continuity, there exist $(q_1, ..., q_{n+1}) \in \mathbb{Q}^{n+1}$ and a $(n+1)$-basic box R such that $\overline{t}$ is strictly positive on the set $(q_1, ..., q_{n+1}) + \in R$.
Therefore, in $\mathcal{N}$ and so in $\mathcal{M}$, $\tilde{t}(q_1 + X_1, ..., q_{n+1} + X_{n+1})$ is strictly positive on $j(R)$. As $\tilde{y}$ satisfies the type p_y, $(j(u_1), ..., j(u_n), \tilde{y}) \in R$; thereby $\tilde{t}(j(u_1), ..., j(u_n), \tilde{y}) > 0$.

We deduce from the study of these two cases, that there exists an $\mathcal{L}$-embedding, which extends j, from the structure generated by D and y into $\mathcal{M}$. We conclude, using the hypothesis of maximality of $(\mathcal{D}, j)$, that $D = \mathbb{R}.\square_{proposition}$

Proposition 4.6 . *Let $\mathcal{N}$ be a model of T and $\mathcal{M}$ be a $\mathcal{L}$-sub-structure of $\mathcal{N}$; $\mathcal{M}$ and $\mathcal{N}$ can be extended into two $\mathcal{L}(\mathbb{R})$-structures $\mathcal{M}'$ and $\mathcal{N}'$ such that :*

- $\mathcal{M}'$ and $\mathcal{N}'$ are $\mathcal{L}(\mathbb{R})$-structures

- $\mathcal{M}'$ is a $\mathcal{L}(\mathbb{R})$-sub-structure of $\mathcal{N}'$ and $\mathcal{N}'$ is a model of $T(\mathbb{R})$

Proof : let i be an $\mathcal{L}$-embedding from $\mathbb{R}$ to $\mathcal{M}$.

We will show that for every term $t \in RT[\overline{\mathcal{E}}(\mathbb{R}), \mathbb{R}](P)$ (P a n-basic box), there exist a n-basic box Q over $\overline{\mathcal{E}}_0$, a r-basic box S over $\overline{\mathcal{E}}_0$, a $\overline{\mathcal{E}}$-transcendent family $(u_1, ..., u_r) \in \overset{\circ}{S}$ (possibly $r = 0$; in this case, this family doesn't play a part) and a term $\hat{t} \in RT(Q \times S)$, where Q contains P, such that :

$$\overline{t}(X_1, ..., X_n) = \overline{\hat{t}}(X_1, ..., X_n, u_1, ..., u_r)\rceil_P$$

on all $\mathbb{R}^n$

$\hat{t}(X_1,...,X_n,i(u_1),...,i(u_r))\rceil_{i(P)}$ will be the ideal candidate for the interpretation of the term t in $\mathcal{M}$ and in $\mathcal{N}$ (we extend canonically the notation $\rceil_P$ to $\mathcal{N}$).

The main problem comes from the fact that we could use some functions of $\overline{\mathcal{E}}(\mathbb{R})$ to construct t ; however $\overline{\mathcal{E}}(\mathbb{R})$ is different from $\overline{\mathcal{E}}$.

The construction of the operator $\hat{}$ (which is not necessary unique : we use implicitly the Axiom of Choice at each step of the construction) is made by induction on the rank of the creation of t. We won't detail each operation and we will focus on certain one : the other cases are obtained by analogy with the considered cases.

1) At first, if $f \in \mathcal{E}$, we simply pose $\hat{{}^*f} = {}^*f$. So we have the initialization.

2) Then, let us consider the closure by implicit functions ; the method is very close to the one used in the proof of the proposition 3.2. Here we suppose that P -and so Q- is a $n+1$-basic box, thus t is $(n+1)$-ary. We suppose too that $\frac{\partial \overline{t}}{\partial X_n} \neq 0$ and $\overline{t}(0) = 0$.
We place ourselves in the conditions of map of point 2) of the definition 2.9. We use the same notations, posing $\Delta = [a, b]$. D is not necessarily a basic box over $\overline{\mathcal{E}}_0$ and a and b are not also necessarily in $\overline{\mathcal{E}}_0$.
As $D \times [a, b] \subset \overset{\circ}{P}$, there exist a n-basic box over $\mathbb{Q}$, D' and $(a', b') \in \mathbb{Q}^2$ such that $D \times [a, b] \subset D' \times [a', b'] \subset \overset{\circ}{P}$ and $\frac{\partial \overline{t}}{\partial X_n} \neq 0$ on all $D' \times [a', b']$; so there exists an implicit function for $\overline{t}$, noted ϕ' -notice : exceptionally, here, ϕ' won't refer to the derivative of ϕ- defined on D' and which extends ϕ. Moreover we can suppose that $\forall (x_1, ..., x_n) \in D' \quad \phi'(x_1, ..., x_n) \in [a', b']$.
By hypothesis of induction, $\overline{t}(X_1, ..., X_n, X_{n+1}) = \overline{\hat{t}}(X_1, ..., X_n, X_{n+1}, u_1, ..., u_r)\rceil_P$

By continuity, there exist a r-basic box B over $\mathbb{Q}$ and $(q_1, ..., q_r) \in \mathbb{Q}^r$ such that :

$(u_1 - q_1, ..., u_r - q_r) \in \overset{\circ}{B}$, $(q_1, ..., q_r) + B \subset S$ and for all $(x_1, ..., x_n, x_{n+1}, y_1, ..., y_r) \in D' \times [a', b'] \times B$, $\frac{\partial \overline{\hat{t}}}{\partial X_{n+1}}(x_1, ..., x_{n+1}, q_1 + y_1, ..., q_r + y_r) \neq 0$

We pose $\overline{\hat{t}}(0, ..., 0, q_1, ..., q_r) = \mu$ and

$\theta(X_1, ..., X_{n+1}, U_1, ..., U_r, Z) = ((\overline{\hat{t}}(X_1, ..., X_{n+1}, q_1 + U_1, ..., q_r + U_r)) - \mu + Z)\rceil_{Q \times B \times [-1-\mu, 1+\mu]}$
Thus $\theta \in \overline{\mathcal{E}}$, $\theta(0) = 0$ and $\frac{\partial \theta}{\partial X_{n+1}} \neq 0$ on all $D' \times [a', b'] \times B \times [-1 - \mu, 1 + \mu]$.
So there exists an implicit map ψ defined on $D' \times B$ such that

$$\overline{\hat{t}}(X_1, ..., X_n, \psi(X_1, ..., X_n, U_1, ..., U_r, Z), q_1 + U_1, ..., q_r + U_r, Z) = 0$$

on all $D' \times B \times [-1 - \mu, 1 + \mu]$
By unicity of the implicit functions, $\phi(X_1, ..., X_n) = \psi(X_1, ..., X_n, u_1 - q_1, ..., u_r -$

$q_r, \overline{\hat{t}}(0, ..., 0, q_1, ..., q_r))\rceil_D$.
Therefore we pose $\hat{\phi}(X_1, ..., X_n) = \psi(X_1, ..., X_n, u_1 - q_1, ..., u_r - q_r)\rceil_D$.

3) If $a \in \mathbb{R}$, let us consider $s = a + t \in \overline{\mathcal{E}}(\mathbb{R})$; let $q \in \mathbb{Q}$ such that $|a| < q$. In this case we pose, (we still use the notations of the hypothesis of induction)

$$h(X_1, ..., X_n, U_1, ..., U_r, Y) = (Y + \overline{\hat{t}}(X_1, .., X_n, U_1, ..., U_r))\rceil_{Q\times S\times[-q,q]}$$

Thus $h \in \overline{\mathcal{E}}$. Moreover, thanks to the proposition 3.2, either the family $(u_1, ..., u_r, a)$ is $\overline{\mathcal{E}}$-transcendent, or $a = \phi(u_1 - q_1, ..., u_r - q_r)$ with $\phi \in \overline{\mathcal{E}} \cap \mathcal{Z}^\infty(S')$ and $(q_1, ..., q_r) \in \mathbb{Q}^r$, such that $(q_1, ..., q_r) + S' \subset S$. In this last case, we consider

$$k(X_1, ..., X_n, V_1, ..., V_r) = h(X_1, ..., X_n, V_1, ..., V_r, \phi(V_1, ..., V_r))\rceil_{Q\times S'}$$

So $k \in \overline{\mathcal{E}}$.
In the first case, we pose $\hat{s} = {}^*h$; indeed, $\overline{s}(X_1, ..., X_n) = h(X_1, ..., X_n, u_1, ..., u_r, a)\rceil_P$ and $(u_1, ..., u_r, a)$ is $\overline{\mathcal{E}}$-transcendent.
In the second case, we pose $\hat{s} = {}^*k$; indeed, $\overline{s}(X_1, ..., X_n) = k(X_1, ..., X_n, u_1 - q_1, ..., u_r - q_r)\rceil_P$ and $(u_1 - q_1, ..., u_r - q_r)$ is $\overline{\mathcal{E}}$-transcendent.

We can use the same method to go on the construction of the operator $\hat{}$ for every point of the definitions 2.7 anf 2.15.
So we interpret in $\mathcal{N}$ and $\mathcal{M}$ the term $t(X_1, ..., X_n)$ by $\hat{t}(X_1, ..., X_n, i(u_1), ..., i(u_r))\rceil_P$.
By construction, the $\mathcal{L}(\mathbb{R})$-structures $\mathcal{M}'$ and $\mathcal{N}'$ obtained by this way are two extensions of $\mathcal{M}$ and $\mathcal{N}$, and $\mathcal{M}'$ is a $\mathcal{L}(\mathbb{R})$-sub-structure of $\mathcal{N}'$.
It only remains to prove that $\mathcal{N}'$ is a model of $T(\mathbb{R})$. In fact all has been done to reach this result ; indeed, if $\overline{t}$ vanishes on all P in $\mathbb{R}$, then $\overline{\hat{t}}(X_1, ..., X_n, u_1, .., u_r)$ vanishes on all Q. As $(u_1, ..., u_r)$ is $\overline{\mathcal{E}}$-transcendent, $\overline{\hat{t}}$ vanishes on all $\mathbb{R}^{n+r}$ and so on all $Q \times B$, where B is a r-basic box over $\mathbb{Q}$ which contains $(u_1, ..., u_r)$.
It is the same for $\mathcal{N}$, since $\mathcal{N}$ is a model of T ; therefore, by definition of the interpretation of t in $\mathcal{N}'$, $\mathcal{N}'$ satisfies that $\tilde{t}$ vanishes identically on $i(P)$.
In the same way, and by analogy with the end of the proof of the proposition 4.4, using the proposition 2.18, we also obtain that $\mathcal{N}'$ satisfies the axiom of the point 2 in the definition of $T(\mathbb{R})$ (sign of the terms).
The other axioms of $T(\mathbb{R})$ are clearly satisfied by $\mathcal{N}'$, thanks to its definition.
So $\mathcal{N}'$ is a model of $T(\mathbb{R})$.$\Box_{proposition}$

Theorem 4.7 . *T is equivalent to the complete theory of $\mathbb{R}$ in the language $\mathcal{L}$ and T is model-complete, so admits the quantifier elimination.*

Proof : let $\mathcal{A}$ be a $\mathcal{L}$-sub-structure of a model $\mathcal{B}$ of T ; by compactness, there exist an elementary $\mathcal{L}$-extension $\mathcal{M}$ of $\mathcal{A}$ and an elementary $\mathcal{L}$-extension $\mathcal{N}$ of $\mathcal{B}$ such that $\mathcal{M}$ is a $\mathcal{L}$-sub-structure of $\mathcal{N}$ and $\mathcal{M}$ realizes $\mathbb{R}$. Thus, thanks to the previous proposition, $\mathcal{M}$ and $\mathcal{N}$ can be extended in two $\mathcal{L}(\mathbb{R})$-structures $\mathcal{M}'$ and $\mathcal{N}'$ such that $\mathcal{M}'$ is a $\mathcal{L}(\mathbb{R})$-sub-structure of $\mathcal{N}'$ and $\mathcal{N}' \models T(\mathbb{R})$. As $T(\mathbb{R})$ is model-complete and universal, $\mathcal{M}'$ is an elementary $\mathcal{L}(\mathbb{R})$-sub-structure of $\mathcal{N}'$.
Therefore $\mathcal{M}$ is an elementary $\mathcal{L}$-sub-structure of $\mathcal{N}$ and so $\mathcal{A}$ is an elementary $\mathcal{L}$-sub-structure of $\mathcal{B}$. Thus T is model-complete and admits quantifier elimination,

because T is universal.
Thanks to the proposition 4.4, for all model $\mathcal{M}$ of T which realizes $\mathbb{R}$, there exists an $\mathcal{L}$-embedding from $\mathbb{R}$ into $\mathcal{M}$. So every model which realizes $\mathbb{R}$ is a model of the complete theory of $\mathbb{R}$ in the language $\mathcal{L}$; thus, by compactness, T is equivalent to the complete theory of $\mathbb{R}$ in the language $\mathcal{L}$. $\square_{theorem}$

Corollary 4.8 . *Let $\mathcal{L}_0$ be the union of the language of ordered rings and of the set $\{\,{}^*f; f \in \mathcal{E}\}$. Let $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}$; the complete theory of $\mathbb{R}$ in the language $\mathcal{L}_1$ has a prime model which is $\overline{\mathcal{E}}_0$.*

Proof : it is sufficient to remark that all the elements of $\overline{\mathcal{E}}$ are definable in the language $\mathcal{L}_0$. $\square_{corollary}$

Bibliography

[C] M. Coste, *An introduction to o-minimal geometry,* Istituti Editoriali e Poligrafici Internazionali, 2000.

[vdD1] L. van den Dries, *Tame topology and o-minimal structures,* Cambridge University Press, 1998.

[vdD2] L. van den Dries, *A generalization of the Tarski-Seidenberg theorem and some nondefinability results,* Bulletin of the American Mathematical Society, 15, 189-193, 1986.

[vdDMM] L. van den Dries, A. Macintyre and D. Marker, *The elementary theory of restricted analytic fields with exponentation,* Annals of Mathematics, 140, 183-205, 1994.

[LR] J-M. Lion, J-P. Rolin, *Théorème de préparation pour les fonctions logarithmico-exponentielles,* Annales de l'Institut Fourier, 47, 859-884, 1997.

[Ra1] A.Rambaud, *Quasi-analycité, o-minimalité et élimination des quantificateurs, Thèse, Université Paris 7,* Paris, 2005.
(http ://www.logique.jussieu.fr/ rambaud/these.pdf)

[Ra2] A.Rambaud, *Quasi-analycité, o-minimalité et élimination des quantificateurs,* Compte-Rendus de l'Académie des Sciences, Ser. I 343, Paris, 2006.

[Ri] P. Ribenboim, *Le théorème des zéros pour les corps ordonnés,* Séminaire Dubreil. Algèbre et Théorie des Nombres, tome 24, exp° 17, n°2, 1-32, 1970-1971.

[RSW] J-P. Rolin, P. Speissegger et A. J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality,* Journal of the American Mathematical Society, 16, 751-777, 2003.

[VP] Ch. de la Vallée Poussin, *Quatre leçons sur les fonctions quasi-analytiques de variables réelles,*Bulletin de la Société Mathématique de France, 52, 175-203, 1924.

[W1] A. Wilkie, *On the theory of the real exponential field,* Illinois Journal of Math., 33, 3,384-408, 1989.

[W2], A. Wilkie, *Schanuel's conjecture and the decidability of the real exponential field,* Algebraic Model Theory, Kluwer,223-230, 1997.