G-COMPACTNESS AND GROUPS

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Abstract. Lascar described E_{KP} as a composition of E_L and the topological closure of E_L ([1]). We generalize this result to some other pairs of equivalence relations.

Motivated by an attempt to construct a new example of a non-G-compact theory, we consider the following example. Assume G is a group definable in a structure M. We define a structure M' consisting of M and X as two sorts, where X is an affine copy of G and in M' we have the structure of M and the action of G on X. We prove that the Lascar group of M' is a semi-direct product of the Lascar group of M and G/G_L. We discuss the relationship between G-compactness of M and M'. This example may yield new examples of non-G-compact theories.

1. Introduction

Let T be a complete theory in language L. We work within a monster model $\mathfrak{C} \models T$. A model $M \models T$ is small if $M \prec \mathfrak{C}$ and $|M| = |T|$. If X is a subset of a topological space, then by $\operatorname{int}(X)$ we denote its interior and by $\operatorname{cl}(X)$ its closure. We recall some well known facts about the Lascar Group and Lascar strong types (see [1, 9]). The group of Lascar strong automorphisms is defined by:

$$\operatorname{Autf_L}(\mathfrak{C}) = \big\langle \operatorname{Aut}(\mathfrak{C}/M)\colon M \text{ is a small model } \big\rangle,$$

and the *Lascar (Galois) group* of T by:

$$\operatorname{Gal_L}(T) = \operatorname{Aut}(\mathfrak{C})/\operatorname{Autf_L}(\mathfrak{C}).$$

This definition does not depend on the choice of the monster model $\mathfrak{C}$ of T (it is enough that $\mathfrak{C}$ is $|T|^+$-saturated and $|T|^+$-strongly homogeneous). We say that $a, b \in \mathfrak{C}^k$ ($k < |T|^+$) have the same Lascar strong type, and write $E_L(a, b)$, if there exists $f \in \operatorname{Autf_L}(\mathfrak{C})$ such that $a = f(b)$. Thus E_L is a $\emptyset$-invariant and bounded equivalence relation on every sort $\mathfrak{C}^k$ (because if $a \underset{M}{\equiv} b$ for some small $M \prec \mathfrak{C}$, then $E_L(a, b)$, so $|\mathfrak{C}^k/E_L| \leq |S_k(M)| \leq 2^{|T|}$).

Definition 1.1. A symmetric formula $\varphi(x, y) \in L_{k+k}(\emptyset)$ is *thick* if for some $n < \omega$, for every sequence $(a_i)_{i<n}$ there exist $i < j < n$ such that $\varphi(a_i, a_j)$. By Θ we denote the conjunction of all thick formulas:

$$\Theta(x, y) = \bigwedge_{\varphi \text{ thick}} \varphi(x, y).$$

In the above definition we can equivalently take an infinite sequence $(a_i)_{i<\omega}$. If φ and θ are thick, then $\psi(x, y) = \varphi(y, x)$ and $\varphi \wedge \theta$ are also thick (this follows from the Ramsey theorem). Θ is a $\emptyset$-invariant relation (not necessarily an equivalence relation) and if $\Theta(a_0, a_1)$, then we can extend (a_0, a_1) to an order indiscernible sequence $(a_i)_{i<\omega}$.

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On the other hand if $(a_i)_{i<\omega}$ is a 2-indiscernible sequence, then $\varphi(a_0, a_1)$ for every thick φ.

Lemma 1.2. [9, Lemma 7]

(i) *If $\Theta(a,b)$, then there is a small M such that $a \underset{M}{\equiv} b$.*

(ii) *If for some small M we have $a \underset{M}{\equiv} b$, then $\Theta^2(a,b)$, i.e. there is c such that $\Theta(a,c) \wedge \Theta(c,b)$.*

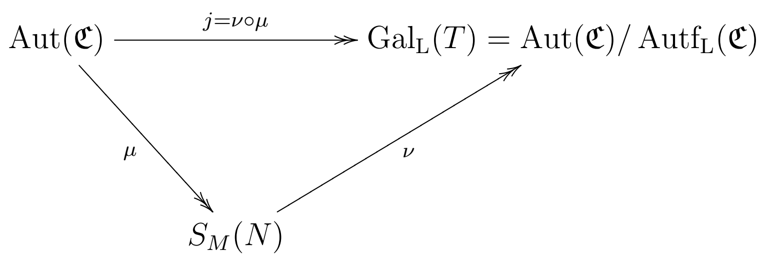
(iii) *E_L is the transitive closure of Θ.*

If π is a type over $\emptyset$, then we can define thick formulas on $\pi(\mathfrak{C})$ and their conjunction Θ_π similarly as in the above definition. Moreover, the last remark also holds for Θ_π, so $E_L|_{\pi(\mathfrak{C})}$ is the transitive closure of Θ_π. One can prove that $E_L|_{\pi(\mathfrak{C})}$ is the finest bounded $\emptyset$-invariant equivalence relation on $\pi(\mathfrak{C})$.

There is a compact (not necessarily Hausdorff) topology on the group $\mathrm{Gal_L}(T)$. Let M and N be arbitrary small models and let

$$S_M(N) = \{\mathrm{tp}(M'/N) \colon \mathrm{tp}(M') = \mathrm{tp}(M)\}$$

be a closed subset of $S_{|T|}(N)$. Thus $S_M(N)$ carries a compact subspace topology. The quotient map $j\colon \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal_L}(T)$ factors as $j = \nu \circ \mu$, where $\mu\colon \mathrm{Aut}(\mathfrak{C}) \to S_M(N)$ maps f to $\mathrm{tp}(f(M)/N)$, and $\mu\colon S_M(N) \to \mathrm{Gal_L}(T)$ maps $\mathrm{tp}(f(M)/N)$ to an appropriate coset of $\mathrm{Autf_L}(\mathfrak{C})$, so we have the following commutative diagram:



We can induce topology on $\mathrm{Gal_L}(T)$ from ν, i.e. $X \subseteq \mathrm{Gal_L}(T)$ is closed if and only if its preimage $\nu^{-1}[X]$ is closed in $S_M(N)$. It can be easily seen that this definition of topology does not depend on the choice of small models M and N ([9, Theorem 4]). With this topology $\mathrm{Gal_L}(T)$ becomes a compact topological group. We say that T is G-compact when $\mathrm{Gal_L}(T)$ is Hausdorff. If we consider $\mathrm{Aut}(\mathfrak{C})$ with the usual topology of pointwise convergence, then all the maps in the diagram are continuous. However ν need not be open, instead ν satisfies some weak kind of openness.

Theorem 1.3. [9, Lemma 12] *For $p \in S_M(N)$ define its Θ-neighbourhood as:*

$$[p]_\Theta = \{q \in S_M(N) \colon p(x) \cup q(y) \cup \Theta(x,y) \text{ is consistent }\}.$$

If we take an arbitrary point $p \in S_M(N)$ and subset $U \subseteq S_M(N)$ such that $[p]_\Theta \subseteq \mathrm{int}(U)$, then $\nu(p) \in \mathrm{int}(\nu[U])$.

The relation E_L is $\emptyset$-invariant, so we may consider E_L as a subset of $S_{|T|+|T|}(\emptyset)$. Using this, we define the relation $\overline{E_L}$ as $\mathrm{cl}(E_L)$. $\overline{E_L}$ is $\emptyset$-invariant and contains E_L. There exists the finest bounded $\bigwedge$-definable over $\emptyset$ equivalence relation, denoted by E_{KP} and known as equality of Kim-Pillay strong types (there is also an appropriate group of automorphisms $\mathrm{Autf_{KP}}(\mathfrak{C})$ such that $E_{KP}(a,b)$ if and only if for some $f \in \mathrm{Autf_{KP}}(\mathfrak{C})$, $a = f(b)$). The next theorem describes some relationship between E_{KP}, Θ and E_L.

Theorem 1.4. [1, Corollary 2.6] $E_{KP} = \Theta \circ \overline{E_L}$

An attempt to understand the proof of this theorem was a starting point of this paper. In particular it was puzzling what properties of E_L, E_{KP} and Θ are responsible for the relationship described in Theorem 1.4. It turnes out that the important point here is that both E_L and E_{KP} are orbit equivalence relations with respect to some groups of automorphisms of $\mathfrak{C}$. We elaborate on this in Section 2. We generalize Theorem 1.4 there and give a new proof of it based on Theorem 1.3. Also in Section 2 we generalize some results about Lascar, Kim-Pillay and Shelah strong types.

Section 3 contains a model-theoretic analysis of a structure $N = (M, X, \cdot)$, where M is a given stucture and X is affine copy of some group G definable in M. We describe the group of automorphisms of N as a semi-direct product of G and the group of automorphisms of M. In particular we reduce the question of G-compactness of N to the question of $\bigwedge$-definability of certain subgroup G_L of G. This motivates us to look for examples of G, where G_L is not $\bigwedge$-definable.

In Section 4 we verify that G_L is $\bigwedge$-definable in several cases, e.g. when M is small or simple or o-minimal and G is definable compact.

In Section 5 we provide an example where a subgroup of G, similar in some sense to G_L, is not $\bigwedge$-definable, and also an example of a group G that is not G-compact.

We assume that the reader is familiar with basic notions of model theory.

The results in Sections 2, 3 and 4 are due to the first author, the proof of Lemma 3.7(1) and the examples in Section 5 are due to the second author.

2. Orbit equivalence relations

In this section G is always a subgroup of $\mathrm{Aut}(\mathfrak{C})$. We can consider the orbit equivalence relation E_G defined as follows: $E_G(a, b)$ if and only if there is some $f \in G$ with $a = f(b)$, where a and b are tuples of elements of $\mathfrak{C}$ of length $\leq |T|$, such tuples are called small. In this paper we consider E_G as an equivalence relation on the sets of small tuples of elements of various sorts of $\mathfrak{C}$.

The results of this section are concerned with various properties of relations of the form E_G. Our motivation is based on the observation that almost all important equivalence relations in model theory (e.g. E_L, E_{KP} and E_{Sh}) are of this form.

Some statements from the next proposition are probably well known (see [1, 5, 7, 9]).

Lemma 2.1. (i) *Let M be an arbitrary small model, then*

$$G \cdot \mathrm{Aut}(\mathfrak{C}/M) = \{f \in \mathrm{Aut}(\mathfrak{C}) \colon E_G(M, f(M))\}.$$

(ii) *The relation E_G is $\emptyset$-invariant on every sort if and only if for every small $M \prec \mathfrak{C}$ and every $F \in \mathrm{Aut}(\mathfrak{C})$*

$$G \subseteq G^F \cdot \mathrm{Aut}(\mathfrak{C}/M).$$

In particular if G contains $\bigcup_{F\in \mathrm{Aut}(\mathfrak{C})} \mathrm{Aut}(\mathfrak{C}/F[M])$ for some small M, then E_G is $\emptyset$-invariant if and only if $G \lhd \mathrm{Aut}(\mathfrak{C})$.

(iii) *If G has bounded index in $\mathrm{Aut}(\mathfrak{C})$, then E_G is bounded and $E_L \subseteq E_G$. If E_G is $\emptyset$-invariant bounded $G \lhd \mathrm{Aut}(\mathfrak{C})$ and G contains $\mathrm{Aut}(\mathfrak{C}/M)$ for some small M, then G has bounded index in $\mathrm{Aut}(\mathfrak{C})$.*

(iv) *Let $j\colon \mathrm{Aut}(\mathfrak{C}) \longrightarrow \mathrm{Gal_L}(T)$ be the quotient map and assume that $\mathrm{Autf_L}(\mathfrak{C}) \subseteq G$.*

(a) *$j[G]$ is closed in $\mathrm{Gal_L}(T)$ if and only if E_G is $\bigwedge$-definable over any small model. If $G \lhd \mathrm{Aut}(\mathfrak{C})$, then $\bigwedge$-definability is over $\emptyset$.*

(b) *$j[G]$ is open in $\mathrm{Gal_L}(T)$ if and only if $G = \mathrm{Aut}(\mathfrak{C}/e)$ for some $e \in \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ (i.e. $e = \overline{m}/F$ for some $\emptyset$-definable finite equivalence relation F on some $\mathfrak{C}^n, n < \omega$).*

Proof. (i) Easy.

(ii) Without loss of generality we may work with small models, because every tuple a may be extended to small model M. Take an arbitrary small $M \prec \mathfrak{C}$, $g \in G$ and $F \in \mathrm{Aut}(\mathfrak{C})$. Then $E_G(M, g(M))$. Assume that E_G is $\emptyset$-invariant. Then $E_G(F(M), F(g(M)))$ holds, so for some $g' \in G$, $F(g(M)) = g'(F(M))$. Thus $F^{-1} \circ g'^{-1} \circ F \circ g \in \mathrm{Aut}(\mathfrak{C}/M)$, so $g \in g'^F \circ \mathrm{Aut}(\mathfrak{C}/M) \subseteq G^F \circ \mathrm{Aut}(\mathfrak{C}/M)$. The other implication is similar.

For the second statement of (ii) assume that $G \subseteq G^F \cdot \mathrm{Aut}(\mathfrak{C}/M)$. Then conjugating by F^{-1} we obtain

$$G^{F^{-1}} \subseteq G \cdot \mathrm{Aut}(\mathfrak{C}/F[M]) = G,$$

for an appropriate small model M.

(iii) If G has bounded index in $\mathrm{Aut}(\mathfrak{C})$, then there is a normal subgroup $H \lhd \mathrm{Aut}(\mathfrak{C})$ of bounded index, with $H \subseteq G$ (an intersection of boundedly many conjugates of G). Thus E_H is bounded and invariant, so $E_L \subseteq E_H \subseteq E_G$.

For the second statement we use (i) to conclude that $G = \mathrm{Aut}(\mathfrak{C}/\ulcorner M/E_G \urcorner)$. G has bounded index, because M/E_G has boundedly many conjugates.

(iv) Note that $j^{-1}[j[G]] = G \cdot \mathrm{Autf_L}(\mathfrak{C}) = G$, thus $\mu[G] = \nu^{-1}[j[G]]$ (because $j = \nu \circ \mu$).

(a) $\Rightarrow$: Let M be an arbitrary small model. If $j[G]$ is closed in $\mathrm{Gal_L}(T)$, then $\mu[G] = \nu^{-1}[j[G]] = \{\mathrm{tp}(M'/M) \colon \Phi(M', M)\}$ for some type $\Phi(x, y)$ over $\emptyset$. We have that

$$E_G(a, b) \iff (\exists f \in \mathrm{Aut}(\mathfrak{C}))\, (a = f(b) \wedge \Phi(f(M), M))\,,$$

and thus E_G is $\bigwedge$-definable over M:

$$E_G(a, b) \iff (\exists z)(\mathrm{tp}(b, M) = \mathrm{tp}(a, z) \wedge \Phi(z, M)).$$

$\Leftarrow$: There is a type $\Phi(x, y)$ over M such that

$$E_G(a, b) \iff \Phi(a, b).$$

Since $\mu[G] = \nu^{-1}[j[G]]$ it is enough to prove that $\mu[G]$ is closed in $S_M(M)$. This is clear, because:

$$\mu[G] = \{\mathrm{tp}(g(M)/M) \colon g \in G\} = \{\mathrm{tp}(M'/M) \colon \Phi(M', M)\}.$$

(b) $\Rightarrow$: First we deal with the case where $G \lhd \mathrm{Aut}(\mathfrak{C})$. Since $\mathrm{Gal_L}(T)$ is a compact topological group, $j[G]$ has finite index in $\mathrm{Gal_L}(T)$, hence it is closed. By (iva) E_G is $\emptyset$-$\bigwedge$-definable. Also G has finite index in $\mathrm{Aut}(\mathfrak{C})$. It follows that E_G has finitely many classes on $\mathrm{tp}(M)(\mathfrak{C})$ (the set of realisations of type $\mathrm{tp}(M)$) and from (i) we have $G = \mathrm{Aut}(\mathfrak{C}/(M/E_G))$. Hence there are a finite $\emptyset$-definable equivalence relation F and $\overline{m} \subset M$ such that $G = \mathrm{Aut}(\mathfrak{C}/(\overline{m}/F))$.

Now we deal with the general case, so $G < \mathrm{Aut}(\mathfrak{C})$ need not be normal. However, still G has finite index in $\mathrm{Aut}(\mathfrak{C})$. Hence there is a normal subgroup $H \lhd \mathrm{Aut}(\mathfrak{C})$ contained in G and such that $j[H]$ is open (an intersection of finitely many conjugates of G). We may apply the first case to H. We get an $e \in \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ such that $H = \mathrm{Aut}(\mathfrak{C}/e)$. An element e has finitely many conjugates, so $e' = \ulcorner\{g \cdot e \colon g \in G\}\urcorner \in \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Now it is obvious that $G = \mathrm{Aut}(\mathfrak{C}/e')$.

$\Leftarrow$: The subset $\nu^{-1}[j[G]] = \mu[G] = \{\mathrm{tp}(f(M)/M)\colon F(\overline{m}, f(\overline{m})), f \in \mathrm{Aut}(\mathfrak{C})\}$ of $S_M(M)$ is clopen. □

Problem 2.2. Consider an equivalence relation E on sorts of $\mathfrak{C}$ which is $\emptyset$-invariant. Then we can build the following growing sequence of $\emptyset$-invariant relations:

(i) $E_0 = E$,
(ii) $E_1 = \mathrm{cl}(E)$ in $S_{k+k}(\emptyset)$,
(iii) for $1 \leq \alpha \in Ord$ let
- $E_{\alpha+1} = \mathrm{cl}(\text{transitive closure of } E_\alpha)$,
- if $\alpha \in Lim$, then $E_\alpha = \bigcup_{\lambda<\alpha} E_\lambda$.

Take $E_\infty = \bigcup_{\alpha \in Ord} E_\alpha$. Then clearly E_∞ is the finest type definable equivalence relation which extends E, so we may ask the question: what is the first α_E for which $E_{\alpha_E} = E_\infty$? If $E = E_G$, where $\mathrm{Autf_L}(\mathfrak{C}) \subseteq G \lhd \mathrm{Aut}(\mathfrak{C})$, then from the next Theorem 2.3(ii) we conclude that $\alpha_E \leq 2$.

It can be proved that $\mathrm{Autf_{KP}}(\mathfrak{C}) = j^{-1}[\mathrm{cl}(\mathrm{id}_{\mathrm{Gal_L}(T)})]$. Recall that $E_{KP} = E_{\mathrm{Autf_{KP}}(\mathfrak{C})}$ is the finest bounded $\bigwedge$-definable over $\emptyset$ equivalence relation. The next Theorem 2.3(i) generalizes this remark and Theorem 1.4 to an arbitrary group of automorphisms containing $\mathrm{Autf_L}(\mathfrak{C})$.

Theorem 2.3. *Let* $\mathrm{Autf_L}(\mathfrak{C}) \subseteq G < \mathrm{Aut}(\mathfrak{C})$ *and consider* $\overline{\overline{G}} = j^{-1}[\mathrm{cl}(j[G])]$*. Then*

(i) *On each sort of* $\mathfrak{C}$ *the relation* $E_{\overline{\overline{G}}}$ *is the finest bounded* $\bigwedge$*-definable over any small model equivalence relation which extends* E_G*.*
(ii) *If additionally* $G \lhd \mathrm{Aut}(\mathfrak{C})$*, then*

$$E_{\overline{\overline{G}}} = \Theta \circ \overline{E_G},$$

where $\overline{E_G}$ *is* $\mathrm{cl}(E_G)$ *in* $S_{k+k}(\emptyset)$*.*

Proof. (i) Let E be a $\bigwedge$-definable over M equivalence relation and $E_G \subseteq E$. Take an arbitrary $f \in \overline{\overline{G}}$ and a small tuple b. We have to prove that $E(f(b), b)$. Consider the following set

$$H = \{f \in \mathrm{Aut}(\mathfrak{C})\colon E(f(b), b)\}$$

(H is not necessarily a group, because E is not necessarily $\emptyset$-invariant). It is enough to show that $\overline{\overline{G}} \subseteq H$.

Note that $j^{-1}[j[H]] = \mathrm{Autf_L}(\mathfrak{C}) \cdot H = H$, because for $f \in \mathrm{Autf_L}(\mathfrak{C})$, $h \in H$ we have $E(h(b), b)$ and $E(f(h(b)), h(b))$ ($E_L \subseteq E$), so $E(f(h(b)), b)$ and $f \circ h \in H$.

Since $E_G \subseteq E$ we have $G \subseteq H$, so we must only prove that $\mathrm{cl}(j[G]) \subseteq j[H]$ (because $j^{-1}[j[H]] = H$). The proof is completed by showing that $j[H]$ is closed in $\mathrm{Gal_L}(T)$. This follows from the fact that the set

$$\nu^{-1}[j[H]] = \mu[H] = \{\mathrm{tp}(f(M')/M')\colon E(f(b), b), f \in \mathrm{Aut}(\mathfrak{C})\}$$

is closed in $S_{M'}(M')$, where $Mb \subseteq M' \prec \mathfrak{C}$.

(ii) The relation $E_{\overline{\overline{G}}}$ is $\bigwedge$-definable over $\emptyset$, so $E_{\overline{\overline{G}}}$ is a closed subset of $S_{k+k}(\emptyset)$, thus $\overline{E_G} \subseteq E_{\overline{\overline{G}}}$. This gives $\Theta \circ \overline{E_G} \subseteq E_{\overline{\overline{G}}}$.

Now we prove that $E_{\overline{\overline{G}}} \subseteq \Theta \circ \overline{E_G}$. Assume that a, b are small tuples such that $E_{\overline{\overline{G}}}(a, b)$, i.e. $a = f(b)$ for some $f \in \overline{\overline{G}}$. Without loss of generality we may assume that $b = M$, for some small $M \prec \mathfrak{C}$, so $a = f(M)$. Let $p = \mu(f) = \mathrm{tp}(f(M)/M)$. Then $\nu(p) = j(f) \in \mathrm{cl}(j[G])$ and

$$[p]_\Theta \cap \mathrm{cl}(\nu^{-1}[j[G]]) \neq \emptyset,$$

because otherwise $[p]_\Theta \subseteq \mathrm{int}(\nu^{-1}[j[G]^c])$, and from Theorem 1.3

$$\nu(p) \in \mathrm{int}(\nu[\nu^{-1}[j[G]^c]]) = \mathrm{int}(j[G]^c) = \mathrm{cl}(j[G])^c,$$

a contradiction.

Let $q \in [p]_\Theta \cap \mathrm{cl}(\nu^{-1}[j[G]])$. There is some $c = M' \models q$ such that $\Theta(f(M), M')$ and $q = \mathrm{tp}(M'/M)$ is in

$$\begin{aligned}\mathrm{cl}(\nu^{-1}[j[G]]) = \mathrm{cl}(\mu[G]) &= \mathrm{cl}\{\mathrm{tp}(g(M)/M) \colon g \in G\} \\ &= \mathrm{cl}\{\mathrm{tp}(g(M)/M) \colon E_G(g(M), M)\}.\end{aligned}$$

Finally $\mathrm{tp}(M', M) \in \mathrm{cl}(E_G) = \overline{E_G}$, and we obtain that $\Theta(a, c)$ and $\overline{E_G}(c, b)$. □

Now we consider the relation E_{Sh} of equality of Shelah strong types:

$$E_{Sh} = \bigcap\{E \colon E \text{ is a } \emptyset\text{-definable finite equivalence relation}\}.$$

It can be proved that $E_{Sh} = E_{j^{-1}[\mathrm{QC}]}$, where $\mathrm{QC} \lhd \mathrm{Gal_L}(T)$ is the intersection of all open subgroups of $\mathrm{Gal_L}(T)$ (the quasi-connected component). When $\mathrm{Gal_L}(T)$ is Hausdorff (i.e. T is G-compact) then QC is just the connected component of $\mathrm{Gal_L}(T)$.

In the next proposition we generalize this property of E_{Sh}, but first we need a definition: if $A \subseteq \mathrm{Gal_L}(T)$, then by $\mathrm{QC}(A)$ we denote the following set

$$\bigcap\{H < \mathrm{Gal_L}(T) \colon A \subseteq H \text{ and } H \text{ is open}\}.$$

Proposition 2.4. *If $H < \mathrm{Gal_L}(T)$, then $E_{j^{-1}[\mathrm{QC}(H)]}$ is the intersection of all $\emptyset$-definable finite equivalence relations which extend $E_{j^{-1}[H]}$:*

$$E_{j^{-1}[\mathrm{QC}(H)]} = \bigcap\{E \colon E \textit{ is a } \emptyset\textit{-definable finite e.r. and } E_{j^{-1}[H]} \subseteq E\}.$$

Moreover $j^{-1}[\mathrm{QC}(H)]$ is equal to the group of all $f \in \mathrm{Aut}(\mathfrak{C})$, satisfying

$$E_{j^{-1}[\mathrm{QC}(H)]}(a, f(a))$$

for arbitrary small tuple a.

Proof. First we prove the equality of relations. ($\subseteq$) Assume that small tuples a, b are $E_{j^{-1}[\mathrm{QC}(H)]}$ equivalent, so $a = f(b)$ for some $f \in j^{-1}[\mathrm{QC}(H)]$, and E is a $\emptyset$-definable finite equivalence relation extending $E_{j^{-1}[H]}$. Define

$$G' = \{f \in \mathrm{Aut}(\mathfrak{C}) \colon E(f(b), b)\} = \mathrm{Aut}(\mathfrak{C}/(b/E)).$$

Then $H \subseteq j[G']$ and $j[G']$ is open as a subset of $\mathrm{Gal_L}(T)$ (Lemma 2.1(iv)(b)). Therefore $\mathrm{QC}(H) \subseteq j[G']$ and

$$f \in j^{-1}[\mathrm{QC}(H)] \subseteq j^{-1}[j[G']] = G' \cdot \mathrm{Autf_L}(\mathfrak{C}) = G',$$

so $E(a, b)$ holds.

($\supseteq$) Let $\mathrm{QC}(H) = \bigcap\{G_i \colon i \in I\}$. Using Lemma 2.1(iv)(b) we can find $(e_i)_{i \in I} \subseteq \mathrm{acl^{eq}}(\emptyset)$ such that $j^{-1}[G_i] = \mathrm{Aut}(\mathfrak{C}/e_i)$. Then $j^{-1}[QC(H)] = \mathrm{Aut}(\mathfrak{C}/\{e_i\}_{i \in I})$. We can assume that $e_i = m_i/F_i$ for some $\emptyset$-definable finite equivalence relations F_i. Assume that (a, b) belongs to

$$\bigcap\{E \colon E \text{ is a } \emptyset\text{-definable finite e.r. and } E_{j^{-1}[H]} \subseteq E\}.$$

We have to find $f \in j^{-1}[QC(H)]$ for which $b = f(a)$. It suffices to prove that the following type in variables $(y_i)_{i\in I}$ is consistent:

$$\text{“}\operatorname{tp}(b, y_i)_{i\in I} = \operatorname{tp}(a, m_i)_{i\in I}\text{”} \wedge \bigwedge_{i\in I} F_i(m_i, y_i).$$

Let $\varphi(x, x_1, \ldots, x_n) \in \operatorname{tp}(a, m_i)_{i\in I}$. It is enough to show that

$$\psi(b, m_1, \ldots, m_n) = (\exists y_1, \ldots, y_n)\left(\varphi(b, y_1, \ldots, y_n) \wedge \bigwedge_{1\le i\le n} F_i(m_i, y_i)\right)$$

holds. The formula $\psi = \psi(x, m_1, \ldots, m_n)$ is almost over $\emptyset$ (because $m_1, \ldots, m_n \in \operatorname{acl}(\emptyset)$). Let $\psi_1, \ldots, \psi_k$ be all conjugates of ψ over $\emptyset$ and take

$$A(x, y) = \bigwedge_{1\le i\le k} (\psi_i(x) \leftrightarrow \psi_i(y)).$$

A is a $\emptyset$-definable finite equivalence relation and $E_{j^{-1}[H]} \subseteq A$ (because $j^{-1}[H] \subseteq j^{-1}[QC(H)] = \operatorname{Aut}(\mathfrak{C}/(m_i/F_i))$). Therefore $A(a, b)$ and we know that $\psi(a, m_1, \ldots, m_n)$ holds, so $\psi(b, m_{\le n})$ also holds.

Now we prove the second part of the proposition. Let G' be the group of all automorphisms preserving $E_{j^{-1}[\mathrm{QC}(H)]}$. Inclusion $j^{-1}[\mathrm{QC}(H)] \subseteq G'$ is obvious.

($\supseteq$) Let $g \in G'$ and $a = M$ be a small model. Then $E_{j^{-1}[\mathrm{QC}(H)]}(M, g(M))$, so $g(M) = f(M)$ for some $f \in j^{-1}[\mathrm{QC}(H)]$. Thus $j(g) = j(f)$ (because $gf^{-1} \in \operatorname{Autf}_{\mathrm{L}}(\mathfrak{C})$) and $j(f) \in \mathrm{QC}(H)$. Therefore $g \in j^{-1}[\mathrm{QC}(H)]$. □

3. An Example

Let M be an arbitrary structure in which we have a $\emptyset$-definable (interpretable) group G. In this section we consider the following two sorted structure: $N = (M, X, \cdot)$, where

- X and M are disjoint sorts,
- $\cdot\colon G \times X \to X$ is a regular (free and transitive) action of G on X i.e. X is an affine copy of G,
- on M we take its original structure.

This structure was already considered e.g. in [9, 7]. Our study of N is based on ideas from [9, Section 7].

In this section we describe various groups of automorphisms of N in terms of appropriate groups of automorphisms of M and groups related to G. We also give a description of the relations E_L, E_{KP} and E_{Sh} on the sort X of N. In particular, in Corollary 3.6 we prove that G-compactness of N is equivalent to G-compactness of M and $\bigwedge$-definability of certain subgroup G_L of G. Thus constructing a group G where the subgroup G_L is not $\bigwedge$-definable may yield a new example of a non-G-compact theory.

Fix an arbitrary point x_0 from X and take $N^* = (M^*, X^*, \cdot)$, a monster model extending N. Then $G \subseteq G^*$ and $X = G \cdot x_0 \subseteq G^* \cdot x_0 = X^*$.

The group G^* acts on itself in two different, but commuting ways, the first one is by left translation $(g, h) \mapsto gh$, and the second one by the following rule $(g, h) \mapsto hg^{-1}$. We define homomorphic embeddings of automorphism groups:

$$\bar{\cdot}\colon \operatorname{Aut}(M^*) \hookrightarrow \operatorname{Aut}(N^*), \quad \tilde{\cdot}\colon G^* \hookrightarrow \operatorname{Aut}(N^*).$$

Let $h \in G^*, f \in \mathrm{Aut}(M^*), g \in G^*$. We define $\overline{f}, \overline{g} \in \mathrm{Aut}(N^*)$ by:

$$\overline{f}|_{M^*} = f,\ \overline{f}(h \cdot x_0) = f(h) \cdot x_0,$$

$$\overline{g}|_{M^*} = \mathrm{id}_{M^*},\ \overline{g}(h \cdot x_0) = (hg^{-1}) \cdot x_0.$$

It is easy to verify the following laws: for $f \in \mathrm{Aut}(M^*)$, $g \in G^*$ we have

$$\overline{f} \circ \overline{g} = \overline{f(g)} \circ \overline{f},\ \ \overline{g} \circ \overline{f} = \overline{f} \circ \overline{f^{-1}(g)}.$$

Using these embeddings we can identify $\mathrm{Aut}(M^*)$ and G^* with their images in $\mathrm{Aut}(N^*)$ and conclude that $G^* \lhd \mathrm{Aut}(N^*)$. In fact we will prove that $\mathrm{Aut}(N^*)$ is a semi-direct product of G^* and $\mathrm{Aut}(M^*)$.

There are two different actions of the group G^* on the set X^*: the first one comes from the above embedding

$$\overline{g}(h \cdot x_0) = (hg^{-1}) \cdot x_0$$

(it is definable over x_0). The second one comes from the regular action

$$g \cdot (h \cdot x_0) = (gh) \cdot x_0.$$

If $A \subseteq G^*$ satisfies $hA^{-1} = Ah$, then the orbits of $h \cdot x_0$ under both actions coincide: $\overline{A}(h \cdot x_0) = A \cdot (h \cdot x_0)$ (in this case we just write $A \cdot (h \cdot x_0)$).

In order to describe properties of N^* in terms of M^* and G^* we need the next definition.

Definition 3.1. For a group G and a binary relation E on G we define the set of E-commutators $X_E = \{a^{-1}b \colon a, b \in G, E(a,b)\}$ and the E-commutant G_E as the subgroup of G generated by X_E

$$G_E = \langle X_E \rangle < G.$$

Remark 3.2. If $E = E_H$ for some $H < \mathrm{Aut}(G, \cdot)$, then $G_{E_H} \lhd G$. If E is $\emptyset$-invariant, then X_E and G_E are also $\emptyset$-invariant. If E is bounded, then G_E has bounded index in G, moreover $[G : G_E] \leq |G/E|$.

Proof. Let $a, x \in G$ and $h \in H$. Then

$$(X_{E_H})^x \ni (a^{-1}h(a))^x = (ax)^{-1}h(a)x = ((ax)^{-1}h(ax))(h(x)^{-1}x) \in X^2_{E_H}.$$

The last statement follows from the observation: if $a^{-1}b \notin G_E$, then $\neg E(a,b)$. ☐

The following example justifies the names "E-commutators" and "E-commutant" from the previous definition. Let E be the conjugation relation in G i.e. $E = E_{\mathrm{Inn}(G)}$ (where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G). Then X_E is the set of all commutators and $G_E = [G,G]$.

In the case where $E = E_L$ [$E = E_{KP}, E_{Sh}$, respectively] we just write X_L and G_L [X_{KP}, X_{Sh}] instead of X_{E_L} and G_{E_L} [$X_{E_{KP}}$, $X_{E_{Sh}}$]. Note that G_L is generated by X_Θ.

In the next proposition we describe $\mathrm{Aut}(N^*), \mathrm{Autf_L}(N^*)$ and $\mathrm{Gal_L}(\mathrm{Th}(N))$ as semidirect products of automorphisms groups of M^* and appropriate groups associated with G.

Proposition 3.3. (1) $\mathrm{Aut}(N^*) = G^* \rtimes \mathrm{Aut}(M^*)$, *more precisely: for* $F \in \mathrm{Aut}(N^*)$, $F = \overline{g} \circ \overline{f}$, *where* $f = F|_{M^*}$ *and* $F(x_0) = g^{-1} \cdot x_0$.

(2) *Let* $(N', X') \prec (N^*, X^*)$ *and* $X' = G' \cdot (h_0 \cdot x_0)$ *for some* $h_0 \cdot x_0 \in X'$. *Then*

$$F \in \mathrm{Aut}(N^*/N') \iff (\exists f \in \mathrm{Aut}(M^*/M')) \left(F = \overline{f}^{\overline{h_0}}\right).$$

(3) $\mathrm{Autf_L}(N^*) = G^*_L \rtimes \mathrm{Autf_L}(M^*)$ *and* $\mathrm{Gal_L}(\mathrm{Th}(N)) = G^*/G^*_L \rtimes \mathrm{Gal_L}(\mathrm{Th}(M))$.

Proof. (1) Let $F \in \mathrm{Aut}(N^*)$ and $f = F|_{M^*}$. Then $F\overline{f}^{-1}$ is the identity on M^*, and on $X^* = G^* \cdot x_0$ we have:

$$F\overline{f}^{-1}(h \cdot x_0) = F(f^{-1}(h) \cdot x_0) = h \cdot F(x_0) = h \cdot (g^{-1} \cdot x_0) = \overline{g}(h \cdot x_0),$$

for some $g \in G^*$. Thus $F = \overline{g} \circ \overline{f}$. The group $\mathrm{Aut}(M^*)$ acts on G^* by conjugation, so for $g \in G^*$ and $f \in \mathrm{Aut}(M^*)$, $\overline{g}^{\overline{f}} = \overline{f(g)} \in G^*$. It is clear that $G^* \cap \mathrm{Aut}(M^*) = \{0\}$.

(2) ($\Leftarrow$) It is clear that $F|_{M'} = \mathrm{id}_{M'}$. Using the fact that $f|_{M'} = \mathrm{id}_{M'}$ we get for $h' \in X'$:

$$\overline{f}^{\overline{h_0}}(h'h_0 \cdot x_0) = \overline{h_0^{-1}} \circ \overline{f}(h' \cdot x_0) = \overline{h_0^{-1}}(h' \cdot x_0) = h'h_0 \cdot x_0.$$

Thus $\overline{f}^{\overline{h_0}}|_{X'} = \mathrm{id}_{X'}$.

($\Rightarrow$) Let $f = F|_{M^*}$. Then $f = \overline{f}^{\overline{h_0}}|_{M^*} \in \mathrm{Aut}(M^*/M')$. By assumptions

$$h_0 \cdot x_0 = F(h_0 \cdot x_0) = F(h_0) \cdot F(x_0) = f(h_0) \cdot F(x_0),$$

and then $F(x_0) = f(h_0^{-1})h_0 \cdot x_0$. By (1), $F = \overline{h_0^{-1}f(h_0)} \circ \overline{f} = \overline{h_0^{-1}} \circ \overline{f} \circ \overline{h_0} = \overline{f}^{\overline{h_0}}$.

(3) It suffices to prove the first equality. $\subseteq$: From (2) we conclude that for every $F \in \mathrm{Autf_L}(N^*)$ there are $h_1, \ldots, h_n \in G^*$ and $f_1, \ldots, f_n \in \mathrm{Autf_L}(M^*)$ such that $F = \overline{f_1}^{\overline{h_1}} \circ \ldots \circ \overline{f_n}^{\overline{h_n}}$. Then

$$F = \overline{h_1^{-1}f_1(h_1)} \circ \overline{f_1} \circ \overline{h_2^{-1}f_2(h_2)} \circ \overline{f_2} \circ \ldots \circ \overline{h_n^{-1}f_n(h_n)} \circ \overline{f_n}.$$

Using the rule $\overline{f} \circ \overline{g} = \overline{f(g)} \circ \overline{f}$, one can prove that $F = \overline{g} \circ \overline{f_1 \ldots f_n}$, for some $g \in G_L$ (for example $\overline{f_1} \circ \overline{h_2^{-1}f_2(h_2)} = \overline{f_1(h_2^{-1})f_1(f_2(h_2))} \circ \overline{f_1}$, and $f_1(h_2^{-1})f_1(f_2(h_2)) = f_1(h_2)^{-1}f_2^{f_1^{-1}}(f_1(h_2)) \in X_L$).

$\supseteq$: It is clear that $\mathrm{Autf_L}(N^*) \supseteq \mathrm{Autf_L}(M^*)$ (use (2)). It is enough to prove that $\mathrm{Autf_L}(N^*) \supseteq X_L$. Assume that small tuples a, b satisfy $b = f(a)$, for some $f \in \mathrm{Autf_L}(M^*)$. We have to prove that $\overline{a^{-1}b} \in \mathrm{Autf_L}(N^*)$. Since $\overline{f}^{\overline{a}} \in \mathrm{Autf_L}(N^*)$, we have $\overline{a^{-1}b} = \overline{a^{-1}f(a)} = \overline{f}^{\overline{a}} \circ \overline{f^{-1}} \in \mathrm{Autf_L}(N^*)$. ☐

Now we characterize some invariant subgroups of G^*: $G^0_\emptyset, G^{00}_\emptyset$ and $G^\infty_\emptyset$, in terms of N^*.

Proposition 3.4. (1) $G^*_L = G^* \cap \mathrm{Autf_L}(N^*)$ *and* G^*_L *is the smallest* $\emptyset$*-invariant subgroup of* G *with bounded index in* G^* *(i.e.* $G^*_L = G^\infty_\emptyset$*).*

(2) *Let* $G'_{KP} = G^* \cap \mathrm{Autf_{KP}}(N^*)$*, then* $G^*_{KP} \subseteq G'_{KP}$ *and* G'_{KP} *is the smallest* $\bigwedge$*-definable over* $\emptyset$ *subgroup with bounded index in* G^* *(i.e.* $G'_{KP} = G^{00}_\emptyset$*).*

(3) *Let* $G'_{Sh} = G^* \cap \mathrm{Autf_{Sh}}(N^*)$*, then* $G^*_{Sh} \subseteq G'_{Sh}$ *and* G'_{Sh} *is the intersection of all* $\emptyset$*-definable subgroups of* G^* *with finite index (i.e.* $G'_{Sh} = G^0_\emptyset$*).*

Proof. (1) The first equality follows directly from Proposition 3.3(3). Let $H < G^*$ be $\emptyset$-invariant with bounded index. It suffices to prove that $X_\Theta \subseteq H$. Take an order inscernible sequence $(a_n)_{n<\omega}$ (so $\Theta(a_0, a_1)$). If $a_0^{-1}a_1 \notin H$, then for every $i < j < \omega$, $a_i^{-1}a_j \notin H$, but we can extend an indiscernible sequence as much as we want, so the index $[G^* : H]$ is unboundedly large, a contradiction.

(2) If $N' \prec N^*$ is an arbitrary small model, then

$$G'_{KP} = \{g \in G^* \colon E_{KP}(N', \overline{g}(N'))\}.$$

Inclusion $\subseteq$ is obvious. $\supseteq$: If $E_{KP}(N', \overline{g}(N'))$, then $\overline{g}(N') = F(N')$ for some $F \in \mathrm{Autf}_{\mathrm{KP}}(N^*)$. Since $\overline{g}|_{M^*} = \mathrm{id}_{M^*}$, $F \in G^*$, and $\overline{g}F^{-1} \in \mathrm{Autf}_{\mathrm{L}}(N^*)$, so $\overline{g} \in G'_{KP}$.

G'_{KP} has bounded index ($G^*_L \subseteq G'_{KP}$) and is $\bigwedge$-definable over $N'x_0$. In fact G'_{KP} is $\emptyset$-invariant. To see this let $F = \overline{g'} \circ \overline{f'} \in \mathrm{Aut}(N^*)$. Then

$$F[G'_{KP}] = \{F(g) \colon E_{KP}(N', \overline{g}(N'))\} = \{F(g) \colon E_{KP}(F(N'), F(\overline{g}(N')))\},$$

but $F \circ \overline{g} = \overline{g'} \circ \overline{f'} \circ \overline{g} = \overline{g' f'(g) g'^{-1}} \circ F$, thus

$$\begin{aligned} F[G'_{KP}] &= \{f'(g) \colon E_{KP}(F(N'), \overline{g'} \circ \overline{f'(g) g'^{-1}}(F(N')))\} \\ &= \{f'(g) \colon E_{KP}(\overline{g'^{-1}}F(N'), \overline{f'(g)}(\overline{g'^{-1}}F(N'))\} = G'_{KP}, \end{aligned}$$

and hence G'_{KP} is $\bigwedge$-definable over $\emptyset$. The relation $E(x,y) = x^{-1}y \in G'_{KP}$ is bounded $\bigwedge$-definable over $\emptyset$, therefore $E_{KP}|_{G^*} \subseteq E$ and $G^*_{KP} \subseteq G'_{KP}$. Take $H < G^*$, another subgroup which is $\bigwedge$-definable over $\emptyset$ and has bounded index in G^*. Then $E_{KP} \subseteq E_H$, so for $g \in G'_{KP}$ we have $E_H(x_0, \overline{g}(x_0))$ and then $g^{-1} \cdot x_0 = \overline{g}(x_0) = \overline{h}(x_0) = h^{-1} \cdot x_0$ for some $h \in H$. By regularity of $\cdot$ we obtain $g = h \in H$.

(3) As in (2) it can be proved that G'_{Sh} is $\bigwedge$-definable over $\emptyset$. Let $g \in G'_{Sh}$, and $H < G^*$ be a $\emptyset$-definable subgroup with finite index in G^*. We show that $g \in H$. Consider the relation $E(x,y) = (\exists h \in H)(x = h \cdot y)$ on X^*. E is a $\emptyset$-invariant, finite equivalence relation on X^*, thus $E_{Sh}|_{X^*} \subseteq E$. By regularity of $\cdot$ we conclude that $g \in H$. If we consider $E(x,y) = x^{-1}y \in H$ on G^*, then $E_{Sh}|_{G^*} \subseteq E$ and therefore $G^*_{Sh} \subseteq H$.

Let g belong to all $\emptyset$-definable subgroups of G^* of finite index. We prove that $\overline{g} \in \mathrm{Autf}_{\mathrm{Sh}}(N^*)$. From Proposition 2.4 we know that $\mathrm{Autf}_{\mathrm{Sh}}(N^*)$ is the preimage under the quotient map j of the quasi-connected component QC of $\mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(N))$. Let $H \lhd \mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(N))$ be an open subgroup. It suffices to show that $\overline{g} \in j^{-1}[H] \lhd \mathrm{Aut}(N^*)$. Note that the group $H' = j^{-1}[H] \cap G^*$ is $\emptyset$-invariant, because for $f \in \mathrm{Aut}(M^*)$ if $\overline{g} \in j^{-1}[H]$, then $\overline{f(g)} = \overline{g}^{\overline{f}^{-1}} \in j^{-1}[H]$. H' is also definable, because by Lemma 2.1(iv)(b), $j^{-1}[H] = \mathrm{Aut}(\mathfrak{C}/\overline{m}/F))$, so $g \in H'$ if and only if $F(\overline{m}, \overline{g}(\overline{m})))$. Hence H' is a $\emptyset$-definable subgroup of G^* of finite index and thus $\overline{g} \in j^{-1}[H]$. ☐

The compact topological group $\mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(N^*))$ contains as a subgroup the group G^*/G^*_L, so we may ask about the induced topology on G^*/G^*_L. The next proposition describes this topology.

Proposition 3.5. (1) *The induced subspace topology on G^*/G^*_L from* $\mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(N))$ *is precisely the logic topology: let $i \colon G^* \to G^*/G^*_L$ be the quotient map, then $X \subseteq G^*/G^*_L$ is closed if and only if its preimage $i^{-1}[X] \subseteq G^*$ is $\bigwedge$-definable over some (equivalently every) small model. With this topology G^*/G^*_L is a compact topological group (this topology is Hausdorff if and only if G^*_L is $\bigwedge$-definable).*

(2) *The topology of* $\mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(M))$ *as the Lascar group of* $\mathrm{Th}(M)$ *and the induced topology on* $\mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(M))$ *as a subspace of* $\mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(N))$ *coincide.*

(3) *If $X \subseteq G^*/G^*_L$ and $Y \subseteq \mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(M))$ are closed, then $X \cdot Y \subseteq \mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(N))$ is also closed. In particular, if $\mathrm{Th}(M)$ is G-compact, then G^*/G^*_L is closed subgroup of* $\mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(N))$.

(4) *The closure of identity in G^*/G^*_L is G'_{KP}/G^*_L.*

(5) *The quasi-connected component (the intersection of all open subgroups) of G^*/G^*_L is G'_{Sh}/G^*_L.*

Proof. (1) Let N' be a small model. Without loss of generality we may assume that $x_0 \in N'$. The restriction of the quotient map j to G^* is precisely the quotient map i.

We have the following commutative diagram:

$$\begin{array}{ccc} G^* & \xrightarrow{\ \ i\ \ } & G^*/G^*_L \\ \Big\downarrow \subseteq & & \subseteq \Big\downarrow \\ \mathrm{Aut}(N^*) & \xrightarrow{\ \ j\ \ } & \mathrm{Gal_L}(\mathrm{Th}(N)) \end{array}$$

Let $X \subseteq G^*/G^*_L$ be closed in the induced subspace topology, i.e. $X = G^*/G^*_L \cap C$, where $C \subseteq \mathrm{Gal_L}(\mathrm{Th}(N))$ is closed. Then $\nu^{-1}[C]$ is closed in $S_{N'}(N')$, so there exists a type $\Phi(x,y)$ over $\emptyset$ for which

$$\nu^{-1}[C] = \mu[j^{-1}[C]] = \{\mathrm{tp}(F(N')/N') \colon F \in j^{-1}[C]\} = \{\mathrm{tp}(N''/N') \colon \Phi(N'', N')\}.$$

The subset $i^{-1}[X] \subseteq G^*$ is $\bigwedge$-definable over N', because for $g \in G^*$

$$g \in i^{-1}[X] \ \Leftrightarrow\ \overline{g} \in j^{-1}[C] \ \Leftrightarrow\ \Phi(\overline{g}(N'), N').$$

The implication $\Leftarrow$ in the last equivalence holds, because if $\Phi(\overline{g}(N'), N')$, then $\overline{g}(N') = F(N')$ for some $F \in j^{-1}[C]$, and thus $j(\overline{g}) = j(F) \in C$.

Now assume that $i^{-1}[X]$ is $\bigwedge$-definable over N', i.e. for $g \in G^*$, $g \in i^{-1}[X]$ if and only if $\Psi(g, N')$, for some type Ψ. Let $C = X \cdot \mathrm{Gal_L}(\mathrm{Th}(M)) \subseteq \mathrm{Gal_L}(\mathrm{Th}(N))$. Then

$$X = G^*/G^*_L \cap C.$$

In order to prove that C is closed in $\mathrm{Gal_L}(\mathrm{Th}(N))$ it is enough to show that

$$\begin{aligned} \nu^{-1}[C] &= \{\mathrm{tp}(F(N')/N') \colon F \in j^{-1}[C]\} \\ &= \{\mathrm{tp}(N''/N') \colon x_0^{N''} = g^{-1} \cdot x_0,\ \Psi(g, N') \text{ holds and } \mathrm{tp}(N'') = \mathrm{tp}(N')\}. \end{aligned}$$

The last equality holds because $j^{-1}[C] = i^{-1}[X] \circ \mathrm{Aut}(M^*)$, and if $F = \overline{g} \circ \overline{f},\ g \in i^{-1}[X]$, then $x_0^{N''} = F(x_0) = \overline{g} \circ \overline{f}(x_0) = g^{-1} \cdot x_0$ (here $N'' = F(N')$).

(2) The proof is similar to the proof in (1) and we leave it to the reader.

(3) The set $\nu^{-1}[X \cdot Y] = \mu[j^{-1}[X \cdot Y]]$ is closed in $S_{N'}(N')$ because it is equal to the following

$$\begin{aligned} &\{\mathrm{tp}(\overline{g} \circ \overline{f}(N')/N') \colon g \in i^{-1}[X], f \in j^{-1}[Y]\} = \\ &\{\mathrm{tp}(N''/N') \colon \mathrm{tp}(M''/M') \in \nu^{-1}[Y],\ x_0^{N''} = g^{-1} \cdot x_0,\ g \in i^{-1}[X] \text{ and } \mathrm{tp}(N'') = \mathrm{tp}(N')\}. \end{aligned}$$

Above we use the fact that $j^{-1}[X \cdot Y] = i^{-1}[X] \circ \nu^{-1}[Y]$.

(4) G'_{KP}/G^*_L contains cl(id), because G'_{KP} is $\bigwedge$-definable over $\emptyset$. The subgroup $i^{-1}[\mathrm{cl(id)}]$ of G^* is $\bigwedge$-definable over $\emptyset$ and of bounded index (because $G^*_L \subseteq i^{-1}[\mathrm{cl(id)}]$), thus $G'_{KP} \subseteq i^{-1}[\mathrm{cl(id)}]$.

(5) The group G'_{Sh} is the intersection of all $\emptyset$-definable subgroups of G^* of finite index, thus G'_{Sh}/G^*_L contains quasi-connected component QC (because if $H < G^*$ is $\emptyset$-definable of finite index, then H/G^*_L is closed of finite index, hence open). Let H be an arbitrary open subgroup of G^*/G^*_L. It suffices to show that $G'_{Sh}/G^*_L \subseteq H$. The group H is closed of finite index, hence $H \cdot \mathrm{Gal_L}(\mathrm{Th}(M))$ is a closed subgroup of $\mathrm{Gal_L}(\mathrm{Th}(N))$ of finite index. Therefore

$$\mathrm{Autf_{Sh}}(N^*) \subseteq j^{-1}[H \cdot \mathrm{Gal_L}(\mathrm{Th}(M))],$$

and then $G'_{Sh} \subseteq i^{-1}[H]$. This gives $G'_{Sh}/G^*_L \subseteq H$. □

The next corollary motivates us to investigate $\bigwedge$-definability of G^*_L. We do this in the next section. If G^*_L is not $\bigwedge$-definable, then N may give us a new kind of not G-compact theory.

Corollary 3.6. $\mathrm{Th}(N)$ *is G-compact if and only if* $\mathrm{Th}(M)$ *is G-compact and* G^*_L *is* $\bigwedge$*-definable.*

Proof. The topological group G is Hausdorff if nad only if $\{e_G\}$ is closed and we can apply the previous proposition. □

Now we describe the relations Θ, E_L, E_{KP} and E_{Sh} on the sort X^* in terms of orbits of the groups G^*_L, G'_{KP} and G'_{Sh} from Proposition 3.3.

Lemma 3.7. *Let* $x \in X^*$ *and* $n < \omega$.
 (1) $\{y \in X^* \colon \Theta^n(x,y)\} = X^n_\Theta \cdot x$
 (2) $x/E_L = G^*_L \cdot x$
 (3) $x/E_{KP} = G'_{KP} \cdot x$
 (4) $x/E_{Sh} = G'_{Sh} \cdot x$

Proof. (1) It is enough to prove this for $n=1$. $\subseteq$: Assume $x, y \in X^*, \Theta(x,y)$ and $y = g_0 x$ for some $g_0 \in G^*$. We may assume that $x = x_0$. We can extend $(x_0, g_0 x_0)$ to an order indiscernible sequence $(x_0, g_0x_0, g_1x_0, \ldots) \subseteq X^*$. Then for $0 \le i_1 < \ldots < i_n < \omega, 0 \le j_1 < \ldots < j_n < \omega$:

$$(x_0, g_{i_1}x_0, g_{i_2}x_0, \ldots) \equiv (g_{j_1}x_0, g_{j_2}x_0, g_{j_3}x_0, \ldots).$$

Applying the automorphism $\overline{g_{j_1}}$ we obtain:

$$(g_{j_1}x_0, g_{j_2}x_0, g_{j_3}x_0, \ldots) \equiv (x_0, g_{j_2}g_{j_1}^{-1}x_0, g_{j_3}g_{j_1}^{-1}x_0, \ldots).$$

Hence from the previous two equivalences we get

$$(g_{i_1}x_0, g_{i_2}x_0, \ldots) \underset{x_0}{\equiv} (g_{j_2}g_{j_1}^{-1}x_0, g_{j_3}g_{j_1}^{-1}x_0, \ldots),$$

so

$$(g_{i_1}, g_{i_2}, \ldots) \equiv (g_{j_2}g_{j_1}^{-1}, g_{j_3}g_{j_1}^{-1}, \ldots).$$

It means that $(g_0, g_1, \ldots) \subseteq G^*$ is also order indiscernible and $g_0 \equiv g_0 g_1^{-1}$, so $g_0 \in X_\Theta$.

$\supseteq$: Let $y = gx_0$ for $g = ab^{-1} \in X_\Theta$, where $\Theta(a,b)$. We can find an indiscernible sequence $(b, gb, \ldots) \subseteq G^*$, and then $(bx_0, gbx_0, \ldots) \subseteq X^*$ is also indiscernible, so $\Theta(bx_0, gbx_0)$. Applying $\overline{b}$, we obtain $\Theta(x_0, gx_0)$.

(2) Inclusion $\supseteq$ follows from Proposition 3.3(3). $\subseteq$: Let $y = F(x)$ for some $F = \overline{g} \circ \overline{f} \in \mathrm{Autf_L}(N^*)$. We may assume that $x = x_0$. Then $y = \overline{gf}(x_0) = \overline{g}x_0 = g^{-1}x_0$ and $g \in G_L$.

(3) $\supseteq$ follows from Proposition 3.4(2). Since $E_{KP}|_{X^*} \subseteq E_{G'_{KP}}|_{X^*}$ we have $\subseteq$.

(4) $\supseteq$ follows from Proposition 3.4(3). $\subseteq$: We know that $G'_{Sh} = \bigcap_{i \in I} H_i$, where H_i is $\emptyset$-definable with finite index. Therefore $E_{G'_{Sh}}|_{X^*} = \bigcap_{i\in I} E_{H_i}|_{X^*}$, so $E_{Sh}|_{X^*} \subseteq E_{G'_{Sh}}|_{X^*}$ and we are done. □

Using Theorems 1.1 and 3.1 from [7] we can give a detailed analysis of Lascar and Kim-Pillay strong types on X^*. This analysis describes also some basic properties of the group G. By $\mathrm{diam}(a)$ we denote the diameter of the Lascar strong type a/E_L (see [7]). Note that every two elements of X^* have the same type over $\emptyset$, thus their Lascar strong types have the same diameter.

Remark 3.8. There are only two possibilities:

Case 1 The diameters of all Lascar strong types on X^* are infinite. The group G^*_L is not $\bigwedge$-definable, $E_L \subsetneqq E_{KP}$, $G^*_L \subsetneqq G'_{KP}$ (i.e. $\mathrm{Th}(N)$ is not G-compact) and $2^{\aleph_0} \le [G^* : G^*_L] = |X^*/E_L| \le 2^{|T|}$.

Case 2 There is $n < \omega$ such that for every $x \in X^*$, $\mathrm{diam}(x) = n$. Then $E_L|_{X^*} = E_{KP}|_{X^*} = \Theta^n|_{X^*}$ and $G^*_L = X^n_\Theta = G'_{KP}$ are $\bigwedge$-definable groups.

Lemma 3.9. (1) *Either $G'_{KP} = G^*_L$, or $[G'_{KP} : G^*_L] \geq 2^{\aleph_0}$.*

(2) *If the language of the structure M is countable, then either*

$$G'_{Sh} = G'_{KP} \text{ or } [G'_{Sh} : G'_{KP}] \geq 2^{\aleph_0}.$$

In the last case the space of $\emptyset$-types $S_G(\emptyset)$ of G is of power $2^{\aleph_0}$.

Proof. (1) follows from preceding remark, Lemma 3.7 and [7, Theorem 1.1].

(2) The proof is very similar to the proof of [4, Theorem 3.5], so we are brief. Consider the group $H = G^*/G'_{KP}$. This group with the logic topology is a compact Hausdorff topological group. Since the language is countable, H is metrizable. Let d_0 be a metric on H. Modifying d_0 as in [4] we obtain an equivalent metric d, which is $\emptyset$-invariant. Since H is Hausdorff, the connected component of H is equal to the quasi-connected component QC, and by Proposition 3.5(5)

$$\mathrm{QC} = G'_{Sh}/G'_{KP}.$$

Assume that $G'_{Sh} \neq G'_{KP}$ and take $g \in G'_{Sh} \setminus G'_{KP}$. Let $r = d(e/G'_{KP}, g/G'_{KP})$. For every δ with $0 < \delta < r$ there is $g_\delta \in G'_{Sh}$ such that $d(e/G'_{KP}, g_\delta/G'_{KP}) = \delta$ (because G'_{Sh}/G'_{KP} is connected). The metric d is $\emptyset$-invariant, hence for $\delta < \delta'$,

$$\mathrm{tp}(g_\delta) \neq \mathrm{tp}(g_{\delta'}) \text{ and } d(g_{\delta'}/G'_{KP}, g_\delta/G'_{KP}) \geq \delta' - \delta > 0.$$

Therefore the power of $S_G(\emptyset)$ is $2^{\aleph_0}$ and $g_{\delta'}g_\delta^{-1} \notin G'_{KP}$, hence $[G'_{Sh} : G'_{KP}] = 2^{\aleph_0}$. □

4. $\bigwedge$-definability in G

In this section we investigate $\bigwedge$-definability of G^*_L in several special cases.

Proposition 4.1. *If the theory of M is small, then $G^*_L = G'_{KP} = G'_{Sh}$. Hence G^*_L is $\bigwedge$-definable.*

Proof. Equality $G^*_L = G'_{KP}$ follows from [7, Theorem 3.1(2)]. Equality $G'_{KP} = G'_{Sh}$ follows from Lemma 3.9. □

Proposition 4.2. *If the theory of M is simple, then the theory of N is also simple and $G^*_L = X^2_\Theta = G'_{KP}$.*

Proof. If $\mathrm{Th}(M)$ is simple, then $\mathrm{Th}(N)$ is also simple, because the structure $N' = (M, G, \cdot)$ (where $\cdot\colon G \times G \to G$ is the group action) is definable in M. Thus N' is simple, and N is obtained from N' by forgetting some structure. Therefore $\mathrm{Th}(N)$ is also simple. In every simple structure $E_L = E_{KP} = \Theta^2$, so $G^*_L = X^2_\Theta$ follows from Lemma 3.7. □

Now we give a criterion for equality $G'_{KP} = G'_{Sh}$, when the theory of M is simple. If in this case $G'_{KP} \subsetneq G'_{Sh}$, then it gives us a solution of an open problem: there exist an example of a structure with simple theory and in which Kim-Pillay and Shelah strong types are different (see Lemma 3.7). To state this criterion we need one definition. We call a subset $P \subseteq G^*$ *thick* if P is symmetric ($P = P^{-1}$) and there exist a natural number $n < \omega$ such that for any sequence $g_0, \ldots, g_{n-1} \in G$ there exist $i < j < n$ such that

$$g_i^{-1} \cdot g_j \in P.$$

When $\varphi(x,y)$ is a thick formula (see Definition 1.1) then X_φ (see Definition 3.1) is thick set. On the other hand if P is definable thick set, then the formula $\varphi_P(x,y) = x^{-1}\cdot y \in P$ is also thick and $P = X_{\varphi_P}$. It is easy to see that for every $n < \omega$ we have

$$X^n_\Theta = \bigcap \{X^n_\varphi \colon \varphi \in L \text{ is thick}\}.$$

Lemma 4.3. *If M has a simple theory, then $E_{KP}|_{X^*} \subsetneq E_{Sh}|_{X^*}$ (i.e. $G'_{KP} \subsetneq G'_{Sh}$) if and only if there exists a $\emptyset$-definable thick set P such that*

$$G'_{Sh} \nsubseteq P^2,$$

i.e. P^2 does not contain any $\emptyset$-definable subgroup of G of finite index (see Proposition 3.4(3)).

Proof. If every thick P satisfies $G'_{Sh} \subseteq P^2$, then clearly $G'_{Sh} \subseteq X^2_\Theta = G^*_L = G'_{KP}$, so $G'_{Sh} = G'_{KP}$.

If $G'_{Sh} = G'_{KP}$, then from 4.2 we have that $G'_{Sh} = X^2_\Theta = \bigcap\{P^2 \colon P \text{ is thick}\}$. Thus every thick P satisfies $G'_{Sh} \subseteq P^2$. □

Example 4.4. There is an example of an abelian group $(G, \cdot, \ldots)$ which has a simple ω-categorical theory and satisfies $X_\Theta \subsetneq X^2_\Theta = G^*$ (Example 6.1.10 in [2], private communication by E. Hrushovski). Consider a countable infinite dimensional vector space V over $\mathbb{F}_2 = \{0,1\}$. Let $\mathcal{B} = \{b_i \colon i < \omega\}$ be its basis and $Q \colon V \to \mathbb{F}_2$ be the following degenerate orthogonal form with the induced scalar product $(\cdot,\cdot)$:

$$Q\left(\sum_i \lambda_i b_i\right) = \lambda_0^2 + \lambda_1\lambda_2 + \lambda_3\lambda_4 + \ldots, \quad (a,b) = Q(a+b) - Q(a) - Q(b),\ a,b \in V.$$

Q is degenerate, because its radical $K = \{v \in V \colon (v,\cdot) \equiv 0\} = \{0, b_0\}$ is nontrivial. The structure $\mathcal{G} = (V, +, Q)$ has simple ω-categorical theory. We show that

$$X_\Theta \subseteq V \setminus \{b_0\}.$$

If $\Theta(v,w)$, then $Q(v) = Q(w)$. Assume on the contrary that $v - w = b_0$, then $v = w + b_0$, so:

$$Q(w) = Q(v) = Q(w+b_0) = Q(w) + Q(b_0) + (w, b_0) = Q(w) + 1,$$

and we reach a contradiction.

There are only 4 types over $\emptyset$: $\mathrm{tp}(0), \mathrm{tp}(b_0), p(x), q(x)$, where p, q are types of elements $v, w \neq 0, b_0$ with $Q(v) = 0, Q(w) = 1$ respectively. The sets X_Θ, X^2_Θ are $\emptyset$-invariant, so they must be a union of some sets described by above types. Consider $V_0 = \mathrm{lin}(b_0, b_k \colon k \geq 5) \prec V$. It is easy to see that

$$(b_1, b_4) \underset{b_2 b_3 V_0}{\equiv} (b_1 + b_3, b_4 + b_2), \quad (b_1, b_3) \underset{b_2 b_4 V_0}{\equiv} (b_1 + b_4, b_3 + b_2).$$

Thus by Lemma 1.2(ii) $b_3 = (b_1 + b_3) - b_1 \in X^2_\Theta$ and $Q(b_3) = 0$. Also $b_1 \underset{V_0}{\equiv} b_1 + b_4 + b_3 + b_2$, so $b_4 + b_3 + b_2 \in X^2_\Theta$ and $Q(b_4 + b_3 + b_2) = 1$. Therefore $V \setminus \{0, b_0\} \subseteq X^2_\Theta$ and then by Proposition 4.2 $X_\Theta \subsetneq X^2_\Theta = V$.

The next proposition gives us $\bigwedge$-definability of G^*_L for some special groups definable in the o-minimal theories.

Proposition 4.5. (1) *If G is definably compact, definable in an o-minimal expansions of a real closed field, then $G^*_L = X^2_\Theta = G'_{KP} = G^{00}$.*

(2) *If* $(G,<,+,\dots)$ *is an o-minimal expansion of an ordered group* $(G,<,+)$, *then* $G^*=G_L^*=X_\Theta^2=G^{00}$.

Proof. (1) In [3] it is proved that under the above assumptions G has *fsg* and there exists G^{00} (the smallest definable subgroup of bounded index in G^*). It is also proved that G^{00} is equal to

$$\mathrm{Stab}(p)=\{g\in G^*\colon g\cdot q=q\},$$

for some (global) generic type $p(x)\in S(G^*)$. Since p is a type over the model G^*, $\mathrm{Stab}(p)\subseteq X_\Theta^2$. Therefore

$$G^{00}=\mathrm{Stab}(p)\subseteq X_\Theta^2\subseteq G_L^*\subseteq G'_{KP}=G_\emptyset^{00}=G^{00}.$$

(2) By [8, Corollary 2.6] we can find a global type $p(x)\in S(G^*)$, satisfying $\mathrm{Stab}(p)=G^*$. Therefore $G^*=G_L^*=X_\Theta^2=G^{00}$. □

Case 1 from Remark 3.8 may lead us to a new example of a non-G-compact theory.

There is a criterion for $\bigwedge$-definability of G_L^* [7, Theorem 3.1]: G_L^* is $\bigwedge$-definable if and only if $G_L^*=X_\Theta^n$ for some $n<\omega$. Thus if X_Θ generates a group in infinitely many steps, then G_L^* is not $\bigwedge$-definable and Case 1 holds.

We have some further partial results concerning $\bigwedge$-definability of G_L^*. These results involve generic subsets of G and measures on G. They will be a part of Ph.D. thesis of the first author and appear in a forthcoming paper.

5. More examples

We were not able to construct an example of a group G, where G_L is not $\bigwedge$-definable. We can try at least to construct a group G, where G_E is not $\bigwedge$-definable for some equivalence relation E other that E_L (which gives rise to G_L).

It is rather easy to find such examples even in the stable case, with the relation E $\bigwedge$-definable and coarser than equality of types $\equiv$.

However even in the stable case we were not able to construct an example of G where $G_\equiv$ is not $\bigwedge$-definable, although we conjecture such an example exists. In this case G_{Sh}^* equals G^0, and is type definable, and equals G_L.

Since we are interested in finding an example where G_L is not $\bigwedge$-definable, naturally we are interested in non-$\bigwedge$-definable G_E, where E is close to E_L.

In this section we give only an example (Example 5.1), where $G_\equiv$ is not $\bigwedge$-definable. We could not come closer to E_L than $\equiv$. We give also an example (Example 5.2) of a group G with non-G-compact theory.

Example 5.1. In [6] there is an example (for every $n<\omega$) of a finite group G_n in which commutators $X_{\mathrm{Inn}(G_n)}$ generate commutant $G_n'=[G_n,G_n]=G_{\mathrm{Inn}(G_n)}$ in precisely n steps. We expand the structure $(G_n,\cdot)$ to obtain a structure $\mathcal{G}_n$ satisfying

$$\mathrm{Aut}(\mathcal{G}_n)=\mathrm{Inn}(G_n),$$

i.e. every automorphism of $\mathcal{G}_n$ is an inner automorphism of G_n. Note that in $\mathcal{G}_n$ the set $X_\equiv$ equals $X_{\mathrm{Inn}(G_n)}$ and generates a group in n steps. Consider the product $\prod_{n<\omega}G_n$ of the groups G_n. We expand $\prod_{n<\omega}G_n$ to a structure $\mathcal{G}$ as follows. For each k let E_k be the equivalence relation on $\prod_{n<\omega}G_n$ given by

$$E_k(u,v)\Leftrightarrow u(k)=v(k).$$

Then $\prod_{n<\omega} G_n/E_k$ is naturally identified with G_k. We expand $\prod_{n<\omega} G_n$ by the relations E_k, $k<\omega$, and the $\mathcal{G}_k$-structure on G_k (identified with $\prod_{n<\omega} \mathcal{G}_n/E_k$). We denote the quotient map

$$\prod_{n<\omega} G_n \to \prod_{n<\omega} G_n/E_k$$

by π_k.

Let $\mathcal{G}^*$ be a large saturated extension of $\mathcal{G}$. We will prove that in $\mathcal{G}^*$, the group $\mathcal{G}^*_{\equiv}$ is not $\bigwedge$-definable. This boils down to proving that

$$(*) \qquad \pi_k[X_{\equiv}^{\mathcal{G}^*}] = X_{\equiv}^{\mathcal{G}_k}.$$

Indeed, suppose the above holds. Then, since $X_{\equiv}^{\mathcal{G}_k}$ generates a group in $\geq n$ steps, also $X_{\equiv}^{\mathcal{G}^*}$ generates a group in $\geq n$ steps. As n is arbitrary, we get that $X_{\equiv}^{\mathcal{G}^*}$ generates the group $\mathcal{G}^*_{\equiv}$ in infinitely many steps. By [7, Theorem 3.1(1)], the group $\mathcal{G}^*_{\equiv}$ is not $\bigwedge$-definable.

Now we prove $(*)$. $\subseteq$ is clear, since every automorphism of $\mathcal{G}^*$ induces an automorphism of $\mathcal{G}_k$. To prove $\supseteq$, consider $a, b \in \mathcal{G}_k$ with $b = f(a)$ for some $f \in \mathrm{Aut}(\mathcal{G}_k)$. We can extend f to an automorphism of $\mathcal{G}$ and then to $\mathcal{G}^*$. If $a = \pi_k(a')$ for $a' \in \mathcal{G}^*$, then

$$b = f(a) = \pi_k(f(a')),$$

and therefore $\pi_k(a'^{-1}f(a')) = a^{-1}b$.

Now we give an example of group G whose theory is not G-compact, but case 2 from Remark 3.8 holds.

Example 5.2. First we construct a group with a large finite diameter of Lascar strong types. Let $\mathcal{M}_0 = (M_0, R, f)$ be a dense circular ordering (with respect to a ternary relation R), equipped with a function f, which is a cyclic bijection of M_0 respecting R, of period 3. This structure was considered in [1] to construct the first example of a non-G-compact theory. Our group M_3 will be the disjoint union of M_0 and $\{0\}$ equipped with a structure of vector space over $\mathbb{F}_2$ (i.e. M_3 will be an abelian group of exponent 2) so that the addition $+$ on M_3 is "independent" of f and R.

To be more specific, let L be the language consisting of a ternary relation symbol R, function symbols $+$ (binary) and f (unary) and a constant 0. To express "independence" of f and $+$ we define inductively a set of terms $\mathcal{T}$ in L as follows.

Definition 5.3. Let $\mathcal{T}$ be the smallest set of terms of L such that:

(1) $v, f(v), f^2(v)$ are in $\mathcal{T}$ for every variable v.

(2) If $\tau_1, \ldots, \tau_k$ $(k \geq 2)$ are distinct terms in $\mathcal{T}$, then the terms $f(\tau_1 + \ldots + \tau_k)$ and $f^2(\tau_1 + \ldots + \tau_k)$ are in $\mathcal{T}$.

Let $\mathcal{T}(\overline{x})$ be the set of terms in $\mathcal{T}$ in variables $\overline{x}$.

$+$ will be interpreted as an associative operation, so we may omit parentheses in $\tau_1 + \ldots + \tau_k$ in condition (2) in Definition 5.3. f will be interpreted as a cyclic function of period 3, so in Definition 5.3 there is no need to consider f^k for $k \geq 3$.

Definition 5.4. Let $\mathcal{C}$ be the class of L-structures $(V, +, 0, R, f)$ such that:

(1) $(V, +, 0)$ is a vector space over $\mathbb{F}_2$, of infinite dimension.

(2) R is a circular order on the set $V^* = V \setminus \{0\}$.

(3) f is a cyclic bijection of V^* of period 3, respecting R. That is, every point of V^* is a cyclic point of f of period 3 and $R(x, y, z)$ implies $R(f(x), f(y), f(z))$. Also, $R(x, f(x), f^2(x))$ holds in V^*.

(4) For every $a \in V^*$, the set $\mathcal{T}(a) = \{\tau(a) : \tau \in \mathcal{T}\}$ is lineary independent.

Condition (4) in Definition 5.4 expresses the fact that in the structures in $\mathcal{C}$ f is "independent" of the vector space structure.

$\mathcal{C}$ is an elementary class with the joint embedding and amalgamation properties. Let T_3 be the model completion of $\mathrm{Th}(\mathcal{C})$. T_3 has quantifier elimination and its models are the existentially closed structures in $\mathcal{C}$. We describe the 1-types in T_3.

Let $\mathfrak{C}$ be a monster model of T_3. Let $a \in \mathfrak{C}^* = \mathfrak{C} \setminus \{0\}$. The type of a is determined by the way in wich the linear span $\mathrm{lin}(\mathcal{T}(a))$ is circularly ordered by R, or even by the way in which the set $\mathrm{lin}(\mathcal{T}(a)) \cap (a, f(a))$ is linearly ordered by R. Here for $a \neq b \in \mathfrak{C}^*$

$$(a, b) = \{c \in \mathfrak{C}^* : R(a, c, b)\}, \quad [a, b) = \{a\} \cup (a, b).$$

R induces on (a, b) a linear ordering that we denote by $<$. So there are $2^{\aleph_0}$ complete 1-types over $\emptyset$ in T_3.

We say a few words about indiscernible sequences in $\mathfrak{C}$. First, if $(a_n)_{n<\omega}$ is an infinite indiscernible sequence in $\mathfrak{C}$, then $a_1 \in (a_0, f(a_0))$ or $a_1 \in (f^2(a_0), a_0)$.

Secondly, we point how to construct an indiscernible sequence in $\mathfrak{C}$. Assume $p(x) = \mathrm{tp}(a)$ for some $a \in \mathfrak{C}^*$. Let $C^-(a), C^+(a)$ be a Dedekind cut in the set $\mathrm{lin}(\mathcal{T}(a)) \cap (a, f(a))$. That is, $C^-(a) < C^+(a)$ and $C^-(a) \cup C^+(a) = \mathrm{lin}(\mathcal{T}(a)) \cap (a, f(a))$.

It follows that for every a' realising p, the corresponding sets $C^-(a'), C^+(a')$ are a Dedekind cut in the set $\mathrm{lin}(\mathcal{T}(a')) \cap (a', f(a'))$ and also the sets $f(C^-(a')), f(C^+(a'))$ and $f^2(C^-(a')), f^2(C^+(a'))$ are Dedekind cuts in the sets $\mathrm{lin}(\mathcal{T}(a')) \cap (f(a'), f^2(a'))$ and $\mathrm{lin}(\mathcal{T}(a')) \cap (f^2(a'), a')$, respectively.

We can find a sequence $(a_n)_{n<\omega}$ of elements of $\mathfrak{C}^*$ such that for every $n > m$,

$$[a_n, f(a_n)) \subseteq (a_m, f(a_m)) \text{ and } C^-(a_m) < [a_n, f(a_n)) < C^+(a_m).$$

Using the Ramsey theorem we can find such a sequence that is moreover indiscernible. Using indiscernible sequences like that we see that $p(x)$ is a strong Lascar type of diameter at least 3 and at most 6.

Similarly, replacing period 3 by period n ($n \geq 3$), we construct a G-compact group $\mathcal{M}_n = (M_n, +, 0, R_n, f_n)$ with the diameter of Lascar strong types $\geq n$.

Now let $\mathcal{M} = \prod_{3 \leq n < \omega} \mathcal{M}_n$ be the product of the groups $\mathcal{M}_n$. Let E_n be the equivalence relation on $\mathcal{M}$ given by $E_n(x, y) \Leftrightarrow x(n) = y(n)$. Then $\mathcal{M}/E_n$ is naturally identified with $\mathcal{M}_n$.

We consider $\mathcal{M}$ as a group expanded by the relations E_n and the relations R_n and functions f_n on $\mathcal{M}/E_n$. Hence in $\mathcal{M}$ there is no finite bound on the diameter of Lascar strong types. By [7], $\mathcal{M}$ is not G-compact.

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