DECIDABILITY OF MODULES OVER A BÉZOUT DOMAIN $D + XQ[X]$ WITH D A PRINCIPAL IDEAL DOMAIN AND Q ITS FIELD OF FRACTIONS.

GENA PUNINSKI AND CARLO TOFFALORI

ABSTRACT. We describe the Ziegler spectrum of a Bézout domain $B = D + XQ[X]$ where D is a principal ideal domain and Q is its field of fractions; in particular we compute the Cantor–Bendixson rank of this space. Using this, we prove the decidability of the theory of B-modules when D is "sufficiently" recursive.

1. INTRODUCTION

The model theory of modules over Bézout domains has been recently developed in [9]. This note is a further contribution to this theory, in which we analyze a particular class of Bézout domains obtained from principal ideal domains using the so-called D+M-construction (see [1, p. 7]).

Recall that a commutative domain B (with identity) is said to be *Bézout* if every 2-generated (and therefore every finitely generated) ideal of B is principal. Thus for every pair of elements $a, b \in B$ one can introduce a *greatest common divisor* $\gcd(a, b)$ as a generator c of the ideal $aB + bB$ (this element is unique up to a multiplicative unit of B). Furthermore the intersection $aB \cap bB$ is again a principal ideal dR (therefore B is coherent), and we call d a *least common multiple* of a and b (again defined up to a multiplicative unit). Under a suitable choice of lcm and gcd we have an equality $\mathrm{lcm}(a, b) \cdot \gcd(a, b) = ab$.

The D+M-construction produces from any principal ideal domain D, which is not a field, a Bézout domain which is not noetherian. In detail let

- $Q = Q(D)$ denote the quotient field of D,
- $B = B(D)$ be the subring of $Q[X]$ consisting of polynomials whose constant term is in D, that is $B = D + XQ[X]$.

Note that in the particular case when D is the ring of integers, $B = \mathbb{Z} + X\mathbb{Q}[X]$.

For basic properties of this construction see [1, pp. 7–8]. For instance (see [1, Example III.1.5]) B is a Bézout domain which is not noetherian. Namely for every prime (= irreducible) $p \in D$, we have a strictly ascending chain $XB \subset p^{-1}XB \subset$

2000 *Mathematics Subject Classification.* 03C60, 13F05, 13F30.

Key words and phrases. Bézout domain, decidability, Ziegler spectrum, Cantor–Bendixson rank, Krull–Gabriel dimension.

$p^{-2}XB \subset \dots$ of ideals of B. It follows that B is not a unique factorization domain (since being a UFD and being noetherian are equivalent for Bézout domains).

Our aim in this note is to examine the decidability of the theory of modules over a Bézout domain $B = B(D)$ for a sufficiently recursive principal ideal domain D. With this purpose in mind, we will study in Section 2 for any such B (sufficiently recursive or not) the Ziegler spectrum of B, $\mathrm{Zg}(B)$, both the points and the topology, and we will compute the Krull–Gabriel dimension of B, equivalently, the Cantor–Bendixson rank of the spectrum. After that we will describe in Section 3 the right setting for analyzing decidability of modules over Bézout domains and we will single out the "effectively" given B for which our decision problem makes sense. Finally in Section 4 we tackle the decidability question for B-modules and we answer it positively when D is effectively given. For instance this is the case for our capital example $D = \mathbb{Z}$ and $B = \mathbb{Z} + X\mathbb{Q}[X]$.

We assume some familiarity with the basic model theory of modules, in particular with pp-formulae, pp-types, (indecomposable) pure injective modules and Ziegler topology. We refer about these premises to [3, 10] or also [6, Chapter 10]. In particular we adopt the following notation: if φ, ψ are pp-formulae in one free variable over a given ring R and M is an R-module, then $\mathrm{Inv}(M, \varphi, \psi)$ denote the index of the subgroup $\varphi(M) \cap \psi(M)$ in $\varphi(M)$, which is either a positive integer k or ∞. Thus $\mathrm{Inv}(\varphi, \psi) = k$ and $\mathrm{Inv}(\varphi, \psi) \geq k$ are first order sentences in the language of R-modules saying that the index (in a given module) is exactly k, or at least k. Such statements are called *invariant sentences*.

For basic facts on model theory over Bézout domains we refer to [9]. Chapter 17 of [3] discusses the topic of decidability of modules. Modules are always assumed to be right.

2. The Ziegler spectrum

In this section we consider the Ziegler spectrum, $\mathrm{Zg}(B)$, of a Bezout domain $B = B(D) = D + XQ[X]$, where D is a principal ideal domain, not a field, and $Q = Q(D)$ denotes its quotient field. Recall that $\mathrm{Zg}(B)$ is a topological space whose points are (isomorphism classes) of indecomposable pure injective B-modules, and a basis of the topology is given by the compact open sets $(\varphi/\psi) \doteq \{M \in \mathrm{Zg}(B) \mid \varphi(M) \cap \psi(M) \subset \varphi(M)\}$ (a strict inclusion), where φ and ψ range over pp-formulae over B in (at most) one free variable.

We want to calculate the Cantor–Bendixson rank of the Ziegler spectrum of B. Because the lattice L of pp-formulae over any Bézout domain is distributive, by [5, Corollary 5.3.29] this ordinal equals the Krull–Gabriel dimension of B, that is the m-dimension of $L(B)$. The latter invariant is determined by iterative factoring L (and what is obtained from it) by congruence relations collapsing intervals of finite length (see [5, Chapter 7]). For instance the m-dimension is undefined exactly when L contains a subchain isomorphic to the ordering of the rationals $(\mathbb{Q}, \leq)$.

As a preliminary step in our analysis let us describe the prime ideals of B.

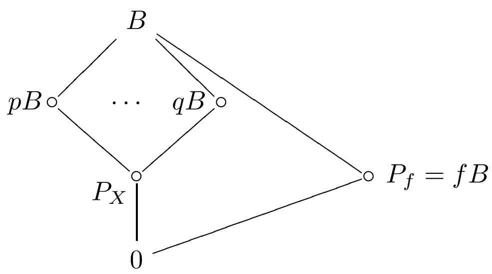
Lemma 2.1. *Any nonzero prime ideal P of B is one of the following:*

1) pB, where p is a prime element of D;

2) for some irreducible polynomial $f(X) \in Q[X]$ whose constant term is 1, the ideal $P_f = f(x)B$;

3) $P_X = XQ[X]$.

Furthermore, these prime ideals satisfy the following inclusion schema:



In particular, P_X is not principal and $P_X = \cap_p pB$.

Proof. The following arguments are standard (for instance, see [1, p. 8]). If A is a multiplicative subset of B (or any commutative domain) then there exists a natural 1-1 correspondence between prime ideals P of B such that $P \cap A = \emptyset$ and prime ideals of the localization B_A. This correspondence is defined as $P \mapsto P_A$ and $I \mapsto I \cap B$ (where I is a prime ideal of B_A).

We take $A = D \setminus 0 \subset B$. Then A is multiplicative and $B_A = Q[X]$. Let P be a prime ideal of B.

If $P \cap A \neq \emptyset$ then, since P is prime, it contains some prime p. Furthermore $X = p \cdot (p^{-1}X) \in pB \subseteq P$. It follows that $B/pB \cong D/pD$ is a field, therefore pB is a maximal ideal of B. But then $P = pB$.

Suppose now that $P \cap A = \emptyset$, therefore P is obtained by restriction from a prime ideal $fQ[X]$ of $Q[X]$, where $f(X)$ is an irreducible polynomial. If the constant term of f is zero, we may assume that $f = X$, therefore $P = XQ[X] \cap B = XQ[X]$. Otherwise we may suppose that the constant term of f equals 1 (therefore $f \in B$) and $P = fQ[X] \cap B$. But clearly this intersection equals fB.

The remaining claims are straightforward. $\square$

Note that the Krull dimension of B (defined as a maximal length of a chain of prime ideals) is 2, and B is not catenary ($0 \subset P_f$ is another saturated chain of prime ideals of length 1).

Now we describe, for each prime P, the corresponding localization B_P. Since B is a Bézout domain, B_P must be a valuation domain.

First consider the case $P = pB$ for a prime $p \in B$. Clearly $B_P = B_p = D_p + XQ[X]$, where D_p stands for localization of D with respect to pD. The principal ideals of B_p form the following chain:

$$B_p \supset pB_p \supset p^2B_p \supset \cdots \supset p^{-1}XB_p \supset XB_p \supset pXB_p \supset \cdots \supset X^2B_p \supset \dots \ .$$

In particular, the Krull dimension of B_p equals 2.

One corollary is immediate.

Lemma 2.2. *The Krull–Gabriel dimension of B is at least 4.*

Proof. It suffices to prove that $\mathrm{KG}(B_p) = 4$ for some (in fact for any) prime p. Since B_p is a valuation domain, this is a standard procedure (see [6, Chapter 5]). Each indecomposable pure injective B_p module M is uniquely determined by a pair of ideals (I, J) of B_p, therefore we will write $M = \mathrm{PE}(I, J)$, where I stands for the annihilator ideal of some element of M and J is its non-divisibility ideal. Furthermore, the Cantor–Bendixson rank of M equals $\mathrm{mdim}(I) \oplus \mathrm{mdim}(J)$, where $\mathrm{mdim}(I)$ is the m-dimension of the cut defined by I on the chain of principal ideals of B_p.

Note that the only cut on this chain of maximal m-dimension 2 corresponds to the zero ideal, and the cut defined by a principal ideal has m-dimension 0.

Thus the unique point of maximal CB-rank in $\mathrm{Zg}(B_p)$ corresponds to the pair $(0, 0)$, hence isomorphic to $Q(X)$ (the generic point). Its CB-rank equals $2 + 2 = 4$. As we have already noticed this value coincides with the Krull–Gabriel dimension of B_p. □

To simplify ongoing considerations let us make some general remarks. If P is a prime ideal of a commutative ring R, then by Zg_P we will denote the closed subspace of $\mathrm{Zg}(R)$ consisting of modules on which each $r \in R \setminus P$ acts as an automorphism. Clearly this set can be identified with the Ziegler spectrum of the localization R_P, that is $\mathrm{Zg}_P = \mathrm{Zg}(R_P)$.

Define a map $P \mapsto \mathrm{Zg}_P$ from the set of prime ideals of R ordered by inclusion to the collection of closed subsets of $\mathrm{Zg}(R)$.

Remark 2.3. *The map $P \mapsto \mathrm{Zg}_P$ preserves the ordering. Furthermore, if the intersection of prime ideals $\cap_{i \in I} P_i$ is a prime ideal (say, if the P_i form a chain), then this map preserves this intersection.*

Proof. Suppose that $P \subseteq Q$ are prime ideals and $M \in \mathrm{Zg}_P$. For any $r \notin Q$ we have $r \notin P$, therefore r acts as an isomorphism on M. But this means that $M \in \mathrm{Zg}_Q$. □

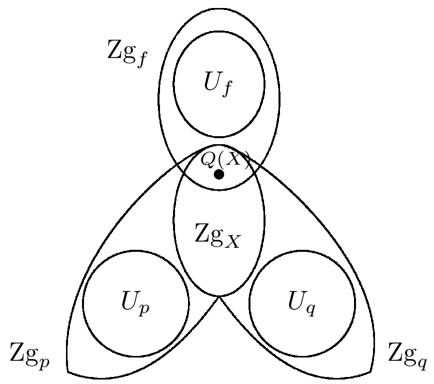
Using this (though we do need this) the above map can be extended to semiprime ideals, therefore (taking radicals) to all ideals of R.

If M is an indecomposable pure injective R-module, then consider the set $P = P(M)$ consisting of $r \in R$ which act as non-isomorphisms on M. It follows from [10, Theorem 5.4] that P is a prime ideal and M has a natural structure of an (indecomposable pure injective) R_P-module. Therefore the whole Ziegler spectrum $\mathrm{Zg}(R)$ is covered by the union of closed subsets Zg_P.

Now we are in a position to show that the above estimate of the Krull–Gabriel dimension of our B is sharp.

Theorem 2.4. *The Krull–Gabriel dimension of B equals 4 with $Q(X)$ being a unique point of maximal* CB*-rank.*

Proof. The following is a schematic diagram of $\mathrm{Zg}(B)$: we imagine it as a bouquet of closed subspaces anchored in the generic point $Q(X)$.



We know the Ziegler spectrum of any valuation domain B_P with P a prime ideal of B, and know the relative CB-ranks of points measured in $\mathrm{Zg}(B_P)$. But $\mathrm{Zg}(B_P)$ is a closed subset of $\mathrm{Zg}(B)$ which is not open. Thus, if $M \in \mathrm{Zg}(B_P)$, the 'global' CB-rank of M could be larger than the CB-rank of M calculated in relative topology. Measuring this jump is the main problem to take care of.

Let M be an indecomposable pure injective B-module and $P = P(M)$, therefore M has a natural structure of a B_P-module.

First assume that $P = fB$ for an irreducible polynomial $f(X) \in Q[X]$ with 1 as a constant term. We have already mentioned that $B_f = B_{fB}$ is a noetherian valuation domain and described its ideals. It follows that either $M = B_f/f^n B_f$ is a finitely generated B_f-module, or M is Prüfer or adic, or $M = Q(X)$, the unique generic module.

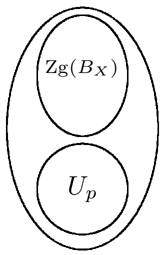
Note that the basic open set $V_f = (xf = 0/x = 0)$ consists of points on which $f(X)$ acts with a nontrivial kernel. We claim that V_f is contained in $\mathrm{Zg}(B_f)$. Indeed, let M be a point in V_f and $P = P(M)$. Then $f \in P(M)$, therefore $P(M) = fB$ and hence every $r \notin fB$ acts as an isomorphism on M.

The same is true for the open set $W_f = (x = x/f \mid x)$. In the union $U_f = V_f \cup W_f$ of these open sets $f(X)$ acts as a non-isomorphism. Observe that U_f contains all points of $\mathrm{Zg}(B_f)$, but the generic, and $U_f \cap \mathrm{Zg}_p = \emptyset$ for any prime p.

Let n be a positive integer. Since $V = (xf^n = 0/f \mid x + xf^{n-1} = 0)$ isolates $B_f/f^n B_f$ in $\mathrm{Zg}(B_f)$, it follows that $V \cap U_f$ isolates this point in $\mathrm{Zg}(B)$. Since the Prüfer module $\mathrm{Pr}(B_f)$ has CB-rank 1 in $\mathrm{Zg}(B_f)$ and f has a nonzero kernel acting on it, it follows that $\mathrm{Pr}(B_f)$ has CB-rank 1 in $\mathrm{Zg}(B)$. Similarly the adic module $\mathrm{PE}(B_f)$ has CB-rank 1 in $\mathrm{Zg}(B)$, as f is not onto when acting on this module.

As we will see later the only remaining point in Zg_f, that is, the generic point $Q(X)$ (whose CB-rank in $\mathrm{Zg}(B_f)$ equals 2) jumps to maximal CB-rank 4 in the whole space.

Now let us consider the points $M \in \mathrm{Zg}(B_p)$ for some prime $p \in D$. We have already mentioned the description of ideals of B_p and the fact that every point of $\mathrm{Zg}(B_p)$ is determined by a pair of ideals (I, J) of B_p; let $\mathrm{PE}(I, J)$ denote this point. Look at the open set U_p consisting of points on which p acts as a non-isomorphism. It is obvious that $U_p \subseteq \mathrm{Zg}(B_p)$ and its complement in Zg_P is Zg_X:

$\mathrm{Zg}(B_p)$ 

It is easily seen that a point $M = \mathrm{PE}(I, J)$ belongs to U_p if and only if either I or J is a principal nonzero ideal of B_p. For instance, if $I = J = pB_p$, then M is a simple B_p-module B_p/pB_p. The m-dimension of the cut defined by a principal nonzero ideal is 0 while the maximal m-dimension (that is, the one of the cut defined by the zero ideal) is 2. Therefore the pairs of m-dimensions of cuts defined by the ideals I, J of B_p are $(0, 0)$, $(0, 1)$, $(1, 0)$, $(2, 0)$ and $(0, 2)$. Their relative CB-ranks are the corresponding sums 0, 1 and 2.

Thus intersecting U_p with an open set which isolates such a point $M = \mathrm{PE}(I, J)$ in $\mathrm{Zg}(B_p)$ at the corresponding level we see that its CB-rank does not change when passing to the ambient space $\mathrm{Zg}(B)$.

All the remaining points of $\mathrm{Zg}(B)$ are not included in U_p for any p. As each p acts as a isomorphism on these points, they belong to $\mathrm{Zg}(B_X)$. For instance if $I = J = \cup_n p^n X B_p$ and $M = \mathrm{PE}(I, J)$, then $M \notin U_p$ and its relative CB-rank equals $1 + 1 = 2$.

Thus, if M is one of the remaining points (and is not generic), then either $M = B_X/X^n B_X$ for some positive integer n or M is Prüfer or adic over B_X.

First we will prove that $M = B_X/X^nB_X$ has CB-rank 2 (by looking at the relative CB-rank, global rank is at least 2). For this we will use the basic open set $(xX^n = 0/X \mid x+xX^{n-1} = 0)$ and intersect it with U_X to avoid the various $\mathrm{Zg}(B_f)$ (clearly $U_X \cap \mathrm{Zg}(B_f) = \emptyset$). It suffices to show that this open set separates M from the points in $\mathrm{Zg}(B_p)$ (p a prime) of CB-rank 2 corresponding to the following pairs of m-dimensions: $(2, 0)$ and $(0, 2)$.

Those are the points $\mathrm{PE}(I, J)$, where one of the ideals is principal and nonzero and the other is zero. Suppose that the above pair opens on an element $m \in \mathrm{PE}(I, J)$, where we may assume that I is the annihilator of m and J is a 'non-divisibility' ideal of m. Since $X^n \in I$, it follows that I is nonzero. Similarly, as $X \in J$, one deduces that J is nonzero. But this contradicts the choice of I and J.

What remains in $\mathrm{Zg}(B_X)$ is the Prüfer point $\mathrm{Pr}(B_X)$, the adic point $\mathrm{PE}(B_X)$, and the generic point $Q(X)$. Clearly $(xX = 0/x = 0)$ separates $\mathrm{Pr}(B_X)$ from $\mathrm{PE}(B_X)$ and $Q(X)$, therefore $\mathrm{CB}(\mathrm{Pr}(B_X)) = 3$. The same is true for $\mathrm{PE}(B_X)$.

The only remaining point $Q(X)$ has CB-rank 4. □

Note that the map in Remark 2.3 does not reflect intersections. Indeed, as follows from this remark, $\mathrm{Zg}_X = \cap_p \mathrm{Zg}_p$. But it is easily seen that for any primes $p \neq q$ we also have $\mathrm{Zg}_X = \mathrm{Zg}_p \cap \mathrm{Zg}_q$, but XB is a proper subset of $pB \cap qB$.

3. Effectively given Bézout domains

We are going to consider decidability of B-modules. It is well known that some natural conditions are to be assumed on an arbitrary ring R (in particular on our B) to ensure that the decision problem of R-modules make sense (see [3, Section 17.1]). Let we briefly discuss this matter. For simplicity we refer to integral domains R with identity. The following definition is a bit informal, but can be easily stated in a rigorous way via Turing Machines and Church Thesis.

Definition 3.1. *A countable integral domain R is said to be effectively given if its elements can be recursively listed (possibly with repetitions) as*

$$r_0 = 0, r_1 = 1, r_2, \ldots, r_k, \ldots \qquad k \in \mathbb{N}$$

so that the following holds:

1) there are algorithms which, given $n, m \in \mathbb{N}$, produce $r_n + r_m$, $-r_n$ and $r_n \cdot r_m$ (more precisely indices for these elements in the list);

2) there is an algorithm which, given $n, m \in \mathbb{N}$, decides whether $r_n = r_m$ or not;

3) there is an algorithm which, given $n, m \in \mathbb{N}$, establishes whether $r_m \mid r_n$ or not.

Notice that, if R is effectively given, then the theory $T(R)$ is recursively enumerable. Here are some further straightforward consequences of the same hypothesis.

Remark 3.2. *Let R be an effectively given integral domain.*

4) There is an algorithm which, given $n, m \in \mathbb{N}$ with $r_m \mid r_n$, provides $r \in R$ such that $r_m \cdot r = r_m$ (that is, an index for this quotient in the list).

5) There is an algorithm which, given $m \in \mathbb{N}$, decides whether r_m is a unit of R or not and, if yes, calculates its inverse.

6) Suppose that R is a Bézout domain. Then there is an algorithm which, given $n, m \in \mathbb{N}$ with $r_n, r_m \neq 0$, calculates a greatest common divisor of r_n, r_m (or rather an index of it).

Proof. 4) Just explore the list r_k, $k \in \mathbb{N}$, for every k calculate $r_m \cdot r_k$ and check whether this product equals r_n. As r_m divides r_n, one eventually finds such an index.

5) Apply 3) and 4) to $n = 1$.

6) Explore the list of all possible 4-types $(a, b, u, v) \in R^4$ (which can be obtained in a standard way from the list of R) looking at the solution of

$$r_n = (r_n \cdot u + r_m \cdot v) \cdot a, \qquad\qquad r_m = (r_n \cdot u + r_m \cdot v) \cdot b\,.$$

As R is Bézout, one eventually finds, after finitely many steps, a successful tuple (a, b, u, v). Then put $\gcd(r_n, r_m) = r_n \cdot u + r_m \cdot v$. $\square$

Note that, given 6) for a Bézout domain R, the conditions 3) (and hence 4)) become excessive. Indeed to check whether r_m divides r_n calculate first $\gcd(r_m, r_n)$, divide r_m by it and look whether the quotient is invertible.

The following result shows that when analyzing pp-formulae over Bézout domains it suffices to consider only divisibility and annihilator conditions. Recall that, up to logical equivalence, if $\chi(x)$ and $\chi'(x)$ are pp-formulas in a single variable x, then also their conjunction $\chi(x) \wedge \chi'(x)$ and their sum $\chi(x) + \chi'(x)$, introduced as $\exists u \exists u' \, (\chi(u) \wedge \chi'(u') \wedge x = u + u')$, are pp-formulas. Moreover the equivalence classes of pp-formulas are a lattice with respect to the corresponding operations.

Lemma 3.3. *Every pp-formula $\chi(x)$ in one variable over a Bézout domain R is equivalent to a finite conjunction of formulae $\varphi_{a,b} \doteq a \mid x + xb = 0$, $a, b \in R$, and also to a finite sum of formulae $\psi_{c,d} \doteq c \mid x \wedge xd = 0$.*

Furthermore, if R is effectively given, then these formulae can be found effectively.

Proof. The existence of such formulas over an arbitrary R follows from [9, Lemma 2.3]. However we have to find them effectively when R is effectively given. To do that, begin producing all the possible implications of $\chi(x)$ and the (recursively enumerable) theory $T(R)$ of R via formal proofs. When this procedure provides a formula $\chi'(x)$ of the desired form – so a suitable combination of divisibility and annihilator conditions – start producing implications from $T(R)$ and $\chi'(x)$, looking for $\chi(x)$.

The existence result ensures that the procedure will eventually halts in a successful way, so producing a formula $\chi'(x)$ equivalent to $\chi(x)$. $\square$

For instance this lemma gives a good basis for the Ziegler topology.

Corollary 3.4. *Let R be an effectively given Bézout domain. Then the open sets $(\psi_{c,d}/\varphi_{a,b})$, $a, b, c, d \in R$ form a basis of the topology of $\mathrm{Zg}(R)$ which can be effectively enumerated.*

4. Decidability

The aim of this section is to prove decidability of modules over Bézout domains B, and we have a range of methods at disposal. Using the fact that the Ziegler spectrum of B is countable and its precise description, we can make an effective list of points M_k, $k \in \mathbb{N}$, of $\mathrm{Zg}(B)$. By Corollary 3.4 we also know an effective basis for this space. According to a general recipe of Ziegler [10, Theorem 9.4] (see also Prest's unpublished preprint [4]) it suffices to provide an algorithm which, given a point M_k, a basic open set (φ_i/ψ_i) and a positive integer l, decides whether $\mathrm{Inv}(\varphi_i, \psi_i) = l$ holds true in M_k.

It is possible to obtain the proof of decidability pursuing this approach, however (being partly logicians) we will produce another proof based on a recent result by Lorna Gregory on the decidability of modules over a valuation domain [2]. To do that, let us introduce some further notation: if V is a valuation domain, then $\mathrm{Jac}(V)$ will denote its Jacobson radical (= the set of non-units) and $F = V/\mathrm{Jac}(V)$ is the residue field of V.

If V is effectively given, then (see [8, p. 273]) the decidability of V-modules yields the knowledge of the size of F (that is whether F is finite or infinite and, if finite, the number of elements in F). By [2] the converse is almost true.

Fact 4.1. [2] *Suppose that V is an effectively given valuation domain with known size of the residue field and with an algorithm checking for given $a, b \in V$ whether $a \in b^n V$ holds for some n. Then the theory of V-modules is decidable.*

Note that, if a principal ideal domain D is effectively given, then it is easily seen that the rings $Q = Q(D)$, $Q[X]$ and $B = D + XQ[X]$ are effectively given. However to reduce decidability to valuation domains, we will require of D some extra effectiveness. We say that (an effectively given) principal ideal domain D is *strongly effectively given* if it satisfies the following extra conditions:

1)′ there is an algorithm that lists all the prime elements of D;

2)′ there is an algorithm that lists all the irreducible polynomials of $Q[X]$;

3)′ for every prime p the size of the field D/pD is known.

For instance, it is well known (say, by an old Kronecker's algorithm checking indecomposability of rational polynomials) that $\mathbb{Z}$ is effectively given.

We do not know whether these extra effectiveness conditions can be formally derived from decidability of B-modules. The problem is that despite a localization of B, say the one B_p at some prime p, is given effectively, the theory of B_p-modules is defined in the theory of B-modules using an infinite set of axioms (so it is not clear in advance that the theory of B_p-modules must be decidable).

However the previous restrictions are natural and satisfied for many examples. In fact the condition $2)'$ rephrases, in the terminology of [3, p. 344], the property that Q has a splitting algorithm, and $\mathbb{Q}$ does admit it. On the basis of $1)'$ and $2)'$, one also gets algorithms to decompose a non-invertible element of D in a product of primes, in particular to decide whether it is irreducible or not; and we can do the same for polynomials in $Q[X]$.

We can even specify this for B, with an obvious proof.

Lemma 4.2. *Let D be strongly effectively given. Every nonzero polynomial $F[X] \in B$ can be effectively decomposed as $r\, s^{-1}\, X^n F'(X)$, where r, s are coprime elements of D, n is a non-negative integer and $F'(X)$ has constant term 1 and is (effectively) written as a product of irreducible polynomials.*

If a principal ideal domain D is effectively given, the same is clearly true for each localization B_p and B_f. Thus in the remainder of this section we will refer to these localizations with a fixed effective enumeration.

Lemma 4.3. *Let D be a strongly effectively given principal ideal domain. Then each localization B_p, p a prime, and B_f, f an irreducible polynomial of $Q[X]$ of constant term 1, has a decidable theory of modules.*

Proof. Each such localization is an effectively given valuation domain. Furthermore, because $B_p/\operatorname{Jac}(B_p) \cong D/pD$, and $B_f/\operatorname{Jac}(B_f)$ is infinite, we know the sizes of residue fields. Using Gregory's result, it suffices to decide, for given elements a, b of any of these localizations V, whether $a \in b^n V$ holds true for some n.

This can be easily checked, because we can reduce a, b to polynomials in B and and then use their presentations from Lemma 4.2. $\square$

Now we are in a position to prove the following.

Theorem 4.4. *Let D be a strongly effectively given principal ideal domain and let $B = D + XQ[X]$ be the corresponding Bezout domain. Then the theory $T(B)$ of B-modules is decidable.*

Proof. Since B is effectively given, from axioms for B-modules we can generate a list of sentences true in any B-module (that is, $T(B)$ is recursively enumerable). To prove decidability we have to enumerate a complement of $T(B)$, which is equivalent to listing in an effective way sentences true in some B-module.

Every indecomposable pure injective B-module localizes, therefore has a natural structure of either a B_p-module for some prime p or a B_f-module for some irreducible polynomial f with 1 as a constant term. Make an effective list of such modules with marks from which localization they stem.

In view of [4], in order to complete our proof, it suffices to restrict to modules M that are finite direct sums of indecomposable pure injective summands, $M = M_0 \oplus \cdots \oplus M_k$, and to produce a set of axioms for the theory of any such M, $T(M)$.

By Baur–Monk theorem and because $T(M)$ is complete, this theory is axiomatized by invariant sentences $\mathrm{Inv}(\varphi, \psi) \geq n$. We will list all such sentences σ and decide whether they are true in M. By additivity $M \models \sigma$ if and only if each $M_i \models \mathrm{Inv}(\varphi, \psi) \geq n_i$ and $n_1 \cdot \ldots \cdot n_k \geq n$, where we may assume that $n_i \leq n$.

Since the theory of each localization of B is decidable, each question $M_i \models \mathrm{Inv}(\varphi, \psi) \geq n_i$ can be answered effectively (using the localization at the marked prime ideal), hence so is σ. $\square$

Thus we obtain the result of our original interest.

Corollary 4.5. *The theory of $\mathbb{Z} + X\mathbb{Q}[X]$-modules is decidable.*

References

[1] L. Fuchs, L. Salce, Modules over non-Noetherian Domains, Mathematical Surveys and Monographs Series 84, American Mathematical Society, 2001.

[2] L. Gregory, Decidability for theories of modules over valuation domains, Preprint, 31 pages, www.math.uni-konstanz.de/ gregory/Decmodulesgregory.pdf

[3] M. Prest, Model Theory and Modules, London Mathematical Society Lecture Note Series 130, Cambridge University Press, 1988.

[4] M. Prest, Decidability for modules — summary, unpublished notes, 1991.

[5] M. Prest, Purity, Spectra and Localization, Encyclopedia of Mathematics and its Applictions 121, Cambridge University Press, 2009.

[6] G. Puninski, Serial Rings, Kluwer, 2001.

[7] G. Puninski, M. Prest, P. Rothmaler, Rings described by various purities, Comm. Algebra 27 (1999), 2127–2162.

[8] G. Puninski, V. Puninskaya, C. Toffalori, Decidability of the theory of modules over commutative valuation domains, Ann. Pure Appl. Logic, 145 (2007), 258–275.

[9] G. Puninski, C. Toffalori, Model theory of modules over Bézout domains. The width, Preprint, 34 pages, http://www.logique.jussieu.fr/modnet/Publications/Preprint%20server/papers/639/639.pdf

[10] M. Ziegler, Model theory of modules, Ann. Pure Appl. Logic 26 (1984), 149–213.

(G. Puninski) Faculty of Mechanics and Mathematics, Belarusian State University, 4, praspekt Nezalezhnosti, Minsk 220030, Belarus

E-mail address: punins@mail.ru

(C. Toffalori) University of Camerino, School of Science and Technologies, Division of Mathematics, Via Madonna delle Carceri 9, 62032 Camerino, Italy

E-mail address: carlo.toffalori@unicam.it