Semisimple torsion in groups of finite Morley rank

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Introduction

A group of finite Morley rank is a group whose definable subsets have a notion of dimension satisfying several basic axioms [BN94, p. 57]. Such groups arise naturally in model theory; initially as $\aleph_1$-categorical groups, i.e. as groups determined up to isomorphism by their first order theory, and here the simple groups correspond exactly. More recently groups of finite Morley rank have appeared in applications of model theory to diophantine problems.

The main examples are algebraic groups over algebraically closed fields, where the notion of dimension is the usual one; and the dominant conjecture is that all such simple groups are algebraic.

Algebraicity Conjecture (Cherlin/Zilber)**.** *A simple group of finite Morley rank is an algebraic group over an algebraically closed field.*

Much work towards this conjecture involves local analysis in an inductive setting reminiscent of the classification of the finite simple groups; but without transfer arguments or character theory.

New methods have emerged recently in the study of groups of finite Morley rank, and have led to a number of advances. Among the characteristic features of the recent work are the systematic use of *generic covering* arguments, which will be met with below, as well as the study of divisible abelian p-subgroups (commonly known as *p-torii*), with which we will also be occupied here.

Such p-tori may always be viewed as *semisimple*. However, there are difficulties when one wishes to view individual p-elements as either semisimple or unipotent. For example, even a connected solvable p-group of a group of finite Morley rank is merely a central product, not necessarily a direct product, of a

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p-torus and a definable connected nilpotent p-subgroups of bounded exponent, (commonly known as *p-unipotent subgroups*).

Our main objective here is to obtain several results concerning that p-torsion in connected groups of finite Morley rank which is semisimple in a robust but technical sense involving the absence of p-unipotent subgroups. We say a group G has *$p^\perp$ type* if it contains no nontrivial unipotent p-subgroup.

For the case $p = 2$, one may always assume $2^\perp$ type, (sometimes referred to *odd* or *degenerate type*) because the Algebraicity Conjecture holds in the presence of a 2-unipotent subgroup [ABC07]. In this case, our results will have numerous applications to classification problems, beginning with the generation theorem of [BC08]—or strictly speaking, beginning with some older papers that could have been shortened had the result been available at the time.

The article is arranged linearly, with each section depending upon the previous ones. We expect the main result of each section to find other applications outside this article, and therefore list each here; however, the first two have a more technical character. Some results are proved for sets of primes π, but here we state only the stared natural reduction to a single prime p.

The first section expands upon [Che05] and clarifies the nature of the generic element of G.

Theorem 1. *Let G be a connected group of finite Morley rank, p a prime, and a be a generic element of G. Then*

1. *a commutes with a unique maximal p-torus T_a of G,*
2. *$d(a)$ contains T_a, and*
3. *If G has $p^\perp$ type then $d(a)$ is p-divisible.*

The next section contains a new genericity argument for cosets.

Theorem 2*. *Let G be a group of finite Morley rank, let a be a p-element in G such that $C(a)$ has $p^\perp$ type, and let T be a maximal p-torus of $C(a)$ (possibly trivial). Then*

$$\bigcup (aC^\circ(a,T))^{G^\circ} \text{ is generic in } aG^\circ.$$

These two technical results are the main ingredients in the following robust criteria for semi-simplicity.

Theorem 3*. *Let G be a connected group of finite Morley rank, p a prime, a any p-element of G, and suppose $C^\circ_G(a)$ has $p^\perp$ type. Then a belongs to a p-torus.*

Theorem 3* has immediate consequence for the structure of Sylow p-subgroups.

Corollary. *Let G be a connected group of finite Morley rank of $p^\perp$ type, and T a maximal p-torus of G. Then any p-element of $C(T)$ belongs to T.*

Our fourth section further exploits the genericity argument for cosets to prove conjugacy of Sylow p-subgroups, i.e. maximal solvable p-subgroups.

Theorem 4. *Let G be a group of finite Morley rank of $p^\perp$ type. Then all Sylow p-subgroups are conjugate.*

We note that the conjugacy of Sylow 2-subgroups in general groups of finite Morley rank is known, and for general p the result is known in solvable groups of finite Morley rank.

Our last topic concerns the so-called *Weyl group*, which for our present purposes may be defined as follows.

Definition 0.1. Let G be a group of finite Morley rank, and T a maximal divisible abelian torsion subgroup of G. The *Weyl group* of G is the group $N(T)/C^\circ(T)$, which can be viewed as a group of automorphisms of T.

The maximal divisible abelian torsion subgroups of G are conjugate by [Che05], so this group is well-defined up to conjugacy and in particular up to isomorphism. Furthermore it is finite since $N^\circ(T) = C^\circ(T)$.

Theorem 5.2. *Let G be a connected group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then G contains a unipotent r-subgroup.*

All of these results will be needed in [BC08], and the torality theorem should be quite useful subsequently in the analysis of particular configurations associated with classification problems in odd type groups. The corollary to Theorem 3 is also given in [BC06] for $p = 2$, and is applied there to the study of generically multiply transitive permutation groups.

Outside material will be introduced as needed, but much of this occurs already in the first section. Any facts used without explicit can be found in [BN94].

1 Generic p-divisibility

We begin by analyzing the generic element of a connected group of finite Morley rank. We use the notation $d(a)$ for the definable hull of an element a. The definable hull of a divisible abelian torsion subgroup of G is called a *decent torus*.

Theorem 1. *Let G be a connected group of finite Morley rank, p a prime, and a be a generic element of G. Then*

1. *a commutes with a unique maximal p-torus T_a of G,*
2. *$d(a)$ contains T_a, and*
3. *If G has $p^\perp$ type then $d(a)$ is p-divisible.*

Here we consider only elements a whose *type* over $\emptyset$ is generic. When G is of $p^\perp$ type there need not be any generic definable *set* X such that $d(a)$ is p-divisible for every element of X. Indeed, with G an algebraic torus in characteristic other than p, that stronger claim fails. In this case the generic element has infinite order, but every infinite definable set contains p-elements of finite order, and for such elements $d(a) = \langle a \rangle$.

The idea of the proof is to replace the group G by the centralizer of one of its maximal decent tori. This depends on the main result of [Che05], which is closely related to point (1) above.

Proposition 1.1 ([Che05]). *Let G be a group of finite Morley rank. Then all maximal decent tori of G are conjugate. Furthermore, if T is a maximal decent torus of G, then there is a generic subset X of the group $C^\circ_G(T)$ such that*

1. *$X \cap X^g = \emptyset$ for $g \notin N_G(T)$, and*
2. *$\bigcup X^G$ is generic in G.*

In particular Proposition 1.1 states that any element of the generic definable set $\bigcup_G(X^G)$ commutes with a unique conjugate of T, or in other words with a unique maximal decent torus of G. So we have our first point:

Lemma 1.2. *Let G be a connected group of finite Morley rank, and let $a \in G$ be generic over $\emptyset$. Then $C^\circ(a)$ contains a unique maximal decent torus T_a of G.*

Our next point is that T_a is contained in $d(a)$, and at this point we must work not with the generic set X, but with the type of a itself generic. In this situation there are two notions of genericity which are relevant: genericity in the group G, and genericity in the subgroup $C^\circ(T_a)$, but again by Proposition 1.1 these two notions can be correlated. Indeed, the next result is a direct reformulation of Lemma 1.2.

Lemma 1.3. *Let G be a connected group of finite Morley rank, and T_0 a maximal decent torus. Then an element $a \in C^\circ(T_0)$ is generic over $\emptyset$ in G if and only if the following hold.*

1. *T_0 is generic over $\emptyset$, in the set of maximal decent tori;*
2. *a is generic in the group $C^\circ(T_0)$ over the canonical parameter for T_0.*

A word on terminology: as a definable set, T_0 can be viewed as an "imaginary element" of G, and the canonical parameter for T_0 is simply this element. As this may be identified with T_0 itself, one may speak of genericity "over T_0". The natural language for discussing the group $C^\circ(T_0)$ treats this parameter as a distinguished constant; it is interdefinable with $C^\circ(T_0)$.

Proof. Suppose first that a is generic. Then $T_0 = T_a$.

We show that T_0 is generic over $\emptyset$. If T_0 belongs to a $\emptyset$-definable family $\mathcal{T}$ in G^{eq} (a uniformly definable family in G) then a belongs to the set $X := \bigcup_{T \in \mathcal{T}} C^\circ(T)$. If the family $\mathcal{T}$ is nongeneric in the set of maximal decent tori, then X is nongeneric in G, a contradiction—the failure of genericity is immediate by a rank calculation.

Now we show a is generic over the canonical parameter t_0 for T_0, in $C(T_0)$. If a belongs to some nongeneric set $Y_{t_0} \subseteq C^\circ(T_0)$, where Y is defined over t_0, then a belongs to the $\emptyset$-definable nongeneric set $\bigcup_{t\in X} Y_t$, where again the nongenericity follows by a direct rank calculation.

Now suppose T_0 is generic over $\emptyset$ and $a \in C^\circ(T_0)$ is generic over the parameter T_0. Suppose that a belongs to the $\emptyset$-definable subset Y of G. Let $Y_0 = Y \cap C^\circ(T_0)$. As $a \in Y_0$, the set Y_0 is generic in $C^\circ(T_0)$. The set $\mathcal{T}$ of conjugates T of T_0 for which $Y \cap C^\circ(T)$ is generic in $C^\circ(T)$ is $\emptyset$-definable and contains T_0, and hence $\mathcal{T}$ is generic in the set of conjugates of T_0. It then follows from Proposition 1.1 that Y is generic in G. □

We will also need some general properties of definable quotients.

Lemma 1.4. *Let G be a group of finite Morley rank, $A \subseteq G$, H a normal A-definable subgroup of G, and $\bar{G} := G/H$.*

1. *If an element $a \in G$ is generic over A then its image $\bar{a}$ in $\bar{G}$ is generic over A.*
2. *If T is a maximal decent torus of G, and H is solvable, then the image $\bar{T}$ of T in $\bar{G}$ is a maximal decent torus of $\bar{G}$.*

The first point has already occurred in a special form above, and is also contained in Lemma 6.2 of [Poi87].

Proof. Ad 1. Suppose $\bar{X}$ is an A-definable subset of $\bar{G}$ containing $\bar{a}$, with preimage X in G. As X contains a it is generic. But $\mathrm{rk}(X) = \mathrm{rk}(\bar{X}) + \mathrm{rk}(H)$ and thus $\bar{X}$ is generic in $\bar{G}$. Thus $\bar{a}$ is generic in $\bar{G}$.

Ad 2. Let T_p be the p-torsion subgroup of T. It suffices to show that $\bar{T}_p$ is a maximal p-torus of $\bar{G}$. Let S_p be the preimage in G of a maximal p-torus $\bar{S}_p$ of $\bar{G}$ containing $\bar{T}_p$. We may suppose that $G = d(S_p)$ and thus G is solvable. Now let P be a Sylow p-subgroup of G containing T_p. Then $\bar{P}$ is a Sylow p-subgroup of $\bar{G}$ [ACCN98] and therefore contains a maximal p-torus of $\bar{G}$. Hence $\bar{P}$ contains a maximal p-torus of $\bar{G}$. But as S is a solvable p-group, $S^\circ = T_p * U$ with U unipotent [BN94, Corollary 6.20], so $\bar{T}_p$ is the maximal p-torus of $\bar{P}$, and hence is a maximal p-torus of $\bar{G}$. □

Lemma 1.5. *Let G be a connected group of finite Morley rank and $a \in G$ generic. Then $d(a)$ contains T_a.*

Proof. Treating the parameter T_a as a constant, and bearing in mind Lemma 1.3, we may suppose that a is a generic element of $C^\circ(T_a)$, and T_a is $\emptyset$-definable. Hence we may replace G by $C^\circ(T_a)$, assuming therefore that

G contains a unique maximal decent torus T, which is central in G.

Let T_1 be the definable hull of the torsion subgroup of $d(a) \cap T$. As T is taken to be $\emptyset$-definable, the torsion subgroup of T is contained in $\mathrm{acl}(\emptyset)$ and hence the definable set T_1, treated as another parameter, is also in $\mathrm{acl}(\emptyset)$. Thus

$\bar{a}$ is generic in the quotient $\bar{G} := G/T_1$, and in this quotient $\bar{T} := T/T_1$ is a maximal decent torus. So replacing G by $\bar{G}$, we may suppose that $d(a) \cap T$ is torsion free. It suffices to show that $T = 1$.

By [BN94, Ex. 10 p. 93], $d(a) = A \oplus C$ is the direct sum of a divisible abelian group A and a finite cyclic group C. If $n = |C|$ then for any multiple N of n, we have $d(a^N) = A$. On the other hand, for any torsion element $c \in T$, the element $a' = ac$ is also generic over $\emptyset$ and hence a' and a realize the same type. Letting N be a multiple of n and the order of c, it follows that $d((a')^n) = d((a')^N) = d(a^N) = d(a^n)$ and thus $c^n \in d(a^n) \leq d(a)$. Now by our reductions $d(a)$ contains no p-torus for any p, and hence $d(a)$ has bounded exponent. Thus c^n has bounded exponent, with c varying and n fixed, and so $T = 1$ as claimed. □

For the final point in Theorem 1 we prepare the following, which is a minor variation on a result of [BBC07].

Lemma 1.6. *Let G be a connected group of finite Morley rank, p a prime, and T a maximal p-torus of G. Suppose that T is central in G and a is a p-element of G not in T. Then $C^\circ(a)$ contains a nontrivial p-unipotent subgroup. Thus if G is of $p^\perp$ type then all p-elements in $C(T)$ belong to T.*

Here we employ one of the main results of [BBC07].

Fact 1.7 ([BBC07, Theorem 4])**.** *Let G be a connected group of finite Morley rank, and let $a \in G$ be a p-element. Then $C(a)$ contains an infinite abelian p-subgroup.*

Proof of Lemma 1.6. Observe first that the p-torsion subgroup of $d(T)$ is T, and thus $a \notin d(T)$. Now passing to a quotient as in the previous argument we may suppose that $T = 1$ and G contains no p-torus. So $C^\circ(a)$ contains a nontrivial p-unipotent subgroup by Fact 1.7. □

We turn to the last point in Theorem 1.

Lemma 1.8. *Let G be a connected group of finite Morley rank of $p^\perp$ type, and $a \in G$ a generic element. Then $d(a)$ is p-divisible.*

Proof. As we have seen above, we may suppose that T_a is central in G and $\emptyset$-definable. The group $d(a)$ is an abelian group of finite Morley rank, and hence has the form $A \times C$for some p-divisible abelian group A, and some p-group C of bounded exponent by [BN94, Ex. 10, p. 93]. Since G is of $p^\perp$ type, $C \leq T_a$. As $T_a \leq d(a)$ by Lemma 1.5, $d(a)$ is p-divisible. So $C \leq T_a$. Now $a = bt$ with $b \in A$ and $t \in C$. It follows easily that $d(b) = A$ is p-divisible. But $b = at^{-1}$ is also generic, so our result holds for b, and hence for any realization of the same type. □

Now Theorem 1 is contained in Lemmas 1.2, 1.5, and 1.8.

2 Coset genericity

In this section we prove a *generic covering theorem*. Theorems of this type have played an increasing role in the analysis of connected groups of finite Morley rank. Our aim here is to show that for a p-element a of a group G of $p^\perp$ type, the union of the conjugates of $C^\circ(a)$ is generic in G. This improves on the analysis carried out in [BBC07] for groups of p-degenerate type. In order to prove this we need to sharpen it substantially and identify a subgroup of $C^\circ(a)$ actually responsible for the genericity. The precise result we aim at is the following, which generalizes the result in several directions, notably by allowing the element a to lie outside the connected component of G.

We formulate this analysis using a more general set of primes π, as opposed to the single prime p used in the introductory statement. Of course, π-torus will mean simply a divisible abelian π-group, and need not realize all primes in π. Likewise, $\pi^\perp$ type means no p-unipotent subgroup for any $p \in \pi$.

Theorem 2. *Let G be a group of finite Morley rank, let a be a π-element in G such that $C(a)$ has $\pi^\perp$ type, and let T be a maximal π-torus of $C(a)$ (possibly trivial). Then*

$$\bigcup (aC^\circ(a,T))^{G^\circ} \text{ is generic in } aG^\circ.$$

In particular

$$\bigcup (aC^\circ(a)^{G^\circ}) \text{ is generic in } aG^\circ.$$

Generic covering theorems have involved definable subgroups more often than cosets. The following covering lemma, given in [BBC07], is well adapted to the case of cosets.

Fact 2.1 ([BBC07, Lemma 4.1])**.** *Let G be a group of finite Morley rank, H a definable subgroup of G, and X a definable subset of G. Suppose that*

$$\mathrm{rk}(X \setminus \bigcup_{g \notin H} X^g) \geq \mathrm{rk}(H)$$

Then $\mathrm{rk}(\bigcup X^G) = \mathrm{rk}(G)$.

The following property of generic subsets of cosets is very well known for subgroups, but occurs more rarely in its general form.

Lemma 2.2. *Let G be a group of finite Morley rank, H a connected definable subgroup, and X a definable generic subset of the coset aH. Then $\langle X \rangle = \langle aH \rangle = \langle a, H \rangle$.*

Proof. The second equality is purely algebraic, and clear. For the first, an application of genericity and connectedness shows that $H \leq \langle X \rangle$, and thus $aH \subseteq \langle X \rangle$. □

Proof of Theorem 2. We will use the notation $N_G(X)$ here for arbitrary subsets of G, not just subgroups, with its usual meaning: the setwise stabilizer of X under the action of G by conjugation.

Let $\mathcal{T}$ be the set of maximal π-tori in $C_G^\circ(a)$. We observe first that $\mathcal{T}$ may be identified with a definable set in G^{eq}. Indeed, it follows from the conjugacy of maximal decent tori that maximal π-tori are conjugate under the action of the group

$$G_a = C_G^\circ(a)$$

so $\mathcal{T}$ corresponds naturally to the right coset space $N_{G_a}(T)\backslash G_a$ for any $T \in \mathcal{T}$, and $N(T) = N(d(T))$ is definable. As the elements of $\mathcal{T}$ are themselves undefinable, this identification should be used with circumspection.

As the maximal π-tori of $C(a)$ are conjugate, we may suppose that the π-torus $T \in \mathcal{T}$ is chosen generic over a. Set

$$H := C^\circ(\langle a, T\rangle)$$

which enters the picture most naturally here as $C^\circ_{C^\circ(a)}(T)$. Then T is the unique maximal π-torus in H, and we aim to show that

$$\mathrm{rk}(\bigcup (aH)^{G^\circ}) = \mathrm{rk}(G^\circ)$$

Let $\hat{H}$ be the generic stabilizer of aH, defined as

$$\{g \in G : \mathrm{rk}((aH) \cap (aH)^g) = \mathrm{rk}(aH)\}$$

This is a definable subgroup of G. We claim

$$\mathrm{rk}(\hat{H}) = \mathrm{rk}(H) \tag{1}$$

Since a is an element of finite order normalizing (even centralizing) H, the group $\langle a, H\rangle$ is definable with $\langle a, H\rangle^\circ = H$. Applying the preceding lemma,

$$\hat{H} \le N_G(\langle a, H\rangle) \le N_G(\langle a, H\rangle^\circ) = N_G(H) \le N(T)$$

Thus $\hat{H}^\circ \le C(T)$.

We claim that any π-element u of $\langle a, H\rangle$ lies in the abelian group $\langle a, T\rangle$: indeed, the π-group $U = \langle u, a\rangle$ has the form $U_0\langle a\rangle$ with $U_0 = U \cap H$. We claim that $U_0 \le T$. For this, it suffices to show that any π-element $u' \in U_0$ belongs to T. But this holds by Lemma 1.6.

Therefore $\langle a, H\rangle$ contains only finitely many elements of the same order as a, and as $\hat{H}$ acts by conjugation on these elements, we have $\hat{H}^\circ \le C^\circ(a)$ and thus $\hat{H}^\circ \le C^\circ(\langle a, T\rangle) = H$. So (1) holds.

We would like to apply the generic covering lemma, Fact 2.1, with $X = aH$ and with H (in the lemma) equal to $\hat{H}$ (here). For this, it suffices to verify the condition

$$\mathrm{rk}(aH \setminus \bigcup_{g \notin \hat{H}} (aH)^g) = \mathrm{rk}(H) \tag{$*$}$$

Now suppose $x \in aH$ is generic over the parameters a and T (really, $d(T)$). We claim that both

(2) x centralizes a unique maximal π-torus of $C^\circ(a)$, namely T, and

$$a \in d(x) \tag{3}$$

Clearly $x = ah$ with $h \in H$ generic over a and T. Since T is itself generic over a, h realizes the type of a generic element of $C^\circ(a)$ over a (Lemma 1.3). By Theorem 1, h centralizes a unique maximal π-torus of $C^\circ(a)$ for $p \in \pi$, and hence so does x. So (2) follows.

As $C_G^\circ(a)$ has $\pi^\perp$ type, $d(h)$ is π-divisible by Theorem 1. So, for any π-number q, the quotient $d(h)/d(h^q)$ is a p-divisible group of exponent at most q, and hence trivial: $d(h) = d(h^q)$. Let q be the order of a. Then $d(x^q) = d(a^q h^q) = d(h^q) = d(h)$. So $h \in d(x)$, and hence also $a \in d(x)$, giving (3).

If $(*)$ fails, then $H \cap \bigcup_{g \notin \hat{H}} (aH)^g$ is generic in aH, so, as $x \in aH$ is generic over the parameters a and T, we have $x^g \in aH$ for some $g \notin \hat{H}$. As $x, x^g \in aH$, $d(x)$ and $d(x^g)$ both commute with T, and therefore $d(x)$ also commutes with $T^{g^{-1}}$. Since $a \in d(x)$, we have $T^{g^{-1}} \leq C_G^\circ(a)$. By (2) it follows that $T = T^{g^{-1}}$, that is $g \in N(T)$.

Again, since $a \in d(x)$, we have $a^g \in d(x^g)$ is an element of order q in $\langle a, H \rangle$. So $a^g \in \langle a, T \rangle$. Since $g \in N(T)$ this gives $g \in N_G(\langle a, T \rangle)$ as well. Now $H = C^\circ(\langle a, T \rangle)$ so

$$H^g = C^\circ(\langle a, T \rangle)^g = C^\circ(\langle a^g, T \rangle) = C^\circ(\langle a, T \rangle) = H$$

Hence $(aH)^g = (xH)^g = x^g H^g = x^g H = aH$, and $g \in \hat{H}$, a contradiction. So $(*)$ holds, and our result follows by Fact 2.1. □

3 Torality

We now prove the main result of the paper. Again, we formulate this in a technical form slightly more general than the original statement using a set of primes π.

Theorem 3. *Let G be a connected group of finite Morley rank, π a set of primes, a any π-element of G, and suppose $C_G^\circ(a)$ has $\pi^\perp$ type. Then a belongs to a π-torus.*

Theorem 3 has the following direct corollary.

Corollary 3.1. *Let G be a connected group of finite Morley rank with a π-element a such that $C(a)$ has $\pi^\perp$ type. Then a belongs to any maximal π-torus of $C(a)$.*

Proof. By Theorem 3, there is a π-torus T containing a. By Proposition 1.1, any maximal π-torus in $C(a)$ is $C^\circ(a)$-conjugate to T, and so contains a. □

These provide very strong restrictions on a simple group G of finite Morley rank. Since, for $\pi = \{2\}$, the outstanding structural problems concern groups of $2^\perp$ type (i.e., odd or degenerate type), this bears directly on these issues and in particular provides constraints on the structure of a Sylow 2-subgroup, which will be developed in [BC08].

For the proof, we use the following variation on Fact 1.7 [BBC07, Theorem 4], due to Tuna Altınel.

Lemma 3.2 (Altınel)**.** *Let G be a connected group of finite Morley rank, and let $a \in G$ be a π-element. Then $C(a)$ contains an infinite abelian p-subgroup for some $p \in \pi$.*

We require the following standard fact.

Fact 3.3 ([ABCC01]; [Bur04, Fact 3.2])**.** *Let $G = H \rtimes T$ be a group of finite Morley rank with H and T definable. Suppose T is a solvable π-group of bounded exponent and $Q \lhd H$ is a definable solvable T-invariant $\pi^\perp$-subgroup. Then*

$$C_H(T)Q/Q = C_{H/Q}(T).$$

Proof of Lemma 3.2. We make take G to be a minimal counterexample; in particular $C^\circ(a)$ is a $\pi^\perp$-group. Of course, G does have an infinite abelian p-group for every $p \in \pi$ by Fact 1.7. So clearly $a \notin Z(G)$.

As $Z^\circ(G)$ has no π-torsion, $C_{G/Z^\circ(G)}(a) = C_G(a)/Z^\circ(G)$ by Fact 3.3. So $C_{G/Z^\circ(G)}(a)$ has no π-torsion by [BN94, Ex. 11 p. 93 or Ex. 13c p. 72]. Thus $Z^\circ(G) = 1$ by minimality of G.

We now observe that $a \in d(x) \cap aC^\circ(a)$ for any $x \in aC^\circ(a)$. Let $K := d(x) \cap C^\circ(a)$. So x is a π-element in $d(x)/K$. By [BN94, Ex. 11 p. 93], $xd^\circ(x)$ contains a π-element b. But a is the unique π-element in $aC^\circ(a) \supseteq xK$. Thus $a = b \in d(x) \cap aC^\circ(a)$, as desired.

By Theorem 2, $\bigcup (aC^\circ(a))^G$ is generic in G. We show that G has no divisible torsion. Otherwise, choose a maximal decent torus T of G. By Fact 1.1, $\bigcup C^\circ(T)^G$ is generic in G too, and hence meets $aC^\circ(a)$ in an element x. So $a \in d(x)$ lies inside some $C^\circ(T)^g$. But $C_{C^\circ(T)^g}(a)$ is still a $\pi^\perp$-group, contradicting the minimality of G.

As $C^\circ(a^{-1}) = C^\circ(a)$, $\bigcup (a^{-1}C^\circ(a))^G$ is also generic in G, by Theorem 2. So there is some $x \in a^{-1}C^\circ(a) \cap (aC^\circ(a))^g$ for some $g \in G$. As above a^g and a^{-1} are the only π-elements in $(aC^\circ(a))^g$ and $a^{-1}C^\circ(a)$, respectively. So $a^g = a^{-1}$. It follows that G has an involution in $d(g)$. As G has no divisible torsion, G has even type, but has no algebraic simple section. So $B(G)$ is a 2-unipotent subgroup normal in G, by the Even Type Theorem [ABC07]. Now $Z^\circ(B(G)) \neq 1$ by cite[Lemma 6.2]BN. But now $Z^\circ(B(G)) \leq Z^\circ(G) = 1$ because again G has no divisible torsion (see [Bur06, §4] or [BBC07, Proposition 8.1]), a contradiction. □

Proof of Theorem 3. By Lemma 3.2, there is a non-trivial π-torus T of $C^\circ(a)$, which we take maximal in $C^\circ(a)$. Set $H := C^\circ(a, T)$. By Theorem 2, the set $(aH)^G$ is generic in G. So after conjugating we may suppose that some $x = ah \in aH$ is generic in G. We claim that $a \in d(x)$.

Since x is generic in G, $C(x)$ contains a unique maximal π-torus S of G, which lies inside $d(x)$, by Theorem 1. Clearly $T \leq S$ since $T \leq C(x)$. The definable hull $d(x)$ contains a p-element x' with $x'H = xH = aH$. So $x'a^{-1} \in H$ is a p-element, and lies inside T. Thus $x'a^{-1} \in S \leq d(x)$ by Lemma 1.6, and so $a \in d(x)$.

Again since x is generic in G, we have $x \in C^\circ(S)$ by Theorem 1, and hence $a \in C^\circ(S)$ and $T = S$ is nontrivial. Since $H \leq C(a)$ has $\pi^\perp$ type, we have $a \in T$ by Lemma 1.6. This proves our claim. □

This theorem has consequences for the structure of Sylow p-subgroups in connected groups of $p^\perp$ type and low Prüfer p-rank, especially Prüfer rank 1, for which see [BC08].

4 Conjugacy of Sylow p-subgroups

We will define *Sylow p-subgroups* of a group G of finite Morley rank as maximal solvable p-subgroups. Here one arrives at the same class of groups by imposing local finiteness or local nilpotence in place of solvability [BN94, §6.4]. If S is a Sylow p-subgroup of G then S° will be a central product of a p-unipotent subgroup and a p-torus, and in particular is nilpotent. So, if S is a Sylow p-subgroup and X a proper subgroup of S, then $N_S(X) > X$.

Our goal in the present section is the following.

Theorem 4. *Let G be a group of finite Morley rank of $p^\perp$ type. Then all Sylow p-subgroups are conjugate.*

The conjugacy result is also known for solvable groups, as a special case of the theory of Hall subgroups ([BN94, Theorem 9.35]) and for arbitrary groups of finite Morley rank when $p = 2$.

As an immediate consequence we can strengthen [BBC07, Theorem 3].

Corollary 4.1. *Let G be a connected group of finite Morley rank and $p^\perp$ type. If G some Sylow p-subgroup of G is finite then G contains no elements of order p.*

The critical case for the proof is the case in which at least one Sylow p-subgroup is finite; which proves the corollary itself. It also shows that Sylow p-subgroups are conjugate if all lie outside G°.

Lemma 4.2. *Suppose G is a group of finite Morley rank and $p^\perp$ type containing a finite Sylow p-subgroup P. Then all Sylow p-subgroups of G are conjugate.*

Proof. Let $O_p(G)$ denote the subgroup of G generated by its solvable normal p-subgroups. Such a p-subgroup must be contained in P and thus $O_p(G) \leq P$ is finite, and is the largest finite normal p-subgroup of G. In $\bar{G} = G/O_p(G)$ we have $O_p(\bar{G}) = 1$ and $\bar{P} = P/O_p(G)$ is a finite Sylow p-subgroup of $\bar{G}$, and if we prove the claim for $\bar{G}$ it follows for G. So we may suppose

$$O_p(G) = 1 \tag{1}$$

Let D be a subgroup of P of maximal order subject to the condition: D is contained in a solvable p-subgroup of G which has no conjugate contained in P. Let R be such a p-subgroup. Let $D_1 = N_P(D)$, $D_2 = N_R(D)$. By the maximality of D, any p-Sylow subgroup P_1 of $N(D)$ containing D_1 is conjugate to a subgroup of P. Let R_1 be a Sylow p-subgroup of $N(D)$ containing D_2. If R_1 is conjugate to P_1, then R_1 is conjugate to a subgroup of P. In particular D_2 is then conjugate to a subgroup of P and R is conjugate to a group meetings P in a subgroup of order greater than $|D|$. But this contradicts the choice of D.

It follows that in $N(D)$ we have nonconjugate Sylow p-subgroups, so by the minimality of G we find $D \triangleleft G$ and thus $D \leq O_p(G) = 1$. Hence any p-subgroup which meets P nontrivially is conjugate to a subgroup of P.

Fix $a \in P$ nontrivial. We claim

$$C^\circ(a) \text{ is a } p^\perp\text{-group} \tag{2}$$

If this fails, take $x \in C^\circ(a)$ a nontrivial p-element. By Fact 1.7, $C(x)$ is an infinite abelian p-subgroup A. As $O_p(G) = 1$, we have $C(x) < G$ and hence the Sylow p-subgroups of $C(x)$ are conjugate. Taking Sylow p-subgroups Q and R of $C(x)$ containing $\langle a, x \rangle$ and A respectively, we find that Q is conjugate to a subgroup of P since Q meets P nontrivially, and hence the infinite group R is conjugate to a subgroup of the finite group P, a contradiction.

Now let b be an arbitrary p-element of the coset aG°, and T_b a maximal torus of $C^\circ(b)$. Then $\bigcup (bC^\circ(b, T_b))^{G^\circ}$ is generic in aG° by Theorem 2. This applies in particular to a, with $T_a = 1$. As we have generic sets associated to a and b in the coset aG°, their intersection is nontrivial, giving

$$aC^\circ(a) \cap b'C^\circ(b', T_{b'}) > 1 \quad \text{for some conjugate } b' \text{ of } b.$$

Fix $h \in aC^\circ(a) \cap b'C^\circ(b', T_{b'})$. Since $h \in aG^\circ$, there is a p-element $h' \in d(h) \cap aG^\circ$. But $d(h) \cap aG^\circ$ is contained in both $aC^\circ(a)$ and $b'C^\circ(b', T_{b'})$. So $h' \in aC^\circ(a)$, and as h' is a p-element we find $h' = a$. Similarly $h' \in b'T_{b'}$. Thus $a \in b'T_{b'}$ and $T_{b'} \leq C^\circ(a)$, so $T_{b'} = 1$. We conclude that $a = b'$ and thus

$$\text{For } a \in P^\#, \text{ any two } p\text{-elements in } aG^\circ \text{ are conjugate.} \tag{3}$$

Now fix an arbitrary Sylow p-subgroup Q of G. We will show that P and Q are conjugate.

Let $\bar{G} = G/G^\circ$ and let $\bar{R}$ be a Sylow p-subgroup of $\bar{R}$ containing $\bar{P}$. We may suppose after conjugating Q that $\bar{Q} \leq \bar{R}$. We claim

$$\bar{R} = \bar{P} \tag{4}$$

Assuming the contrary, let R be the preimage in G of $\bar{R}$. We have $N_{\bar{R}}(\bar{P}) > \bar{P}$ and thus $N_R(PG^\circ) > PG^\circ$. If $PG^\circ < G$ then Sylow p-subgroups of PG° are conjugate and therefore $N_R(PG^\circ) = G^\circ N_R(P)$. Thus $N_R(P)$ covers $N_{\bar{R}}(\bar{P})$ and therefore there is a p-element $x \in N_R(P) \setminus P$. But then P is not a Sylow p-subgroup, a contradiction. So (4) holds.

Hence $PG^\circ = QG^\circ$. In particular there are $a \in P^\#$, $b \in Q$ with $aG^\circ = bG^\circ$ and thus a, b are conjugate. Hence some conjugate of Q meets P, and as we have shown this conjugate of Q must itself be conjugate to P. □

Proof of Theorem 4. We have G a group of finite Morley rank of $p^\perp$ type and P_1, P_2 Sylow p-subgroups. We may suppose that Sylow p-subgroups in proper definable subgroups of G are conjugate, and we wish to prove the same for G.

Let T_1, T_2 be the maximal p-tori in P_1, P_2 respectively. We may suppose $T_1 \le T_2$. If P_1 is finite the preceding lemma applies. So we may suppose T_1 is nontrivial.

If $N_G(T_1) < G$ then conjugacy holds in $N_G(T_1)$ and thus T_2 is conjugate to a subgroup of P_1. In this case $T_1 = T_2$, so $P_1, P_2 \le N(T_1)$ and our claim follows.

So suppose $T_1 \lhd G$. Then passing to $\bar{G} = G/d(T_1)$, the image $\bar{P}_1$ of P_1 is finite. We claim that $\bar{P}_1$ is a Sylow p-subgroup of $\bar{G}$. Let $\bar{Q}_1$ be a solvable p-group containing $\bar{P}_1$, set $\bar{Q} = d(\bar{Q}_1)$, and let Q be the preimage in G of $\bar{Q}$. Then Q is solvable. By [ACCN98], $\bar{P}_1$ is a Sylow p-subgroup of $\bar{Q}$, and hence $\bar{P}_1 = \bar{Q}_1$. That is, $\bar{P}_1$ is a Sylow p-subgroup of $\bar{G}$.

By the previous lemma $\bar{P}_1$ and $\bar{P}_2$ are conjugate, and we may suppose they are equal. Let $\bar{P} = d(\bar{P}_1)$ and let P be the preimage in G of $\bar{P}$. Then P is solvable and $P_1, P_2 \le P$, so by [ACCN98] the groups P_1, P_2 are conjugate, as claimed. □

5 Weyl groups

A suitable notion of "Weyl group" in the context of groups of finite Morley rank is the following.

Definition 5.1. Let G be a group of finite Morley rank. Then the *Weyl group* associated to G is the abstract group $W = N(T)/C^\circ(T)$ where T is a maximal decent torus, viewed as a group of automorphisms of T.

This is well-defined up to conjugacy in G, and finite.

In algebraic groups, Weyl groups are Coxeter groups, and in particular are generated by involutions. We note that, by a Fratini argument (Proposition 1.1), the "Weyl group" associated to some non-maximal decent torus is a quotient of the Weyl group associated to a maximal decent torus,

Theorem 5.2. *Let G be a connected group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then G contains a unipotent r-subgroup.*

In fact, we prove that either

(H1) an r-element representing an element of order r in W centralizes a unipotent r-subgroup, or else

(H2) some toral r-element centralizes a unipotent r-subgroup.

One should briefly consider a group G whose proper simple sections have $r^\perp$ type; which obviously includes minimal connected simple groups. Here if some toral r-element centralizes a unipotent r-subgroup, then the whole torus centralizes the unipotent r-subgroup. Of course, any Weyl group element whose centralizer has $r^\perp$ type is toral by Theorem 3. So the first of these two conditions suffices in this case, and we have the following as a corollary of the proof.

Corollary 5.3. *Let G be a minimal connected simple group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then G contains a unipotent r-subgroup in the centralizer of any r-element representing an element of order r in W.*

Proof of Theorem 5.2. We consider a counterexample G with maximal decent torus T, set $W := N(T)/C^\circ(T)$, and take $r \mid |W|$ minimal. Suppose also that both (H1) and (H2) fail in G, or simply that G has $r^\perp$ type for the simplified statment. We take G to have minimal Morley rank subject to these conditions.

We first reduce to the case

$$Z(G) = 1$$

In $\bar{G} = G/Z(G)$ the image $\bar{T}$ of T is a maximal decent torus by Lemma 1.4, with preimage $TZ(G)$ in G, and T is the unique maximal decent torus of $TZ(G)$. Hence $N(\bar{T})$ is the image of $N(T)$ and so $C^\circ(\bar{T}) = N^\circ(\bar{T})$ is the image of $C^\circ(T) = N^\circ(T)$. Thus $N(\bar{T})/C^\circ(\bar{T})$ is isomorphic with W, So we may assume $Z(G) = 1$ after replacing G by $\bar{G}$.

Now let T_r be the maximal r-torus of T; which is nontrivial by [BBC07, Theorem 3]. Fix an element w of order r in W, and choose a representative $a \in N(T)$ for w which is itself an r-element. We now assume that $C^\circ(a)$ has $r^\perp$ type since (H1) fails. Then $a \in N(T_r) \setminus T_r$.

We claim

$$C_{T_r}(a) \text{ is finite.}$$

Otherwise, set $U := C^\circ_{T_r}(a) \neq 1$. Let $\hat{U}$ be a maximal r-torus of $C^\circ_G(a)$ which contains U. Then $a \in \hat{U}$ by Theorem 3. So $H = C^\circ_G(U)$ contains both T and a, since $a \in \hat{U}$. As $Z(G) = 1$, we have $H < G$. The Weyl group of H is $N_H(T)/C_H(T)$, and a represents an r-element of this group, This contradicts the supposed minimality of G. So indeed $C_{T_r}(a)$ is finite.

Now commutation with a produces an endomorphism of T_r with finite kernel, and it is easy to see that any such endomorphism is surjective (working either which large invariant finite subgroups of T_r, or equivalently with the action of the endomorphism ring of T_r on the dual "Tate module"). So $[a, T_r] = T_r$, and multiplying by a on the left gives

$$(*) \qquad a^{T_r} = a \cdot T_r$$

Now there is some r-element $b \in C_{T_r}(a)^{\#}$ since $\Omega_1(T_p) \cdot \langle a \rangle$ is a finite p-group. Our goal is to play with $(*)$ and variations of $(*)$ to show that b and b^2 are conjugate, under the action of the Weyl group, which will contradict our hypothesis on the minimality of r. Observe to begin with that a, ab, and ab^2 are all T_r-conjugate, as they are in the coset aT_r.

We show next that $a \notin C^\circ(b)$ and $b \notin C^\circ(a)$.

If $a \in C^\circ(b)$ then, as $T \leq C^\circ(b)$, we again contradict the minimality of G. Hence $a \notin C^\circ(b)$. On the other hand we may now assume, by failure of (H2), that $C^\circ(b)$ has $r^\perp$ type since b is toral. So, if $b \in C^\circ(a)$ then, by Theorem 3 and its corollary, b belongs to a maximal r-torus U of $C^\circ(a)$, and also $a \in U$; so $a \in C^\circ(b)$, a contradiction. Thus $b \notin C^\circ(a)$.

As $C(a)$ has $r^\perp$ type, Theorem 4 says that its Sylow r-subgroups are conjugate. So, as b lies inside $C(a)$, there is a maximal decent torus U of $C(a)$ normalized by b. Now $a \in U$ by Corollary 3.1, and U is a maximal decent torus of G. Thus b represents a nontrivial r-element of the Weyl group relative to U. Now, with U_r the r-torsion in U, we can reverse the roles of a and b, and conclude that b, ab are conjugate under the action of U_r. Thus a, b are conjugate in G. As $r > 2$ we may argue similarly that a, b^2 are conjugate in G. So b, b^2 are conjugate in G.

Now observe, by [BN94, Lemma 10.22], that $N(T)$ controls fusion in T: if $X \subseteq T$ and $X^g \subseteq T$ then $T, T^g \leq C(X^g)$ and hence T^g is conjugate to T in $C(X^g)$, thus T is conjugate to T^g in $N(T)$. So b and b^2 are conjugate under the action of $N(T)$. In other words, we have a Weyl group element carrying b to b^2. Thus we have elements in W whose order is some prime dividing the order of 2 in the multiplicative group modulo r. Such a prime is a factor of $r-1$, and hence less than r. This contradicts the minimization of r. □

Corollary 5.4. *Let G be a connected group of finite Morley rank without unipotent torsion. If the Weyl group is nontrivial then it has even order; in particular, the group G is not of degenerate type in this case.*

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