A generation theorem for groups of finite Morley rank

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Introduction

Groups of finite Morley rank

A group of finite Morley rank is a group equipped with a notion of dimension satisfying various natural axioms [BN94, p. 57]; These groups arise naturally in model theory, expecially geometrical stability theory. The main examples are algebraic groups over algebraically closed fields, where the notion of dimension is the usual one, as well as certain groups arising in applications of model theory to diophantine problems, where the notion of dimension comes from differential algebra rather than algebraic geometry.

The structural analysis of groups of finite Morley rank is dominated by the Algebraicity Conjecture (Cherlin/Zilber), which states that a simple group of finite Morley rank should, in fact, be a Chevalley group over an algerbaically closed field, i.e., an algebraic group. This is a strong conjecture, which asserts that the classification of the simple algebraic groups can be carried out using only their coarsest properties.

It is known that a counterexample to the Algebraicity Conjecture containing at least one involution (i.e., element of order two) must have *odd type*. This means that the connected component of a Sylow 2-subgroup is a 2-torus (i.e., a divisible abelian group). This reduction to odd type is a result of a large body of work presented in [ABC07]. An equivalent condition is that the Sylow subgroups have finite 2-*rank*, where the 2-rank $m_2(G)$ is the supremum of the ranks of elementary abelian 2-subgroups of G.

Generic vs. thin

Our aim in the present work is to lay the foundations of the theory of generic simple groups of finite Morley rank as broadly as possible, taking all groups with $m_p(G) \geq 3$ as generic. A larger goal, not reached here, is the following.

Generic Algebraicity Conjecture. *Let G be a simple K^*-group of finite Morley rank with $m_2(G) \geq 3$. Then G is a Chevalley group over an algebraically closed field, of Lie rank at least two.*

A K^*-group is one whose proper definable simple sections are algebraic. A minimal nonalgebraic simple group of finite Morley rank would be a K^*-group.

The point of stating this conjecture separately under the hypothesis $m_2(G) \geq 3$ is that this is the natural target for methods of a *general* character. Some results of this type are already known, but under more restrictive assumptions. Building on this work, we aim to broaden the class of groups which may be treated as generic to its widest natural limit, leaving over as little as possible for separate consideration on a case by case basis.

The philosophy in the study of finite simple groups, which has been transferred also to the case of simple groups of finite Morley rank, is that large groups are those which can be handled by reasonably general methods, and small groups are those which require close analysis by ad hoc methods. The terms in use in finite group theory for "large" and "small" are "generic" and "thin", with an intermediate category, "quasithin". The precise definitions even in finite group theory remain in flux; they are chosen to match the details of specific proof strategies, and in the two generations of the classification of finite simple groups there has been considerable variation in the treatment of borderline cases. But tinkering aside, these are robust notions: generic groups are for the most part groups which turn out to be Chevalley groups of Lie rank at least 3, quasithin more or less Lie rank 2, and the thin groups are populated by Lie rank 1 and more pathological groups which offer comparably little internal structure.

Our goal *here* is to treat a group G as *generic* if $m_2(G) \geq 3$; thus also covering some quasithin cases. Stricter notions of genericity are defined by considering the structure of the Sylow 2-subgroup S, whose *normal* 2-rank $n_2(S)$ and *Prüfer* 2-rank $\mathrm{pr}_2(S)$ are defined by taking the maximal rank of a normal elementary abelian 2-subgroup, or a 2-subgroup contained in a divisible abelian 2-subgroup, respectively. One sets $n_2(G) = n_2(S)$, $\mathrm{pr}_2(G) = \mathrm{pr}_2(S)$, which by conjugacy of 2-Sylow subgroups is well-defined. It is easy to see that

$$m_2(G) \geq n_2(G) \geq \mathrm{pr}_2(G)$$

Generic groups and Uniqueness Cases

The following excellent approximation to the Generic Algebraicity Conjecture is proved in [Bur07]; an earlier version, assuming the absence of "bad fields", is due to Borovik.

High Prüfer rank Case. *A simple K^*-group of finite Morley rank with Prüfer 2-rank at least three is algebraic.*

Pushing the same result through in the case of $m_2(G) \geq 3$ must involves substantial technical complications. The general thrust of the argument in [Bur07] is to first exclude a "uniqueness case" involving the *2-generated core*, and then to apply systematic methods—signalizer functor theory and the Curtis-Tits theorem, primarily—to carry through the necessary structural analysis and recognition process.

Broadly speaking, a *uniqueness subgroup* of a simple group G is a very large definable subgroup which does not interact much with other subgroups of G. A typical example is the Borel subgroup of PSL_2. In generic type groups one aims to show that there can be no proper uniqueness subgroup in one or another sense. This then becomes the point of departure for a systematic study of interactions between subgroups, and eventually for the recognition of the group, or a geometry on which the group acts.

In our context there are four notions of uniqueness subgroup that come into play:

1. The groups Γ_V where V is elementary abelian 2-subgroup of rank 2 are

$$\langle C^\circ(v) : v \in V^{\#} \rangle$$

2. The weak 2-generated core, denoted $\Gamma_{S,2}^0$: for $S \leq G$ a Sylow 2-subgroup of G, this is the definable hull of the group generated by all subgroups

$$N(A)$$

where A varies over rank 2 elementary abelian subgroups of S which are contained in rank 3 elementary abelian subgroups of S.

3. The 2-generated core, denoted $\Gamma_{S,2}$, is defined similarly, but taking into account all elementary abelian subgroups A of S of rank 2, including those that are maximal in S.

4. Strongly embedded subgroups: a proper definable subgroup M of G is strongly embedded if M contains an involution while $M \cap M^g$ contains no involution for $g \notin M$.

In the definition of the weak 2-generated core, the superscript 0 is not a reference to the connectedness of any group, but rather an allusion to a connected component in an associated graph of elementary abelian p-subgroups. This graph is introduced explicitly in §2.

The results we aim at are the following. The second turns out to be a corollary of the first.

Generation Theorem I. *Let G be a simple K^*-group of finite Morley rank and odd type, with $m_2(G) \geq 3$. Then $\Gamma_V = G$ whenever V is an elementary abelian subgroup of rank 2.*

2-Generated Core Theorem I. *Let G be a simple K^*-group of finite Morley rank and odd type, with $m_2(G) \geq 3$. Then $\Gamma^0_{S,2} = G$ for any Sylow 2-subgroup S of G.*

These results will be given in a more general form as Theorem 6.4, and in the forms just stated as Corollary 6.5.

The first of these results is a generation theorem: G is generated by the connected components of centralizers $C^\circ(v)$ when v varies over involutions in an elementary abelian 2-subgroup of order 4. This points the way toward the identification of the group G in terms of the structure of centralizers of involutions. The meaning of the second result is less transparent, but using signalizer functor theory it can be shown that a proper weak 2-generated core $\Gamma^0_{S,2}$ arises whenever the centralizer $C(i)$ of in involution is "not sufficiently reductive" (meaning that $O(C(i))$ is "large" enough to be "unipotnet"). So for generic G, these statements give both a generation property by related centralizers of involutions, and a weak reductivity condition applying to those centralizers. Combined, in Prüfer rank at least three these lead eventually to generation of G by quasisimple components of centralizers of involutions, and then to a full description of G. Here the Curtis-Tits theorem simplifies matters considerably; in Prüfer rank two other methods come into play (one may consider this a quasithin case: generic arguments carry one some distance, then one must enter into the consideration of a certain number of special configurations).

In the proofs of our results we follow the line of [BBN06], showing that our hypothetical counterexample G is a minimal simple group of Prüfer 2-rank at least two, with a strongly embedded subgroup. Since such a configuration is impossible by [BCJ07], this suffices. The line of argument goes as follows: if Γ_V is the offending uniqueness subgroup (namely, $\Gamma_V < G$) then $N(\Gamma_V)$ turns out to be strongly embedded; this is the archetypal uniqueness condition. Further analysis then shows that G is a minimal simple group, reaching a contradiction to [BCJ07].

A technical point of considerable importance is some new information about "semisimple torsion" (e.g., 2-elements in odd type groups) which has become available only recently. This development begins with [BBC07] and has been developed further in [BC08], with particular attention to results which are useful in the present analysis. We are able to put aside a number of pathological configurations on this basis, greatly simplifying the flow of the argument.

Variation: p

We will take some pains to place our two main theorems in a substantially broader context. In the first place, we will replace the prime 2 by a general prime p. This requires some additional work, some of it already carried out in [BC08]. Our two uniqueness case theorems take on the following forms in general.

Theorem 6.4. *Let G be a simple K^*-group of finite Morley rank and $p^\perp$ type, with $m_p(G) \geq 3$ and $\mathrm{pr}_2(G) \geq 2$. Then $\Gamma_V = G$ for any elementary abelian*

p-subgroup of rank two. In particular, $\Gamma^0_{S,2} = G$ for any Sylow p-subgroup S of G.

Two important technical points which come up here are the following.

Theorem 1.1. *Let G be a connected group of finite Morley rank and $p^\perp$ type with $m_p(G) \geq 3$. Then $\mathrm{pr}_p(G) \geq 2$.*

This is particularly important for $p = 2$.

Theorem 1.2. *Let G be a connected group of finite Morley rank and odd type. with $m_p(G) \geq 3$. Then $m_p(V) \geq 3$ for any maximal elementary abelian p-subgroup $V \leq G$.*

Theorem 1.1 should always be borne in mind. We would not relish being forced to treat cases of Prüfer rank 1 within the "generic" framework.

Variation: L^*

Returning to the case $p = 2$, we would also like to dispense with the restriction to K^*-groups. If all simple K^*-groups are algebraic, then the same follows for all groups of finite Morley rank. But we would prefer to deal with groups of odd type in a way which does not require a prior analysis of groups of degenerate type. So the natural class of groups to consider, called L^*-*groups*, are those whose proper definable connected simple sections are either algebraic or of degenerate type (hence without involutions by [BBC07]). The analysis of L^*-groups then presupposes some prior analysis of L-groups, where an L-group satisfies the same condition for *all* definable connected simple sections.

Versions of our main results may be proved under the L^* hypothesis in a form that then yields the stated results immediately under the stronger K^*-hypothesis. We give this with an eye to the future—we are a long way from having the same kind of theory in an L^* context that we have for K^*-groups, but at least the present chapter has a satisfactory development at that level of generality.

Theorem 4.3. *Let G be a simple L^*-group of finite Morley rank and odd type, with $m_2(G) \geq 3$. Suppose that*

$$\Gamma_V < G \text{ for some } V \leq G \text{ with } m_2(V) \geq 2$$

Then $N(\Gamma_V)$ is a strongly embedded subgroup of G. Furthermore $N(\Gamma_V) = \Gamma_{S,2}(G)$ and $\Gamma_V = \Gamma_{S,2}{}^\circ$.

Actually, the statement of Theorem 4.3 below is slightly more general. We replace the L^* condition by a an assumption that a generation statement applies to proper connected sections of G. This is done simply to lay out more plainly the structure of the argument. The corresponding generation statement for L-groups, which is needed to justify the result as phrased above, is found in Theorem 2.1.

This result may be elaborated on further. We will call a group of finite Morley rank a *D-group* if all its definable connected simple sections are of degenerate type, and a *D*-group* if the same applies to all proper definable connected simple sections. In the context of K^* groups of odd type, a D^* group would be a minimal connected simple group.

Theorem 6.6. *Let G be a simple L^*-group of finite Morley rank and odd type, with $m_2(G) \geq 3$. Let V be an elementary abelian 2-group of 2-rank two with $\Gamma_V < G$. Then*

1. *G is a D^*-group.*

2. *$N(\Gamma_V)$ is a strongly embedded subgroup of G.*

3. *The Sylow 2-subgroups of G are connected.*

4. *If r is the least prime divisor of $|M/M^\circ|$, then G contains a nontrivial unipotent q-subgroup for some $q \leq r$.*

In this formulation, the result as stated is very close to the version given in [BBN06] under more restrictive conditions. In the K^* case one reaches a contradiction by applying [BCJ07], which says that minimal connected simple groups of finite Morley rank and odd type with $\mathrm{pr}_2(G) \geq 2$ never have strongly embedded subgroups. The corresponding result is not known in the D^* context, and should be difficult.

Finally, to close the gap between Γ_V and $\Gamma_{S,2}$, one more result is required.

Corollary 4.2. *Let G be a simple L^*-group of finite Morley rank and odd type, with $m_2(G) \geq 3$. Then $\Gamma_V \leq \Gamma_{S,2}$.*

Again, the statement actually given as Corollary 4.2 replaces the L^*-group hypothesis by an inductive condition on sections.

We have the following "characteristic zero" corollary.

Corollary. *Let G be a simple L^*-group of finite Morley rank with $m_2(G) \geq 3$, which has no unipotent torsion. Then*

1. *$\Gamma_V = G$ whenever $m_2(V) \geq 2$; equivalently,*

2. *$\Gamma_{S,2}^0 = G$ for any Sylow 2-subgroup S of G.*

At this stage it is natural to focus on the K^* case, and to ignore the further subtleties that intervene at a more general level, but even in that case something very much like our version II must be proved on the way toward a contradiction via [BCJ07].

Contents

We begin by proving Theorems 1.1 and 1.2. These come quickly out of the general theory developed in [BBC07], We also give some L-group theory which is parallel to what is known in the K-group case, in preparation for the proof of our generation theorems.

After that we give a preparatory generation theorem for L-groups when $p = 2$, and for K-groups when $p \neq 2$. This takes considerable analysis, involving a reduction to the simple algebraic case. The statement runs as follows.

Theorem 2.1. *Fix a prime p. Let H be a connected group of finite Morley rank. If $p = 2$ suppose that H is an L-group of $2^\perp$ type; if $p > 2$ suppose that H is a K-group of $p^\perp$ type. Let E be a finite elementary abelian p-group acting definably on H. Then*

$$H = \langle C_H{}^\circ(E_0) : E_0 \leq E, [E : E_0] = p \rangle.$$

After that we turn to the generation theorem for simple K^*-groups and L^*-groups. We also show that the corresponding results for the weak 2-generated core $\Gamma_{S,2}^0$ follow from the Γ_V versions.

The first result is strong embedding for $N(\Gamma_V)$ when $\Gamma_V < G$. In section §5 we prove that a strongly embedded subgroup of our group G is a D-group. The argument is based in part on one given by Christine Altseimer in [Alt] in the K^* context.

Theorem 5.1. *Let G be a simple L^*-group of finite Morley rank and odd type. Then a strongly embedded subgroup M of G is either a D-group, or else satisfies $M^\circ/O(M^\circ) \cong \mathrm{SL}_2$.*

In the last section we complete our analysis by combining Theorems 4.3 and 5.1, finally proving that G is a D^*-group. Then [BCJ07] provides a contradiction in the K^* case, proving the generation theorem in that case.

1 Preliminaries

We will prove the following two results in the present section.

Theorem 1.1. *Let G be a connected group of finite Morley rank and $p^\perp$ type with $m_p(G) \geq 3$. If $p > 2$ then $\mathrm{pr}_p(G) \geq 3$. If $p = 2$ then $\mathrm{pr}_2(G) \geq 2$.*

Theorem 1.2. *Let G be a connected group of finite Morley rank and $p^\perp$ type with $m_p(G) \geq 3$. Then any maximal elementary abelian p-subgroup $V \leq G$ has p-rank at least 3.*

We will also develop some general L-group theory which comes into play in our analysis.

1.1 Semisimple torsion

We will rely on results on p-torsion in groups of $p^\perp$ type developed in [BC08], notably the following.

Fact 1.3 ([BC08]). *Let G be a connected group of finite Morley rank of $p^\perp$ type. Let $a \in G$ be a p-element, and $T \leq C(a)$ a maximal p-torus in $C(a)$. Then $a \in T$.*

This is commonly applied in two different ways. First and foremost, any p-element will be *toral* (lie in a p-torus). Secondly, any p-element in the centralizer of a maximal p-torus T of G will lie in T, or in other words any p-element in $N(T) \setminus T$ will act nontrivially on T (in the latter formulation, this is a statement about the *Weyl group* $N(T)/C(T)$).

We also have a conjugacy theorem for Sylow p-subgroups in the $p^\perp$ case. By definition, a Sylow p-subgroup of a group of finite Morley rank is a maximal solvable p-subgroup (equivalently, a maximal locally finite p-subgroup—which makes existence clearer).

Fact 1.4 ([BC08, Theorem 4]). *Let G be a group of finite Morley rank of $p^\perp$ type. Then Sylow p-subgroups of G are conjugate.*

For $p = 2$ this is known for arbitrary groups of finite Morley rank.

We will also require some elementary facts about automorphisms of p-tori.

Fact 1.5 ([Ber01, BB04, §3.3]). *Let T be a p-torus of Prüfer p-rank d. Then $\mathrm{End}(T)$ can be faithfully represented as the ring $M_d(\mathbb{Z}_p)$ of $d \times d$ matrices over the p-adic integers $\mathbb{Z}_p$.*

Underlying this fact is a duality between T and its so-called *Tate module*, which is a free $\mathbb{Z}_p$-module of rank d naturally associated with T. We remark that the action of $M_2(\mathbb{Z}_p)$ on T is natural if T is represented as C_p^n with C_p quasicylic (the direct limit of cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$); multiplication of elements of C_p by elements of $\mathbb{Z}_p$ is well defined, since $\mathbb{Z}_p/p^n\mathbb{Z}_p = \mathbb{Z}/p^n\mathbb{Z}$ for any n.

Lemma 1.6. *Let T be a nontrivial 2-torus and let i be an involution acting on T. Then either i inverts T or $C_T(i)$ is infinite.*

Proof. Viewing i as given by a matrix E in $M_d(\mathbb{Z}_p)$, if $E - I$ is noninvertible then $E - I$ has a kernel in its natural action on $\mathbb{Z}_p^d$. Fixing a nonzero vector $v \in \mathbb{Z}_p^d$ annihilated by $E - I$ and working modulo p^n for large n, we obtain arbitrarily large cyclic subgroups of T centralized by i. So in this case $C_T(i)$ is infinite.

On the other hand if $E - I$ is invertible then since $(E - I)(E + I) = 0$ we have $E = -I$ and i acts by inversion. $\square$

This proof can be given just as easily in the language of finite group theory, but as we have a description of $\mathrm{End}(T)$ available we prefer to use it.

Lemma 1.7. *Let G be a connected group of finite Morley rank, p an odd prime, T a maximal p-torus of G, and $a \in N(T)$ a p-element centralizing $\Omega_1(T)$. Then a centralizes T.*

Proof. Suppose that $a \notin C(T)$. Passing to a power of a we may suppose that a^p centralizes T. So a induces an automorphism of T of order p. Now the action of a on T is represented by a matrix M over $\mathbb{Z}_p$ which acts trivially on $\Omega_1(T)$ and therefore is congruent to the identity I modulo p. Write

$$M = I + p^k A$$

with k maximal and A a matrix over $\mathbb{Z}_p$. Now $M^p = I$, so as $p > 2$ we have by the binomial theorem

$$I \equiv I + p^{k+1} A \mod p^{2k+1}$$

This forces A to be divisible by p, a contradiction. $\square$

Lemma 1.8. *Let T be a p-torus of Prüfer rank 1 (a quasicyclic p-group), with $p > 2$. Then $\mathrm{Aut}(T)$ contains no element of order p.*

Proof. Such an element would correspond to a primitive p-th root of unity in $\mathbb{Z}_p$, and by the usual argument the cyclotomic polynomial of degree $p - 1$ is irreducible over $\mathbb{Z}_p$, forcing $p - 1 \le 1$. $\square$

Fact 1.9. *Let P be an infinite solvable p-subgroup of a group G of finite Morley rank and Q a normal subgroup of P. Then $Q \cap Z(P) > 1$.*

Proof. The group P is nilpotent by finite [BN94], and the claim is easily proved for nilpotent by finite p-groups. $\square$

1.2 Prüfer rank

We treat the two claims of Theorem 1.1 separately.

Lemma 1.10. *Let G be a connected group of finite Morley rank and $p^\perp$ type, with $m_p(G) \ge 3$ and $p > 2$. Then $\mathrm{pr}_p(G) \ge 3$.*

Proof. We let T be a maximal p-torus of G and S a Sylow p-subgroup of G containing T. By the conjugacy of Sylow p-subgroups of G, S contains an elementary abelian p-subgroup E with $m_p(E) = 3$. We choose E so as to maximize $A = E \cap \Omega_1(T)$. Suppose toward a contradiction that $m_p(T) \le 2$.

Suppose first that $A < \Omega_1(T)$. Fix $t \in \Omega_1(T) \setminus A$ such that the image of t in $(\Omega_1(T)E)/A$ is in the center of this group. Looking at the commutator map $[t, \cdot] : E \to A$, since A has p-rank at most 1 we find $m_p(C_E(t)) \ge 2$. Thus we may replace E by a subgroup of p-rank 3 containing A and t, contradicting the maximality of A.

So $A = \Omega_1(T)$. Hence E acts trivially on $\Omega_1(T)$, and by Lemma 1.7 E centralizes T. Then by Fact 1.3 we have $E \le T$, and our claim follows. $\square$

Lemma 1.11. *Let G be a connected group of finite Morley rank and odd type, with $m_2(G) \geq 3$. Then $\mathrm{pr}_2(G) \geq 2$.*

Proof. As in the previous argument we begin with T a maximal 2-torus of G and $E \leq N(T)$ an elementary abelian 2-subgroup of 2-rank 3, chosen so as to maximize $A = E \cap \Omega_1(T)$.

If $T = 1$ then we contradict Fact 1.3 (as well as older results of [BBC07] on which the proof of this fact depends), so we need only consider the case in which

$$m_2(T) = 1$$

Now each element of E either centralizes or inverts T by Lemma 1.6. So $|C_E(T)| \geq 4$. On the other hand $C_E(T) \leq T$ by Fact 1.3, so $|C_E(T)| \leq 2$, a contradiction. □

So Theorem 1.1 is proved.

1.3   Maximal elementary abelian subgroups

Proof of Theorem 1.2. We have a connected group G of finite Morley rank and $p^\perp$ type, with $m_p(G) \geq 3$. We claim that any elementary abelian p-subgroup is contained in one of p-rank at least three. We know that $\mathrm{pr}_p(G) \geq 2$.

Suppose toward a contradiction that some maximal elementary abelian p-subgroup V of G has $m_p(V) \leq 2$. Let S be a Sylow p-subgroup of G containing V. Let $T = S^\circ$. By the conjugacy of Sylow p-subgroups, we have $m_p(S) \geq 3$.

Let $A = \Omega_1(Z(S))$. By maximality of V, we have $A \leq V$. If $m_p(A) \geq 2$ we then have $V = A \leq Z(S)$, and as $m_p(S) \geq 3$ we reach a contradiction to the maximality of V. So A is cyclic of order p. As $\Omega_1(T) \cap Z(S) > 1$ by Lemma 1.9, we have $\Omega_1(T) \cap Z(S) = A \leq V$.

If $V \leq T$, then by maximality of V we have $V = \Omega_1(T) \lhd S$. Choose $E \leq S$ elementary abelian with p-rank 3. Then $[E, V] \leq A$ and viewing this as a bilinear map from $E \times V$ to A, we find that $C_E(V)$ has p-rank at least 2. By maximality of V we have $C_E(V) \leq V$ and thus $V = C_E(V) \leq E$, which again contradicts the maximality of V.

So we find

$$(*) \qquad\qquad\qquad V \nleq T$$

Choose $v \in V \setminus T$. Then $V = \langle A, v \rangle$. If $m_p(T) \geq 3$ then looking at commutation with v as a map from $\Omega_1(T)/A$ to A, we find an element $t \in \Omega_1(T) \setminus A$ centralizing V and contradict the maximality of A. So $m_p(T) = 2$ and in particular $p = 2$ by Theorem 1.1.

By maximality of V, v does not centralize $\Omega_1(T)$, and hence v does not invert T. By Fact 1.6, $T_0 := C_T{}^\circ(v)$ is nontrivial. Note that $A = \Omega_1(T_0)$. Let T_1 be a maximal torus of $C(v)$ containing T_0. By Theorem 1.3, $v \in T_1$. So $V = \langle A, v \rangle \leq T_1$.

We have shown that V lies inside a p-torus of G. But then we could take T to be such a p-torus, contradicting $(*)$. □

1.4 Structure of L-groups

In this section, we give some variations on [Bor95, Thm. 5.12], adapted to an L-group context. We use the notation $\hat{O}(H)$ for the largest definable connected normal subgroup of H without involutions, $E(H)$ for the product of the quasisimple subnormal subgroups L of H, $F(H)$ for the Fitting subgroup, and $F^*(H) = F(H)E(H)$ for the generalized Fitting subgroup. By [BN94, Lemmas 7.9, 7.10, 7.13], the product $F(H)E(H)$ is a central product, the group $E(H)$ is a central product of finitely many normal quasisimple factors, each definable in G, and

$$C_H(F^*(H)) \le F^*(H)$$

Lemma 1.12. *Let H be a connected L-group of finite Morley rank and odd type with $\hat{O}(H) = 1$. Let $E_{\mathrm{alg}}(H)$ be the product of the algebraic components of $E(H)$. Then*

$$H = E_{\mathrm{alg}}(H) * H_0$$

where $H_0 = C_H{}^\circ(E_{\mathrm{alg}}(H))$, and $E_{\mathrm{alg}}(H_0) = 1$.

Proof. If L is a component of $E(H)$ with $L/Z(L)$ algebraic, then L is algebraic by [AC99]. Furthermore, a definable group of automorphisms of a quasisimple algebraic group, in a finite Morley rank context, must itself consist of algebraic automorphisms, that is graph automorphisms combined with inner automorphisms ([BN94, Thm. 8.4]). So H, being connected, induces only inner automorphisms on $E_{\mathrm{alg}}(H)$, and this yields

$$H = E_{\mathrm{alg}}(H) * H_0$$

with $H_0 = C_H{}^\circ(E_{\mathrm{alg}}(H))$. It is clear that $E_{\mathrm{alg}}(H_0) \le E_{\mathrm{alg}}(H) \cap H_0 \le Z(H_0)$, so $E_{\mathrm{alg}}(H_0) = 1$. $\qquad\square$

Theorem 1.13. *Let H be a connected L-group of finite Morley rank and odd type, with no normal definable connected solvable subgroup and no normal definable connected subgroup of degenerate type. Then $H = H_1 * \cdots * H_n$ is isomorphic to a central product of quasisimple algebraic groups $H_1, \ldots, H_n$ over algebraically closed fields of characteristic not equal to 2.*

Proof. In this context we have $E(H) = E_{\mathrm{alg}}(H)$ and H has no nontrivial definable connected solvable subgroup, so $F^*(H) = F(H)E_{\mathrm{alg}}(H)$ with $F^\circ(H) = 1$. Thus $C_H(E_{\mathrm{alg}}(H)) = C_H(F^*(H)) = Z(F^*(H))$ is finite and $C_H{}^\circ(E_{\mathrm{alg}}(H)) = 1$. So by the preceding lemma, $H = E_{\mathrm{alg}}(H)$. $\qquad\square$

Lemma 1.14. *The Sylow 2-subgroup of a connected D-group G of finite Morley rank is connected.*

Proof. We proceed by induction on the rank of G. Suppose first that G has a definable connected normal subgroup H with $1 < H < G$. Let S be a Sylow 2-subgroup of H and S_1 a Sylow 2-subgroup of G containing S. By [Wag97,

Corollary 1.5.5], the image of S_1 in G/H is a Sylow 2-subgroup of G/H, and hence connected by induction. Hence this is also the image of $S_1{}^\circ$ in G/H. So $S_1 \leq S_1{}^\circ H$, and $S_1 = S_1{}^\circ (S_1 \cap H)$. But $S_1 \cap H$ is a Sylow 2-subgroup of H, and is therefore connected. So S_1 is connected.

Now suppose G has no definable connected proper and nontrivial normal subgroup. If G is abelian, our claim is straightforward (and more generally if G is solvable it falls under known results). So we may suppose $Z(G)$ is finite, and then $G/Z(G)$ is simple. By the definition of D-group, $G/Z(G)$ has degenerate type, and hence so does G. But groups of degenerate type contain no involutions, so again the Sylow 2-subgroup is connected. $\qquad\square$

2 p-Generation

The proof of our generation result for simple K^*-groups and L^*-groups will be inductive, and depends therefore on generation results for K-groups and L-groups. The latter can be put in a stronger form. The full result runs as follows.

Theorem 2.1. *Let G be a connected group of finite Morley rank, p a prime. Suppose that G has $p^\perp$ type and V is an elementary abelian p-group of rank 2 acting definably on G. Suppose further that one of the following holds.*

1. G is a K-group;

2. G is an L-group, and $p = 2$;

Then $\Gamma_V = G$.

This generation theorem can be strengthened further, but the argument for this is purely formal. Consider the following property.

Definition 2.2. A group G of finite Morley rank has the *p-generation property* if for every elementary abelian p-group E of rank 2 acting definably on G, we have
$$G = \langle C_G(v) : v \in E^\# \rangle$$

We observe that this property is trivial when the action of V is not faithful, so we concern ourselves only with faithful actions.

Lemma 2.3. *Let G be a group of finite Morley rank such that every connected definable subgroup of G has the p-generation property, and let E be a nontrivial elementary abelian p-group of definable automorphisms of G. Then*
$$G = \langle C_G{}^\circ(E_0) : E_0 \leq E, [E : E_0] = p \rangle$$

Proof. If E is cyclic this is vacuous. For E of p-rank at least 2, fix $V \leq E$ of p-rank 2. Then
$$G = \langle C_G{}^\circ(v) : v \in V^\# \rangle$$

For $v \in V^\#$ let $E = \langle v \rangle \oplus E_v$. Inductively,

$$C_G{}^\circ(v) = C_G{}^\circ(\langle v, E_1 \rangle : E_1 \le E_v, [E_v : E_1] = p)$$

and our claim follows. $\qquad\square$

Note that in a counterexample to Theorem 2.1 of minimal Morley rank, all proper definable connected subgroups and all quotients by nontrivial definable connected normal subgroups will have the p-generation property.

The aim of the present section is to reduce the treatment of Theorem 2.1 to the simple algebraic case, and to show that the elementary abelian p-groups V involved can be taken to be contained in the original group G.

Reductions

We first treat a "base case" of Theorem 2.1.

Lemma 2.4. *Let G be a connected group of finite Morley rank of $p^\perp$ type and V an elementary abelian p-group of rank 2 consisting of definable automorphisms of G. Suppose that one of the following holds.*

1. *G is abelian and contains no nontrivial proper definable connected V-invariant subgroup;*

2. *G has degenerate type and $p = 2$.*

Then $G = \Gamma_V(G)$.

Proof. If G has degenerate type and $p = 2$, this is [BBC07, Theorem 4].

So suppose G is abelian. If G contains no p-torsion, the result is given as [Bur04, Fact 3.7].

Now suppose that G contains p-torsion. Being of $p^\perp$ type, connected, and minimal V-invariant, G is then the definable closure of a p-torus T. It suffices to show that $T \le \Gamma_V$.

Now $V \le \operatorname{End}(T) \cong M_n(\mathbb{Z}_p)$ acts on a vector space W over $\mathbb{Q}_p$ and in this representation the action of V is completely reducible. Consider an irreducible summand W_0 of W. The image of V in $\operatorname{End}(W_0)$ is cyclic, by Schur's lemma. Thus W_0 is centralized by some nontrivial $v \in V$. In matrix terms, this means that the identity matrix I can be expressed as a sum of projection matrices P_i with coefficients in $\mathbb{Q}_p$, each annihilated by an element corresponding to the endomorphism $1 - v$ for some $v \in V^\#$. Thus some multiple $p^n I$ can be similarly expressed in $M_n(\mathbb{Z}_p)$. In T this says that every element of $T = p^n T$ can be expressed as a linear combination of elements annihilated by $1 - v$ for various $v \in V^\#$ and thus $T \le \Gamma_V$. $\qquad\square$

The next result prepares for an inductive argument relative to normal subgroups.

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Lemma 2.5. *Let G be a group of finite Morley rank, p a prime, v a definable automorphism of G of order a power of p, and K a definable normal p-invariant connected subgroup of $p^\perp$ type. Suppose one of the following holds.*

1. *K is solvable.*

2. *K is of degenerate type and $p = 2$.*

Then $C_G{}^\circ(v \bmod K) = C_G{}^\circ(v)K/K$.

Here we use the notation $C_G{}^\circ(v \bmod K)$ for the connected component of the preimage in G of $C_{G/K}(v)$.

Proof. We may suppose that $G = C_G{}^\circ(v \bmod K)$. Then G is connected, and v acts trivially on G/K. Furthermore it suffices to treat the case in which v has order p, since we can then argue inductively, replacing G by $C_G{}^\circ(v^p)$ and v by its restriction to $C_G{}^\circ(v^p)$.

Now in the case $p = 2$ with K of degenerate type, this is essentially [BBC07, Lemma 9.3], together with the fact that connected groups of degenerate type have no involutions ([BBC07, Theorem 1]).

So we may suppose K is solvable. Proceeding by induction on $\mathrm{rk}(K)$, we may suppose that K contains no proper nontrivial definable and definably characteristic connected subgroup. Then K is abelian.

If K contains no p-torsion, then $\langle v \rangle$ is a Sylow p-subgroup of $K\langle v \rangle$, and as $[G, v] \subseteq K$ we find $K\langle v \rangle \lhd G\langle v \rangle$. As Sylow p-subgroups are conjugate in $K\langle v \rangle$, the Frattini argument gives $G \leq K\langle v \rangle \cdot N_G(\langle v \rangle)$. But v normalizes G, so $N_G(\langle v \rangle) = C_G(v)$. Thus $G \leq KC_G(v)\langle v \rangle = KC_{G\langle v \rangle}(v)$, and therefore $G = KC_G(v)$.

Finally, suppose that K contains nontrivial p-torsion. By the minimality of K, as K is contains no p-unipotent subgroup, it is the definable closure of a p-torus. As G is connected, we have $K \leq Z(G)$. So the commutation map $h \mapsto [v, h]$ is a homomorphism from G into K. If v centralizes K then the image of this map has exponent p and is connected, hence trivial. Otherwise, our hypothesis on K implies that $C_K(v)$ is finite, and thus commutation with v is surjective from K to K. Then for $h \in G$ there is $k \in K$ with $[v, h] = [v, k]$, and hence $hk^{-1} \in C(v)$, $h \in KC_G(v)$. $\qquad\square$

Now we may carry out an inductive argument.

Lemma 2.6. *Let G be a connected group of finite Morley rank of $p^\perp$ type, for some prime p. Suppose that G contains a nontrivial definable connected normal definably characteristic subgroup K satisfying one of the following conditions.*

1. *K is solvable.*

2. *K has degenerate type, and $p = 2$.*

If G/K has the p-generation property, then so does G.

Proof. Fix V a p-group of rank 2 consisting of definable automorphisms of G. By Lemma 2.4 we have $K \leq \Gamma_V$. By hypothesis $G = \langle C_G^{\circ}(v \mod K) : v \in V^{\#} \rangle$, and by the previous lemma $C_G^{\circ}(v) \leq K\Gamma_V(G)$ for $v \in V^{\#}$. So putting this all together, $G = \Gamma_V$. $\qquad \square$

We need a variation of this lemma to handle finite normal subgroups.

Lemma 2.7. *Suppose that G is a connected group of finite Morley rank with $Z(G)$ finite, p a prime, and V an elementary abelian p-group of rank 2 acting definably on G. If $\Gamma_V(G/Z(G)) = G/Z(G)$, then $\Gamma_V(G) = G$.*

Proof. For $v \in V^{\#}$, let $G_v = C_G^{\circ}(v \mod Z(G))$ Then $G = \langle G_v Z(G) \rangle$ and it suffices to show that $G_v \leq \Gamma_V$.

Now $[v, G_v] \leq Z(G)$ is connected, hence trivial. So $G_v \leq C^{\circ}(v) \leq \Gamma_V$. $\qquad \square$

Summing up the reductions so far, we have the following.

Lemma 2.8. *Let G be a connected group of finite Morley rank of $p^{\perp}$ type for some prime p. Suppose that every proper definable connected section of G has the p-generation property, and G does not. Then $F^*(G) = E(G)$ is a direct product of simple groups, and if $G \neq E(G)$ then $E(G)$ contains no p-torsion, and $p > 2$.*

Proof. Fix an elementary abelian p-group V of rank 2 acting definably on G for which $\Gamma_V < G$.

By Lemmas 2.6 and 2.7 we have

$$Z(G) = 1$$

Then by Lemma 2.6 again, we have $F(G) = 1$ and $F^*(G) = E(G)$ is a direct product of connected simple groups.

Now suppose

$$E(G) < G$$

Then by hypothesis $E(G) \leq \Gamma_V$.

If $E(G)$ contains p-torsion, then as $E(G)$ is connected of $p^{\perp}$ type, $E(G)$ contains a nontrivial p-torus. Let S be a Sylow p-subgroup of GV. As GV has $p^{\perp}$ type its Sylow p-subgroups are conjugate. Therefore S contains some maximal p-torus T of $E(G)$. By the Frattini argument $G = E(G)N(T)$, and as G is connected we find

$$G = E(G)N^{\circ}(T)$$

As $N^{\circ}(T)$ is V-invariant and $N^{\circ}(T) < G$ we also have $N^{\circ}(T) \leq \Gamma_V$, so $G = \Gamma_V$, a contradiction. So $E(G)$ contains no p-torsion.

Now if $p = 2$ then $E(G)$ has degenerate type and we violate Lemma 2.6. $\qquad \square$

This amounts to a substantial reduction of the proof of Theorem 2.1. Indeed, if G is a counterexample of minimal Morley rank, then $Z(G)$ is finite by Lemma 2.6, and $G/Z(G)$ is again a counterexample to the same theorem by Lemma

2.7. Then replacing G by $\bar{G} = G/Z(G)$ we find $Z(\bar{G}) = 1$ and every proper connected section of G has lower rank, so the previous lemma applies to $\bar{G}$.

Furthermore, under the hypotheses of Theorem 2.1, if $E(G) < G$ then either $E(G)$ is a K-group, hence a product of simple algebraic groups, and therefore contains p-torsion for all primes p, or else $p = 2$. In either case Lemma 2.8 is violated. So in this context, we come down to the case $G = E(G)$ with $Z(G) = 1$.

2.1 Replacement

An important tool in the proof of Theorem 2.1 will be our ability to replace one elementary abelian p-group V by another, as follows.

Lemma 2.9. *Let G be a connected centerless group of finite Morley rank and $p^\perp$ type for some prime p. Suppose that every proper definable connected subgroup of G has the p-generation property, and let E, V be two elementary abelian p-groups of p-rank at least two, acting definably on G in such a way that the actions commute. Then*

$$\Gamma_E(G) = \Gamma_V(G)$$

Proof. For $v \in V^{\#}$ we have $C_G{}^\circ(v) < G$ and E acts on $C_G{}^\circ(v)$, so by hypothesis $C_G{}^\circ(v) \leq \Gamma_E$, and hence $\Gamma_V \leq \Gamma_E$. Similarly $\Gamma_E \leq \Gamma_V$. $\qquad\square$

We note that if G is both of $p^\perp$-type and $q^\perp$ type for two primes p, q then the same result holds with V an elementary abelian p-groups and W an elementary abelian q-group. This can be useful with $p > 2$ and $q = 2$.

It is natural at this point to look at the graph $\mathcal{E}_{p,2}$ whose vertices are the elementary abelian p-groups acting definably on G (and for definiteness, living in the normalizer of G in some larger group), and whose edges are commuting pairs of such groups. Then Γ_V is associated to a connected component of this graph, rather than to a single vertex. This graph is frequently connected, in which case Γ_V is a canonical normal subgroup of G—which is not surprising, as we expect it to be the group G itself in the cases of interest to us. We will write $\mathcal{E}_{p,2}(X)$ for the corresponding graph with vertices contained in X where X is a fixed group normalizing G.

The following is [Bur07, Fact 1.20].

Fact 2.10. *Let S be a locally finite p-group. Then*

    1. The induced subgraph $\mathcal{E}_{p,2}^0(S) := \{X \in \mathcal{E}_{p,2}(S) : m_p(C_S(X)) > 2\}$ is connected, i.e., all nonsingleton vertices of $\mathcal{E}_{p,2}(S)$ are linked by paths.

    2. $\mathcal{E}_{p,2}^0(S)$ contains $\Omega_1(S^\circ)$.

    3. If $n_p(S) > p$ then S is 2-connected, i.e. if S contains an abelian normal subgroup of p-rank at least $p + 1$ then $\mathcal{E}_{p,2}(S)$ is connected.

This has the following immediate consequence.

Lemma 2.11. *Let G be a connected semisimple group of finite Morley rank (that is, a direct product of connected simple groups) of $p^\perp$ type for some prime p, and suppose that G does not have the p-generation property, but every proper definable connected subgroup of G does. Then one of the following holds:*

1. G is simple;

2. G is the product of exactly p isomorphic simple components, and each component has an automorphism of order p which normalizes no nontrivial proper definable connected subgroup of L. In particular, G contains no p-torsion.

Proof. Let $V = \langle a, b \rangle$, be an elementary abelian p-group of p-rank 2 acting definably on G so that $\Gamma_V < G$. Suppose G contains a proper nontrivial normal V-invariant subgroup H. Then $G = H \times C_G(H)$ and by hypothesis both H and $C_G(H)$ are contained in Γ_V, so $G = \Gamma_V$, a contradiction.

Accordingly V acts transitively on the simple components of G. Suppose first that V acts regularly on the set of components. Let L be a component of G, and let H, K be the normal closures of L under the action of $\langle a \rangle$ and $\langle b \rangle$ respectively. Then $C_H(a)$, $C_K(b)$ are diagonally embedded copies of L in H and K respectively, and $H \cap K = L$. It follows that $[C_H(a), C_K(b)] = L$. Thus $L \leq \Gamma_V$ and so $G = \Gamma_V$, a contradiction.

So V does not act regularly on the components of G and hence we may suppose the component L of G is normalized by a. If L is normalized by V then $G = L$ is simple. So we remain with the case in which G is the product of precisely p conjugates of L. Now suppose a normalizes a nontrivial proper definable connected subgroup L_0 of L. Then the product of the conjugates of L_0 under V lies in Γ_V by hypothesis, and in particular $L_0 \leq \Gamma_V$. On the other hand, $C_G{}^\circ(b)$ is a diagonal subgroup of G covering L in this case and $[C_G{}^\circ(b), L_0] = L$ and so $L \leq \Gamma_V$ and then $G = \Gamma_V$, a contradiction.

So a normalizes no nontrivial proper definable connected subgroup of L. In particular L contains no p-torsion, as otherwise L would contain an a-invariant maximal p-torus, using the conjugacy of Sylow p-subgroups in $L\langle a \rangle$. So G contains no p-torsion. $\square$

Corollary 2.12. *For the proof of Theorem 2.1 it suffices to treat the case of simple groups G, under the hypothesis that every proper definable connected section of G has the p-generation property.*

Proof. Indeed, we have already seen that after minimization and factoring out the center, the group G will be semisimple. If G is not simple, then the previous lemma shows that G contains no p-torsion. On the other hand, under the hypotheses of Theorem 2.1, G is either a K-group, or an L-group with $p = 2$. In the former case as the factors of G are algebraic, G contains p-torsion for all p, and we contradict the lemma. In the latter case as $p = 2$ and G contains no p-torsion, G is of degenerate type and Lemma 2.4 applies. $\square$

Finally, we argue that we may move the elementary abelian p-groups under consideration inside the group G.

Lemma 2.13. *Let G be a simple algebraic group of $p^\perp$ type, possibly in an enriched language, such that all proper definable connected sections of G have the p-generation property. Let V be an elementary abelian p-group of rank two acting definably and faithfully on G, with $\Gamma_V < G$. Then there is an elementary abelian p-group U of rank two contained in G such that $\Gamma_V = \Gamma_U$.*

Proof. By Fact 2.10, if the Lie rank of G is at least $p+1$ and S is a Sylow p-subgroup of GV, then the graph $\mathcal{E}_{p,2}(S)$ is connected. Then by Lemma 2.9 we have $\Gamma_V = \Gamma_U$ for any $U \leq \Omega_1(S^\circ)$ of rank two.

On the other hand, if G has no graph automorphisms of order p then any definable automorphism of G of order p will be inner, by [BN94, Thm. 8.4], in which case we have nothing to prove. As the graph automorphisms have orders $2, 3$ we consider the two cases $p = 2, 3$.

If $p = 3$ we are dealing with the group D_4 which has Lie rank $p + 1$, and as noted we are done in this case.

If $p = 2$ we need to deal only with groups of Lie rank at most two and the only one with a graph automorphism is A_2, namely PSL_3. In this case a graph automorphism $v \in V$ is conjugate to the inverse-transpose automorphism, with centralizer and $C_G(v) \cong \mathrm{PSL}_2$, by [GLS98, Table 4.3.1 p. 145 & Table 4.3.3 p. 151]. On the other hand V must also contain an element u inducing an inner automorphism; taking u without loss of generality to be an element of G we have $u \in C_G(v)$. Now u belongs to a 4-group U in $C_G(v)$, and then U commutes with V, so by Lemma 2.9 $\Gamma_V = \Gamma_U$. $\qquad\square$

3 p-Generation in simple algebraic groups

Our goal now is to prove Theorem 2.1 for simple algebraic groups. As we have seen, we may suppose that all proper definable connected subgroups satisfy p-generation, and we may restrict our attention to Γ_V for V contained in the given group.

It will be convenient to dispose of PSL_2 at the outset.

Lemma 3.1. *If $G = \mathrm{PSL}_2$ over a field of characteristic other than p, then G satisfies p-generation.*

Proof. We are dealing with an elementary abelian p-group V of rank 2 with $V \leq G$. Each element of V belongs to a maximal torus of G and if these tori do not lie in a Borel subgroup they generate G. But if they do lie in a Borel subgroup then so does V, contradicting the structure of Borel subgroups in this case. $\qquad\square$

Now we undertake some structural analysis. The following extreme case arises frequently.

Lemma 3.2. *Let G be an algebraic group of $p^\perp$ type such that every proper connected Zariski closed subgroup has the p-generation property. Let V be an elementary abelian subgroup of rank 2 contained in a Borel subgroup of G. Then $\Gamma_V = G$.*

Proof. As G is not PSL_2, the maximal parabolic subgroups of G containing B generate G, and by hypothesis each such subgroup is contained in Γ_V. So $G = \Gamma_V$. □

Lemma 3.3. *Let G be a simple algebraic group of $p^\perp$ type, and $V \leq G$ an elementary abelian p-subgroup of p-rank two. Suppose that every proper connected Zariski closed subgroup of G has the p-generation property. Then Γ_V is a Zariski closed reductive subgroup of G containing a maximal torus of G, and with finite center; the center has order relatively prime to p.*

Proof. Every element of V lies in a maximal torus of G and thus Γ_V contains a maximal torus of G. Let H be the subgroup of Γ_V generated by maximal tori of G. Then H is Zariski closed and $\Gamma_V \leq N^\circ(H) < G$. By generation $\Gamma_V = N^\circ(H)$, a Zariski closed subgroup.

If U is the unipotent radical of Γ_V then for $v \in V^\#$ we have $U \cap C^\circ(v)$ contained in the unipotent radical of $C^\circ(v)$; but v being a semisimple element, $C^\circ(v)$ is reductive [Car93, Thm. 3.5.4], and thus $C_U{}^\circ(v) = 1$. But by hypothesis $U = \langle C_U{}^\circ(v) : v \in V^\# \rangle$ and thus $U = 1$, Γ_V is reductive.

Now we claim that $Z(\Gamma_V)$ contains no element of order p. As G is of $p^\perp$ type it then follows that $Z(\Gamma_V)$ contains no torus, and hence $Z(\Gamma_V)$ is finite of order prime to p. So this will complete the analysis.

So suppose toward a contradiction that $u \in Z(\Gamma_V)$ has order p. We can replace V by a subgroup of $\langle V, u \rangle$ containing u. So we may suppose $u \in V$, say $V = \langle u, v \rangle$. Now v belongs to a p-torus T of Γ_V and u commutes with T. Take a maximal p-torus T_1 of $C(u)$ containing T. Then $v \in T_1$ by definition and $u \in T_1$ by Fact 1.3.

Thus V is contained in a torus of G, and in particular in a Borel subgroup of G, so $\Gamma_V = G$, and $Z(\Gamma_V) = 1$, a contradiction. □

Now Lemma 3.3 puts us in a position to control Γ_V even more tightly.

Lemma 3.4. *Let G be a simple algebraic group of $p^\perp$ type, and $V \leq G$ an elementary abelian p-subgroup of p-rank two. Suppose that every proper connected Zariski closed subgroup of G has the p-generation property. Then Γ_V is quasisimple.*

Proof. We know that Γ_V is a central product of quasisimple factors, and that the center of Γ_V contains no elements of order p. Therefore the Sylow p-subgroup of Γ_V is the direct product of the Sylow p-subgroups of the factors.

Now as every element of V belongs to a torus we have $V \leq \Gamma_V$ and thus V is contained in a Sylow p-subgroup of Γ_V. Let $V = \langle u, v \rangle$ and choose quasisimple components L_u, L_v of G so that the projection u' of u into L_u and v' of v into L_v is nontrivial. Let $U = \langle u', v' \rangle$. Then U, V commute and we may replace V by U. Evidently $\Gamma_U = G$ unless $L_u = L_v$. In this case writing $L = L_u = L_v$, we have $U \leq L$ and if $L < G$ then by assumption $L \leq \Gamma_U$, so again $G = \Gamma_V$. We are left with the possibility $L = G$. □

Now we require some precise numerical information relating to the universal cover of a simple algebraic group.

Lemma 3.5. *Let $\hat{G}$ be the universal central extension of a quasisimple algebraic group G. Then for any semisimple element $s \in G$ we have*

$$[C_G(s) : C_G{}^\circ(s)] \ \text{divides} \ |Z(\hat{G})|/|Z(G)|$$

Proof. A maximal torus of $\hat{G}$ covers a maximal torus of G, and hence there is a semisimple element $t \in \hat{G}$ corresponding to s under the quotient map $\hat{G} \to G$. In simply connected quasisimple algebraic groups the centralizers of semisimple elements are connected ([Car93, Theorem 3.5.6]). Thus $C_{\hat{G}}(t)$ is connected.

Let Z be the kernel of the map $\hat{G} \to G$. Then $|Z| = |Z(\hat{G})|/|Z(G)|$. Now $K = C_G(s)$ pulls back to the subgroup $\hat{K} = C_{\hat{G} \mod Z}(t)$ of $\hat{G}$. Taking the commutator with t we get a homomorphism $\hat{K} \to Z$ with kernel $C_{\hat{G}}(t)$. As $C_{\hat{G}}(t)$ is connected we have $\hat{K}^\circ = C_{\hat{G}}(t)$ and $|\hat{K}/\hat{K}^\circ|$ divides $|Z| = |Z(\hat{G})|/|Z(G)|$.

Now as $Z(\hat{G}) \leq \hat{K}^\circ$ we have $|K/K^\circ| = |\hat{K}/\hat{K}^\circ|$ and our claim follows. $\square$

The Schur multiplier of a finite simple group is the center of its universal central extension.

Lemma 3.6. *Let G be an algebraic group of $p^\perp$ type, and assume that proper Zariski closed subgroups of G have the p-generation property. If p does not divide the order of the Schur multiplier then G has the p-generation property.*

Proof. As we have shown this comes down to the case of an elementary abelian p-group of p-rank two contained in G. We show that V is contained in a torus of G. Write $V = \langle u, v \rangle$ and consider $C^\circ(u)$. By Lemma 3.5, the index $[C(u) : C^\circ(u)]$ is not divisible by p and therefore $v \in C^\circ(u)$. Now v belongs to a maximal torus T of $C^\circ(u)$ and u also belongs to T by Fact 1.3, so V is contained in a torus of G. $\square$

The orders of the Schur multipliers of simple algebraic groups are given in [Car93, §1.11, p. 25–26]. Namely: $n + 1$ for A_n; either 2 or 4 for each of B_n, C_n, D_n, and E_7; 3 for E_6; and finally, 1 for G_2, F_4, and E_8.

Proof of Theorem 2.1. After considering a minimal case, factoring out the center, and using our replacement lemma, we arrive at the following situation:

1. G is a simple algebraic group of $p^\perp$ type.

2. $V \leq G$ is an elementary abelian p-subgroup of p-rank two.

3. Every proper definable connected section of G has the p-generation property.

4. $\Gamma_V < G$.

Furthermore Γ_V contains a maximal torus of G and is quasisimple. By considerations of dimension this already eliminates the groups of type A_n (PSL_{n+1}).

Furthermore the prime p involved has to divide the order of the Schur multiplier of G, and by the calculations of Schur multipliers in the remaining cases this brings us down to the cases $p = 2, 3$.

On the other hand, by Fact 2.10, if the Lie rank of G is at least $p + 1$ we can replace V by a subgroup of a torus and conclude by Lemma 3.2. So looking at the orders of the Schur multipliers again we eliminate the case $p = 3$ and for $p = 2$ we come down to type B_2. But then as Γ_V is itself quasisimple of Lie rank two, the only possibility compatible with the dimensions would be an embedding of SL_3 into B_2, which does not occur (for example, in view of the Weyl groups). $\qquad\square$

4 Generation in K^*-group and L^*-groups

We turn now to the proof of our main generation results.

Lemma 4.1. *Let G be a connected simple group of finite Morley rank of $p^\perp$ type. Suppose that any proper definable connected subgroup of G has the p-generation property and that V is an elementary abelian p-subgroup of G of rank two. Let S be a Sylow p-subgroup of G containing V, and suppose that $m_p(C_S(V)) \geq 3$. Then*

1. $N(\Gamma_V) = \Gamma^0_{S,2}$

2. $\Gamma_V = (\Gamma^0_{S,2})^\circ$.

Proof. Let $M = N(\Gamma_V)$. Then V normalizes M°. If $M^\circ < G$ then M° satisfies p-generation and hence $M^\circ \leq \Gamma_V$. If $M^\circ = G$ then $\Gamma_V \lhd G$ and hence $\Gamma_V = G$. So in either case $M^\circ \leq \Gamma_V$. Thus

$$M^\circ = \Gamma_V$$

We show next

$$\Gamma_V \leq \Gamma^0_{S,2}$$

We take $v \in V^{\#}$ and we take $E \leq S$ an elementary abelian p-group of rank three, and write $E = \langle v \rangle \oplus E_v$ with E_v elementary abelian of rank two. Then by p-generation we have

$$C^\circ(v) = \langle C^\circ(\langle v, w \rangle) : w \in E_v \rangle$$

and as $C^\circ(v, w) \leq N(\langle v, w \rangle) \leq \Gamma^0_{S,2}$ for such pairs v, w we find $\Gamma_V \leq \Gamma^0_{S,2}$.

On the other hand by Lemma 2.10 we have $\Gamma_V = \Gamma_W$ for $W \leq \Omega_1(S)$ and in particular we find $S \leq \Gamma_V$. So by the Frattini argument $N(\Gamma_V) \leq \Gamma_V \cdot N(S)$ and as $N(S) \leq N(\Omega_1(S)) \leq \Gamma^0_{S,2}$ we find

$$N(\Gamma_V) \leq \Gamma^0_{S,2}$$

Now we show $\Gamma_{S,2}^0 \leq N(\Gamma_V)$. We take $W \leq S$ an elementary abelian p-group with $m_p(C_S(W)) \geq 3$ and observe that $\Gamma_V = \Gamma_W$ by Lemma 2.9. Therefore $N(W) \leq N(\Gamma_V)$ for all such W, and $\Gamma_{S,2}^0 \leq N(\Gamma_V)$. With this all claims are proved. □

Corollary 4.2. *Let G be a connected simple group of finite Morley rank of $p^\perp$ type with $m_p(G) \geq 3$. Suppose that any proper definable connected subgroup of G has the p-generation property and the 2-generation property. Then the following are equivalent.*

1. For some elementary abelian p-subgroup V of G of p-rank two, $\Gamma_V < G$.

2. For S a Sylow p-subgroup of G, we have $\Gamma_{S,2}^0 < G$.

Proof. By the conjugacy of Sylow p-subgroups and Lemma 1.2, every elementary abelian p-subgroup V of p-rank two in G is contained in a Sylow p-subgroup S satisfying $m_p(C_S(V)) \geq 3$, and every Sylow p-subgroup S of G contains such an elementary abelian p-subgroup V. Since $N(\Gamma_V) = \Gamma_{S,2}^0$ under these conditions, our claim follows. □

Now we state the key result.

Theorem 4.3. *Let G be a connected simple group of finite Morley rank of $p^\perp$ type with $m_p(G) \geq 3$ and $m_2(G) \geq 2$. Suppose that any proper definable connected subgroup of G has the p-generation property and the 2-generation property. Suppose that V is an elementary abelian p-subgroup of G and $\Gamma_V < G$. Then $N(\Gamma_V)$ is strongly embedded in G.*

By Theorem 2.1, our condition on the sections of G will hold if G is a K^*-group, or if it is an L^*-group with $p = 2$, giving the two cases of concrete interest.

We record a standard criterion for strong embedding ([BN94, Thm. 10.20]).

Fact 4.4. *A subgroup M of a group G is strongly embedded if and only if M satisfies the following two conditions.*

1. M contains the normalizer of a Sylow 2-subgroup of G;

2. M contains $C(i)$ for each involution $i \in M$.

Proof of Theorem 4.3. Let $M = \Gamma_V$.

If G is not of odd type then it is algebraic by [ABC07], and this falls under our previous results. So we assume G has odd type.

Now we may suppose $V \leq S$ with S a Sylow p-subgroup chosen so that $m_p(C_S(V)) \geq 3$.

We show first that Γ_V contains a Sylow° 2-subgroup of G. Let T be a maximal divisible abelian torsion subgroup of G, and T_p, T_2 the p-torsion and 2-torsion subgroups of T. The maximal divisible abelian torsion subgroups of G are conjugate [ABC07] and thus T_p and T_2 are respectively a maximal p-torus and a maximal 2-torus of G. We may suppose after conjugating that $T_p \leq S$.

Take $A \leq \Omega_1(T_p)$ and $B \leq \Omega_1(T_2)$ elementary abelian subgroups of rank 2. Then by Fact 2.10 and Lemma 2.9 we have $\Gamma_V = \Gamma_A$. By the variant of Lemma 2.9 mentioned after its proof, we also have $\Gamma_A = \Gamma_B$.

Embed T_2 in a Sylow 2-subgroup P of G. Then $P^\circ = T_2$ and $N(P) \leq N(B) \leq N(\Gamma_B) = N(\Gamma_V)$. Thus $N(P) \leq M$.

Next we show that for i an involution of M, we have $C^\circ(i) \leq M$. We may suppose that i normalizes T_2. If i centralizes B, then i lies in an elementary abelian 2-subgroup U of 2-rank two that commutes with B, and hence $\Gamma_V = \Gamma_B = \Gamma_U$ and $C^\circ(i) \leq \Gamma_U \leq M$. So suppose that i does not centralize B and hence does not invert T_2. Therefore i centralizes a nontrivial 2-torus $T_0 \leq T_2$ by Lemma 1.6. By Lemma 1.3 if $\hat{T}_0$ is a maximal 2-torus in $C(i)$, then $i \in \hat{T}_0$. Now $\hat{T}_0 \leq \Gamma_V$ since $\hat{T}_0$ contains an involution of T_2. Hence after conjugation in Γ_V we may suppose $\hat{T}_0 \leq T_2$ and thus $i \in T_2$. But then as above our claim follows.

Now for i an involution of M it remains to show that $C(i) \leq M$. By Lemma 1.3 the element i lies in some maximal 2-torus T_i of G, and then $T_i \leq C^\circ(i) \leq \Gamma_V$ and after conjugation we may suppose $T_i = T_2$. Then by the Frattini argument

$$C(i) \leq C^\circ(i) \cdot N(T_2)$$

and as $N(T_2) \leq N(\Gamma_V) = M$ our claim follows. $\square$

5   Strongly Embedded Subgroups

We next treat the structure of strongly embedded subgroups of L^*-groups, by a method used by Altseimer [Alt] in the K^* context.

Theorem 5.1. *Let G be a simple L^*-group of finite Morley rank and odd type. Then a strongly embedded subgroup M of G is either a D-group, or else satisfies $M^\circ/O(M^\circ) \cong \mathrm{SL}_2$.*

We begin by recalling an elementary property of groups with strongly embedded subgroups.

Fact 5.2 ([BN94, Theorem 10.19]). *Let G be a group of finite Morley rank with a proper definable strongly embedded subgroup M. Then G and M each have only one conjugacy class of involutions.*

For the proof, one shows first that any involution in M is conjugate to any involution not in M, after which the conjugacy in G is clear and the conjugacy in M follows from the definition of strong embedding.

Lemma 5.3. *Let M be an L-group of finite Morley rank and odd type with exactly one conjugacy class of involutions. Let $H = M^\circ/\hat{O}(M)$. Then either*

1. *M is a D-group, and $H/Z(H)$ contains no involutions; or else*

2. *H is quasisimple, and and $H/Z(H)$ is one of the following: PSL_2, PSL_3, or G_2.*

Proof. We need a few facts about algebraic groups. First, it is well known, and can be extracted from Theorem 1.12.5d of [GLS98], that SL_2 is the only quasisimple algebraic group whose involutions all lie in its center. Second, as can be seen in Table 4.3.1 on p. 145 of [GLS98], any algebraic group which is simple as an abstract group and has a unique conjugacy class of involutions must be one of the following: PSL_2, PSL_3, or G_2. All others contain involutions with nonisomorphic centralizers, and hence have distinct conjugacy classes of involutions that do not even fuse under the action of $\mathrm{Aut}(G)$.

Now suppose first that

(I) H contains no nontrivial connected normal definable D-subgroup.

Then by Theorem 1.13 $H = E_{\mathrm{alg}}(H)$.

If $Z(H)$ contains an involution then all involutions in H are central and the components are all of the form SL_2. Then using once more the fact that all involutions lie in $Z(H)$, it follows easily that there is a unique component and $H = SL_2$. For example, in the case in which the central involutions of two components coincide, a product of two elements of order 4 from the two components will produce a noncentral involution. As the case $H \cong SL_2$ is one of those envisioned, we may put this aside.

So suppose $Z(H)$ contains no involution. Then each component contains noncentral involutions and hence every involution lies in a unique component. This again implies that there is only one component. Furthermore, all of its involutions are conjugate. So H is again one of those listed.

Now suppose

(II) H contains some nontrivial connected normal definable D-subgroup K.

We may take K minimal. Note that $\hat{O}(K) = 1$. If $F^\circ(K) = 1$ then $E(K)$ must be nontrivial, and for each quasisimple component L of $E(K)$ we have $L/Z(L)$ of degenerate type. But also $Z(L) \leq F(K)$ is finite, and thus L is of degenerate type, and connected. So L contains no involutions and $L \leq \hat{O}(K) = 1$, a contradiction. Accordingly

$$F^\circ(K) > 1$$

Again, $O(F^\circ(K)) = 1$, so the 2-torsion in $F^\circ(K)$ forms a nontrivial 2-torus T. As $N^\circ(T) = C^\circ(T)$, this 2-torus is central in H. Furthermore every involution of H lies in T. It then follows that all 2-elements of H lie in T, and thus $H/Z(H)$ contains no involutions, and H is a D-group.   □

This moves us substantially in the direction of Theorem 5.1. But we still need to eliminate the configurations in which $M^\circ/\hat{O}(M)$ is PSL_2, PSL_3, SL_3, or G_2. For this we must further exploit strong embedding in the manner of [Alt, §5], and also (in a different direction) [AC03, §5].

A powerful idea in this context is to study the distribution of involutions in cosets of M (or M°). This leads to the following.

Fact 5.4 ([Alt96, Lemma 3.8]). *Let G be a group of finite Morley rank with a strongly embedded subgroup M. Then there is an involution $w \in G \setminus M$ such that $\mathrm{rk}(I(wM)) \geq \mathrm{rk}(I(M))$.*

Fix such an involution w, set $Y := \{uw \mid u \in I(wM)\}$, and let $K := d(Y)$.

Fact 5.5 (see [Alt96, Propositions 3.9,3.10]). *Let G be a group of finite Morley rank with a strongly embedded subgroup M. Then the group K contains no involutions, and for any involution $i \in M$, we have*

$$M^\circ = C^\circ(i)K^\circ$$

This decomposition will then pass to the quotient $M^\circ/O(M)$, or for that matter a further quotient by the center, if the center has odd order. So we require the following.

Lemma 5.6. *Let H be one of the groups PSL_2, PSL_3, or G_2, over a field of characteristic not 2. Then there is no connected definable subgroup K of H without involutions for which we have a decomposition*

$$H = C(i)K$$

for each involution $i \in H$.

The following general fact will be useful in our analysis.

Lemma 5.7. *Any definable unipotent subgroup U in an algebraic group H over a field k of finite Morley rank and characteristic zero is Zariski closed.*

We note that the language in which U is supposed to be definable (the language of k) is arbitrary, subject to the finite rank hypothesis.

Proof. The additive group of a field k of finite Morley rank and characteristic zero has no proper (infinite) definable subgroup [Poi87, Cor. 3.3]. It follows that a definable endomorphism α of $(k, +)$ is given by multiplication in k. Indeed, if $\alpha(1) = a$ then $\alpha - a$ has a nontrivial kernel, which must then be k.

Since any such group U would be nilpotent, we may assume, proceeding inductively, that U is abelian, and minimal. Let $\hat{U}$ be the Zariski closure of U. Then $\hat{U}$ can be identified with a vector space over k (cf. [Hum75, Lemma 15.1C]). Consider a projection into any 1-dimensional subspace V of $\hat{U}$. Since $\hat{U}$ is the Zariski closure of U, U is not contained in the kernel. Hence U maps injectively into V. So the projection gives an isomorphism. Writing $\hat{U} = \oplus_i V_i$, we can identify the V_i with k and view U as being given by a series of definable automorphisms of $(k, +)$. These are represented by multiplication maps on k and thus U is defined by linear equations. $\square$

Proof of Lemma 5.6. We make use of rank computations for algebraic subgroups of H. If r is the rank of the base field, the rank of such a subgroup is just r times its dimension. So we may work directly with dimensions.

Suppose first that K is solvable, and let B be a Borel subgroup of H containing K. (We remark that the model theoretic and algebraic notions of Borel subgroup coincide here, but in any case we will need the structure theory of algebraic groups, so the algebraic sense is dominant.) Taking i in B, and discounting the overlap caused by a maximal torus T of B containing i, we find

$$\dim(H) \leq \dim(C(i)) + \dim(B) - \dim(T)$$

where one may replace $\dim(B) - \dim(T)$ by $\dim(U)$ with U the unipotent radical of B, so that this term is the number of positive roots. The relevant numbers are found in the following table.

H	Lie rank	$\dim(C(i))$	$\dim(U)$	Sum	$\dim(H)$
PSL_2	1	1	1	2	3
$\mathrm{(P)SL}_3$	2	4	3	7	8
G_2	2	6	6	12	14

Table 1: Dimensions of some subgroups

Looking at the last two columns, we see that our conditions are not met. So we conclude

$$\text{The group } K \text{ is not solvable.}$$

By a subtle result of Poizat, if the base field for H has positive characteristic then every connected definable simple section of H is algebraic (cf. [AC03, 4.19] for the form quoted here). This then forces K to contain involutions and gives a contradiction. So the base field has characteristic 0.

Now let $\hat{K}$ be the Zariski closure of K. Observe that $\hat{K}$ inherits from H the key property:

$$\text{If } i \text{ is an involution in } \hat{K} \text{ then } \hat{K} = C_{\hat{K}}(i)K.$$

Furthermore $\hat{K}/U(\hat{K})$ (factoring out the unipotent radical) inherits this property. This is a reductive group. There are two possibilities: either it is quasisimple of Lie rank 2, or every quasisimple component, modulo its center, is PSL_2.

In the latter case, look at the image of K in $\hat{K}/U(\hat{K})$ and then projected into the simple components modulo the center. These give subgroups of PSL_2 containing no involutions, which are solvable by Poizat's result. In this case K itself is solvable and we have a contradiction. Thus

$$\hat{K}/U(\hat{K}) \text{ is quasisimple of Lie rank 2.}$$

Replacing H by $\hat{K}/U(\hat{K})$ and K by its image in $\hat{K}/U(\hat{K})$, we may therefore suppose that K is Zariski dense in H.

As H is simple, K contains no nontrivial abelian normal subgroup. Hence $F^*(K) = E(K)$.

We claim that K contains no nontrivial Zariski closed subgroup A of H. Supposing the contrary, $[E(K), A]$ would be a Zariski closed and connected subgroup of H normal in $E(K)$. Such a group, if nontrivial, would contain an involution, giving a contradiction. So $[E(K), A] = 1$. But as $C_K(F^*(K)) \leq Z(F^*(K))$, this shows $A = 1$.

In particular, since the characteristic is zero and unipotent subgroups are therefore Zariski closed, all elements of K are semisimple. So a Borel subgroup of K is contained in a maximal torus T of H.

If B is a Borel subgroup of H containing T, then $\mathrm{rk}(K/T) \leq \mathrm{rk}(H/B)$. Consider an involution $i \in T$. Then $H = KC_H(i)$, so

$$\mathrm{rk}(H) \leq \mathrm{rk}(C_H(i)) + \mathrm{rk}(H/B)$$

After cancellation and rearrangement, and replacement of $\mathrm{rk}(B)$ by $2 + \mathrm{rk}(U)$, this says

$$\dim(U) \leq \dim(C_H(i)) - 2$$

contradicting Table 1. $\qquad\square$

Proof of Theorem 5.1. We have G a simple L^*-group of finite Morley rank and odd type, with M strongly embedded. By Fact 5.2 and Lemma 5.3 we find one of the following

1. M is a D-group;

2. $M^\circ/\hat{O}(M)$ is quasisimple, with central quotient of the form $\mathrm{PSL}_2, \mathrm{PSL}_3$, or G_2.

In the first case, or in the second case with $M^\circ/\hat{O}(M)$ of the form SL_2, we have what we claim. We are left therefore with the following possibilities for the structure of $H = M^\circ/O(M)$:

$$\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{SL}_3, \mathrm{G}_2$$

By Fact 5.5 we have a decomposition of H of the type refuted in Lemma 5.6 in the cases $H = \mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{G}_2$, and if $H = \mathrm{SL}_3$ we can pass to PSL_3 by factoring out the center (of odd order). So in all of these remaining cases we arrive at a contradiction. $\qquad\square$

6   Strong embedding and D-groups

We aim now at the following. Recall that in the context of K^*-groups of odd type, a simple D^*-group is a minimal simple connected group.

Theorem 6.1. *Let G be a simple L^*-group of finite Morley rank and $p^\perp$ type, with $\mathrm{pr}_2(G) \geq 2$. Suppose that G has a strongly embedded subgroup M. Then the following hold.*

    1. G is a D^*-group

2. Sylow 2-subgroups of G are connected.

We break the proof into two lemmas.

Lemma 6.2. *Let G be a simple L^*-group of finite Morley rank and $p^\perp$ type, with $\mathrm{pr}_2(G) \geq 2$. Suppose that G has a strongly embedded subgroup M. Then M is a D-group, and Sylow 2-subgroups of G are connected.*

Proof. There is a Sylow 2-subgroup S contained in M by Fact 4.4. For any $t \in S$, there is a 2-torus T of G containing t by Fact 1.3. So $T \leq M$ by Fact 4.4, and $t \in M^\circ$. Thus $S \leq M^\circ$ and $\mathrm{pr}_2(M^\circ) \geq 2$. So $M^\circ/O(M)$ cannot be SL_2. Therefore M is a D-group by Theorem 5.1. Now S is connected by Lemma 1.14. $\qquad\square$

Lemma 6.3 (compare [BBN06, Claim 5.5]). *Let G be an L^*-group of finite Morley rank and odd type. Suppose that G has a strongly embedded subgroup M which is a D-group. Then G is a D^*-group.*

Proof. Suppose toward a contradiction that G has a proper definable connected algebraic simple section, and consider a minimal proper definable subgroup K of G which has such an algebraic section. Then K is not a D-group, so K is not contained in M and K contains involutions. So after conjugation we may suppose that $M_0 := M \cap K$ contains an involution. Then M_0 is strongly embedded in K.

If $K = M_0\hat{O}(K)$ then K is a D-group. So $K > M_0\hat{O}(K)$. We pass to $\bar{K} = K/\hat{O}(K)$. Since $M_0\hat{O}(K)$ is strongly embedded in K, it follows that $\bar{M}_0$ is strongly embedded in $\bar{K}$. So replacing K by $\bar{K}$ we may suppose

$$\hat{O}(K) = 1$$

Suppose $Z^\circ(K) > 1$. As $\hat{O}(K) = 1$ and K has odd type, the 2-torsion in $Z^\circ(K)$ forms a nontrivial 2-torus T. But K has a strongly embedded subgroup, so all involutions are conjugate, and hence all involutions of K belong to $T \leq Z(K)$. It follows easily that T is a Sylow 2-subgroup of K. But then K is a D-group after all. So we will suppose

$$Z^\circ(K) = 1$$

Now suppose $F^\circ(K) \neq 1$. As $\hat{O}(K) = 1$, the 2-torsion in $F^\circ(K)$ forms a nontrivial 2-torus T, normalized by the connected group K. So $Z^\circ(K) > 1$ and we have a contradiction. So we now have

$$F^\circ(K) = 1$$

Thus $F(K)$ is finite and $C_K(E(K)) = C_K(F^*(K)) = Z(K)$. In particular $E(K)$ is nontrivial. Let L be a quasisimple component of $E(K)$. Then $Z(L)$ is finite, so if $L/Z(L)$ is of degenerate type then L is of degenerate type, hence $L \leq \hat{O}(K) = 1$, a contradiction. So $L/Z(L)$ is algebraic and L is algebraic by [AC99]. By minimality of K, we have $K = L$ quasisimple and algebraic. But

then K contains a subgroup of the form SL_2 or PSL_2, and we may suppose this group meets M after conjugation. As M is a D-group this intersection is proper. So, by minimality of K, the group K is itself of the form SL_2 or PSL_2.

Now if $K = \mathrm{SL}_2$ then $K \leq C_G^\circ(i)$ for i an involution, and we contradict strong embedding. Similarly if $K = \mathrm{PSL}_2$ then $M \cap K < K$ contains a Sylow 2-subgroup of K, but, for each involution in M, the corresponding torus is in M. So M° contains K, which is again a contradiction. $\qquad\square$

Conclusion

We may now deduce our main results.

Theorem 6.4 (K^* Generation Theorem)**.** *Let G be a simple K^*-group of finite Morley rank and $p^\perp$ type, with $m_p(G) \geq 3$ and $\mathrm{pr}_2(G) \geq 2$. Let V be an elementary abelian p-group of p-rank two. Then $\Gamma_V = G$. In particular $\Gamma_{S,2}^0 = G$ for any Sylow p-subgroup S of G.*

Proof. Suppose $\Gamma_V < G$. By Theorem 4.3, $N(\Gamma_V)$ is strongly embedded in G. By Theorem 6.1, G is a D^*-group, hence in this context a minimal connected simple group. By the main result of [BCJ07], a minimal connected simple group of finite Morley rank and odd type with a proper definable strongly embedded subgroup has Prüfer 2-rank one. This contradiction proves

$$\Gamma_V = G$$

Now by Corollary 4.2 we have also $\Gamma_{S,2}^0 = G$ for any Sylow 2-subgroup S of G. $\qquad\square$

We state the most important case separately.

Corollary 6.5. *Let G be a simple K^*-group of finite Morley rank and odd type, with $m_2(G) \geq 3$. Let V be an elementary abelian 2-group of 2-rank two. Then $\Gamma_V = G$. In particular $\Gamma_{S,2}^0 = G$ for any Sylow 2-subgroup S of G.*

Proof. We have $\mathrm{pr}_2(G) \geq 2$ by Theorem 1.1 and now Theorem 6.4 applies. $\quad\square$

We continue the analysis further in the L^* case.

Theorem 6.6. *Let G be a simple L^*-group of finite Morley rank and odd type, with $m_2(G) \geq 3$. Let V be an elementary abelian 2-group of 2-rank two with $\Gamma_V < G$. Then*

1. *G is a D^*-group.*

2. *$M := N(\Gamma_V)$ is a strongly embedded subgroup of G.*

3. *The Sylow 2-subgroups of G are connected.*

4. *G contains a nontrivial unipotent r-subgroup where r is the least prime divisor of the Weyl group W, which is nontrivial.*

For the final point we define and use the *Weyl groups* as in [BC08].

Definition 6.7. The *Weyl group* W of a group G of finite Morley rank is the abstract group $W = N(T)/C^\circ(T)$ where T is a maximal decent torus, which can be viewed as a group of automorphisms of T.

As all maximal decent tori are conjugate by [Che05], the Weyl group is well defined up to conjugacy.

Fact 6.8 ([BC08, Theorem 5]). *Let G be a connected group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then G contains a unipotent r-subgroup.*

Here one might prefer M/M° over the Weyl group W. Let T denote a Sylow 2-subgroup of M. By a Frattini argument, $W_T := N(T)/C^\circ(T)$ naturally embeds into W. But $|W| > |W_T|$ seems plausible if M is not solvable. In particular the minimal prime divisor of $|W|$ might be strictly less than that of $|M/M^\circ|$. Of course, one may argue that $N_{M^\circ}(T)/C^\circ{}_{M^\circ}(T) = 1$ because M° is still a D-group. A pair of Frattini arguments then shows that $M/M^\circ \cong W_T$. So one may prove that M/M° is nontrivial using Fact 5.2. However, Fact 6.8 seems tricky to localize. Fortunately one cares most about such detailed information when $\mathrm{pr}_2(G) = 2$, and 3 is the minimal prime divisor everywhere.

Proof of Theorem 6.6. By Theorem 1.1 we have $\mathrm{pr}_2(G) \geq 2$. By Theorem 4.3, $M := N(\Gamma_V)$ is strongly embedded in G. By Theorem 6.1, G is a D^*-group and the Sylow 2-subgroups of G are connected.

We now turn our attention to the final point. Let T be a maximal 2-torus of M. By Fact 5.2, M has only one conjugacy class of involutions. So $W_T := N(T)/C^\circ(T)$ is nontrivial. By a Frattini argument using [Che05], W_T naturally embeds into W. Hence W is nontrivial as well. Now $|W|$ has odd order because T is a Sylow 2-subgroup of G. So the last point follows from Fact 6.8. □

Acknowledgments

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