A NEW TRICHOTOMY THEOREM FOR GROUPS OF FINITE MORLEY RANK

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ABSTRACT

There is a longstanding conjecture, due to Gregory Cherlin and Boris Zilber, that all simple groups of finite Morley rank are simple algebraic groups. Here we will conclude that a simple K^*-group of finite Morley rank and odd type has normal rank at most two. To this end, it suffices, by [**14**, **10**, **15**], to produce a proper 2-generated core in groups with Prüfer rank two and normal rank at least three; which is what is proven here. Our final conclusion constrains the Sylow 2-subgroups available to a minimal counterexample, and finally proves Borovik's trichotomy theorem in the nontame context.

The Algebraicity Conjecture for simple groups of finite Morley rank, also known as the Cherlin-Zilber conjecture, states that simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields. In the last 15 years, considerable progress has been made by transferring methods from finite group theory; however, the conjecture itself remains decidedly open. In the formulation of this approach, groups of finite Morley rank are divided into four types, odd, even, mixed, and degenerate, according to the structure of their Sylow 2-subgroup. For even and mixed type the Algebraicity Conjecture has been proven, and connected degenerate type groups are now known to have trivial Sylow 2-subgroups [**9**]. Here we concern ourselves with the ongoing program to analyze a minimal counterexample to the conjecture in *odd type*, where the Sylow 2-subgroup is divisible-abelian-by-finite.

The present paper lies between the high Prüfer rank, or generic, where general methods are used heavily, and the "end game" where general methods give way to consideration of special cases. In the first part, the Generic Trichotomy Theorem [**14**] says that a *minimal* non-algebraic simple group of finite Morley rank has Prüfer rank at most two. Thus we may consider small groups whose simple sections are restricted to PSp_4, G_2, PSL_3, and PSL_2. In the next stage, we hope to proceed via the analysis of components in the centralizers of *toral* involutions; however, the existence of such a component requires further constraints on the Sylow 2-subgroup [**12**, p. 196]. In particular, the normal 2-rank provides another analog of Lie rank, more delicate and traditional than the Prüfer 2-rank, which needs to be controlled.

Therefore the present paper argues that

THEOREM 1. *Any simple K^*-group G of finite Morely rank and odd type, with Prüfer 2-rank at most two and normal 2-rank at least 3, has a proper 2-generated core, i.e. $\Gamma^0_{S,2}(G) < G$.*

For tradition and technical reasons, we prefer to record this result as a tameness free version of the old trichotomy theorem from [8], using the Generic Trichotomy Theorem [14] (page 8) to handle .

TRICHOTOMY THEOREM 2. *Let G be a simple K^*-group of finite Morley rank and odd type. Suppose either that G has normal 2-rank ≥ 3, or else has an eight-group centralizing the 2-torus S°. Then one of the following holds.*
   1. G has a proper weak 2-generated core, i.e. $\Gamma^0_{S,2}(G) < G$.
   2. G is an algebraic group over an algebraically closed field of characteristic not 2.

It follows immediately from normal 2-rank ≥ 3 that some eight-group $\cong (\mathbb{Z}/2\mathbb{Z})^3$ centralizes some maximal 2-torus. So we work throughout with this more general condition, which we plan to exploit in the later stages described above. By Fact 1.2 below, normal 2-rank ≥ 3 also implies that the 2-generated core $\Gamma_{S,2}(G)$ from [8] and our weak 2-generated core $\Gamma^0_{S,2}(G)$ coincide.

We summarize the present status of the classification as follows.

THEOREM 0.1. *Let G be a simple K^*-group of finite Morley rank and odd type, with normal 2-rank ≥ 3 and Prüfer 2-rank ≥ 2. Then G is an algebraic group over an algebraically closed field of characteristic not 2.*

This result follows immediately from two previous papers, our Trichotomy Theorem, and Fact 1.2 below.

STRONG EMBEDDING THEOREM 3 [10]. *Let G be a simple K^*-group of finite Morley rank and odd type, with normal 2-rank ≥ 3 and Prüfer 2-rank ≥ 2. Suppose that G has a proper 2-generated core $M = \Gamma_{S,2}(G) < G$. Then G is a minimal connected simple group, and M is strongly embedded.*

MINIMAL SIMPLE THEOREM 4 [15]. *Let G be a minimal connected simple group of finite Morley rank and of odd type. Suppose that G contains a proper definable strongly embedded subgroup M. Then G has Prüfer 2-rank one.*

1. *Cores and components*

We begin by recalling the various consequences of the absences of a proper 2-generated core, as laid out in [14, §2]. Of primary importance is the role of quasisimple components of the centralizers of toral involutions.

Much of our time will be spent analyzing a so-called simple K^*-group of finite Morley rank. A K^*-group is a group whose *proper* definable simple sections are all algebraic. K^*-groups are analyzed by examining various proper subgroups, especially the centralizers of involutions. A group is said to be a K-group if *all* definable simple sections are algebraic, and this property is holds for proper subgroups of our K^*-group.

An algebraic group is said to be *reductive* if it has no unipotent radical, and a reductive group is a central product of semisimple algebraic groups and algebraic tori. In a simple (even reductive) algebraic group, over a field of characteristic not 2, the centralizer of an involution is itself reductive. In this section, we establish,

in the absence of a proper 2-generated core, that the centralizers of involutions in our simple K^*-group are "somewhat reductive".

1.1. *Proper 2-generated cores*

DEFINITION 1.1. *Consider a group G of finite Morley rank and a 2-subgroup S of G. We define the* 2-generated core $\Gamma_{S,2}(G)$ *of G (associated to S) to be the definable hull of the group generated by all normalizers of four-subgroups in S. We also define the* weak 2-generated core $\Gamma^0_{S,2}(G)$ *of G (associated to S) to be the definable hull of all normalizers of four-subgroups $U \leq S$ with $m(C_S(U)) > 2$. We say that G has a* proper 2-generated core *or a* proper weak 2-generated core *when, for a Sylow 2-subgroup S, $\Gamma_{S,2}(G) < G$ or $\Gamma^0_{S,2}(G) < G$, respectively.*

Both notions are well-defined because the Sylow 2-subgroups of a group of finite Morley rank are conjugate [**7**, **11**, Thm. 10.11].

FACT 1.2 [**14**, Fact 1.20-2]; compare [**2**, 46.2]. *Let G be a group of finite Morley rank, and let S be a 2-subgroup of G. If S has normal 2-rank ≥ 3, then $\Gamma^0_{S,2}(G) = \Gamma_{S,2}(G)$.*

For an elementary abelian 2-group V acting definably on G, we define $\Gamma_V(G)$ to be the group generated by the connected components of centralizers of involutions in V.

$$\Gamma_V(G) = \langle C^\circ_G(v) : v \in V^\# \rangle.$$

Out most basic tool for producing a proper 2-generated core is the following.

FACT 1.3 [**14**, Prop. 1.22]. *Let G be a simple K^*-group of finite Morley rank and odd type, with $m(G) \geq 3$, and let S be a Sylow 2-subgroup of G. Suppose that $\Gamma_E(G) < G$ for some four-group $E \leq G$ with $m(C_G(E)) > 2$. Then G has a proper weak 2-generated core.*

1.2. *Components and descent*

DEFINITION 1.4. *A quasisimple subnormal subgroup of a group G is called a* component *of G (see [**11**, p. 118 ii]). We define $E(G)$ to be the connected part of the product of components of G, or equivalently the product of the components of G° (see [**11**, Lemma 7.10iv]). Such components are normal in G° by [**11**, Lemma 7.1iii], and indeed $E(G) \triangleleft G$.*

A connected reductive algebraic group G is a central product of $E(G)$ and an algebraic torus.

We require several plausible "unipotent radicals" to define our notion of partial reductivity.

DEFINITION 1.5. *The* Fitting subgroup $F(G)$ *of a group G is the subgroup generated by all its nilpotent normal subgroups.*

In any group of finite Morley rank, the Fitting subgroup is itself nilpotent and

definable [**3**, **21**, **11**, Theorem 7.3], and serves as a notion of unipotence in some contexts.

DEFINITION 1.6. *A connected definable p-subgroup of bounded exponent inside a group H of finite Morley rank is said to be p-unipotent. We write $U_p(H)$ for the subgroup generated by all p-unipotent subgroups of H.*

If H is solvable, then $U_p(H) \leq F^\circ(H)$ is p-unipotent itself (see [**17**, Cor. 2.16] and [**1**, Fact 2.36]).

DEFINITION 1.7. *We say that a connected abelian group of finite Morley rank is indecomposable if it has a unique maximal proper definable connected subgroup, denoted $J(A)$ (see [**13**, Lemma 2.4]). We define the reduced rank $\bar{r}(A)$ of a definable indecomposable abelian group A to be the Morley rank of the quotient $A/J(A)$, i.e. $\bar{r}(A) = \mathrm{rk}(A/J(A))$. For a group G of finite Morley rank, and any integer r, we define*

$$U_{0,r}(G) = \left\langle \, A \leq G \; \middle| \; \begin{array}{l} A \text{ is a definable indecomposable group,} \\ \quad \bar{r}(A) = r, \text{ and } A/J(A) \text{ is torsion-} \\ \text{free} \end{array} \right\rangle.$$

We say that G is a $U_{0,r}$-group if $U_{0,r}(G) = G$, and set $\bar{r}_0(G) = \max\{r \mid U_{0,r}(G) \neq 1\}$.

We view the reduced rank parameter r as a *scale of unipotence*, with larger values being more unipotent. By [**13**, Thm. 2.16], the "most unipotent" groups, in this scale, are nilpotent.

DEFINITION 1.8. *In a group H of finite Morley rank, we write $O(H)$ for the subgroup generate by the deinable connected normal subgroups without involutions.*

If H is a K-group of finite Morley rank, then $O(H)$ is solvable, as simple algebraic groups always contain 2-torsion.

Our approach to reductivity begins with the following fact.

FACT 1.9 [**8**, Thm. 5.12]. *Let G be a connected K-group of finite Morley rank and odd type with $O(G) = 1$. Then $G = F^\circ(G) * E(G)$ is isomorphic to a central product of quasisimple algebraic groups over algebraically closed fields of characteristic not 2 and of a definable normal divisible abelian group $F^\circ(G)$.*

However, a more subtle definition is required to find an applicable version of this fact. The following definition was applied in [**14**], under the assumption of Prüfer rank ≥ 3.

DEFINITION 1.10. *Consider a simple group G of finite Morley rank and let X be a subgroup of G with $m(X) \geq 3$. We write $I^0(X) := \{i \in I(G) : m(C_X(i)) \geq 3\}$ for the set of involutions from eight-groups in X. We define $\bar{r}^*(X)$ to be the supremum of $\bar{r}_0(k^*)$ as k ranges over the base fields of the algebraic components of the quotients $C_G^\circ(i)/O(C_G(i))$ associated to involutions $i \in I^0(X)$.*

Clearly $\bar{r}^*(G)$ is the maximum of $\bar{r}^*(E)$ as E ranges over eight-groups in G. We recall that, for a nonsolvable group L of finite Morley rank, $U_{0,r}(L)$ and $U_p(L)$ need not be solvable, as quasisimple algebraic groups are generated by the unipotent radicals of their Borel subgroups. We exploit this in the following central definition.

DEFINITION 1.11. *We continue in the notation of Definition 1.10. For a definable subgroup H of G, we define $\tilde{U}_X(H)$ to be the subgroup of H generated by $U_p(H)$ for p prime as well as by $U_{0,r}(H)$ for $r > \bar{r}^*(X)$. As an abbreviation, we use $\tilde{F}_X(H)$ to denote $F^\circ(\tilde{U}_X(H))$, and $\tilde{E}_X(H)$ to denote $E(\tilde{U}_X(H))$. We use $\tilde{\mathcal{E}}_Y^X$ to denote the set of components of $\tilde{E}_X(C_G(i)) = E(\tilde{U}_X(C_G(i)))$ for $i \in I^0(Y)$ with $Y \leq X$, and set $\tilde{\mathcal{E}}_X = \tilde{\mathcal{E}}_X^X$.*

$\tilde{U}_X(H)$ is the subgroup of H which is generated by its unmistakably unipotent subgroups. These definitions are all sensitive to the choice of X, which is usually a fixed eight-group.

DEFINITION 1.12. *We say that a simple K^*-group G with $m(G) \geq 3$ satisfies the $\tilde{B}$-property if, for every 2-subgroup $X \leq G$ with $m(X) \geq 3$ and every $t \in I^0(X)$, the group $\tilde{U}_X(O(C_G^\circ(t)))$ is trivial.*

We recall from [14] that the $\tilde{B}$-property holds in the absence of a proper 2-generated core. The proof of this result explains the definition of $\bar{r}^*$.

FACT 1.13 [14, Thm. 2.9]. *Let G be a simple K^*-group of finite Morley rank and odd type with $m(G) \geq 3$. Then either*
1. *G has a proper weak 2-generated core, or*
2. *G satisfies the $\tilde{B}$-property.*

For us, the point of the $\tilde{B}$-property is that it ensures the existence of a well behaved family of components in the centralizers of involutions.

DEFINITION 1.14. *Given a simple group G of finite Morley rank and a set of involutions J, we say a family of components $\mathcal{E}$ from the centralizers of involutions in J is descent inducing for J if*

> *For every $K \in \mathcal{E}$ and every involution $i \in J$ which normalizes K, there are components $L_1, \ldots, L_n \in \mathcal{E}$ with $L_k \lhd C_G^\circ(i)$ such that $E(C_K(i)) \leq L_1 * \cdots * L_k$.*

In this situation we say that K, or just $E(C_K(i))$, descends.

The $\tilde{B}$-property provides us with such a family of components.

LEMMA 1.15. *Let G be a simple K^*-group of finite Morley rank and odd type with $m(G) \geq 3$ which satisfies the $\tilde{B}$-property. Then, for every 2-subgroup $X \leq G$ with $m(X) \geq 3$, the family of components $\tilde{\mathcal{E}}_X$ is descent inducing for $I^0(X)$.*

We will extract this result from the following facts.

FACT 1.16 [14, Cor. 2.11]. *Let G be a simple K^*-group of finite Morley rank and odd type with $m(G) \geq 3$ which satisfies the $\tilde{B}$-property. Then, for every 2-*

subgroup $X \leq G$ with $m(X) \geq 3$ and every $i \in I^0(X)$, we have $\tilde{U}_X(C_G(i)) = \tilde{E}_X(C_G(i)) * \tilde{F}_X(C_G(i))$ and $\tilde{F}_X(C_G(i))$ is abelian.

FACT 1.17 [**14**, Prop. 2.13]. *Let H be a groups of finite Morley rank which is isomorphic to a linear algebraic groups over an algebraically closed field k. Then*
  1. If $U_{0,r}(H) \neq 1$ for some $r > \bar{r}_0(k^*)$ then $\mathrm{char}(k) = 0$ and $\mathrm{rk}(k) = r$.
  2. If $U_p(H) \neq 1$ then $\mathrm{char}(k) = p$.
If H is quasisimple, these conditions imply $U_p(H) = H$ and $U_{0,r}(H) = H$, respectively.

FACT 1.18 [**23**, Thm. 8.1]; [**14**, Fact 1.6]. *Let G be a quasisimple algebraic group over an algebraically closed field. Let ϕ be an algebraic automorphism of G whose order is finite and relatively prime to the characteristic of the field. Then $C_G^\circ(\phi)$ is nontrivial and reductive.*

Proof of Lemma 1.15. Consider a component $K \in \tilde{\mathcal{E}}_X$ and an involution $i \in I^0(X)$ which normalizes K. By Fact 1.18, $C_K(i)$ is reductive. By Fact 1.17 $\tilde{U}_X(C_G(i)) \geq \tilde{E}_X(C_K(i)) = E(C_K(i))$. As $E(C_K(i))$ is nonabelian, Fact 1.16 yields a set of components $L_1, \ldots, L_n \in \mathcal{E}_X$ with $L_k \lhd C_G^\circ(i)$ such that

$$E(C_K(i)) \leq L_1 * \cdots * L_n.$$

□

Our descent inducing family of components generates G.

FACT 1.19 [**14**, Thm. 2.18]. *Let G be a K^*-group of finite Morley rank and odd type with $m(G) \geq 3$. Suppose that there is a four-group $E \leq G$ which centralizes a Sylow$^\circ$ 2-subgroup T of G, and that there is an eight-group X in $C_G(T)$ which contains E. Then either*
  1. G has a proper weak 2-generated core, or else
  2. $\langle \tilde{\mathcal{E}}_E^X \rangle = \langle \tilde{E}_X(C_G(z)) : z \in E^\# \rangle = G$.

1.3. *Automorphisms*

The following two facts ensure that involutions acting upon quasisimple components are understood.

DEFINITION 1.20. *Given an algebraic group G, a maximal torus T of G, and a Borel subgroup B of G which contains T, we define the group Γ of* graph automorphisms *associated to T and B, to be the group of algebraic automorphisms of G which normalize both T and B.*

FACT 1.21 [**11**, Thm. 8.4]. *Let $G \rtimes H$ be a group of finite Morley rank, with G and H definable. Suppose that G is a quasisimple algebraic group over an algebraically closed field, and $C_H(G)$ is trivial. Let T be a maximal torus of G and let B be a Borel subgroup of G which contains T. Then, viewing H as a subgroup of $\mathrm{Aut}(G)$, we have $H \leq \mathrm{Inn}(G)\Gamma$, where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms of G associated to T and B.*

FACT 1.22. *Let G be an infinite quasisimple algebraic group over an alge-braically closed field of characteristic not 2 and let ϕ be an involutive automorphism centralizing some Sylow° 2-subgroup T of G. Then ϕ is an inner automorphism in-duced by some element of T.*

Proof. We observe that $H := C_G(T)$ is a maximal torus of G containing T. Let B be a Borel subgroup containing H. By Fact 1.21, $\phi = \alpha \circ \gamma$ where γ is a graph automorphism normalizing H and B, and $\alpha(x) = x^t$ is an inner automorphism induced by some element $t \in G$. Since ϕ and γ normalize the H, we know that α does as well, so $t \in N_G(H)$.

Following [16, p. 17–18], we consider the root system Φ to be a subset of the set of algebraic homomorphisms $\text{Hom}(H, k^*)$ from H to the multiplicative group of the field k^*. The action of both the Weyl group and the graph automorphisms of G on the root system is the natural action on $\text{Hom}(H, k^*)$. If $t \notin H$, then t is a representative for a nontrivial Weyl group element, would not preserve the set of positive roots in Φ. On the other hand, ϕ centralizes T, so it preserves the set of positive root of Φ. Since γ normalizes B, it preserves the set of positive roots too. So $t \in H$. Now $\phi \equiv \gamma \pmod{H}$ is a graph automorphism itself. By [20, Table 4.3.1 p. 145], a nontrivial involutive graph automorphism of a quasisimple algebraic group G never centralizes a maximal torus of G. So ϕ must be an inner automorphism. $\square$

Table 1 contains necessary information about conjugacy classes of involutions and their centralizers, in Lie rank two quasi-simple groups (see [20, Table 4.3.1 p. 145 & Table 4.3.3 p. 151]).

G	$\bar{\Gamma}$	Z	i	$C_G^\circ(i)$
SL_2	1	$\mathbb{Z}/2\mathbb{Z}$	inner	k^*
PSL_2	1	1	inner	k^*
SL_3	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	inner	$\text{SL}_2 * k^*$
			graph	PSL_2
PSL_3	$\mathbb{Z}/2\mathbb{Z}$	1	inner	$\text{SL}_2 * k^*$
			graph	PSL_2
Sp_4	1	$\mathbb{Z}/2\mathbb{Z}$	inner	$\text{SL}_2 \times \text{SL}_2$
			inner	$\text{SL}_2 * k^*$
PSp_4	1	1	inner	$\text{SL}_2 * \text{SL}_2$
			inner	$\text{PSL}_2 \times k^*$
G_2	1	1	inner	$\text{SL}_2 * \text{SL}_2$

TABLE 1. *Data on Chevalley Groups*

2. *Proof of the Trichotomy Theorem*

We prove our main result in this section.

TRICHOTOMY THEOREM 5. *Let G be a simple K^*-group of finite Morley rank and odd type. Suppose either that G has normal 2-rank ≥ 3, or else has an eight-group centralizing the 2-torus S°. Then one of the following holds.*

1. G has a proper 2-generated core.

2. G is an algebraic group over an algebraically closed field of characteristic not 2.

In high Prüfer rank, [**14**] produces an algebraic group. In other situations, we will reach a contradiction below.

2.1. *Consequences of* $\Gamma^0_{S,2}(G) = G$.

We observe that an elementary abelian 2-group which is normal in a Sylow 2-subgroup S is centralized by S°. So we may begin proof of the trichotomy theorem by assuming the following.

HYPOTHESIS 2.1. *Let G be a simple K^*-group of finite Morley rank and odd type which satisfies the following.*
a. G has an eight-group centralizing the 2-torus S°.
b. G does not have a proper weak 2-generated core, i.e. $\Gamma^0_{S,2}(G) = G$.

Let S be a Sylow 2-subgroup of G, and let $A \leq S$ be an eight-group such that $[A, S^\circ] = 1$. We may also assume $\Omega_1(S^\circ) \leq A$. The high Prüfer rank case is covered by the following.

GENERIC TRICHOTOMY THEOREM 6 [**14**]. *Let G be a simple K^*-group of finite Morley rank and odd type with Prüfer 2-rank ≥ 3. Then either*
1. G has a proper weak 2-generated core, or
2. G is an algebraic group over an algebraically closed field of characteristic not 2.

As we have assumed that $\Gamma^0_{S,2}(G) = G$, the Generic Trichotomy Theorem allows us to assume that

$$G \text{ has Prüfer 2-rank 1 or 2.}$$

By Fact 1.13, G satisfies the $\tilde{B}$-property too. In particular, the family of components $\tilde{\mathcal{E}}_A$ is descent inducing for $A^\#$ by Lemma 1.15. We will make extensive use of this family.

DEFINITION 2.2. *Let $\mathcal{E} := \tilde{\mathcal{E}}_A$, and let $\mathcal{E}^* \subset \mathcal{E}$ be the set of those components in $\mathcal{E}$ with Lie rank two.*

Since $m(A) \geq 2$ and $[A, S^\circ] = 1$, we have $\langle \tilde{\mathcal{E}}^A_{\Omega_1(S^\circ)} \rangle = G$ by Fact 1.19. In particular, $\tilde{\mathcal{E}}^A_{\Omega_1(S^\circ)} \neq \emptyset$. As G is simple, we also obtain the following.

$$\text{For } M \in \mathcal{E}, \text{ there is a } v \in \Omega_1(S^\circ)^\# \text{ such that } E(C_G(v)) \not\leq M. \qquad (\star)$$

We next show that G has a Lie rank two component, i.e. $\mathcal{E}^* \neq \emptyset$. This contains our final application of $\Gamma^0_{S,2}(G) = G$.

LEMMA 2.3. *For any component $K \in \mathcal{E}_A$, there is some component $M \in \mathcal{E}^*_A$ containing K.*

We will need the following fact.

FACT 2.4 [**8**, Lemma 8.1]; compare [**24**, Proposition I.1.1]. *Let H be a connected*

K-group of finite Morley rank with $H = F(H)E(H)$. Let t be a definable involutive automorphism of H and let L be a component of $E(C_H(t))$. Then there is a component $K \lhd E(H)$ such that one of the following holds.

1. $K = K^t$ and $L \lhd E(C_K(t))$.
2. $K \neq K^t$ and $L \lhd E(C_{K^t K}(t))$.

Proof. We may assume $H = E(H)$ as $L \leq E(H)$. For any component K of $E(H)$, either $K^t = K$, or else $C_{K^t K}(t)$ is contained in the diagonal $\Delta(K)$ of the central product $K^t * K$. We take $\Delta(K) := K$ in the first situation, to simplify notation. Now $C_H(t) \leq \Delta(K_1) * \cdots * \Delta(K_n)$. As all $\Delta(K_i)$s are fixed by i, $C_H(t) = C_{\Delta(K_1)}(t) * \cdots * C_{\Delta(K_n)}(t)$ too. So there is a unique $\Delta(K_i)$ containing L, as desired. □

Proof Proof of Lemma 2.3. We may assume that $K \notin \mathcal{E}^*$ has type (P)SL$_2$. Since A centralizes S°, S° is a Sylow$^\circ$ 2-subgroup of $C_G^\circ(y)$ for any $y \in A^\#$, and A fixes all components of $E(C_G(y))$. In particular, A normalizes K.

Since K has no graph automorphisms, A acts on K via inner automorphisms by Fact 1.21, and these inner automorphisms normalize the 2-torus $K \cap S^\circ$. Since a four-group in $\mathrm{Aut}(K) \cong \mathrm{PSL}_2$ normalizing this 2-torus has an involution inverting the torus and A centralizes the 2-torus, we have that $m(C_A(K)) \geq m(A) - 1 \geq 2$. Since $K \in \mathcal{E}$ induces descent for $A^\#$, $K \leq E(C_G(x))$ for all $x \in C_A(K)$.

Now suppose that $K \lhd E(C_G(x))$ for all $x \in C_A(K)$. So $K \lhd \Gamma_{C_A(K)}(G)$ and hence $\Gamma_{C_A(K)}(G) < G$. By Fact 1.3, G must have a proper weak 2-generated core. Thus

There is an $x \in C_A(K)$ such that K is not normal in $E(C_G(x))$.

Fix such an x. Suppose toward a contradiction that $E(C_G(x))$ is not quasisimple. Since $\mathrm{pr}(G) \leq 2$, we have $E(C_G(x)) = L_1 * L_2$ with L_i of (P)SL$_2$ type. Since $K \lhd E(C_G(y))$ for some $y \in A^\#$, we have $K \lhd E(C_{L_1 * L_2}(y))$ too. By Fact 2.4, there is a component $L \lhd E(L_1 * L_2) = L_1 * L_2$ such that either

(i) $L^y = L$ and $K \lhd E(C_L(y))$, or
(ii) $L^y \neq L$ and $K \lhd E(C_{L^y L}(y))$

Since $L \cong$ (P)SL$_2$, y acts on L by an inner automorphism by Fact 1.21, so $C_L(y)$ is an algebraic subgroup of L. In case (i), either $C_L(y) = L$ or $C_L(y)$ is solvable, and both are contradictions. In case (ii), we have $L^y \neq L$, contradicting $[A, S^\circ] = 1$. □

Since $\mathcal{E} \neq \emptyset$, we now have $\mathcal{E}^* \neq \emptyset$, and thus

$$\mathrm{pr}(G) = 2.$$

2.2. *Lie rank two components*

A brief inspection of [**20**, Table 4.3.1 p. 145] will reveal that quasisimple algebraic groups almost never have centralizers of involutions which are themselves quasisimple of the same Lie rank, the only exceptions being B_4 inside F_4, A_7 inside E_7, and D_8 inside E_8. In particular, the Lie rank two quasisimple algebraic groups (P)SL$_3$, (P)Sp$_4$, and G_2 never have Lie rank two quasisimple centralizers. We view this observation as inspirational, and proceed to reach a contradiction by considering the

various possible types of components in $\mathcal{E}^*$. Our hypotheses are as follows, given the analysis of the preceding subsection.

HYPOTHESIS 2.5.   *Let G be a simple K^*-group of finite Morley rank and odd type which satisfies the following.*
  a. *G has an eight-group A centralizing the 2-torus S°.*
  b. *G has a nonempty family of components $\mathcal{E}$, from the centralizers of involutions in $A^\#$, which is descent inducing for $A^\#$.*
  c. *The set $\mathcal{E}^* \subset \mathcal{E}$ of components with Lie rank two is nonempty.*
  d. *For all $M \in \mathcal{E}$, there is a $v \in \Omega_1(S^\circ)^\#$ such that $E(C_G(v)) \not\leq M$.*

Fix some $M \in \mathcal{E}^*$. As M must be isomorphic to one of $(P)SL_3$, $(P)Sp_4$, or G_2, we proceed by analyzing each of these cases separately. Each case will either reach a contradiction, or arrive at a previous treated case.

CASE 1.   *$\mathcal{E}^*$ contains a component $M \cong Sp_4$.*

Proof Analysis.   Since S° is a Sylow$^\circ$ 2-subgroup of M, and the central involution of Sp_4 is toral, there is an $x \in \Omega_1(S^\circ)$ with $M \leq E(C_G(x))$, and x is the central involution of M. It follows that the group $E(C_G(x))$ is quasisimple, so $M = E(C_G(x)) \cong Sp_4$. Since $M \not\leq C_M(u)$ for some $u \in \Omega_1(S^\circ) \setminus \langle x \rangle$, u is not conjugate to x in M. Since Sp_4 has exactly two conjugacy classes of involutions by Table 1, the two distinct involutions $u, v \in \Omega_1(S^\circ) \setminus \langle x \rangle$ are conjugate. By Table 1, $C_M^\circ(u) \cong SL_2 \times SL_2$.

By $(\star)$ and the conjugacy of u and v, we have $E(C_G(u)) \not\leq M$. Since $u \in S^\circ \leq C_M^\circ(u)$ and $C_M^\circ(u) \cong SL_2 \times SL_2$, we find $u \in Z(E(C_M(u)))$. By Table 1, $K_u := E(C_G(u)) \cong Sp_4$. Since u and v are conjugate, $K_v := E(C_G(v)) \cong Sp_4$ as well. Since M is descent inducing for $A^\#$, we have $K_u, K_v \in \mathcal{E}^*$.

Since $[A, S^\circ] = 1$, the group A acts on M by inner automorphisms induced by elements of S° by Fact 1.22. So there is an involution $w \in A \setminus \Omega_1(S^\circ)$ such that $M \leq E(C_G(w))$. By Table 1, only Sp_4 has Lie rank two and $SL_2 \times SL_2$ inside the centralizer of an involution, so $E(C_G(w)) \cong Sp_4$. Thus $x \in Z(E(C_G(w)))$. Since M induces descent for $A^\#$, we have $E(C_G(w)) \in \mathcal{E}$. Since $E(C_G(w))$ induces descent for $A^\#$, we have $E(C_G(w)) \leq E(C_G(x)) = M$. So $E(C_G(w)) = E(C_G(x)) = M$ and $E(C_G(w)) = M$.

Since w centralizes u, the involution w normalizes K_u. Since K_u is descent inducing for $A^\#$, we have $E(C_{K_u}(w)) \leq M \cap K_u$ and $E(C_{K_u}(w)) \geq C_M^\circ(u) \cong SL_2 \times SL_2$ by Table 1. Since w centralizes S°, Fact 1.22 says that w acts by an inner automorphism on K_u. Since $K_u \not\leq M = E(C_G(w))$, we know that w does not centralize K_u. By Table 1, $C_{K_u}^\circ(w) \cong SL_2 \times SL_2$ or $SL_2 * k^*$, so $C_{K_u}^\circ(w) \cong SL_2 \times SL_2$, and $C_{K_u}^\circ(w) = E(C_{K_u}(w))$. Since $C_M^\circ(u)$ is perfect, $C_M^\circ(u) \leq K_u$, so $C_{K_u}^\circ(w) = C_M^\circ(u)$.

By Table 1, $E(C_M(u))/\langle x \rangle \cong SL_2 * SL_2$ and the centralizer of an element of order four which acts by an involutive automorphism is $SL_2 * k^*$. As this does not match the centralizer of w, the action of w is induced via conjugation by either x or v. So $E(C_M(u)) \cong L_1 \times L_2$ with $L_i \cong SL_2$, $u \in Z(L_1)$, and $v \in L_2$, while x belongs to neither L_1 nor L_2. On the other hand, $E(C_M(u)) = E(C_{K_u}(w))$ can be viewed as either $E(C_{K_u}(x))$ or $E(C_{K_u}(v))$. Since u is the central involution of K_u, the same argument shows that u belongs to neither L_1 nor L_2, a contradiction. $\diamondsuit$

CASE 2. $\mathcal{E}^*$ *contains a component* $M \cong \mathrm{G}_2$.

Proof Analysis. By Table 1, the three involutions in $\Omega_1(S^\circ)$ are conjugate in M, and

$$E(C_M(x)) \cong \mathrm{SL}_2 * \mathrm{SL}_2 \quad \text{for every } x \in \Omega_1(S^\circ)^\#.$$

By $(\star)$, $E(C_G(z)) \not\leq M$ for some $z \in \Omega_1(S^\circ)$. Since $\mathrm{pr}(E(C_G(z))) = 2$, the group $E(C_G(z))$ must be quasisimple. Since $S^\circ \leq C_G^\circ(z)$, we find $z \in Z(E(C_G(z)))$. By Table 1, $E(C_G(z)) \cong \mathrm{Sp}_4$. Since $E(C_M(z))$ is perfect and $M \in \mathcal{E}$ induces descent for $A^\#$, $E(C_G(z)) \in \mathcal{E}^*$ too. The result now follows by reduction to Case 1. $\quad\Diamond$

Before attacking the final two cases, we observe that pairs of components of (P)SL_2 type always generate a proper subgroup.

LEMMA 2.6. *For any two components* $L, J \in \mathcal{E}$ *of* (P)SL_2 *type, one of the following holds.*
1. $E(C_G(i)) = L * J$ *for some* $i \in A^\#$.
2. L *and* J *are algebraic subgroups of* $\langle L, J \rangle \in \mathcal{E}^*$.

Proof. Since A centralizes T, A fixes all components of $E(C_G(y))$ for any $y \in A^\#$, and A normalizes L and J. Since L and J have no graph automorphisms, A acts on L and J via inner automorphisms by Fact 1.21. We observe that $T \cap L$ is a Sylow$^\circ$ 2-subgroup of L which is centralized by A. Since a four-group in PSL_2 normalizing a 2-torus has an involution inverting the torus and A centralizes the 2-torus, we have that $m(C_A(L)) \geq m(A) - 1 \geq 2$ and $m(C_A(J)) \geq 2$. Thus $C_A(L) \cap C_A(J) \neq 1$ and $\langle L, J \rangle \leq C_G(i)$ for some involution $i \in C_A(L) \cap C_A(J)$. Since L and J induce descent for $A^\#$, they must lie inside components of $C_G(i)$ coming from $\mathcal{E}$.

Now there is a $y \in A^\#$ such that $L \lhd E(C_G(y))$. Since y normalizes the component $L' \in \mathcal{E}$ with $L' \lhd E(C_G(i))$ which contains L, Fact 1.21 says that y acts on L' by a graph automorphism composed with an inner automorphism. So $C_{L'}^\circ(y)$ is a reductive algebraic subgroup of L' by Fact 1.18, which has L as a normal subgroup, so L is a component of $C_{L'}^\circ(y)$. If $[L, J] = 1$ then $L = L'$ and $J = J'$ (defined similarly), so the first conclusion follows. If $[L, J] \neq 1$ then L' contains J as an algebraic subgroup too, so the second conclusion follows. $\quad\square$

We now return to our case analysis.

CASE 3. $\mathcal{E}^*$ *contains a component* $M \cong \mathrm{PSp}_4$.

Proof Analysis. By Table 1, the three involutions $x, y, z \in \Omega_1(S^\circ)$ have centralizers $C_M^\circ(z) \cong \mathrm{SL}_2 * \mathrm{SL}_2$ and $C_M^\circ(x) \cong C_M^\circ(y) \cong \mathrm{PSL}_2 * k^*$ with x and y conjugate in M.

Suppose first that $E(C_G(z)) \not\leq M$. There is an $i \in A^\#$ with $M = E(C_G(i))$. Since M induces descent for $A^\#$, the two components of $C_M^\circ(z) \cong \mathrm{SL}_2 * \mathrm{SL}_2$ are found in $\mathcal{E}$. Since these two components induce descent for $A^\#$, $C_M^\circ(z) \leq K_z := E(C_G(z))$ and the components of K_z are found in $\mathcal{E}$. Since i normalizes $K_z := E(C_G(z))$, the group $C_{K_z}(i)$ is an algebraic subgroup of K_z by Fact 1.21. So $C_{K_z}(i) \cong \mathrm{SL}_2 * \mathrm{SL}_2$ too, and K_z is quasisimple. By Table 1, $E(C_G(z)) \cong \mathrm{Sp}_4$. Now $E(C_G(z)) \in \mathcal{E}^*$,

and the result follows by reduction to Case 1. So $E(C_G(z)) \leq M$ and $E(C_G(z)) \cong$ $\mathrm{SL}_2 * \mathrm{SL}_2$.

Therefore, $E(C_G(x)) \not\leq M$ by $(\star)$. Now $E(C_G(x))$ must be either Sp_4 or $\mathrm{SL}_2 \times \mathrm{PSL}_2$. Since $E(C_M(x))$ is perfect and $M \in \mathcal{E}$ induces descent for $A^{\#}$, the Sp_4 case reduces to Case 1. So $E(C_G(x)) \cong \mathrm{SL}_2 \times \mathrm{PSL}_2$.

Let J be the component of $E(C_G(x))$ which is isomorphic to SL_2. If J commuted with either SL_2 component of $E(C_G(z))$, then $J \leq E(C_G(z))$ since $[x, z] = 1$. Now J does not commute with either SL_2 component of $E(C_G(z))$, because otherwise it would be one of them. Now let L be a component of $E(C_G(z))$ with $L \cong \mathrm{SL}_2$. By Lemma 2.6, $K := \langle L, J \rangle \in \mathcal{E}^*$. In this group, the two involutions $x, z \in \Omega_1(S^\circ)$ have SL_2 type components J and L in $C_G(x)$ and $C_G(z)$, respectively. Thus $K \not\cong \mathrm{PSp}_4$ where only one toral involution may have a component of type SL_2 by Table 1. We may also assume that K is not isomorphic to either Sp_4 or G_2 by reduction to Cases 1 and 2 since $L \in \mathcal{E}$ induces descent for $A^{\#}$. So $K \cong (\mathrm{P})\mathrm{SL}_3$. Now the involutions of $\Omega_1(S^\circ)$ are K-conjugate by Table 1, contradicting the fact that $E(C_G(z)) \not\cong E(C_G(x))$. $\diamond$

Therefore we have only two possibilities.

CASE 4.   *Every component $M \in \mathcal{E}^*$ is isomorphic to SL_3 or PSL_3.*

Proof Analysis.   Fix $M \in \mathcal{E}^*$. Since $M \cong (\mathrm{P})\mathrm{SL}_3$, the three distinct involutions $x, y, z \in \Omega_1(S^\circ)$ are conjugate in M. For any one of these involutions $v \in \Omega_1(S^\circ)$, we define $L_v = E(C_M(v)) \cong \mathrm{SL}_2$. Since $M \cong (\mathrm{P})\mathrm{SL}_3$, we have $v \in Z(E(C_M(v)))$ and $L_u \neq L_v$ for $u \neq v$ (see Table 1). Since $E(C_G(v)) \not\cong \mathrm{Sp}_4$ by Case 1, we have $L_v \lhd E(C_G(v))$ and there is a second component in $E(C_G(v))$, which we denote by J_v. Since $L_v \leq E(C_M(\Omega_1(S^\circ) \cap J_v))$, we find $\Omega_1(S^\circ) \cap J_v = \langle v \rangle$, and thus $J_u \neq J_v$ for $u \neq v$. Since $J_x, J_y, J_z \not\leq M$, the set $\mathcal{L} = \{L_x, J_x, L_y, J_y, L_z, J_z\}$ of these subgroups has exactly 6 elements. We also see that $J_v \cong \mathrm{SL}_2$ and $E(C_G(v)) \cong \mathrm{SL}_2 * \mathrm{SL}_2$ for $v \in \Omega_1(S^\circ)^{\#}$.

Let $H := C_G^\circ(S^\circ)$. Since H centralizes $\Omega_1(S^\circ)$, H normalizes any $L \in \mathcal{L}$, H normalizes $M = \langle L_x, L_y, L_z \rangle$. Since H is connected and definable, $H = C_H^\circ(M)(M \cap H)^\circ$ by Fact 1.21. So for any component $K \in \mathcal{E}^*$, there is a natural embedding of the Weyl group $W(K) := N_K(C_K(S^\circ))/C_K(S^\circ)$ of K into $N_G(H)/H$. We define the "Weyl group" W of G to be the subgroup of $N_G(H)/H$ which is generated by the Weyl groups $N_K(H)/H$ of each component $K \in \mathcal{E}^*$. We observe that the subgroups $L \in \mathcal{L}$ are normalized by H, and therefore are the root SL_2-subgroups of those components $K \in \mathcal{E}^*$ to which they belong. Since $K \cong (\mathrm{P})\mathrm{SL}_3$ by previous cases, $W(K)$ is generated by $N_{HL}(H)/H$ for some $L \in \mathcal{L}$, so W is generated by $N_{HL}(H)/H$ for $L \in \mathcal{L}$. Now $|N_{HL}(H)/H| = |N_L(H \cap L)/(H \cap L)| = 2$. For any $L \in \mathcal{L}$, we let r_L denote the involution of $N_{HL}(H)/H$. We observe that r_L acts on S° by reflection in the sense that $S^\circ = C_{S^\circ}(r_L) \times [S^\circ, r_L]$ with $\mathrm{pr}(C_{S^\circ}(r_L)) = 1$ and $\mathrm{pr}([S^\circ, r_L]) = 1$.

By the Tate module argument [4, 6, §3.3] (see also [5, §2.4]), there is a (not necessarily faithful) representation of W by 2×2 matrices over the 2-adic integers $\mathbb{Z}_2$, and over $\mathbb{C}$ after tensoring. We let $\overline{W}$ denote the image of W inside $\mathrm{End}(\mathbb{C}^2)$. The involutions r_L act on $\mathbb{C}^2$ as reflections with one $+1$ eigenvalue and one -1 eigenvalue. The kernel U of the natural action of W on $\mathcal{L}$ leaves invariant the three subgroups $S^\circ \cap L_x$, $S^\circ \cap L_y$, and $S^\circ \cap L_z$, and hence elements of U act either trivially

or by inverting S°. So the action of W on $\mathcal{L}$ can be factored through $\overline{W}$, and $\overline{U}$ consists of the scalar matrices ± 1.

We now consider the Weyl group $W(M) = N_M(C_M(S^\circ))/C_M(S^\circ)$ of M. Let $r \in N_M(C_M(S^\circ)) \cap L_x$ be a representative for the Weyl group element of L_x associated to the maximal torus $C_M(S^\circ)$. We may assume $r \in S$ by [**11**, Ex. 11 p. 93]. Since $C_M(x) \cong \mathrm{SL}_2 * k^*$ with amalgamation by Table 1, conjugation by r must swap y and z. So $[L_x, J_y] \neq 1$. By Lemma 2.6, $\langle L_x, J_y \rangle \in \mathcal{E}^*$. Similarly, $\langle L_y, J_z \rangle, \langle L_z, J_x \rangle \in \mathcal{E}^*$. For any $K \in \mathcal{E}^*$, we have $K \cong (\mathrm{P})\mathrm{SL}_3$ by the previous cases. Since $\mathcal{L}$ is independent of the choice of M, K has the form $\langle L, L' \rangle$ for two $L, L' \in \mathcal{L}$. There is no $\mathrm{SL}_2 * \mathrm{SL}_2$ contained in the centralizer of an involution in $K \cong (\mathrm{P})\mathrm{SL}_3$ by Table 1, so $\langle L_v, J_v \rangle \notin \mathcal{E}^*$ for any $v \in \Omega_1(S^\circ)^\#$. For any $u, v, w \in \Omega_1(S^\circ)^\#$ with $u \neq v$, the group J_w is not contained in $M = \langle L_u, L_v \rangle$.

Assembling these various facts, we discover that

$$\mathcal{E}^* = \{\langle L_x, L_y, L_z \rangle, \langle L_x, J_y, J_z \rangle, \langle J_x, L_y, J_z \rangle, \langle J_x, J_y, L_z \rangle\}.$$

Thus we have the following geometry.
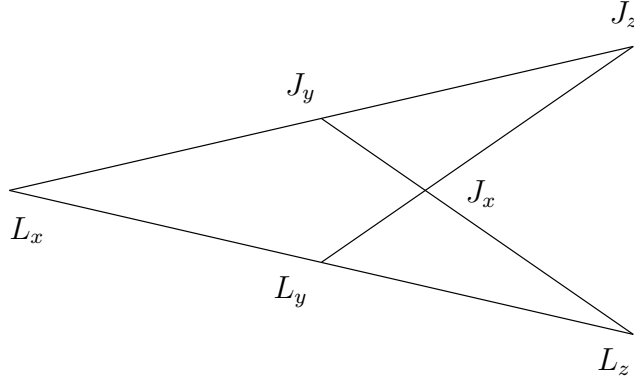
For each $K \in \mathcal{E}^*$, the group $N_K(H)/H$ acts 2-transitively on the three subgroups from $\mathcal{L}$ which are contained in K. Recall that U is the kernel of the action of W on $\mathcal{L}$. So the permutation group W/U preserves the geometry consisting of the six point in $\mathcal{L}$ and the four lines in $\mathcal{E}^*$, and permutes 2-transitively the points on each line. Since a permutation of the lines $\mathcal{E}^*$ determines a permutation of the points $\mathcal{L}$, we have an injective homomorphism

$$\overline{W}/\overline{U} \hookrightarrow \mathrm{Sym}_{\mathcal{E}^*} \cong \mathrm{Sym}_4.$$

Since W/U acts 2-transitively on the points on each line, $\overline{W}/\overline{U} \cong \mathrm{Sym}_4$.

Let $\overline{W}_0 \leq \overline{W}$ be the preimage of Alt_4 under the quotient map. We recall that $\overline{U}$ consists of scalar ± 1 matrices. If $\overline{U} = 1$ then $\overline{W}_0 \cong \mathrm{Alt}_4$ is a subgroup of $\mathrm{GL}_2(\mathbb{C})$, so the subgroup $\mathbb{Z}/4\mathbb{Z} \leq \mathrm{Alt}_4$ is diagonalizable and contains the scalar -1 matrix, a contradiction because Alt_4 is centerless. Thus $\overline{U} \neq 1$.

Let $\overline{Q}$ be the pull-back of the group $(\mathbb{Z}/2\mathbb{Z})^2 \lhd \mathrm{Alt}_4$ under the quotient map $\overline{W}_0 \to W_0/\overline{U}$. Then $\overline{Q}$ is acted upon by $\mathbb{Z}/3\mathbb{Z}$, and the elements of $\overline{Q} \setminus \overline{U}$ have order 2 or 4. If the elements have order 2, then $\overline{Q} \cong (\mathbb{Z}/2\mathbb{Z})^3$, which is not a subgroup of $\mathrm{GL}_2(\mathbb{C})$. So these elements have order 4, and $\overline{Q} \cong Q_8$. Now $\overline{W}_0$ is the semidirect product $Q_8 \rtimes (\mathbb{Z}/3\mathbb{Z})$ of the quaternion group Q_8 of order 8 and the cyclic group

of order 3. We also find that the subgroup $\bar{Q} \cong Q_8$ from $\bar{W}_0$ is normal in $\bar{W}$ and $\bar{W}/\bar{Q} \cong \mathrm{Sym}_3$.

Now the group $\bar{W}$ has order $2 \cdot 4! = 48$, contains a conjugacy class of reflections r_L for $L \in \mathcal{L}$, and has $\left|Z(\bar{W})\right| = 2$. Such a group does not exist, as we now show in a couple ways.

By the classification of irreducible complex reflection groups [**22**, Table VII, p. 301], $\bar{W}$ must be one of $G(12, 6, 2)$, $G(24, 24, 2)$, or have Shephard–Todd number either 6 or 12, as only these have order 48. We can determine from [**22**, Table VII p. 301] that $G(12, 6, 2)$ and number 6 have centers or order 2, while $G(24, 24, 2)$ and number 12 have no conjugacy class of exactly six reflections. One can preform these computations with the following GAP [**19**] commands.

```
RequirePackage("chevie");
Size(Centre(ComplexReflectionGroup(12,6,2)));
Size(Centre(ComplexReflectionGroup(6)));
W := ComplexReflectionGroup(24,24,2);
ForAny(Reflections(W), x -> Size(ConjugacyClass(W,x))=6);
W := ComplexReflectionGroup(12);
ForAny(Reflections(W), x -> Size(ConjugacyClass(W,x))=6);
```

To check the result by hand as follows. There are six such reflections, so $|C_{\bar{W}}(r_L)| = 8$, $\left|C_{\bar{Q}}(r_L)\right| = 4$, and $C_{\bar{Q}}(r_L) \cong \mathbb{Z}/4\mathbb{Z}$. We can choose a basis in the Tate module V so that r_L is represented by $\mathrm{diag}(-1, 1)$. So $C_{\bar{Q}}(r_L)$ consists of diagonal matrices too, and must be generated by either $t = \mathrm{diag}(-i, i)$ or $\mathrm{diag}(i, -i)$. So $r_L t$ is a scalar matrix $\mathrm{diag}(\pm 1, \pm 1)$ and belongs to $Z(\bar{W})$. This means $r_L \in Z(\bar{W})\bar{Q} \lhd \bar{W}$ and the conjugates of r_L can not generate the group, a contradiction. $\diamond$

This concludes the proof of the Trichotomy Theorem.

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