Topological dynamics of stable groups

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Abstract

Assume G is a group definable in a model M of a stable theory T. We prove that the semigroup $S_G(M)$ of complete G-types over M is an inverse limit of some semigroups type-definable in M^{eq}. We prove that the maximal subgroups of $S_G(M)$ are inverse limits of some definable quotients of subgroups of G. We consider the powers of types in the semigroup $S_G(M)$ and prove that in a way every type in $S_G(M)$ is pro-finitely many steps away from a type in a subgroup of $S_G(M)$.

Introduction

Assume H is a group and X is a compact topological space upon which H acts by homeomorphisms. In this case X is called an H-flow. We call an H-flow X point-transitive if X contains a dense H-orbit. This is the basic set-up of topological dynamics [E, A].

In several papers [N4, N5, N6] I proposed to apply the language and tools of topological dynamics in model theory. Specifically, assume T is a complete theory in language L, $\mathfrak{C}$ is a monster model of T, G is a group 0-definable in $\mathfrak{C}$ and $M \prec \mathfrak{C}$ is a (small) model of T. Then the group $G(M)$ acts by left translation on the space $S_G(M)$ of complete G-types over M and $S_G(M)$ is a point-transitive $G(M)$-flow.

In the stable case the crucial role is played by generic types in $S_G(M)$. In general, generic types may not exist. Topological dynamics provides us with a natural surrogate for this notion, namely that of an almost periodic and of weakly generic type in $S_G(M)$. Also, the Ellis semigroup of the flow $S_{G,ext}(M)$ of complete external G-types over M has interesting model-theoretic connotations.

Although [N4, N5, N6] contain some applications of topological dynamics in model theory, the topological-dynamic set-up seems too general for model theory. In order to investigate some specific model-theoretic phenomena we need some additional assumptions and it is not clear yet what the reasonable assumptions should be. In particular, it is not clear which topological-dynamic properties of G have

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model-theoretic nature (and for example transfer between models in elementary extensions).

Stability theory is the core of model theory. The goal of this paper is to investigate the topological dynamics of G in the stable case. This may bring the topological-dynamic approach closer to the core notions of model theory, like forking of types, and suggest the correct additional assumptions to impose on the topological dynamics of G to make it a meaningful tool in model theory.

So in this paper we assume T is stable (unless specified otherwise). In this case the Ellis semigroup of the $G(M)$-flow $S_G(M)$ is naturally isomorphic (as a $G(M)$-flow) to the flow $S_G(M)$ itself, inducing on it a semigroup structure. The semigroup operation in $S_G(M)$ is the free multiplication of types [N3]. We prove that the semigroup $S_G(M)$ is an inverse limit of an inverse system of semigroups $S_{G,\Delta}(M), \Delta \in Inv$, that are type-definable in M^{eq} (the notation is explained later). This result (and many other results of this paper) heavily relies on the functional interpretation of the semigroups $S_G(M)$ and $S_{G,\Delta}(M), \Delta \in Inv$, as the semigroups of endomorphisms of certain $G(M)$-algebras of sets. A bulk of the paper is devoted to analysis of definability (in M^{eq}) of various objects arising in the topological dynamics of $G(M)$. We prove that the maximal subgroups of $S_{G,\Delta}(M)$ are definable in M^{eq}, they are definably isomorphic to the groups $N_{G(M)}(H)/H$ for Δ-definable Δ-connected subgroups H of $G(M)$. We describe also the maximal subgroups of $S_G(M)$ as inverse limits of definable inverse systems of the groups $N_{G(M)}(H)/H$.

The functional interpretation of types p in $S_G(M)$ and $S_{G,\Delta}(M), \Delta \in Inv$, provides us with some new objects related to p, namely the kernel and image of the endomorphism d_p corresponding to p. The size of these kernels and images may be used to measure the size of types p. We compare this new way of measuring p to forking in the particular case of $*$-powers p^{*n} of p. In this case we prove that the growth of the local Morley ranks of p^{*n} is strictly correlated with the growth of kernels and shrinking of images of the functions $d_{p^{*n}}$. We prove that for every $p \in S_{G,\Delta}(M), \Delta \in Inv$, there is a maximal subgroup S of $S_{G,\Delta}(M)$ such that eventually the $*$-powers of p belong to S. In particular, there is a Δ-definable Δ-connected subgroup H of $G(M)$ such that eventually the $*$-powers of p are left translates of the generic type of H in $S_{G,\Delta}(M)$ by elements of $N_{G(M)}(H)$. This may be rephrased by saying that regarding raising to $*$-power, every type in $S_{G,\Delta}(M)$ is finitely many steps away from a translate of a generic type of a subgroup of $G(M)$.

Since $S_G(M)$ is an inverse limit of the semigroups $S_{G,\Delta}(M), \Delta \in Inv$, for every type $p \in S_G(M)$ there is a (unique) connected M-type-definable subgroup H of G, say with the generic type $q \in S_G(M)$, such that for every finite set $\Delta \subseteq L$ eventually the $*$-powers of $p|_\Delta$ belong to the maximal subgroup of $S_{G,\Delta}(M)$ containing $q|_\Delta$, hence are some left translates of $q|_\Delta$. This may be rephrased by saying that, regarding raising to $*$-power, every type in $S_G(M)$ is pro-finitely many steps away from a translate of a generic type of an M-type-definable connected subgroup H of G. Recall that in the 1-based case every type $p \in S_G(M)$ itself is a translate of the generic type of an M-type-definable subgroup H of G (provided M is $|T|^+$-saturated) [HP].

The paper is organized as follows. In Section 1 we recall the basic notions of topological dynamics and set up the model-theoretic context wherein they are used in this paper. In Section 2 we describe the semigroup $S_G(M)$ as an inverse limit of semigroups $S_{G,\Delta}(M), \Delta \in Inv$ and prove that the semigroups $S_{G,\Delta}(M)$ are type-definable in M^{eq}. In Section 3 we describe the maximal subgroups of $S_G(M)$ and $S_{G,\Delta}(M), \Delta \in Inv$. In Section 4 we deal with $*$-powers of types.

1 Preliminaries

In this section we recall the basic notions of topological dynamics and put them into a model-theoretic context. The general references are [E, A, N4]. In our model-theoretic notation we follow [Pi].

In particular, we regard formulas of L as formulas with separated variables. This means that given a formula φ of L we separate its free variables into a tuple of object variables x and a tuple of parameter variables y and write it down as $\varphi(x, y)$. By an instance of φ we mean a formula $\varphi(x, a)$, where the variables y are substituted by parameters a from $\mathfrak{C}$. We will be freely working in M^{eq}, an L^{eq}-structure obtained by adjoining to M its imaginary elements. The next definition essentially appears in [N3].

Definition 1.1 *For $p, q \in S_G(M)$ we define $p * q$ as the type $tp(a \cdot b/M)$, where $a \models p, b \models q$ and $a \perp_M b$.*

So $*$ is the free multiplication of types induced by the group operation of $G(M)$ and $(S_G(M), *)$ is a semigroup, with $*$ continuous in each coordinate separately. This semigroup was considered already in [N3]. Here we will consider it in the context of topological dynamics.

Assume H is a group and X is a point-transitive H-flow. In topological dynamics of particular interest are minimal subflows of X, their elements are called almost periodic (in X). Any $h \in H$ determines a homeomorphism $\pi_h : X \to X$ given by $\pi_h(x) = hx$. Let $E(X)$ be the topological closure of the set $\{\pi_h : h \in H\}$ in the space X^X with the Tychonov product topology. $E(X)$ with the operation of composition of functions is a semigroup, called the Ellis enveloping semigroup of X. $E(X)$ is also an H-flow itself: for $h \in H$ and $f \in E(X)$, $(hf)(x) = h \cdot f(x)$.

A set $I \subseteq E(X)$ is called a left ideal if I is non-empty and closed under left multiplication by elements of $E(X)$. It turns out that the minimal subflows of $E(X)$ are exactly the minimal left ideals $I \subseteq E(X)$. Every minimal left ideal $I \subseteq E(X)$ splits into a disjoint union of groups, called ideal subgroups of $E(X)$. All ideal subgroups of $E(X)$ are isomorphic.

If X, Y are H-flows, then we say that a continuous function $f : X \to Y$ is an H-mapping, if f respects the action of H. H-flows form a category, with H-mappings as morphisms.

The largest point-transitive H-flow is the space βH of ultrafilters on H. The action of H on βH is the left translation. It turns out that the Ellis semigroup of βH is isomorphic (as an H-flow) to βH itself.

In [N4, N5, N6] these topological-dynamic notions were applied in a model-theoretic setting. While in model theory it is natural to consider the $G(M)$-flow $S_G(M)$, the role of the maximal point-transitive $G(M)$-flow there is played by the space $S_{G,ext}(M)$ of complete external G-types over M (instead of $\beta(G(M))$). $S_{G,ext}(M)$ is also isomorphic to its Ellis semigroup (as a $G(M)$-flow). This induces a semigroup operation on $S_{G,ext}(M)$ itself. In this paper we will consider several $G(M)$-flows isomorphic to their Ellis semigroups. Below we present a general setting for this.

Assume H is a group and $\mathcal{A}$ is an algebra of sets. We say that $\mathcal{A}$ is an H-algebra if there is an action of H on $\mathcal{A}$ by Boolean automorphisms. By an H-endomorphism of an H-algebra $\mathcal{A}$ we mean a Boolean endomorphism of $\mathcal{A}$ respecting the action of H. Let $\mathrm{End}(\mathcal{A})$ denote the semigroup of H-endomorphisms of $\mathcal{A}$ (the semigroup operation is composition of functions).

The action of H on $\mathcal{A}$ induces an action of H on the Stone space $S(\mathcal{A})$ by homeomorphisms, making $S(\mathcal{A})$ an H-flow.

In this paper we will consider H-algebras $\mathcal{A}$ of subsets of H, where the action of H on $\mathcal{A}$ is induced by left translation in the group H (this means just that $\mathcal{A}$ is closed under left translation). For example $\mathcal{A} = \mathcal{P}(H)$ is an H-algebra and $S(\mathcal{A}) = \beta H$ is an H-flow. Ellis proved [E] that the Ellis semigroup $E(\beta H)$ is isomorphic to $\mathrm{End}(\mathcal{P}(H))$ (as a semigroup) and to βH (as an H-flow).

In the model-theoretic setting let $\mathcal{A} = \mathrm{Def}_{G,ext}(M)$ be the algebra of externally definable subsets of $G(M)$, that is sets of the form $U \cap G(M)$, where U is a definable subset of $\mathfrak{C}$. $\mathrm{Def}_{G,ext}(M)$ is closed under left translation by elements of $G(M)$, hence it is a $G(M)$-algebra of subsets o $G(M)$. The space of external G-types $S_{G,ext}(M)$ is just the Stone space of ultrafilters on $\mathrm{Def}_{G,ext}(M)$.

Following Ellis we proved that also the Ellis semigroup $E(S_{G,ext}(M))$ is isomorphic to $\mathrm{End}(\mathrm{Def}_{G,ext}(M))$ as a semigroup [N5] and to $S_{G,ext}(M)$ as a $G(M)$-flow [N4]. Here we will generalize this result.

Definition 1.2 *Let H be a group and $\mathcal{A}$ be an H-algebra of subsets of H (invariant under left translation).*
(1) For $p \in S(\mathcal{A})$ we define a function $d_p : \mathcal{A} \to \mathcal{P}(H)$ by

$$d_p(U) = \{g \in H : g^{-1}U \in p\}.$$

Clearly $d_p : \mathcal{A} \to \mathcal{P}(H)$ is a homomorphism of H-algebras.
(2) We say that $\mathcal{A}$ is d-closed if $\mathcal{A}$ is closed under d_p for every $p \in S(\mathcal{A})$, that is $d_p[\mathcal{A}] \subseteq \mathcal{A}$. Notice that in this case $d_p \in End(\mathcal{A})$.
(3) If $\mathcal{A}$ is d-closed, then let $d : S(\mathcal{A}) \to End(\mathcal{A})$ be the function mapping p to d_p.

Remark 1.3 *(1) The H-algebra $\mathcal{P}(H)$ is d-closed.*
(2) The $G(M)$-algebra $Def_{G,ext}(M)$ is d-closed.

Proof. (1) is obvious. (2) is [N5, Lemma 1.2]. □

Proposition 1.4 *Assume H is a group and $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{P}(H)$ are d-closed H-subalgebras of $\mathcal{P}(H)$.*

(1) Assume $p \in S(\mathcal{A}), q \in S(\mathcal{B})$ and $p \subseteq q$. Then $d_p = d_q|_{\mathcal{A}}$.
(2) The function $d : S(\mathcal{A}) \to End(\mathcal{A})$ is a bijection.
(3) The function $d : S(\mathcal{A}) \to End(\mathcal{A})$ induces on $S(\mathcal{A})$ a semigroup operation $*$ so that d becomes an isomorphism of semigroups. So for $p, q \in S(\mathcal{A})$ we have $d_{p*q} = d_p \circ d_q$. Also, for $U \in \mathcal{A}$ we have

$$U \in p * q \iff d_q(U) \in p.$$

(4) The restriction function $S(\mathcal{B}) \to S(\mathcal{A})$ is an H-mapping and an epimorphism of semigroups. The following diagram commutes

$$\begin{array}{ccc} S(\mathcal{B}) & \longrightarrow & S(\mathcal{A}) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ End(\mathcal{B}) & \longrightarrow & End(\mathcal{A}) \end{array}$$

where the horizontal arrows are restrictions.
(5) For $p \in S(\mathcal{A})$ let $l_p : S(\mathcal{A}) \to S(\mathcal{A})$ be the function mapping $q \in S(\mathcal{A})$ to $p * q$. Then $E(S(\mathcal{A}))$, the Ellis semigroup of $S(\mathcal{A})$, equals $\{l_p : p \in S(\mathcal{A})\}$.
(6) Let $l : S(\mathcal{A}) \to E(S(\mathcal{A}))$ be the function mapping p to l_p. Then l is an isomorphism of H-flows and of semigroups.

Proof. (1) is obvious.

The proof of (2) is analogous to the proof of [N5, Proposition 1.6]. To see that d is 1-1 consider $p \neq q \in S(\mathcal{A})$. Choose $U \in p$ with $U^c \in q$. Then $1 \in d_p(U)$ and $1 \notin d_q(U)$, hence $d_p(U) \neq d_q(U)$ and $d_p \neq d_q$.

To see that d is "onto" consider any $f \in \mathrm{End}(\mathcal{A})$. Let

$$p = \{U \in \mathcal{A} : 1 \in f(U)\}.$$

Clearly $p \in S(\mathcal{A})$ and it is easy to see that $f = d_p$ (or see the proof of [N5, Proposition 1.6]).

(3) Let $U \in \mathcal{A}$. Since $d_{p*q} = d_p \circ d_q$, we have that

$$U \in p * q \Leftrightarrow 1 \in d_{p*q}(U) \Leftrightarrow 1 \in d_p(d_q(U)) \Leftrightarrow d_q(U) \in p.$$

(4) follows from (1)-(3).

(5) Let $\pi_g : S(\mathcal{A}) \to S(\mathcal{A}), g \in H$, be the family of homeomorphisms given by the action of H on $\mathcal{A}$. So $E(S(\mathcal{A}))$ is the topological closure of the set $\{\pi_g : g \in H\}$ in the topology of pointwise convergence in the space of functions $S(\mathcal{A}) \to S(\mathcal{A})$. For an ultrafilter $\mathcal{U} \in \beta H$ let $\pi_{\mathcal{U}} = \lim_{\mathcal{U}} \pi_g$. This means that for $q \in S(\mathcal{A})$ and $U \in \mathcal{A}$ we have

$$(*) \quad U \in \pi_{\mathcal{U}}(q) \iff \text{the set } X := \{g \in H : U \in \pi_g(q)\} \text{ belongs to } \mathcal{U}.$$

Hence $E(S(\mathcal{A})) = \{\pi_{\mathcal{U}} : \mathcal{U} \in \beta H\}$. Notice that the set X appearing in $(*)$ equals $d_q(U)$, hence for $U \in \mathcal{A}$ we have that

$$U \in \pi_{\mathcal{U}}(q) \Leftrightarrow d_q(U) \in \mathcal{U} \Leftrightarrow d_q(U) \in p \Leftrightarrow U \in p * q,$$

where $p = \mathcal{U} \cap \mathcal{A} \in S(\mathcal{A})$ (here we use the assumption that $\mathcal{A}$ is d-closed). Hence $\pi_{\mathcal{U}} = l_p$ and $E(S(\mathcal{A})) = \{l_p : p \in S(\mathcal{A})\}$.

(6) By (5) we have that the function $l : S(\mathcal{A}) \to E(S(\mathcal{A}))$ is "onto". To see that l is 1-1 consider $p \neq q \in S(\mathcal{A})$. For $g \in H$ let $\mathcal{U}_g = \{U \in \mathcal{A} : g \in U\}$. So $\mathcal{U}_g \in S(\mathcal{A})$. Notice that for $g = 1$, $d_{\mathcal{U}_1} : \mathcal{A} \to \mathcal{A}$ is the identity function, hence

$$d_p = d_p \circ d_{\mathcal{U}_1} = d_{p*\mathcal{U}_1} \text{ and } d_q = d_q \circ d_{\mathcal{U}_1} = d_{q*\mathcal{U}_1}.$$

By (3), $p = p * \mathcal{U}_1 = l_p(\mathcal{U}_1)$ and $q = q * \mathcal{U}_1 = l_q(\mathcal{U}_1)$, hence $l_p \neq l_q$.

To see that the function $l : S(\mathcal{A}) \to E(S(\mathcal{A}))$ is a homeomorphism, consider $U \in \mathcal{A}$ and the basic open set $[U] = \{p \in S(\mathcal{A}) : U \in p\}$. For $p \in S(\mathcal{A})$ we have that

$$U \in p \Leftrightarrow U \in p * \mathcal{U}_1 \Leftrightarrow U \in l_p(\mathcal{U}_1),$$

hence l maps the set $[U]$ onto the open set $\{f \in E(S(\mathcal{A})) : f(\mathcal{U}_1) \in [U]\}$. Since both spaces $S(\mathcal{A})$ and $E(S(\mathcal{A}))$ are compact, l is a homeomorphism.

For every $g \in H$ and $p \in S(\mathcal{A})$ we have $gp = \mathcal{U}_g * p$, hence for every $q \in S(\mathcal{A})$ we have

$$l_{gp}(q) = \pi_g * p * q = \pi_g(l_p(q)) = gl_p(q).$$

Therefore $l_{gp} = \pi_g \circ l_p$. It is obvious that l is a semigroup isomorphism. $\Box$

In particular, by Remark 1.3 Proposition 1.4 applies to the H-algebra $\mathcal{A} = \mathcal{P}(H)$ and the $G(M)$-algebra $\mathcal{A} = \mathrm{Def}_{G,ext}(M)$.

Since in the stable case externally definable subsets of M are internally definable, the above picture is simplified: $\mathrm{Def}_{G,ext}(M)$ equals $\mathrm{Def}_G(M)$, the $G(M)$-algebra of definable subsets of $G(M)$, and $S_{G,ext}(M) = S_G(M)$. Also, the semigroup operation on $S_G(M)$ defined in Proposition 1.4(3) is just the free multiplication of types from Definition 1.1. Hence $(S_G(M), *)$ is the Ellis semigroup of the $G(M)$-flow $S(G)$ and $d : S_G(M) \to \mathrm{End}(\mathrm{Def}_G(M))$ is an isomorphism of semigroups. We will apply Proposition 1.4 to several $G(M)$-subalgebras of $\mathrm{Def}_G(M)$.

Assume Δ is a set of formulas of L (with separated variables). By a relatively Δ-definable subset of $G(M)$ we mean a set of the form $G(M) \cap U$, where $U \subseteq M$ is Δ-definable. Besides the algebra $\mathrm{Def}_G(M)$ we will consider also its subalgebras $\mathrm{Def}_{G,\Delta}(M)$, consisting of the relatively Δ-definable subsets of $G(M)$. Also, $S_{G,\Delta}(M)$ denotes the set of complete Δ-types of elements of G, over M. So $S_{G,\Delta}(M)$ is just the Stone space of the algebra $\mathrm{Def}_{G,\Delta}(M)$.

2 The semigroup $S_G(M)$

In this section we will prove that the semigroup $S_G(M)$ is an inverse limit of a definable inverse system of some semigroups type-definable in M^{eq}.

Definition 2.1 *Assume $\Delta \subseteq L$.*
(1) We say that Δ is left-invariant if the family of subsets of $G(M)$ relatively definable by instances of formulas from Δ is invariant under left translation in $G(M)$. Similarly we define the notion of a right-invariant set Δ.
(2) We say that Δ is invariant if it is both left- and right-invariant.

It is well-known how to modify a given set $\Delta \subseteq L$ to make it invariant. Given a formula $\varphi(x, y)$ let

$$\varphi'(x, yz) = \varphi(z \cdot x, y) \text{ and } \varphi''(x, yzv) = \varphi(z \cdot x \cdot v, y).$$

Here $\cdot$ denotes the group operation in G. For $\Delta \subseteq L$ let $\Delta' = \{\varphi' : \varphi \in \Delta\}$ and $\Delta'' = \{\varphi'' : \varphi \in \Delta\}$. Clearly, Δ' is left-invariant and Δ'' is invariant.

Remark 2.2 *Every subset of $G(M)$ relatively definable by an instance of a formula from Δ is relatively definable by an instance of a formula from Δ' and an instance of a formula from Δ''. So $Def_{G,\Delta}(M) \subseteq Def_{G,\Delta'}(M) \subseteq Def_{G,\Delta''}(M)$.*

Assume $\Delta \subseteq L$ is left-invariant. Then $\mathrm{Def}_{G,\Delta}(M)$ is closed under left translation in $G(M)$, hence it is a $G(M)$-subalgebra of $\mathrm{Def}_G(M)$. The left translation in $\mathrm{Def}_{G,\Delta}(M)$ makes $S_{G,\Delta}(M)$ a $G(M)$-flow and the restriction function $r_\Delta : S_G(M) \to S_{G,\Delta}(M)$ is an epimorphism of $G(M)$-flows.

Given an family $\mathcal{U}$ of uniformly definable subsets of $G(M)$ we regard $\mathcal{U}$ as a definable subset of M^{eq}, identifying elements of $\mathcal{U}$ with their canonical names, uniformly.

Assume $\Delta \subseteq L$ is finite. We may consider $S_{G,\Delta}(M)$ as a type-definable subset of M^{eq}. Namely, for every $\varphi(x, y) \in \Delta$ we pick a formula $d_\varphi(y, z)$ such that every type $p(x) \in S_{G,\varphi}(M)$ had a φ-definition that is an instance of $d_\varphi(y, z)$. This means that for some $c_{p,\varphi} \subseteq M$ we have that

$$d_\varphi(M, c_{p,\varphi}) = \{a \subseteq M : \varphi(x, a) \in p(x)\}.$$

We may assume that $c_{p,\varphi} \in M^{eq}$ is a canonical name of $d_\varphi(M, c_{p,\varphi})$.

Let Z_φ be the set of canonical names of subsets of M definable by instances of $d_\varphi(y, z)$ (where z is the tuple of parameter variables). For $c \in Z_\varphi$ let

$$p^0_{\varphi,c} = \{\varphi(x, a) : a \subseteq M \text{ and } M \models d_\varphi(a, c)\} \cup \{\neg\varphi(x, a) : a \subseteq M \text{ and } M \not\models d_\varphi(a, c)\}.$$

Let $Z^0 = \prod_{\varphi \in \Delta} Z_\varphi$ and for $c = \langle c_\varphi \rangle_{\varphi \in \Delta} \in Z^0$ let $p^0_c = \bigcup_{\varphi \in \Delta} p^0_{\varphi,c_\varphi}$. For $n < \omega$ let $Z_n = \{c \in Z^0 : p^0_c \text{ is } n\text{-consistent with } G(x)\}$ and let $Z = \bigcap_{n<\omega} Z_n$.

Remark 2.3 *Assume $\Delta \subseteq L$ is finite.*
(1) The sets $Z_\varphi, \varphi \in \Delta, Z^0$ and $Z_n, n < \omega$, are definable in M^{eq} and Z is type-definable in M^{eq}.
(2) The function $p \mapsto \langle c_{p,\varphi} \rangle_{\varphi \in \Delta}$ is a bijection $S_{G,\Delta}(M) \to Z$.
(3) For $c \in Z$ the set of formulas p^0_c generates a type in $S_{G,\Delta}(M)$, denoted by p_c. The mapping $c \mapsto p_c$ is a bijection $Z \to S_{G,\Delta}(M)$ inverse to the bijection from (2).

By Remark 2.3 we regard $S_{G,\Delta}(M)$ as a type-definable subset of M^{eq}.

Let Inv_l denote the family of finite left-invariant sets $\Delta \subseteq L$, directed by inclusion. For $\Delta_1, \Delta_2 \in Inv_l$ with $\Delta_1 \subseteq \Delta_2$ let $r^{\Delta_2}_{\Delta_1} : S_{G,\Delta_2}(M) \to S_{G,\Delta_1}(M)$ be restriction. We consider the inverse system $\mathcal{D} = (S_{G,\Delta}(M))_{\Delta \in Inv_l}$ of $G(M)$-flows, with connecting functions $r^{\Delta_2}_{\Delta_1}$ for $\Delta_1, \Delta_2 \in Inv_l$ with $\Delta_1 \subseteq \Delta_2$.

Remark 2.4 *(1) For $\Delta_1, \Delta_2 \in Inv_l$ with $\Delta_1 \subseteq \Delta_2$, the functions $r^{\Delta_2}_{\Delta_1} : S_{G,\Delta_2}(M) \to S_{G,\Delta_1}(M)$ are definable $G(M)$-mappings.*
(2) $S_G(M)$ with restriction functions $r_\Delta : S_G(M) \to S_{G,\Delta}(M), \Delta \in Inv_l$, is an inverse limit of $\mathcal{D}$.

Proof. Straightforward. □

We call a compact topological space X scattered if X contains no perfect subset. In this case the Cantor-Bendixson rank CB_X on X has ordinal values. The next remark follows from basic stability theory. It justifies our interest in scattered flows.

Remark 2.5 *Assume $\Delta \in Inv_l$. Then the $G(M)$-flow $S_{G,\Delta}(M)$ is scattered and its CB-rank is finite.*

Lemma 2.6 *Assume H is a group and X is a scattered H-flow.*
(1) If $Y \subseteq X$ is a minimal subflow, then Y is finite.
(2) If X is point-transitive, then the dense H-orbit in X is unique.

Proof. (1) Choose $p \in Y$ with maximal CB_X-rank. The orbit Hp is dense in Y and $CB_X(q) = CB_X(p)$ for every $q \in Hp$ (since H acts on X by homeomorphisms). If Hp is infinite, then it has an accumulation point $p' \in Y \setminus Hp$ and $CB_X(p') > CB_X(p)$, a contradiction.

(2) The dense H-orbit in X consists of all isolated points. □

By Remarks 2.4 and 2.5 we see that the $G(M)$-flow $S_G(M)$ is pro-scattered, that is it is an inverse limit of scattered flows. In $S_G(M)$ there is a unique minimal subflow $Gen_G(M)$, consisting of the generic types of G. We know that $Gen_G(M)$ is a profinite closed subgroup of $S_G(M)$. More generally, if H is a group and X is a pro-scattered H-flow, then by Lemma 2.6 every minimal subflow of X is profinite. However even if X is additionally point-transitive, there need not be a unique dense H-orbit contained in X. Such an orbit is unique in the model-theoretic setting, provided M is sufficiently saturated.

Proposition 2.7 *Assume $p \in S_G(M)$ and let $X = cl(G(M)p)$ be the subflow of $S_G(M)$ generated by p. For $\Delta \in Inv_l$ let $p_\Delta = p|\Delta$ and $X_\Delta = cl(G(M)p_\Delta) \subseteq S_{G,\Delta}(M)$.*
(1) $(X, (r_\Delta)_{\Delta \in Inv_l})$ is an inverse limit of the system $(X_\Delta)_{\Delta \in Inv_l}$ with the connecting functions $r^{\Delta_2}_{\Delta_1}$, where $\Delta_1, \Delta_2 \in Inv_l$ and $\Delta_1 \subseteq \Delta_2$. In particular, X is pro-scattered.
(2) If M is $|T|^+$-saturated, then the set $G(M)p$ is the unique dense $G(M)$-orbit in X.

Proof. (1) is obvious.

(2) Choose $q \in X$ and for $\Delta \in Inv_l$ let $q_\Delta = q|_\Delta$. Assume the orbit $G(M)q$ is dense in X. Then for every $\Delta \in Inv_l$ we have that the orbit $G(M)q_\Delta$ is dense in X_Δ, just like the orbit $G(M)p_\Delta$. Hence by Lemma 2.6, both orbits coincide and there is a $g_\Delta \in G(M)$ with $q_\Delta = g_\Delta p_\Delta$.

Since $S_{G,\Delta}(M)$ is a type-definable subset of M^{eq}, we can use p_Δ and q_Δ as parameters in formulas of L^{eq}. The set of formulas

$$\Phi(x) = \{q_\Delta = x \cdot p_\Delta : \Delta \in Inv_l\} \cup \{G(x)\}$$

is a type over $|T|$-many parameters, hence by the saturation of M it is realized by some $g \in G(M)$. We see that $q = g \cdot p$, hence the orbits $G(M)p$ and $G(M)q$ are equal. □

A special feature of topological dynamics of a stable group G is the existence of generic types in $S_G(M)$. More generally, we define the notion of a generic point in an arbitrary point-transitive H-flow X [N4]. Then the existence of a generic point in X is equivalent to there being just one minimal subflow of X [N4, Corollary 1.9].

One could wonder if there is a topological property of the flow $S_G(M)$ (in our stable setting) responsible for existence of generic types in $S_G(M)$. We do not know any such property and the next example shows that the property of being pro-scattered would not work.

Let $M = G(M) = (\mathbb{Z}, +, \leq)$ be the ordered group of integers. Every definable subset of M is a Boolean combination of co-sets of the groups $k\mathbb{Z}, k > 0$, and the $\leq$-intervals in $\mathbb{Z}$. For $k > 0$ let Δ_k be the set of formulas $\{k|(x-y), x \leq y\}$ in the language of M. Then every Δ_k is invariant and the $\mathbb{Z}$-flow $S_{G,\Delta_k}(M)$ is scattered, of CB-rank 1. There are two minimal subflows of $S_{\mathcal{G},\Delta_k}(M)$, at $+\infty$ and $-\infty$, both of size k. There are no generic types in $S_{G,\Delta_k}(M)$. $S_G(M)$ is an inverse limit of the flows $S_{G,\Delta_k}(M)$, so it is pro-scattered. There are no generic types in $S_G(M)$.

Now we return to the stable setting. It turns out that $S_G(M)$ is pro-scattered not just as a $G(M)$-flow, but also as an Ellis semigroup. We have already used definability of types in a stable theory to interpret $S_{G,\Delta}(M), \Delta \in Inv_l$, as a type-definable set in M^{eq}. We shall need the following deep result on definability of types in local stability theory.

Lemma 2.8 ([Pi, Lemma I.2.2]) *Suppose $\delta(x, y)$ is a stable formula. Let M be a model and let $p(x) \in S_\delta(M)$. Then there is a formula $\chi(y)$ which is a positive Boolean combination of formulas $\psi(c, y), c \in M$, such that $\chi(y)$ is a δ-definition of $p(x)$, meaning that*

$$\chi(M) = \{b \subseteq M : \delta(x, b) \in p(x)\}.$$

By compactness we get the following remark.

Remark 2.9 *Assume $\delta(x, y)$ is a stable formula. Then there is a natural number n such that for every model M and $p(x) \in S_\delta(M)$ the set $\{b \subseteq M : \delta(x, b) \in p(x)\}$ is M-definable by an instance of the formula*

$$\chi(y, z) = \bigvee_{i<n} \bigwedge_{j<n} \delta(z_{ij}, y),$$

where $z = \langle z_{ij} \rangle$ is the tuple of parameter variables in $\chi(y, z)$.

Let $\mathcal{D}'$ be the subsystem of the inverse system $\mathcal{D} = (S_{G,\Delta}(M))_{\Delta \in Inv_l}$ consisting of the flows $S_{G,\Delta}(M), \Delta \in Inv$. Notice that since essentially Inv is co-final in Inv_l, still $S_G(M)$ is an inverse limit of $\mathcal{D}'$. The next lemma explains the reason why we restrict ourselves to $\Delta \in Inv$.

Lemma 2.10 *(1) Assume $p \in S_G(M)$ and $U \subseteq G(M)$ is definable. Then $d_p(U)$ is a positive Boolean combination of some right translates $Uc, c \in G(M)$, of U.*
(2) Assume $\Delta \subseteq L$ is invariant. Then $Def_{G,\Delta}(M)$ is a d-closed $G(M)$-subalgebra of $Def_G(M)$.

Proof. (1) Choose a formula $\varphi(x)$ over M, defining U. Let $\psi(x, y) = \varphi(y \cdot x)$. So for every $g \in G(M)$ we have that $g^{-1}U = \psi(M, g)$. Hence

$$d_p(U) = \{g \in G(M) : g^{-1}U \in p\} = \{g \in G : \psi(x, g) \in p(x)\}.$$

By Lemma 2.8, the set $d_p(U)$ is defined by a formula $\chi(y)$ that is a positive Boolean combination of formulas $\psi(c, y), c \in G(M)$. But for $c \in G(M)$ we have that $\psi(c, M) = Uc^{-1}$, so we are done.

(2) follows from (1). □

Assume $\Delta \in Inv$. By Lemma 2.10 the $G(M)$-algebra $\mathrm{Def}_{G,\Delta}(M)$ is d-closed, hence by Proposition 1.4 the set $S_{G,\Delta}(M)$ carries a semigroup operation $*$ defined by:

$$U \in p * q \iff d_q(U) \in p$$

and the function $d : S_{G,\Delta}(M) \to \mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ mapping p to d_p is a semigroup isomorphism. Also $S_{G,\Delta}(M)$ is isomorphic to the Ellis semigroup $E(S_{G,\Delta}(M))$, as a semigroup and as a $G(M)$-flow.

By Proposition 1.4, the connecting functions $r^{\Delta_2}_{\Delta_1} : S_{G,\Delta_2}(M) \to S_{G,\Delta_1}(M)$ of the inverse system $\mathcal{D}'$ and the functions $r_\Delta : S_G(M) \to S_{G,\Delta}(M), \Delta \in Inv$, respect the semigroup operations, hence $\mathcal{D}'$ is an inverse system of $G(M)$-flows and semigroups and $S_G(M)$ is an inverse limit of $\mathcal{D}'$ as a $G(M)$-flow and as a semigroup.

Proposition 2.11 *Assume $\Delta \in Inv$. Then the semigroup operation $*$ in $S_{G,\Delta}(M)$ is definable in M^{eq}. Hence the semigroup $S_{G,\Delta}(M)$ is type-definable in M^{eq}.*

Proof. First we put the algebra $\mathrm{Def}_{G,\Delta}(M)$ within the context of definable sets in M^{eq}. Let X_Δ be the family of subsets of $G(M)$ relatively definable by instances of formulas from Δ. Hence X_Δ is a family of uniformly definable subsets of $G(M)$ and we may consider X_Δ a definable subset of M^{eq}. Clearly X_Δ is invariant under both left- and right-translation in the group $G(M)$. So X_Δ generates $\mathrm{Def}_{G,\Delta}(M)$ as an algebra of sets.

For $0 < n < \omega$ let $\mathcal{B}_n(X_\Delta)$ be the family of sets in $\mathrm{Def}_{G,\Delta}(M)$ of the form $\tau(\bar{a})$, where $\tau(\bar{x})$ is a Boolean term of length $\leq n$ and $\bar{a}$ is an n-tuple of elements of X_Δ. Clearly, $\mathcal{B}_n(X_\Delta)$ is uniformly definable, hence we regard $\mathcal{B}_n(X_\Delta)$ as a definable subset of M^{eq}. Also $\mathrm{Def}_{G,\Delta}(M) = \bigcup_n \mathcal{B}_n(X_\Delta)$, hence $\mathrm{Def}_{G,\Delta}(M)$ is $\bigvee$-definable in M^{eq}. Notice that the Boolean operations on $\mathcal{B}_n(X_\Delta)$ are definable in M^{eq}, with values in $\mathcal{B}_{2n}(X_\Delta)$. Also every $\mathcal{B}_n(X_\Delta)$ is closed under translation in $G(M)$ and this translation is an operation definable in M^{eq}.

Since X_Δ generates $\mathrm{Def}_{G,\Delta}(M)$ as an algebra of sets, every $f \in \mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ is determined by its restriction $f|_{X_\Delta}$. By Remark 2.9 and the proof of Lemma 2.10(1) there is an $n < \omega$ (independent of M) such that for every $p \in S_{G,\Delta}(M)$, the function $d_p|X_\Delta$ has values in $\mathcal{B}_n(X_\Delta)$. Also, $d_p|_{\mathcal{B}_n(X_\Delta)}$ maps $\mathcal{B}_n(X_\Delta)$ to $\mathcal{B}_{n^2}(X_\Delta)$.

To see this, consider $U \in \mathcal{B}_n(X_\Delta)$. Hence $U = \tau(V_1, \ldots, V_n)$ for some $V_1, \ldots, V_n \in X_\Delta$ and a Boolean term $\tau(\bar{x})$ of length $\leq n$, where $\bar{x} = \langle x_1, \ldots, x_n \rangle$. We have that $d_p(U) = \tau(d_p(V_1), \ldots, d_p(V_n))$. Since $d_p(V_1), \ldots, d_p(V_n) \in \mathcal{B}_n(X_\Delta)$, there are Boolean terms $\tau_1(\bar{x}_1), \ldots, \tau_n(\bar{x}_n)$ of length $\leq n$ such that

$$d_p(V_i) = \tau_i(V_{i,1}, \ldots, V_{i,n}) \text{ for some } V_{i,j} \in X_\Delta.$$

Hence $d_p(U) = \tau'(V_{i,j})_{1 \leq i,j \leq n}$, where $\tau' = \tau(\tau_1(\bar{x}_1), \ldots, \tau_n(\bar{x}_n))$. We see that $d_p(U) \in \mathcal{B}_{n^2}(X_\Delta)$.

The functions $d_p|_{X_\Delta} : X_\Delta \to \mathcal{B}_n(X_\Delta)$ and $d_p|_{\mathcal{B}_n(X_\Delta)} : \mathcal{B}_n(X_\Delta) \to \mathcal{B}_{n^2}(X_\Delta)$ are definable (in M^{eq}, uniformly in $p \in S_{G,\Delta}(M)$). Identifying $f \in \mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ with $f|_{X_\Delta}$ we have that the set $\mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ is type-definable in M^{eq} (since every such f is of the form d_p, hence f maps X_Δ into $\mathcal{B}_n(X_\Delta)$) and also the bijection $d : S_{G,\Delta}(M) \to \mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ is definable in M^{eq}. To finish the proof it is enough to show that composition of functions in $\mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ is definable in M^{eq}.

Assume $f_1, f_2 \in \mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ and $f = f_1 \circ f_2$. The function $f|_{X_\Delta}$ is uniformly definable in M^{eq} from $f_1|_{X_\Delta}$ and $f_2|_{X_\Delta}$ as follows.

For $U \in X_\Delta$ we have a uniform description of $f_1(f_2(U))$ as an element of $\mathcal{B}_{n^2}(X_\Delta)$. Also we know that $f_1(f_2(U))$ belongs to $\mathcal{B}_n(X_\Delta)$. So we define $f|_{X_\Delta}(U)$ as the unique $V \in \mathcal{B}_n(X_\Delta)$ equal to $f_1(f_2(U))$. □

Corollary 2.12 *The Ellis semigroup $S_G(M)$ is an inverse limit of the definable inverse system $\mathcal{D}' = (S_{G,\Delta}(M))_{\Delta \in Inv}$ of semigroups type-definable in M^{eq}.*

In this way in the stable case we have located the Ellis semigroup $S_G(M)$ in the definable realm of M. Unfortunately, the type-definable semigroups $S_{G,\Delta}(M), \Delta \in Inv$, need not be definable.

Notice that the definition of the inverse system $\mathcal{D}' = (S_{G,\Delta}(M))_{\Delta \in Inv}$ is uniform in M. If we go to an elementary extension M' of M, then the system $\mathcal{D}'(M') = (S_{G,\Delta}(M'))_{\Delta \in Inv}$ is related to $\mathcal{D}'$ as follows.

There are natural embeddings $j_\Delta : S_{G,\Delta}(M) \to S_{G,\Delta}(M')$, mapping $p \in S_{G,\Delta}(M)$ to its heir in $S_{G,\Delta}(M')$. These embeddings are monomorphisms of semigroups and commute with the connecting functions of the systems $\mathcal{D}'$ and $\mathcal{D}'(M')$. They yield a $*$-monomorphism $j : S_G(M) \to S_G(M')$ mapping p to $p|M'$, its non-forking extension to M'.

3 Subgroups of semigroups of types

In this section we will investigate subgroups of the semigroups $S_{G,\Delta}(M), \Delta \in Inv$ and $S_G(M)$. These semigroups are isomorphic (via the functions d) to the semigroups

End($\mathrm{Def}_{G,\Delta}(M)$) and End($\mathrm{Def}_G(M)$). This will be crucial in our analysis. We start with a general background on such subgroups and then proceed with a more specific description in our stable model-theoretic context.

Assume H is a group, X is a point-transitive H-flow and $E(X)$ is its Ellis semigroup. Subgroups of $E(X)$ are interesting on their own. Indeed, the minimal subflows $I \subseteq E(X)$ split into disjoint unions of isomorphic "ideal groups".

Assume $\mathcal{A}$ is a d-closed H-subalgebra of $\mathcal{P}(H)$ and $X = S(\mathcal{A})$. We consider X as an H-flow, the action being left translation. By Proposition 1.4, $S(\mathcal{A})$ carries a semigroup structure isomorphic (via the function d) to the semigroup $\mathrm{End}(\mathcal{A})$. $S(\mathcal{A})$ is isomorphic to its Ellis semigroup (both as a semigroup and as an H-flow). The next lemma appears in [N5, Lemma 1.8].

Lemma 3.1 *Assume $\mathcal{A}$ is an H-algebra of sets and S is a subgroup of $End(\mathcal{A})$.*
(1) There is an H-ideal $K \subseteq \mathcal{A}$, the common kernel of all $f \in S$.
(2) There is an H-subalgebra $\mathcal{B} \subseteq \mathcal{A}$, the common image of all $f \in S$.
(3) $K \cap \mathcal{B} = \{\emptyset\}$, $\mathcal{B}$ is a section of the family of K-cosets in $\mathcal{A}$, $\mathcal{A}/K \cong \mathcal{B}$ and for every $f \in S$ we have that $f|_{\mathcal{B}}$ is an H-automorphism of $\mathcal{B}$.
(4) The mapping $f \mapsto f|_{\mathcal{B}}$ is a group monomorphism $S \to Aut(\mathcal{B}) \cong Aut(\mathcal{A}/K)$.

The next corollary describes the maximal subgroups of $End(\mathcal{A})$.

Corollary 3.2 *Assume $\mathcal{A}$ is an H-algebra of sets and S is a subgroup of $End(\mathcal{A})$. Let K and $\mathcal{B}$ be the common kernel and image of all $f \in S$. Let*

$$S_{K,\mathcal{B}} = \{f \in End(\mathcal{A}) : K = Ker(f), \mathcal{B} = Im(f) \text{ and } f|_{\mathcal{B}} \in Aut(\mathcal{B})\}.$$

Then $S_{K,\mathcal{B}}$ is a unique maximal subgroup of $End(\mathcal{A})$ containing S. Also, every maximal subgroup of $End(\mathcal{A})$ is of this form.

Proof. We need only to see that $S_{K,\mathcal{B}}$ is a group. It is obviously closed under composition of functions and has a neutral element (namely, the neutral element of S). We need to check that every $f \in S_{K,\mathcal{B}}$ has a group inverse in $S_{K,B}$.

So let $f \in S_{K,\mathcal{B}}$. Let $f_0' \in \mathrm{Aut}(\mathcal{B})$ be the inverse of $f|_{\mathcal{B}}$ in the group $\mathrm{Aut}(\mathcal{B})$. Since $\mathcal{B}$ is a section of the family of cosets of K in $\mathcal{A}$, we can define $f' \in \mathrm{End}(\mathcal{A})$ putting $f'(U) = f_0'(U')$, where $U' \in \mathcal{B}$ belongs to the K-coset of U. Clearly $f' \in S_{K,\mathcal{B}}$ is the group inverse of f. □

Assume S is a subgroup of $\mathrm{End}(\mathcal{A})$ and $e \in S$ is its neutral element. Then e is an idempotent (that is, $e^2 = e$). Vice versa, every idempotent $e \in \mathrm{End}(\mathcal{A})$ forms a trivial subgroup $S = \{e\}$ of $\mathrm{End}(\mathcal{A})$, hence it belongs to a unique maximal subgroup $S_{K,\mathcal{B}}$ of $\mathrm{End}(\mathcal{A})$, where $K = \mathrm{Ker}(e)$ and $\mathcal{B} = \mathrm{Im}(e)$. The fact that $\mathcal{B} = \mathrm{Im}(e)$ for an idempotent $e \in \mathrm{End}(\mathcal{A})$ yields additional properties of $\mathcal{B}$.

Assume $\mathcal{A}$ is a Boolean algebra. We say that $\mathcal{B}$ is a complete subalgebra of $\mathcal{A}$ if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and for every set $X \subseteq \mathcal{B}$, if X has the supremum in $\mathcal{A}$, then this supremum belongs to $\mathcal{B}$ (and is the supremum of X in $\mathcal{B}$). Also, $\mathrm{At}(\mathcal{A})$ denotes the set of atoms of $\mathcal{A}$. Assume $\mathcal{B}$ is an atomic subalgebra of $\mathcal{A}$. We say that $U \in \mathcal{A}$ is compatible with $\mathrm{At}(\mathcal{B})$ if for every $V \in \mathrm{At}(\mathcal{B})$ we have that $V \leq U$ or $V \leq U^c$.

Lemma 3.3 *Assume $\mathcal{A}$ is a Boolean algebra, e is an endomorphism of $\mathcal{A}$ and $e^2 = e$. Let $\mathcal{B} = Im(e)$. Assume $\mathcal{B}$ is atomic and $\Sigma^{\mathcal{A}} At(\mathcal{B}) = \mathbf{1}_{\mathcal{A}}$.*
(1) $\mathcal{B}$ consists of the elements of $\mathcal{A}$ compatible with $At(\mathcal{B})$. In this way $\mathcal{B}$ is determined by $At(\mathcal{B})$.
(2) $\mathcal{B}$ is a complete subalgebra of $\mathcal{A}$.

Proof. We regard $\mathcal{A}$ as an algebra of subsets of a set Z.

(1) Clearly, every $V \in \mathcal{B}$ is compatible with $\mathrm{At}(\mathcal{B})$. Assume $V \in \mathcal{A}$ is compatible with $\mathrm{At}(\mathcal{B})$. Let $A = \{U \in \mathrm{At}(\mathcal{B}) : U \subseteq V\}$. We claim that

$$(*) \quad V = \Sigma^{\mathcal{A}} A = \bigcup A.$$

Indeed, $\bigcup A \subseteq V$. Suppose $U \in \mathcal{A}$ meets V. Since $\Sigma^{\mathcal{A}}\mathrm{At}(\mathcal{B}) = \mathbf{1}_{\mathcal{A}}$, we have that $U \cap V$ meets an atom of $\mathcal{B}$, necessarily from A (as V is compatible with $\mathrm{At}(\mathcal{B})$). This proves $(*)$.

Using the fact that $e|_{\mathcal{B}} = id_{\mathcal{B}}$ one can prove similarly that $e(V) = \Sigma^{\mathcal{A}} A = \bigcup A$. Hence $V = e(V) \in \mathcal{B}$.

(2) Assume $X \subseteq \mathcal{B}$, $V \in \mathcal{A}$ and $V = \Sigma^{\mathcal{A}} X$. By (1) it is enough to show that V is compatible with $\mathrm{At}(\mathcal{B})$. So let $U \in \mathrm{At}(\mathcal{B})$. If U is contained in a set from X, then clearly $U \subseteq V$. If U is disjoint from any set in X, then $U \subseteq V^c$. So we are done. $\square$.

In our model-theoretic setting the semigroup $S_G(M)$ is an inverse limit of the semigroups $S_{G,\Delta}(M), \Delta \in Inv$, and since the corresponding $G(M)$-algebras $\mathrm{Def}_G(M)$ and $\mathrm{Def}_{G,\Delta}(M)$ are d-closed, $S_G(M)$ and $S_{G,\Delta}(M)$ are isomorphic with the semigroups $\mathrm{End}(\mathrm{Def}_G(M))$ and $\mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$, respectively (via the functions d). We denote by r_Δ both the (surjective) restriction functions $S_G(M) \to S_{G,\Delta}(M)$ and $\mathrm{End}(\mathrm{Def}_G(M)) \to \mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$. Also, for $\Delta_1, \Delta_2 \in Inv$ with $\Delta_1 \subseteq \Delta_2$, $r^{\Delta_2}_{\Delta_1}$ denotes both the restriction functions $S_{G,\Delta_2}(M) \to S_{G,\Delta_1}(M)$ and $\mathrm{End}(\mathrm{Def}_{G,\Delta_2}(M)) \to \mathrm{End}(\mathrm{Def}_{G,\Delta_1}(M))$.

Assume S is a maximal subgroup of $\mathrm{End}(\mathrm{Def}_G(M))$. So $S = S_{K,\mathcal{B}}$ for some $G(M)$-ideal K and $G(M)$-subalgebra $\mathcal{B}$ of $\mathrm{Def}_G(M)$. For $\Delta \in Inv$ let $S_\Delta = S_{K_\Delta, \mathcal{B}_\Delta}$, where $K_\Delta = K \cap \mathrm{Def}_{G,\Delta}(M)$ and $\mathcal{B}_\Delta = \mathcal{B} \cap \mathrm{Def}_{G,\Delta}(M)$. So every S_Δ is a maximal subgroup of $\mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$.

Remark 3.4 *$r_\Delta[S] \subseteq S_\Delta$ for every $\Delta \in Inv$. Also, S is an inverse limit of the groups $S_\Delta, \Delta \in Inv$.*

One could wonder when the restriction functions $r_\Delta : S \to S_\Delta$ are surjective. This is partially clarified in the next lemma.

Lemma 3.5 *Assume L is countable. The following conditions are equivalent.*
(1) The functions $r_\Delta : S \to S_\Delta$ are surjective for all $\Delta \in Inv$.
(2) The functions $r^{\Delta_2}_{\Delta_1} : S_{\Delta_2} \to S_{\Delta_1}$ are surjective for all $\Delta_1, \Delta_2 \in Inv$ with $\Delta_1 \subseteq \Delta_2$.

Proof. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (1): Clearly $K = \bigcup_{\Delta \in Inv} K_\Delta$ and $\mathcal{B} = \bigcup_{\Delta \in Inv} \mathcal{B}_\Delta$. Let $\Delta \in Inv$ and $f \in S_\Delta$. Choose an increasing cofinal sequence $\Delta_n \in Inv, n < \omega$, with $\Delta_0 = \Delta$. By (2) we find an increasing sequence $f_n \in S_{\Delta_n}, n < \omega$, with $f_0 = f$. Let $f' = \bigcup_{n<\omega} f_n$. Clearly

$$\mathrm{Ker}(f) = \bigcup_{n<\omega} \mathrm{Ker}(f_n) = \bigcup_{n<\omega} K_{\Delta_n} = K,\ \mathrm{Im}(f) = \bigcup_{n<\omega} \mathrm{Im}(f_n) = \bigcup_{n<\omega} \mathcal{B}_{\Delta_n} = \mathcal{B}$$

and $f'|_{\mathcal{B}} = \bigcup_{n<\omega} f_n|_{\mathcal{B}_n}$ is an automorphism of $\mathcal{B}$. So $f' \in S$ and $r_\Delta(f') = f$. This shows that $r_\Delta[S] = S_\Delta$. $\square$

Later in this section we will prove that Lemma 3.5 holds also in some other cases. This will follow from our description of the maximal subgroups of $S_G(M)$.

The maximal subgroups of $S_G(M)$ and $S_{G,\Delta}(M), \Delta \in Inv$, are determined by the idempotents in $S_G(M)$ and $S_{G,\Delta}(M)$. These idempotents are related to each other.

Remark 3.6 *Assume $p \in S_G(M)$ is an idempotent and $\Delta \in Inv$. Then $p|_\Delta \in S_{G,\Delta}(M)$ is also an idempotent. Conversely, every idempotent in $S_{G,\Delta}(M)$ extends to an idempotent in $S_G(M)$.*

Proof. Immediate. To see the second clause, consider an idempotent $q \in S_{G,\Delta}(M)$. Let $X = \{q' \in S_G(M) : q \subseteq q'\}$. Hence X is a closed sub-semigroup of $S_G(M)$. By [E], every closed sub-semigroup of an Ellis semigroup contains an idempotent, hence we are done. $\square$

Assume $\Delta \in Inv$. Subgroups of $S_{G,\Delta}(M)$ are related to $G(M)$-subalgebras of $\mathrm{Def}_{G,\Delta}(M)$. In the next lemma we describe some properties of $G(M)$-subalgebras of $\mathrm{Def}_{G,\Delta}(M)$.

Lemma 3.7 *Assume $\Delta \in Inv$ and $\mathcal{B}$ is a $G(M)$-subalgebra of $Def_{G,\Delta}(M)$.*
(1) $\mathcal{B}$ is atomic.
(2) For $g \in G(M)$ let $\mathcal{U}_{g,\mathcal{B}} = \{U \in \mathcal{B} : g \in U\}$. Then $\mathcal{U}_{g,\mathcal{B}}$ is a principal ultrafilter on $\mathcal{B}$, generated by the atom $U_{g,\mathcal{B}}$ of $\mathcal{B}$ containing g.
(3) $U_{1,\mathcal{B}}$ is a definable subgroup of $G(M)$, denoted by $G_{\mathcal{B}}$, and the atoms $U_{g,\mathcal{B}}, g \in G(M)$, are the left cosets of $G_{\mathcal{B}}$ in $G(M)$.
(4) $G_{\mathcal{B}}$ equals $\{g \in G(M) : g\mathcal{U}_{1,\mathcal{B}} = \mathcal{U}_{1,\mathcal{B}}\}$, the stabilizer of $\mathcal{U}_{1,\mathcal{B}}$.

Proof. (1) The restriction function $S_{G,\Delta}(M) \to S(\mathcal{B})$ is surjective and $S_{G,\Delta}(M)$ is scattered, so also $S(\mathcal{B})$ is scattered. $\mathcal{B}$ is isomorphic to the algebra of clopen subsets of $S(\mathcal{B})$, hence $\mathcal{B}$ is atomic.

(2) Let $U \in \mathcal{B}$ be an atom and let $g \in U$. Then U generates $\mathcal{U}_{g,\mathcal{B}}$. For $h \in G(M)$ the set $hg^{-1}U$ is an atom of $\mathcal{B}$ containing h and generating $\mathcal{U}_{h,\mathcal{B}}$.

(3), (4) is [N6, Remark 3.2]. $\square$

Let RM_Δ, Mlt_Δ denote the local Morley Δ-rank and Δ-multiplicity. The next two lemmas describe further properties of the idempotents in $S_G(M)$ and $S_{G,\Delta}(M)$.

Lemma 3.8 *Assume $p \in S_G(M)$ is an idempotent and for $\Delta \in Inv$ let $p_\Delta = p|_\Delta \in S_{G,\Delta}(M)$.*
(1) p is the generic type of a type-definable connected subgroup S^0 of $G(M)$.

(2) $S^0 = Stab(p)$, where $Stab(p) = \{g \in G(M) : gp = p\}$.
(3) $S^0 = \bigcap_{\Delta \in Inv} Stab(p_\Delta)$, where $Stab(p_\Delta) = \{g \in G(M) : gp_\Delta = p_\Delta\}$.
(4) For $\Delta \in Inv$, $Stab(p_\Delta)$ is a definable subgroup of $G(M)$, in fact $Stab(p_\Delta) \in Def_{G,\Delta}(M)$.

Proof. (1) is by [N3], (2) is by [Pi], (3) is obvious.

(4) Choose $U \in p_\Delta(x)$ with $RM_\Delta(U) = RM_\Delta(p_\Delta)$ and $Mlt_\Delta(U) = Mlt_\Delta(p_\Delta)$. Then

$$\mathrm{Stab}(p_\Delta) = \{g \in G(M) : U \in gp_\Delta\} = d_{p_\Delta}(U) \in \mathrm{Def}_{G,\Delta}(M).$$

□

Lemma 3.9 *Under the assumptions of Lemma 3.8, let $\mathcal{B} = Im(d_p)$ and for $\Delta \in Inv$ let $\mathcal{B}_\Delta = Im(p_\Delta) = \mathcal{B} \cap Def_{G,\Delta}(M)$.*
(1) $p_\Delta \cap \mathcal{B}_\Delta = \mathcal{U}_{1,\mathcal{B}_\Delta}$.
(2) $Stab(p_\Delta) = G_{\mathcal{B}_\Delta}$.
(3) $RM_\Delta(p_\Delta) = RM_\Delta(G_{\mathcal{B}_\Delta})$ and $Mlt_\Delta(p_\Delta) = Mlt_\Delta(G_{\mathcal{B}_\Delta}) = 1$.
(4) $G_{\mathcal{B}_\Delta}$ is Δ-connected, that is it has no proper Δ-definable subgroup of finite index.
(5) p_Δ is the only generic type of $G_{\mathcal{B}_\Delta}$ in $S_{G,\Delta}(M)$.

Proof. (1) By Remark 3.6 p_Δ is an idempotent, hence d_{p_Δ} is an idempotent in $\mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ and $d_{p_\Delta}|_{\mathcal{B}_\Delta}$ is the identity. So for every $U \in \mathcal{B}_\Delta$ with $U \in p_\Delta$ we have that $1 \in d_{p_\Delta}(U)$ and $U = d_{p_\Delta}(U)$, hence $1 \in U$. Therefore $p_\Delta \cap \mathcal{B}_\Delta \subseteq \mathcal{U}_{1,\mathcal{B}_\Delta}$. The equality follows since $p_\Delta \cap \mathcal{B}_\Delta \in S(\mathcal{B}_\Delta)$.

(2) By Lemma 3.7(4), $G_{\mathcal{B}_\Delta} = \mathrm{Stab}(p_\Delta \cap \mathcal{B}_\Delta)$, hence $\mathrm{Stab}(p_\Delta) \subseteq G_{\mathcal{B}_\Delta}$. By Lemma 3.8(4), $\mathrm{Stab}(p_\Delta) \in \mathrm{Def}_{G,\Delta}(M)$. By Lemma 3.6(3), $G_{\mathcal{B}_\delta}$ is an atom of $\mathcal{B}_\Delta$, hence $\mathrm{Stab}(p_\Delta) = G_{\mathcal{B}_\Delta}$.

(3) First notice that

$(*)$ if $q_\Delta \in S_{G,\Delta}(M)$ and $G_{\mathcal{B}_\Delta} \in q_\Delta$, then $q_\Delta * p_\Delta = p_\Delta$.

Indeed, $q_\Delta * p_\Delta = \lim_{q_\Delta} gp_\Delta = p_\Delta$, because $G_{\mathcal{B}_\Delta} \in q_\Delta$ and for $g \in G_{\mathcal{B}_\Delta}$ we have $gp_\Delta = p_\Delta$ (see the proof of Proposition 1.4).

Now we prove that $RM_\Delta(p_\Delta) = RM_\Delta(G_{\mathcal{B}_\Delta})$. Let $q \in S_G(M)$ be a generic type of $G_{\mathcal{B}_\Delta}$ and let $q_\Delta = q|_\Delta \in S_{G,\Delta}(M)$. By $(*)$ we have $(q*p)|_\Delta = q_\Delta * p_\Delta = p_\Delta$, hence $G_{\mathcal{B}_\Delta} \in q * p$.

By [N3] we have that $RM_\Delta(q*p) \geq RM_\Delta(q)$, but $RM_\Delta(q) = RM_\Delta(G_{\mathcal{B}_\Delta})$ and also $RM_\Delta(G_{\mathcal{B}_\Delta}) \geq RM_\Delta(q * p)$, hence $RM_\Delta(q * p) = RM_\Delta(G_{\mathcal{B}_\Delta})$. Hence also $RM_\Delta((q * p)|_\Delta) = RM_\Delta(G_{\mathcal{B}_\Delta})$.

Since $(q * p)|_\Delta = p_\Delta$, we get $RM_\Delta(p_\Delta) = RM_\Delta(G_{\mathcal{B}_\Delta})$.

By [Pi, Lemma I.2.11], $Mlt_\Delta(p_\Delta) = 1$. Suppose for a contradiction that $Mlt_\Delta(G_{\mathcal{B}_\Delta}) > 1$. Then there is a generic type $q \in S_{G,\Delta}(M)$ of $G_{\mathcal{B}_\Delta}$ with $q_\Delta := q|_\Delta \neq p_\Delta$. Choose a generic type $p' \in S_G(M)$ of $G_{\mathcal{B}_\Delta}$ with $p'|_\Delta = p_\Delta$.

Let $r \in S_G(M)$ be the generic type of the connected component of $G_{\mathcal{B}_\Delta}$. By [N3], the set of generic types of $G_{\mathcal{B}_\Delta}$ is a subgroup of $S_G(M)$, with neutral element r. Let $r_\Delta = r|_\Delta$. We consider two cases.

Case 1. $r_\Delta = p_\Delta$. We have that $q * r = q$, hence by $(*)$ we get $q_\Delta = q_\Delta * r_\Delta = q_\Delta * p_\Delta = p_\Delta$, a contradiction.

Case 2. $r_\Delta \neq p_\Delta$. Then we may assume $q = r$. Let p'' be the group inverse of p' in the group of generic types of $G_{\mathcal{B}_\Delta}$ in $S_G(M)$ and let $p''_\Delta = p''|_\Delta$. Then $p'' * p' = q$, hence $p''_\Delta * p_\Delta = q_\Delta$ and by $(*)$ we have $p''_\Delta * p_\Delta = p_\Delta$, again a contradiction.

Therefore $Mlt_\Delta(G_{\mathcal{B}_\Delta}) = 1$.

(4),(5) follow from (3), since the generic types of $G_{\mathcal{B}_\Delta}$ in $S_{G,\Delta}(M)$ have RM_Δ-rank equal to $RM_\Delta(G_{\mathcal{B}_\Delta})$. □

Corollary 3.10 *Under the assumptions of Lemmas 3.8 and 3.9 (and with their notation) we have the following.*
(1) $p \cap \mathcal{B} = \mathcal{U}_{1,\mathcal{B}} := \{U \in \mathcal{B} : 1 \in U\}$.
(2) Let $G_\mathcal{B} = \bigcap \mathcal{U}_{1,\mathcal{B}}$. *Then* $G_\mathcal{B} = \bigcap_{\Delta \in Inv} G_{\mathcal{B}_\Delta} = Stab(p)$.
(3) For $\Delta \in Inv$, p_Δ *is the only idempotent* $q \in S_{G,\Delta}(M)$ *with* $\mathcal{B}_\Delta = Im(d_q)$.
(4) p *is the only idempotent* $q \in S_G(M)$ *with* $\mathcal{B} = Im(d_q)$.

Proof. (1),(2) are easy. (3) If $q \in S_{G,\Delta}(M)$ is an idempotent with $\mathcal{B}_\Delta = \mathrm{Im}(d_q)$, then by Remark 3.6 there is an idempotent $q' \in S_G(M)$ extending q. By Lemma 3.9(5), q is the only generic type of $G_{\mathcal{B}_\Delta}$ in $S_{G,\Delta}(M)$, hence $q = p_\Delta$.

(4) A similar proof. □

In general, consider a group H and an H-algebra of sets $\mathcal{A}$. Let $\mathcal{B}$ be an H-subalgebra of $\mathcal{A}$. If there is a maximal subgroup S of $\mathrm{End}(\mathcal{A})$ with $\mathcal{B}$ being the common image of all $f \in S$, then $S = S_{K,\mathcal{B}}$, where K is the common kernel of all $f \in \mathcal{B}$. However in this situation there may be many H-ideals $K \subseteq \mathcal{A}$ yielding distinct groups $S_{K,\mathcal{B}}$ (although, for a fixed $\mathcal{B}$ these groups are isomorphic). This may happen also in the model-theoretic context, where $H = G(M)$, $\mathcal{A} = \mathrm{Def}_{G,ext}(M)$ and $S(\mathcal{A}) = S_{G,ext}(M)$, for example if there are no generic types in $S_{G,ext}(M)$.

However stable groups do have generic types. In the stable case Corollary 3.10 says more. Assume $S_{K,\mathcal{B}}$ is a maximal subgroup of $\mathrm{End}(\mathrm{Def}_G(M))$ (or $\mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$, where $\Delta \in Inv$). Then K in $S_{K,\mathcal{B}}$ is determined by $\mathcal{B}$. Namely, $K = \mathrm{Ker}(d_p)$ for the unique idempotent $p \in S_G(M)$ ($S_{G,\Delta}(M)$, respectively) with $\mathrm{Im}(d_p) = \mathcal{B}$.

Assume $\Delta \in Inv$. Now we are going to describe the maximal subgroups of $S_{G,\Delta}(M)$. So let $p \in S_{G,\Delta}(M)$ be an idempotent and let $\mathcal{B} = \mathrm{Im}(d_p)$ and $K = \mathrm{Ker}(d_p)$. Let S_p be the maximal subgroup of $S_{G,\Delta}(M)$ containing p. So

$$S_p = \{q \in S_{G,\Delta}(M) : d_q \in S_{K,\mathcal{B}}\}.$$

Let $X = cl(G(M)p)$ be the $G(M)$-subflow of $S_{G,\Delta}(M)$ generated by p. Let $G_\mathcal{B} = \mathrm{Stab}(p)$. So $G_\mathcal{B}$ is the atom of $\mathcal{B}$ containing 1. Notice that K and X are determined by each other as follows.

Lemma 3.11 ([N5, Lemma 1.9]) *Assume* $q \in S_{G,\Delta}(M)$. *Then for every* $U \in Def_{G,\Delta}(M)$ *we have that*

$$U \in Ker(d_q) \iff [U] \cap cl(G(M)q) = \emptyset$$

where $[U] = \{r \in S_{G,\Delta}(M) : U \in r\}$.

Proposition 3.12 *(1) $S_p = \{gp : g \in N_{G(M)}(G_{\mathcal{B}})\}$. In particular, S_p is definable in M^{eq} (as a group).*
(2) The function $f : N_{G(M)}(G_{\mathcal{B}}) \to S_p$ mapping g to gp is a definable group homomorphism, with kernel $G_{\mathcal{B}}$. Hence S_p is definably isomorphic to the quotient group $N_{G(M)}(G_{\mathcal{B}})/G_{\mathcal{B}}$.

Proof. (1) Assume $q \in S_p$. In particular, $K = \mathrm{Ker}(d_q) = \mathrm{Ker}(d_p)$. By Lemma 3.11, $cl(G(M)q) = cl(G(M)p) = X$, hence the orbit $G(M)q$ is dense in X. By Remark 2.5 and Lemma 2.6(2), there is a unique dense $G(M)$-orbit contained in X, hence $G(M)q = G(M)p$ and $q \in G(M)p$.

We see that the group S_p is contained in the set $\{gp : g \in G(M)\}$. Assume $g \in G(M)$. We identify g with $tp(g/M)$. Then $gp = g * p$. We shall prove that the following conditions are equivalent.

(a) $gp \in S_p$

(b) $\mathrm{Im}(d_{gp}) = \mathcal{B}$

(c) $g\mathcal{B}g^{-1} = \mathcal{B}$

(d) $g \in N_{G(M)}(G_{\mathcal{B}})$

$(a) \Leftrightarrow (b)$: Since for $g \in G(M)$ we have that $\mathrm{Ker}(d_{gp}) = \mathrm{Ker}(d_p) = K$, we see that (a) is equivalent to the conjunction of (b) and the statement that $d_{gp}|_{\mathcal{B}}$ permutes $\mathcal{B}$. So $(a) \Rightarrow (b)$ is clear.

For $(b) \Rightarrow (a)$ notice that the function $d_{gp}|_{\mathcal{B}}$ is the composition of the functions $d_p|_{\mathcal{B}}$ and $d_g|_{\mathcal{B}}$. The function $d_g : \mathrm{Def}_{G,\Delta}(M) \to \mathrm{Def}_{G,\Delta}(M)$ maps $U \in \mathrm{Def}_{G,\Delta}(M)$ to Ug^{-1}, hence it is a bijection. So if $\mathrm{Im}(d_{gp}) = \mathcal{B}$, then $d_{gp}|_{\mathcal{B}}$ permutes $\mathcal{B}$.

$(b) \Leftrightarrow (c)$: Since $d_{gp} = d_g \circ d_p$, we have that

$$\mathrm{Im}(d_{gp}) = d_g[\mathrm{Im}(d_p)] = d_g[\mathcal{B}] = \mathcal{B}g^{-1} = g\mathcal{B}g^{-1}.$$

The last equality holds since $\mathcal{B}$ is a $G(M)$-algebra.

$(c) \Leftrightarrow (d)$: By Lemmas 3.3 and 3.7, $\mathcal{B}$ is determined by the set of atoms $\mathrm{At}(\mathcal{B})$, and in turn $\mathrm{At}(\mathcal{B})$ is determined by $G_{\mathcal{B}}$ as the set of left cosets of $G_{\mathcal{B}}$ in $G(M)$. Likewise $g\mathcal{B}g^{-1}$ is determined by $gG_{\mathcal{B}}g^{-1}$. So we are done.

This proves the first clause of (1). The second clause is immediate.

(2) To see that f is a group homomorphism consider $g, h \in N_{G(M)}(G_{\mathcal{B}})$. Since p is the neutral element of S_p and $hp \in S_p$, we have that $p * hp = hp$. Hence

$$f(g) * f(h) = gp * hp = ghp = f(gh).$$

Since $G_{\mathcal{B}} = \mathrm{Stab}(p)$ we get that $G_{\mathcal{B}} = \mathrm{Ker}(f)$. □

By Lemma 3.9 the group $G_{\mathcal{B}}$ is Δ-definable and Δ-connected and p is the generic type of $G_{\mathcal{B}}$ in $S_{G,\Delta}(M)$. Proposition 3.12 shows that the maximal subgroup S_p of $S_{G,\Delta}(M)$ containing p consists of the left translates of p by the elements of

$N_{G(M)}(G_{\mathcal{B}})$. Conversely, if H is a Δ-definable Δ-connected subgroup of $G(M)$ and $N = N_{G(M)}(H)$, then the quotient group N/H is definably isomorphic to the maximal subgroup of $S_{G,\Delta}(M)$ containing the generic type $p_H \in S_{G,\Delta}(M)$ of H and consisting of the left translates of p_H by the elements of N/H.

Next we describe the maximal subgroups of $S_G(M)$. Assume $p \in S_G(M)$ is an idempotent and for all $\Delta \in Inv$ let $p_\Delta = p|_\Delta \in S_{G,\Delta}(M)$. Let S_p be the maximal subgroup of $S_G(M)$ containing p and S_{p_Δ} be the maximal subgroup of $S_{G,\Delta}(M)$ containing p_Δ. For $\Delta \in Inv$ let $H_\Delta = \mathrm{Stab}(p_\Delta)$ and $N_\Delta = N_{G(M)}(H_\Delta)$. Let $H = \mathrm{Stab}(p)$ and $N = \bigcap_{\Delta \in Inv} N_\Delta$. So $H = \bigcap_{\Delta \in Inv} H_\Delta$ and $N = N_{G(M)}(H)$.

Corollary 3.13 *(1) The group S_p is an inverse limit of the groups S_{p_Δ}.*
(2) Assume M is $|T|^+$-saturated. Then the function $g \mapsto gp$ is a group epimorphism $N \to S_p$ with kernel H, inducing a group isomorphism $N/H \to S_p$.
(3) Assume T is totally transcendental. Then for some $\Delta \in Inv$ we have that $N = N_\Delta$ and $H = H_\Delta$. Consequently the conclusion of (2) holds and $S_p \cong N/H \cong S_{p_\Delta}$.

Proof. (1) is easy. For (2) it is enough to prove that $S_p = \{gp : g \in N\}$.

$\supseteq$ is obvious, since by (1) $S_p = \mathrm{invlim}_{\Delta \in Inv} S_{p_\Delta}$ and by Proposition 3.12, $S_{p_\Delta} = \{gp_\Delta : g \in N_\Delta\}$.

For $\subseteq$ consider any $q \in S_p$ and for $\Delta \in Inv$ let $q_\Delta = q|_\Delta$. By (1), $q_\Delta \in S_{p_\Delta}$, hence there is a $g_\Delta \in N_\Delta$ with $q_\Delta = g_\Delta p_\Delta$. By the saturation of M we find $g \in N$ such that $q_\Delta = gp_\Delta$ for every $\Delta \in Inv$. Hence $q = gp$.

(3) If $\Delta_1, \Delta_2 \in Inv$ and $\Delta_1 \subseteq \Delta_2$, then $H_{\Delta_2} \subseteq H_{\Delta_1}$ and $N_{\Delta_2} \subseteq N_{\Delta_1}$. By the descending chain condition for definable groups in a totally transcendental theory we get a $\Delta \in Inv$ such that $H = H_\Delta$ and $N = N_\Delta$. By Proposition 3.12, $S_{p_\Delta} = \{gp_\Delta : g \in N_\Delta\}$. The types $gp_\Delta, g \in N_\Delta$, are the generic Δ-types of their H_Δ-cosets. They extend uniquely to the generic types in $S_G(M)$ of these cosets. So the restriction $S_p \to S_{p_\Delta}$ is an isomorphism. □

Notice that every connected type-definable subgroup H of $G(M)$ corresponds in this way to the group S_p, where $p \in S_G(M)$ is the generic type of H.

Earlier in this section we discussed when the restriction functions $S_p \to S_{p_\Delta}, \Delta \in Inv$, are surjective. In Lemma 3.5 we provided a criterion for this in the case where L is countable. Here we extend this result, using our description of the groups S_p and S_{p_Δ}. We keep the notation from Corollary 3.13

Lemma 3.14 *Assume T is totally transcendental or M is $|T|$-compact. Then the following conditions are equivalent.*
(1) The restriction functions $S_p \to S_{p_\Delta}$ are surjective for every $\Delta \in Inv$.
(2) The connecting maps of the inverse system of groups $(S_{p_\Delta})_{\Delta \in Inv}$ are surjective.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): If T is totally transcendental, then we are done by Corollary 3.13(3). Next, assume M is $|T|^+$-saturated. Let $\Delta_0 \in Inv$. We want to prove that the restriction function $S_p \to S_{p_{\Delta_0}}$ is surjective.

So let $q_{\Delta_0} \in S_{p_{\Delta_0}}$. Let $\kappa = |T|$. For $\Delta \in Inv$ let $N_\Delta = N_{G(M)}(\mathrm{Stab}(p_\Delta))$. By Proposition 3.12, $S_{p_\Delta} = \{gp_\Delta : g \in N_\Delta\}$.

Extend $\langle \Delta_0 \rangle$ to an increasing continuous sequence $\langle \Delta_\alpha, \alpha < \kappa \rangle$ of invariant subsets of L with $|\Delta_\alpha| < \kappa$ and $\bigcup_{\alpha<\kappa} \Delta_\alpha = \bigcup Inv$. For $\alpha < \kappa$ let $p_\alpha = p|_{\Delta_\alpha}$ and $N_\alpha = \bigcap \{N_\Delta : \Delta \in Inv \text{ and } \Delta \subseteq \Delta_\alpha\}$.

We find recursively elements $g_\alpha \in N_\alpha, \alpha < \kappa$, such that for every $\alpha < \kappa$

(∗) the types $g_\beta p_\beta, \beta \leq \alpha$, are compatible.

We begin the construction with a $g_0 \in N_0$ such that $q_{\Delta_0} = g_0 p_{\Delta_0}$. g_0 exists by Proposition 3.12.

Next suppose $\alpha < \kappa$ and for every $\alpha' < \alpha$ we have picked $g_{\alpha'}$ such that (∗) holds with α' in place of α. Let $\Phi(x)$ consist of the following formulas:

- $x \in N_\Delta$ (for all $\Delta \in Inv$ with $\Delta \subseteq \Delta_\alpha$),
- $g_\beta p_\Delta = x p_\Delta$ (for every $\beta < \alpha$ and $\Delta \in Inv$ with $\Delta \subseteq \Delta_\beta$).

We see that $\Phi(x)$ is a type over M^{eq} of power $< |T|$. By the compactness of M we find $g_\alpha \in N_\alpha$ realizing $\Phi(x)$. It is clear that (∗) holds.

Let $q(x)$ be the union of the types $g_\alpha p_\alpha, \alpha < \kappa$. By (∗) we have that $q|_\Delta \in S_{p_\Delta}$ for every $\Delta \in Inv$, hence $q \in S_p$. $q|_{\Delta_0} = q_{\Delta_0}$ by the choice of g_0. □

Lemmas 3.5 and 3.14 resemble the situation that occurred around two-cardinal theorem for stable theories. Assume $\varphi(x)$ is a non-algebraic formula of L. Recall that $\varphi(x)$ has Vaught property if there are models $M \prec N$ of T with $M \neq N$ and $\varphi(M) = \varphi(N)$. We say that a model M of T has the extension property (with respect to $\varphi(x)$) if there is a proper elementary extension N of M with $\varphi(M) = \varphi(N)$. Lachlan proved [La] that if $\varphi(x)$ has Vaught property, then every $M \models T$ has an extension property, provided L is countable (and T is stable). There was a question if this result really needs the countability assumption. Harnik removed the countability assumption from it, instead adding the assumption that M is $|T|$-compact. In [N1, N2] it was proved that it is consistent with $ZFC + \neg CH$ that the Lachlan's result holds for every superstable theory assuming $|L| < 2^{\aleph_0}$. The crucial point of the proof was a construction of locally isolated types and locally atomic models.

In Lemma 3.5 we also have a countability assumption, that is partially removed in Lemma 3.14 at the cost of assuming that M is $|T|$-compact. Is it consistent with $ZFC + \neg CH$ that Lemma 3.5 holds for a superstable T, assuming just that $|L| < 2^{\aleph_0}$?

Proposition 3.12 shows that the maximal subgroups of $S_{G,\Delta}(M), \Delta \in Inv$, are definable in M^{eq}. The next remark shows they are also definable in the pure semigroup $(S_{G,\Delta}(M), *)$.

Remark 3.15 *Let $\varphi(x, y)$ be the formula*

$$x * y = y * x = x \wedge \exists z(z * x = x * z = y)$$

Assume $p \in S_G(M)$ [or $p \in S_{G,\Delta}(M)$, where $\Delta \in Inv$] is an idempotent and $S \subseteq S_G(M)$ [$S \subseteq S_{G,\Delta}(M)$, respectively] is the maximal subgroup containing p. Then the formula $\varphi(x, p)$ defines S in the structure $(S_G(M), *)$ [$(S_{G,\Delta}(M), *)$, respectively].

In the next section we shall need the following lemma.

Lemma 3.16 *Assume $\Delta \in Inv$ and $\mathcal{B}$ is a $G(M)$-subalgebra of $Def_{G,\Delta}(M)$ such that the set*

$$S_{\mathcal{B}} = \{p \in S_{G,\Delta}(M) : \mathcal{B} = Im(d_p) \text{ and } d_p|_{\mathcal{B}} \text{ permutes } \mathcal{B}\}$$

is non-empty. Then $S_{\mathcal{B}}$ is a maximal subgroup of $Def_{G,\Delta}(M)$.

Proof. Let $p \in S_{\mathcal{B}}$ and let $K = \mathrm{Ker}(d_p)$. By the proof of Corollary 3.2 we have that

$$S_{K,\mathcal{B}} = \{f \in \mathrm{End}(\mathrm{Def}_{G,\Delta}(M)) : \mathrm{Ker}(f) = K, \mathrm{Im}(f) = \mathcal{B} \text{ and } f|_{\mathcal{B}} \text{ permutes } \mathcal{B}\}$$

is a maximal subgroup of $\mathrm{End}(\mathrm{Def}_{G,\Delta}(M))$ containing d_p, hence $S := d^{-1}[S_{K,\mathcal{B}}]$ is a maximal subgroup of $S_{G,\Delta}(M)$ containing p.

Since $p \in S_{\mathcal{B}}$ was arbitrary and by Corollary 3.10, K in $S_{K,\mathcal{B}}$ is determined by $\mathcal{B}$, we get that $S = S_{\mathcal{B}}$, that is K is the common kernel of the functions $d_p, p \in S_{\mathcal{B}}$. □

4 $*$-powers of types

In stability theory forking and local Morley ranks are the main tools to measure the size of types. In our context topological dynamics provides some additional tools. The largest types $p \in S_G(M)$ are the generic ones. They have the largest local Morley ranks and also the smallest orbits, meaning that for $p \in S_G(M)$, p is generic if and only if the $G(M)$-subflow $cl(G(M)p)$ is minimal.

So for a type $p \in S_G(M)$ the size of the set $cl(G(M)p)$ may indicate how large p is: the smaller $cl(G(M)p)$, the larger p. Notice that $cl(G(M)p)$ is determined by the kernel $\mathrm{Ker}(d_p)$:

$$cl(G(M)p) = \bigcup\{S_G(M) \cap [U^c] : U \in \mathrm{Ker}(d_p)\}.$$

So the size of $\mathrm{Ker}(d_p)$ is correlated with the size of p. Another object related to p is $\mathrm{Im}(d_p)$. Here the size of $\mathrm{Im}(d_p)$ is inversely correlated with the size of p (we explain it later).

The goal of this section is to compare the three ways of measuring $p \in S_G(M)$: by local Morley ranks, by the size of $\mathrm{Ker}(d_p)$ and by the size of $\mathrm{Im}(d_p)$. First we recall the fundamental lemma connecting forking and local Morley ranks. It appears in [Pi] as Lemmas I.3.4, I.3.6 and Corrolary I.3.5.

Lemma 4.1 *(1) Let $\Delta \subseteq L$ be finite, $A \subseteq B \subseteq \mathfrak{C}$, $q(x) \in S_\Delta(B)$ and $p(x) = q(x)|_A \in S_\Delta(A)$. Then q does not fork over A if and only if $RM_\Delta(q) = RM_\Delta(p)$.*
(2) Let $A \subseteq B$, $q(x) \in S(B)$, $p(x) = q(x)|_A$. Then q does not fork over A if and only if $RM_\Delta(p|_\Delta) = RM_\Delta(q|_\Delta)$ for co-finally many (equivalently: all) finite sets $\Delta \subseteq L$.
(3) With the hypotheses of (2), q does not fork over A if and only if $RM_\Delta(p) = RM_\Delta(q)$ for co-finally many (equivalently: all) finite sets $\Delta \subseteq L$.

For a type $p \in S_G(M)$ let

$$\vec{R}(p) = \langle RM_\Delta(p) : \Delta \in Inv \rangle \text{ and } \vec{R}'(p) = \langle RM_\Delta(p|_\Delta) : \Delta \in Inv \rangle.$$

For $p, q \in S_G(M)$ we write $\vec{R}(p) \leq \vec{R}(q)$ when $RM_\Delta(p) \leq RM_\Delta(q)$ for every $\Delta \in Inv$. The notation $\vec{R}'(p) \leq \vec{R}'(q)$ has an analogous meaning. The next lemma indicates that the independent multiplication of types $*$ increases the size of types. Its items are essentially proved in [N3] or follow from Lemma 4.1

Lemma 4.2 *Assume $p, q \in S_G(M)$.*
(1) $\vec{R}(p * q) \geq \vec{R}(p)$ and $\vec{R}(p * q) \geq \vec{R}(q)$.
(2) (1) holds also with $\vec{R}'$ in place of $\vec{R}$.
(3) The following conditions are equivalent
(a) $\vec{R}(p * q) = \vec{R}(q)$
(b) $\vec{R}'(p * q) = \vec{R}'(q)$
(c) For $a \models p$ and $b \models q$, $a \downarrow_M b$ implies $a \downarrow_M ab$.
(4) The following conditions are equivalent
(a) $\vec{R}(p * q) = \vec{R}(p)$
(b) $\vec{R}'(p * q) = \vec{R}'(p)$
(c) For $a \models p$ and $b \models q$, $a \downarrow_M b$ implies $ab \downarrow_M b$.

$*$ affects also the size of $\mathrm{Ker}(d_p)$ and $\mathrm{Im}(d_p)$ for $p \in S_G(M)$.

Remark 4.3 *Assume $p, q \in S_G(M)$ or $p, q \in S_{G,\Delta}(M)$, where $\Delta \in Inv$. Then $Ker(d_{p*q}) \supseteq Ker(d_q)$ and $Im(d_{p*q}) \subseteq Im(d_p)$.*

Proof. By Proposition 1.4, $d_{p*q} = d_p \circ d_q$. □

This remark justifies our claim above that the size of a type $p \in S_G(M)$ is inversely correlated with the size of $\mathrm{Im}(d_p)$. The next lemma relates the growth of ranks, kernels and images.

Lemma 4.4 *(1) Assume $p, q \in S_G(M)$.*
(a) If $\vec{R}(p * q) = \vec{R}(p)$, then $Im(d_{p*q}) = Im(d_p)$.
(b) If $\vec{R}(p * q) = \vec{R}(q)$, then $Ker(d_{p*q}) = Ker(d_q)$.
(2) Assume $p, q \in S_{G,\Delta}(M)$, where $\Delta \in Inv$.
(a) If $RM_\Delta(p * q) = RM_\Delta(p)$, then $Im(d_{p*q}) = Im(d_p)$.
(b) If $RM_\Delta(p * q) = RM_\Delta(q)$, then $Ker(d_{p*q}) = Ker(d_q)$.

Proof. (1)(a) Let $a, b \in \mathfrak{C}$ be independent realizations of p, q, respectively. Then $ab \models p * q$. Let $q^{-1} = tp(b^{-1}/M)$. By Lemma 4.2 we have that $ab \downarrow_M b$, hence $ab \downarrow_M b^{-1}$. So $a = (ab)b^{-1}$ realizes both p and $(p * q) * q^{-1}$. Hence $p = (p * q) * q^{-1}$. By Remark 4.3 it follows that

$$\mathrm{Im}(d_p) \supseteq \mathrm{Im}(d_{p*q}) \supseteq \mathrm{Im}(d_{p*q*q^{-1}}) = \mathrm{Im}(d_p),$$

hence all these inclusions are equalities and we are done.

(1)(b) A similar proof.

(2)(a) Choose $p', q' \in S_G(M)$ extending p, q, respectively. Let $a, b \in \mathfrak{C}$ be independent realizations of p', q', respectively. Let $(q')^{-1} = tp(b^{-1}/M)$ and $q^{-1} = tp_\Delta(b^{-1}/M)$. So $q^{-1} = (q')^{-1}|_\Delta$.

We have that ab realizes both $p * q$ and $p' * q'$. We claim that

$$p * q * q^{-1} = p.$$

To compute $p*q*q^{-1} = (p*q)*q^{-1}$ we pick a $b' \models q'$ with $b' \perp_M ab$. Then $ab(b')^{-1} \models p * q * q^{-1}$. Since $ab \perp_M b'$, we have that $RM_\Delta(tp_\Delta(ab/Mb')) = RM_\Delta(tp_\Delta(ab/M))$. Since $RM_\Delta(p * q) = RM_\Delta(p)$, we have that

$$RM_\Delta(tp_\Delta(ab/Mb)) = RM_\Delta(tp_\Delta(a/Mb)) = RM_\Delta(tp_\Delta(a/M)) = RM_\Delta(tp_\Delta(ab/M)).$$

Let $r_b = tp_\Delta(ab/Mb)$ and $r_{b'} = tp_\Delta(ab/Mb')$. Since both b and b' realize q' and $Mlt_\Delta(tp_\Delta(ab/M)) = 1$, we have that r_b and $r_{b'}$ are conjugate over M. Hence

$$p = tp_\Delta(abb^{-1}/M) = tp_\Delta(ab(b')^{-1}/M) = p * q * q^{-1}.$$

The rest is as in the proof of (1)(a).

(2)(b) A similar proof. □

Assume $p \in S_G(M)$ (or $p \in S_{G,\Delta}(M)$, where $\Delta \in Inv$). Consider the sequence of types $p^{*n} = p * \ldots * p$ (n-times), $n > 0$. By Lemma 4.2 we get a non-decreasing sequence of ranks $\vec{R}(p^{*n}), n > 0$ (or $RM_\Delta(p^{*n})$). By Remark 4.3 we get a non-decreasing sequence of kernels $\mathrm{Ker}(d_{p^{*n}})$ and non-increasing sequence of images $\mathrm{Im}(d_{p^{*n}})$. We are going to compare the growth properties of these three sequences. We will use the following lemma, which seems also to be of independent interest.

Lemma 4.5 *Let $\mathcal{B} = Im(d_p)$. If the function $d_p|_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ is 1-1, then it is "onto".*

Proof. First we assume that $p \in S_{G,\Delta}(M)$. Suppose $d_p|_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ is 1-1, but not "onto". Then the sequence of algebras $\mathrm{Im}(d_{p^{*n}}), n > 0$, is strictly decreasing. But the sequence of ranks $RM_\Delta(p^{*n}), n > 0$, eventually stabilizes (since $RM_\Delta(x = x)$ is finite), hence by Lemma 4.4 also the sequence $\mathrm{Im}(d_{p^{*n}})$ eventually stabilizes, a contradiction.

Now we deal with the case where $p \in S_G(M)$. If $d_p|_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ is 1-1, but not "onto", then for some $\Delta \in Inv$, some $U \in \mathcal{B} \cap \mathrm{Def}_{G,\Delta}(M)$ lies outside $\mathrm{Im}(d_p|_\mathcal{B})$. We consider $p_\Delta = p|_\Delta$. Let $\mathcal{B}_\Delta = \mathrm{Im}(d_{p_\Delta})$. Then $\mathcal{B}_\Delta = \mathcal{B} \cap \mathrm{Def}_{G,\Delta}(M)$ and $d_{p_\Delta} : \mathcal{B}_\Delta \to \mathcal{B}_\Delta$ is 1-1, but not "onto", contradicting the case, where $p \in S_{G,\Delta}(M)$. □

The next proposition shows that the sequences of kernels and images of the functions $d_{p^{*n}}, n > 0$, are strictly correlated. Later we shall see they are strictly correlated also to the sequence of ranks.

Proposition 4.6 *Assume $n > 0$. The following conditions are equivalent.*
(1) $Ker(d_{p^{*n}}) = Ker(d_{p^{*(n+1)}})$.
(2) $Ker(d_p) \cap Im(d_{p^{*n}}) = \{\emptyset\}$.

(3) Im($d_{p^{*n}}$) = Im($d_{p^{*(n+1)}}$).
If conditions (1)-(3) hold, then Ker($d_{p^{*n}}$) = Ker($d_{p^{*m}}$) and Im($d_{p^{*n}}$) = Im($d_{p^{*m}}$) for all $m \geq n$.

Proof. (1) $\Leftrightarrow$ (2) is trivial. For the rest of the proof let $q = p^{*n}$, $\mathcal{B} = \mathrm{Im}(d_q)$ and $K = \mathrm{Ker}(d_q)$. Notice that $d_q|_{\mathcal{B}} = (d_p|_{\mathcal{B}})^n$.

(2) $\Rightarrow$ (3). By (2), $d_p|_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is 1-1. Hence also $d_q|_{\mathcal{B}}$ is 1-1. It is enough to show that $d_p|_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is "onto". Suppose not. Then also $d_q|_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is not "onto". This contradicts Lemma 4.5.

(3) $\Rightarrow$ (2). Suppose (3) holds and (2) fails. Then also $\mathrm{Im}(d_{q^{*2}}) = \mathcal{B}$ and $K \cap \mathcal{B} \neq \{\emptyset\}$. It follows that for every $m > 0$, $\mathrm{Im}(d_{q^{*m}}) = \mathcal{B}$ and the sequence $\mathrm{Ker}(d_{q^{*m}}), m > 0$, is strictly increasing.

Case 1. $p \in S_{G,\Delta}(M)$. Then $q \in S_{G,\Delta}(M)$ and by Lemma 4.4, the sequence of kernels $\mathrm{Ker}(d_{q^{*m}}), m > 0$, is eventually constant (similarly as in the proof of Lemma 4.5), a contradiction.

Case 2. $p \in S_G(M)$. Then $q \in S_G(M)$. Choose a non-empty $U \in K \cap \mathcal{B}$. $U \in \mathrm{Def}_{G,\Delta}(M)$ for some $\Delta \in Inv$. Let $q_\Delta = q|_\Delta$. We see that $\mathrm{Im}(d_{q_\Delta}) = \mathrm{Im}(d_{q_\Delta^{*2}})$ and $U \in \mathrm{Ker}(d_{q_\Delta}) \cap \mathrm{Im}(d_{q_\Delta})$, contradicting (3) $\Rightarrow$ (2) in the case where $p \in S_{G,\Delta}(M)$ and $n = 1$.

The last clause of the proposition is easy. □

Now we focus our attention on the case where $p \in S_{G,\Delta}(M)$ and $\Delta \in Inv$.

Theorem 4.7 *Assume $\Delta \in Inv$, $p \in S_{G,\Delta}(M)$ and $n_0 > 0$ is minimal such that Ker($d_{p^{*n_0}}$) = Ker($d_{p^{*(n_0+1)}}$). Then there is a Δ-definable Δ-connected subgroup H of $G(M)$ such that the types $p^{*n}, n \geq n_0$, belong to the maximal subgroup S of $S_{G,\Delta}(M)$ containing the generic type $q \in S_{G,\Delta}(M)$ of H. In particular, the types $p^{*n}, n \geq n_0$, are of the form gq, where $g \in N_{G(M)}(H)$.*

Proof. Let $\mathcal{B} = \mathrm{Im}(d_{p^{*n_0}})$. By Proposition 4.6, for every $n \geq n_0$ we have that $\mathcal{B} = \mathrm{Im}(d_{p^{*n}})$ and $d_{p^{*n}}|_{\mathcal{B}}$ permutes $\mathcal{B}$. Hence by Lemma 3.16 all such types p^{*n} belong to the maximal subgroup $S_{\mathcal{B}}$ of $S_{G,\Delta}(M)$. Our theorem follows from Proposition 3.12.
□

In particular, if $RM_\Delta(G(M)) = d$, then for every $p \in S_{G,\Delta}(M)$ and every $m \geq d$ we have that p^{*m} belongs to a maximal subgroup of $S_{G,\Delta}(M)$, the same one that contains p^{*d}.

Corollary 4.8 *Assume $n > 0$ and $p \in S_{G,\Delta}(M)$, where $\Delta \in Inv$. The following conditions are equivalent.*
(1) Ker($d_{p^{*n}}$) = Ker($d_{p^{*(n+1)}}$).
(2) $RM_\Delta(p^{*n}) = RM_\Delta(p^{*(n+1)})$.
In particular, if (1) and (2) hold, then $RM_\Delta(p^{*m}) = RM_\Delta(p^{*m})$ for all $m \geq n$.

Proof. (2) $\Rightarrow$ (1) follows by Lemma 4.4.

(1) $\Rightarrow$ (2). If $\mathrm{Ker}(d_{p^{*n}}) = \mathrm{Ker}(d_{p^{*(n+1)}})$, then by Theorem 4.7 the types p^{*n} and $p^{*(n+1)}$ are both left translates of the generic type $q \in S_{G,\Delta}(M)$ of some Δ-definable Δ-connected subgroup H of $G(M)$. Hence $RM_\Delta(p^{*n}) = RM_\Delta(p^{*(n+1)})$.

The last clause of the corollary is easy. □

Corollary 4.8 show that also the sequence of ranks of $p^{*n}, n > 0$, is strictly correlated to the sequences of kernels and images of $d_{p^{*n}}$.

Corollary 4.9 *Assume $\Delta \in Inv$, $p \in S_{G,\Delta}(M)$ [or $p \in S_G(M)$], $n > 0$ and p^{*n} belongs to a maximal subgroup S of $S_{G,\Delta}(M)$ [$S_G(M)$, respectively]. Then $p^{*m} \in S$ for every $m \geq n$.*

In the case where $p \in S_G(M)$ we get a sequence of Δ-definable Δ-connected subgroups H_Δ of $G(M)$ such that for every $\Delta \in Inv$ the types $(p|_\Delta)^{*n}$ eventually are left translates of the generic Δ-type of H_Δ. Let $H = \bigcap_{\Delta \in Inv} H_\Delta$. So H is a connected type-definable subgroup of $G(M)$ and the sequence $p^{*n}, n > 0$, "converges" to translates of the generic type q of H. Namely, for every $\Delta \in Inv$, eventually the types $(p|_\Delta)^{*n}$ are left translates of $q|_\Delta$. Hence we could say that, considering the operation of raising p to $*$-powers, the type p is pro-finitely steps away from a translate of a generic type of a subgroup of $G(M)$.

In the special case where $U(G)$ is finite, say $U(G) = d$, we get a real convergence: there is an $n \leq d$ such that for every $m \geq n$, p^{*m} is a left translate of q, provided M is $|T|^+$-saturated. This last fact essentially follows also from [Ko].

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